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EXISTENCE OF ANTI-PERIODIC BOUNDARY VALUE PROBLEM FOR A CLASS OF NONLINEAR HILFER FRACTIONAL LANGEVIN EQUATIONS

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Abstract. In this work, we study anti-periodic boundary value problem of Hilfer Langevin equations. Applying generalized Mittag-Leffler functions, we prove the equivalence between the boundary value problem under consideration and an integral equation. Moreover, we obtain existence of solutions for the above boundary value problem in a weighted space. An example is given to demonstrate the consistency with our theoretical results.

Keywords: fractional Langevin equation, Hilfer fractional derivative, anti-periodic problem, weighted space.

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1 Introduction

Fractional differential equations extend traditional calculus to arbitrary real orders, overcoming the limitations of integer-order models that only capture local and instantaneous dynamics. They offer an intrinsic mathematical framework for describing anomalous diffusion, viscoelastic behavior, and long-range dependence in various systems. Research in this domain enriches both theoretical analysis and computational techniques while serving as a vital link between mathematics and disciplines such as physics, engineering, and biology establishing itself as a fundamental approach for tackling complex system modeling challenges [1, 2, 3].

The evolution of fractional-order boundary value problems is deeply connected to fractional calculus. As a key branch, anti-periodic problems have advanced from foundational existence re-

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sults [4, 5, 6] to complex scenarios using nonlinear analysis. Research has expanded from classical Riemann-Liouville/Caputo operators to Hadamard and Langevin types, from single equations to coupled systems and inclusions, and from simple to multi-point/integral conditions. Recent integration with p-Laplace operators and fractional Langevin equations provides theoretical support for modeling memory-dependent systems with periodic characteristics in viscoelasticity and anomalous diffusion.

In 1908, Langevin first proposed the equation named after him to describe the random motion (Brownian motion) of particles suspended in a fluid [7, 8]. This equation has become a cornerstone of statistical physics [9]. With the development of fractional calculus theory, researchers have extended it to the fractional-order case. In 2010, researchers studied anti-periodic boundary value problems, which are significant for describing systems with periodic oscillation characteristics, such as certain circuits or vibration systems [10].

In this paper, we study the following anti-periodic boundary value problem for generalized Langevin equations with Hilfer derivatives:

$$\begin{cases} {}^{H}D_{a^+}^{\alpha_2,\beta_2}({}^{H}D_{a^+}^{\alpha_1,\beta_1}+\lambda)(x(t)-g(t,x(t)))=f(t,x(t)),\ t\in J:=(a,T], & (1.1)\\ (I_{a^+}^{1-\gamma_1}x)(a^+)+(I_{a^+}^{1-\gamma_1}x)(T)=0, & (1.2)\end{cases}$$

where $0<\alpha_2<\gamma_2<\alpha_1<\gamma_1<1$ and $\lambda>0$ is a constant, $0\le a<T$, $g(t,x(t))$ and $f(t,x(t))$ are given later. To the best of the authors' knowledge, studies on boundary value problems (BVPs) for equations involving Hilfer derivatives in weighted spaces are still scarce.

The structure of this paper is as follows. In Section 2, we review and prove some conclusions that will be used in the subsequent study. In Section 3, by applying fixed point theorems, we establish the existence of solutions for the problem under consideration. In Section 4, we present an example to verify the theoretical results.

2 Preliminaries

In this paper, we denote by $C[a,b]$ the space of continuous functions φ on $[a,b]$ with the norm $\|\varphi\|_C=\max\limits_{t\in[a,b]}|\varphi(t)|$, $C_\sigma[a,b]$ the weighted space:

$$C_\sigma[a,b]:=\{\varphi:(a,b]\to\mathbb{R};\ (t-a)^\sigma\varphi(t)\in C[a,b]\}\quad\text{and}\quad\|\varphi\|_{C_\sigma}=\max_{t\in[a,b]}\big|(t-a)^\sigma\varphi(t)\big|.$$

Clearly, when $\sigma=0$, $C_0[a,b]=C[a,b]$. We abbreviate $C[a,b]$, $C_\sigma[a,b]$ with $\mathcal{C}$, C_σ respectively.

Definition 2.1 *[3] Let $\alpha\in(0,1)$. The Riemann-Liouville fractional integral $I_{a^+}^{\alpha}\varphi$ and derivative ${}^{RL}D_{a^+}^{\alpha}\varphi$ are defined by*

$$(I_{a^+}^{\alpha}\varphi)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}\varphi(s)\,\mathrm{d}s,\quad t>a,$$
$$({}^{RL}D_{a^+}^{\alpha}\varphi)(t)=\frac{\mathrm{d}}{\mathrm{d}t}(I_{a^+}^{1-\alpha}\varphi)(t)=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t(t-s)^{-\alpha}\varphi(s)\,\mathrm{d}s,\quad t>a,$$

where $\Gamma(\cdot)$ is the Gamma function. In particular, $({}^{RL}D_{a^+}^{1}\varphi)(t)=(\frac{\mathrm{d}\varphi}{\mathrm{d}t})(t)$.

Lemma 2.2 *[3] Let $0 < \alpha < 1$ and $\omega > 0$. Then*

$$I^{\alpha}_{a^+}(t-a)^{\omega-1} = \frac{\Gamma(\omega)}{\Gamma(\alpha+\omega)}(t-a)^{\alpha+\omega-1}, \tag{2.1}$$

$$^{RL}D^{\alpha}_{a^+}(t-a)^{\omega-1} = \frac{\Gamma(\omega)}{\Gamma(\omega-\alpha)}(t-a)^{\omega-\alpha-1}. \tag{2.2}$$

Lemma 2.3 *[3] Let $\eta_1,\ \eta_2 > 0$, $0 \le \sigma < 1$. For $\varphi \in C_\sigma$, the following assertions are valid.*

$$\begin{aligned}(I^{\eta_1}_{a^+}I^{\eta_2}_{a^+}\varphi)(t) &= (I^{\eta_1+\eta_2}_{a^+}\varphi)(t),\\ (\,^{RL}D^{\eta_1}_{a^+}\,I^{\eta_1}_{a^+}\varphi)(t) &= \varphi(t),\\ (\,^{RL}D^{\eta_2}_{a^+}\,I^{\eta_1}_{a^+}\eta)(t) &= (\,I^{\eta_1-\eta_2}_{a^+}\varphi)(t), \quad \textit{for} \quad \eta_1 > \eta_2.\end{aligned}$$

Lemma 2.4 *[3] Let $\eta, \sigma > 0$. If $\sigma > \eta$ and $\varphi \in C_\sigma$, then $(I^{\eta}_{a^+}\varphi)(t) \in C_{\sigma-\eta}$.*

Lemma 2.5 *[3] Let $\eta > 0$, $\sigma \in [0,1)$ and $\sigma \le \eta$. If $\varphi \in C_\sigma$, then $(I^{\eta}_{a^+}\varphi)(t) \in \mathcal{C}$.*

Lemma 2.6 *Let $\eta \in (0,1)$ and $\eta > \sigma$. If $\varphi \in C_\sigma$, then $\big(I^{\eta}_{a^+}\varphi\big)(a^+) = \lim\limits_{t\to a^+}\big(I^{\eta}_{a^+}\varphi\big)(t) = 0$.*

Lemma 2.7 *[3] Let $0 < \alpha < 1$. Then $^{RL}D^{\alpha}_{a^+}\varphi(t) = 0$ holds if, and only if, $\varphi(t) = c(t-a)^{\alpha-1}$ with any $c \in \mathbb{R}$.*

In [11], the author presented the following modified Hilfer derivative.

Definition 2.8 *[11] The left-sided Hilfer fractional derivative of order $\alpha \in (0,1), \beta \in [0,1]$ of $\varphi(t)$ is defined by:*

$$^{H}D^{\alpha,\beta}_{a^+}\varphi(t) = \,^{RL}D^{1-\gamma+\alpha}_{a^+}[(I^{1-\gamma}_{a^+}\varphi)(t) - (I^{1-\gamma}_{a^+}\varphi)(a^+)],$$

where $\gamma = \alpha + \beta(1-\alpha)$.

Remark 2.9 *(i) When $\beta = 0$, $\gamma = \alpha$, then $^{H}D^{\alpha,0}_{a^+}\varphi(t) = \frac{\mathrm{d}}{\mathrm{d}t}[(I^{1-\alpha}_{a^+}\varphi)(t) - (I^{1-\alpha}_{a^+}\varphi)(a^+)] = (\,^{RL}D^{\alpha}_{a^+}\varphi)(t)$ is the Riemann-Liouville fractional derivative;*

(ii) When $\beta = 1$, $\gamma = 1$, then $^{H}D^{\alpha,1}_{a^+}\varphi(t) = \,^{RL}D^{\alpha}_{a^+}[\varphi(t) - \varphi(a^+)] = \,^{c}D^{\alpha}_{a^+}(t)$ is the Caputo derivative(see [3]).

Lemma 2.10 *Let $0 < \sigma < 1$, for $\varphi \in C_\sigma$,*

$$^{H}D^{\alpha,\beta}_{a^+}I^{\alpha}_{a^+}\varphi(t) = \varphi(t) - \frac{(t-a)^{\gamma-\alpha-1}}{\Gamma(\gamma-\alpha)}(I^{1-\gamma+\alpha}_{a^+}\varphi)(a^+).$$

If $\sigma < 1-\gamma+\alpha$, then $^{H}D^{\alpha,\beta}_{a^+}I^{\alpha}_{a^+}\varphi(t) = \varphi(t)$.

Proof. From Definition 2.8, Lemma 2.3 and Lemma 2.2, we have

$$\begin{aligned}^{H}D^{\alpha,\beta}_{a^+}I^{\alpha}_{a^+}\varphi(t) &= \,^{RL}D^{1-\gamma+\alpha}_{a^+}[(I^{1-\gamma+\alpha}_{a^+}\varphi)(t) - (I^{1-\gamma+\alpha}_{a^+}\varphi)(a^+)]\\ &= \varphi(t) - \frac{(t-a)^{\gamma-\alpha-1}}{\Gamma(\gamma-\alpha)}(I^{1-\gamma+\alpha}_{a^+}\varphi)(a^+).\end{aligned}$$

If $\sigma < 1-\gamma+\alpha$, by Lemma 2.6, one obtains $(I^{1-\gamma+\alpha}_{a^+}\varphi)(a^+) = 0$, hence the conclusion holds. □

Lemma 2.11 *Let $0 < \alpha < 1$ and $0 < \beta < 1$. Then ${}^{H}D_{a^+}^{\alpha,\beta}\varphi(t) = 0$ holds if, and only if, $\varphi(t) = c(t-a)^{\gamma-1}$ with any $c \in \mathbb{R}$.*

Proof. From Definition 2.8, we have

$$ {}^{RL}D_{a^+}^{1-\gamma+\alpha}[(I_{a^+}^{1-\gamma}\varphi)(t) - (I_{a^+}^{1-\gamma}\varphi)(a^+)] = 0.$$

By Lemma 2.7, for constant $\tilde{c} \in \mathbb{R}$, we obtain $(I_{a^+}^{1-\gamma}\varphi)(t) - (I_{a^+}^{1-\gamma}\varphi)(a^+) = \tilde{c}(t-a)^{-\gamma+\alpha}$. Let $t \to a^+$, one has $\tilde{c} = 0$, and hence $(I_{a^+}^{1-\gamma}\varphi)(t) = (I_{a^+}^{1-\gamma}\varphi)(a^+)$. Applying ${}^{RL}D_{a^+}^{1-\gamma}$ to this equality and using Lemmas 2.3 and 2.2, we obtain

$$\varphi(t) = \frac{(I_{a^+}^{1-\gamma}\varphi)(a^+)}{\Gamma(\gamma)}(t-a)^{\gamma-1}.$$

On the other hand, since $I_{a^+}^{1-\gamma}(t-a)^{\gamma-1} = \Gamma(\gamma)$, from Definition 2.8, ${}^{H}D_{a^+}^{\alpha,\beta}(t-a)^{\gamma-1} = 0$. □

Remark 2.12 *From the proof of Lemma 2.11, it can be seen that if ${}^{H}D_{a^+}^{\alpha,\beta}\varphi(t) = 0$, then $\varphi(t) = \frac{(I_{a^+}^{1-\gamma}\varphi)(a^+)}{\Gamma(\gamma)}(t-a)^{\gamma-1}$.*

We denote the weighted space

$$C_\sigma^{\alpha,\beta}[a,b] = \{y \in C_\sigma[a,b];\ {}^{H}D_{a^+}^{\alpha,\beta}y \in C_\sigma[a,b]\}, \quad 0 \le \sigma < 1-\gamma,\ \gamma = \alpha + \beta(1-\alpha),$$

with the norm $\|y\|_{C_\sigma^{\alpha,\beta}} = \|y\|_{C_\sigma} + \|{}^{H}D_{a^+}^{\alpha,\beta}y\|_{C_\sigma}$. $C_\sigma^{\alpha,\beta}[a,b] \subset C_\sigma$ is a Banach space. We abbreviate $C_\sigma^{\alpha,\beta}[a,b]$ with $C_\sigma^{\alpha,\beta}$. When $\sigma = 0$, $C_0^{\alpha,\beta}$ will be abbreviated to $C^{\alpha,\beta}$.

Theorem 2.13 *The space $C_{1-\gamma}^{\alpha,\beta}$ consists of those and only those functions $y(t)$, which can be represented in the form*

$$y(t) = I_{a^+}^{\alpha}\phi(t) + c(t-a)^{\gamma-1}, \tag{2.3}$$

where $\phi(t) \in C_{1-\gamma}$, $c = \frac{I_{a^+}^{1-\gamma}y(a^+)}{\Gamma(\gamma)}$.

Proof. If $y \in C_{1-\gamma}^{\alpha,\beta}$, then $y(t) \in C_{1-\gamma}$ and ${}^{H}D_{a^+}^{\alpha,\beta}y \in C_{1-\gamma}$. This implies that there exists a function $\phi(t) \in C_{1-\gamma}$ such that ${}^{H}D_{a^+}^{\alpha,\beta}y(t) = \phi(t)$. By Lemmas 2.10 and 2.11, one obtains (2.3). In fact, $c = \frac{I_{a^+}^{1-\gamma}y(a^+)}{\Gamma(\gamma)}$.

Conversely, if $y(t)$ satisfies (2.3), clearly, $y(t) \in C_{1-\gamma}$. Applying ${}^{H}D_{a^+}^{\alpha,\beta}$ on both sides of (2.3), by Lemmas 2.10 and 2.11, we obtain ${}^{H}D_{a^+}^{\alpha,\beta}y(t) = \phi(t) \in C_{1-\gamma}$. □

Lemma 2.14 *Let $a < t_1 < t_2 \le T$, $\theta_1 > 0$ and $\theta_2 \in (0,1)$. Then the following assertions hold.*

$$\int_{t_1}^{t_2}(t_2-s)^{\theta_1-1}(s-a)^{-\theta_2}\,\mathrm{d}s \to 0, \quad as \quad t_1 \to t_2, \tag{2.4}$$

$$\int_{a}^{t_1}|(t_1-s)^{\theta_1-1} - (t_2-s)^{\theta_1-1}|(s-a)^{-\theta_2}\,\mathrm{d}s \to 0, \quad as \quad t_1 \to t_2. \tag{2.5}$$

$$\Big|\int_{a}^{t_1}(t_1-s)^{\theta_1-1}(s-a)^{-\theta_2}\,\mathrm{d}s - \int_{a}^{t_2}(t_2-s)^{\theta_1-1}(s-a)^{-\theta_2}\,\mathrm{d}s\Big| \to 0, \quad as \quad t_1 \to t_2. \tag{2.6}$$

Proof. For $a < t_1 < t_2 < T$, we have

$$\int_{t_1}^{t_2}(t_2-s)^{\theta_1-1}(s-a)^{-\theta_2}\,\mathrm{d}s \le (t_1-a)^{-\theta_2}\int_{t_1}^{t_2}(t_2-s)^{\theta_1-1}\,\mathrm{d}s$$
$$\le \frac{(t_1-a)^{-\theta_2}}{\theta_1}(t_2-t_1)^{\theta_1} \to 0, \quad \text{as} \quad t_1 \to t_2. \tag{2.7}$$

It is not difficult to see that

$$\int_a^{t_1}|(t_1-s)^{\theta_1-1}-(t_2-s)^{\theta_1-1}|(s-a)^{-\theta_2}\,\mathrm{d}s$$
$$= \begin{cases} \displaystyle\int_a^{t_1}[(t_2-s)^{\theta_1-1}-(t_1-s)^{\theta_1-1}](s-a)^{-\theta_2}\,\mathrm{d}s, & \text{if } \theta_1 > 1, \\ \displaystyle\int_a^{t_1}[(t_1-s)^{\theta_1-1}-(t_2-s)^{\theta_1-1}](s-a)^{-\theta_2}\,\mathrm{d}s, & \text{if } 0 < \theta_1 \le 1. \end{cases}$$

If $0 < \theta_1 < 1$, by (2.7), we get

$$\int_a^{t_1}|(t_1-s)^{\theta_1-1}-(t_2-s)^{\theta_1-1}|(s-a)^{-\theta_2}\,\mathrm{d}s$$
$$= \int_a^{t_1}[(t_1-s)^{\theta_1-1}-(t_2-s)^{\theta_1-1}](s-a)^{-\theta_2}\,\mathrm{d}s$$
$$= \int_a^{t_1}(t_1-s)^{\theta_1-1}(s-a)^{-\theta_2}\,\mathrm{d}s - \int_a^{t_2}(t_2-s)^{\theta_1-1}(s-a)^{-\theta_2}\,\mathrm{d}s$$
$$+ \int_{t_1}^{t_2}(t_2-s)^{\theta_1-1}(s-a)^{-\theta_2}\,\mathrm{d}s$$
$$\le \frac{\Gamma(\theta_1)\Gamma(1-\theta_2)}{\Gamma(\theta_1+1-\theta_2)}[(t_1-a)^{\theta_1-\theta_2}-(t_2-a)^{\theta_1-\theta_2}] + \frac{(t_1-a)^{-\theta_2}}{\theta_1}(t_2-t_1)^{\theta_1}$$
$$\to 0, \quad \text{as} \quad t_1 \to t_2.$$

Similarly, for the case $\theta_1 > 1$, we can obtain the same conclusion.

Combing with (2.4) and (2.5), we deduce that

$$\Big|\int_a^{t_1}(t_1-s)^{\theta_1-1}(s-a)^{-\theta_2}\,\mathrm{d}s - \int_a^{t_2}(t_2-s)^{\theta_1-1}(s-a)^{-\theta_2}\,\mathrm{d}s\Big|$$
$$\le \int_a^{t_1}|(t_1-s)^{\theta_1-1}-(t_2-s)^{\theta_1-1}|(s-a)^{-\theta_2}\,\mathrm{d}s + \int_{t_1}^{t_2}(t_2-s)^{\theta_1-1}(s-a)^{-\theta_2}\,\mathrm{d}s$$
$$\to 0, \quad \text{as} \quad t_1 \to t_2.$$

Thus, it shows that (2.6) holds true. □

Definition 2.15 *[3, 12] Let $\rho_1, \rho_2 > 0$, $z \in \mathbb{C}$. The generalized Mittag-Leffler function E_{ρ_1,ρ_2} is defined by*

$$E_{\rho_1,\rho_2}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\rho_1 k+\rho_2)}.$$

In particular, when $\rho_2 = 1$, $E_{\rho_1,\rho_2}(z)$ becomes the classical Mittag-Leffler function $E_{\rho_1}(z)$.

Lemma 2.16 *[3, 12, 13] Let $\rho, \tilde{\sigma} \in (0,1), \lambda > 0$. Then E_ρ, $E_{\rho,\rho}$ and $E_{\rho,\rho+\tilde{\sigma}}$ are nonnegative and*

$$E_{\rho,\rho}(-\lambda t^\rho) \le \frac{1}{\Gamma(\rho)}, \qquad E_{\rho,\rho+\tilde{\sigma}}(-\lambda t^\rho) \le \frac{1}{\Gamma(\rho+\tilde{\sigma})}, \quad t \in [0,T].$$

Moreover, for $t_1, t_2 \in J$, $\tilde{\sigma} \in (0,1)$,

$$|E_{\rho,\rho+\tilde{\sigma}}(-\lambda t_2^\rho) - E_{\rho,\rho+\tilde{\sigma}}(-\lambda t_1^\rho)| := O(|t_2 - t_1|^\rho), \quad \text{as} \quad t_1 \to t_2. \tag{2.8}$$

We now recall the definition of the Prabhakar integral operator [14]:

$$\big(\mathbf{E}_{\rho_1,\rho_2;\lambda}\,\phi\big)(t) = \int_a^t (t-s)^{\rho_2-1} E_{\rho_1,\rho_2}[-\lambda(t-s)^{\rho_1}]\,\phi(s)\,\mathrm{d}s,$$

where $\rho_1, \rho_2, \lambda > 0$, $t > s > a$, and ϕ is a suitable integrable function.

Theorem 2.17 *[14] For $\rho_1, \rho_2, \omega > 0$ and $\lambda > 0$,*

$$\begin{aligned} I_{a^+}^{\omega}\big[(\mathbf{E}_{\rho_1,\rho_2;\lambda}\,\phi)(s)\big](t) &= (\mathbf{E}_{\rho_1,\rho_2+\omega;\lambda}\,\phi)(t) = (\mathbf{E}_{\rho_1,\rho_2;\lambda}\, I_{a^+}^{\omega}\phi)(t), \\ (\mathbf{E}_{\rho_1,\rho_2;\lambda}(s-a)^{\omega-1})(t) &= \Gamma(\omega)(t-a)^{\rho_2+\omega-1} E_{\rho_1,\rho_2+\omega}(-\lambda(t-a)^{\rho_1}), \end{aligned} \tag{2.9}$$

where ϕ is a suitable integrable function.

Theorem 2.18 *Let $\rho, \sigma \in (0,1)$, $\lambda > 0$ and $\phi(t) \in C_\sigma$. Then $(\mathbf{E}_{\rho,\rho+\sigma;\lambda}\,\phi)(t)$ has the following property:*

$$|\big(\mathbf{E}_{\rho,\rho+\sigma;\lambda}\,\phi\big)(t_1) - \big(\mathbf{E}_{\rho,\rho+\sigma;\lambda}\,\phi\big)(t_2)| \to 0, \quad \text{as} \quad t_1 \to t_2.$$

Proof. For $a < t_1 < t_2 \le T$, we have

$$\begin{aligned} &|\big(\mathbf{E}_{\rho,\rho+\sigma;\lambda}\,\phi\big)(t_1) - \big(\mathbf{E}_{\rho,\rho+\sigma;\lambda}\,\phi\big)(t_2)| \\ &\le \int_a^{t_1} \big|(t_1-s)^{\rho+\sigma-1} - (t_2-s)^{\rho+\sigma-1}\big| E_{\rho,\rho+\sigma}[-\lambda(t_1-s)^\rho]\,|\phi(s)|\,\mathrm{d}s \\ &\quad + \int_a^{t_1} (t_2-s)^{\rho+\sigma-1} |E_{\rho,\rho+\sigma}[-\lambda(t_1-s)^\rho] - E_{\rho,\rho+\sigma}[-\lambda(t_2-s)^\rho]\,|\phi(s)|\,\mathrm{d}s \\ &\quad + \int_{t_1}^{t_2} (t_2-s)^{\rho+\sigma-1} E_{\rho,\rho+\sigma}[-\lambda(t_2-s)^\rho]\,|\phi(s)|\,\mathrm{d}s \\ &\le \|\phi\|_{C_\sigma} \left[\frac{1}{\Gamma(\rho+\sigma)} \int_a^{t_1} |(t_1-s)^{\rho+\sigma-1} - (t_2-s)^{\rho+\sigma-1}| (s-a)^{-\sigma}\,\mathrm{d}s \right. \\ &\quad + \int_a^{t_1} (t_2-s)^{\rho+\sigma-1}(s-a)^{-\sigma}\,\mathrm{d}s \cdot O(|t_2-t_1|^\rho) \\ &\quad \left. + \frac{1}{\Gamma(\rho+\sigma)} \int_{t_1}^{t_2} (t_2-s)^{\rho+\sigma-1}(s-a)^{-\sigma}\,\mathrm{d}s \right]. \end{aligned} \tag{2.10}$$

Since

$$\int_a^{t_1} (t_2-s)^{\rho+\sigma-1}(s-a)^{-\sigma}\,\mathrm{d}s < \int_a^{t_2} (t_2-s)^{\rho+\sigma-1}(s-a)^{-\sigma}\,\mathrm{d}s = \frac{(t_2-a)^\rho \Gamma(\rho+\sigma)\Gamma(1-\sigma)}{\Gamma(\rho+1)},$$

and combining with (2.4) and (2.5), we deduce that the right-hand side of (2.10) tends to zero, as $t_1 \to t_2$. □

Theorem 2.19 (Krasnoselskii's fixed point theorem) *[15] Let Ω be a closed, convex, and nonempty subset of a Banach space X, and let A, B be operators such that:*

(i) $Au + Bv \in \Omega$ *whenever* $u, v \in \Omega$,

(ii) A *is compact and continuous,*

(iii) B *is a contraction mapping.*

Then there exists $z \in \Omega$ *such that* $z = Az + Bz$.

3 Existence result

We firstly recall the following Lemma.

Lemma 3.1 *[16, Theorem 3.1] For any $\mu \in \mathbb{R}$, $\alpha > 0$, we have the following results.*

(i) For any $h \in C_{1-\gamma}(J, \mathbb{R})$, the series $\sum_{k=0}^{\infty}(-\mu)^k I_{a^+}^{k\alpha} h(t)$ is convergent and the sum is given by:

$$\sum_{k=0}^{\infty}(-\mu)^k I_{a^+}^{k\alpha} h(t) = h(t) - \mu \int_a^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\mu(t-s)^{\alpha}\big] h(s)\,\mathrm{d}s, \qquad t \in J.$$

(ii) The operator $I + \mu I_{a^+}^{\alpha} : C_{1-\gamma}(J, \mathbb{R}) \to C_{1-\gamma}(J, \mathbb{R})$ is invertible and continuous, and satisfies:

$$\big(I + \mu I_{a^+}^{\alpha}\big)^{-1} h(t) = \sum_{k=0}^{\infty}(-\mu)^k I_{a^+}^{k\alpha} h(t), \qquad t \in J.$$

Theorem 3.2 *If for any $x(t) \in C_{1-\gamma_1}$, $f(t, x(t)) \in C_{1-\gamma_1}$ and $g(t, x(t)) \in C^{\alpha_1,\beta_1}$, then $x \in C_{1-\gamma_1}^{\alpha_1,\beta_1}$ satisfies (1.1) and (1.2) if and only if $x(t)$ satisfies the following equation*

$$\begin{aligned} x(t) &= g(t, x(t)) + (\mathbf{E}_{\alpha_1,\alpha_1+\alpha_2;\lambda} f)(t) \\ &\quad - (t-a)^{\gamma_1-1} E_{\alpha_1,\gamma_1}(-\lambda(t-a)^{\alpha_1}) \Big[\frac{(I_{a^+}^{1-\gamma_1} g)(T) + (\mathbf{E}_{\alpha_1,\alpha_1+\alpha_2-\gamma_1+1;\lambda} f)(T)}{1 + E_{\alpha_1}(-\lambda(T-a)^{\alpha_1})}\Big]. \end{aligned} \tag{3.1}$$

Proof. Let $x \in C_{1-\gamma_1}^{\alpha_1,\beta_1}$ satisfy (1.1) and (1.2). Since $1 - \gamma_1 < 1 - \gamma_2 + \alpha_2$, from Lemma 2.10, ${}^H D_{a^+}^{\alpha_2,\beta_2} I_{a^+}^{\alpha_2} f(t, x(t)) = f(t, x(t))$. Hence, (1.1) can be written as

$${}^H D_{a^+}^{\alpha_2,\beta_2}\big[({}^H D_{a^+}^{\alpha_1,\beta_1} + \lambda)(x(t) - g(t, x(t))) - I_{a^+}^{\alpha_2} f(t, x(t))\big] = 0. \tag{3.2}$$

Obviously, $({}^H D_{a^+}^{\alpha_1,\beta_1} + \lambda)(x(t) - g(t, x(t))) \in C_{1-\gamma_1}$. If $1 - \gamma_1 < \alpha_2$, by Lemma 2.5, $I_{a^+}^{\alpha_2} f(t, x(t)) \in \mathcal{C}$; if $1 - \gamma_1 > \alpha_2$, by Lemma 2.4, $I_{a^+}^{\alpha_2} f(t, x(t)) \in C_{1-\gamma_1-\alpha_2} \subset C_{1-\gamma_1}$. Therefore, we always have $I_{a^+}^{\alpha_2} f(t, x(t)) \in C_{1-\gamma_1}$. Thus,

$$I_{a^+}^{1-\gamma_2}\Big[({}^H D_{a^+}^{\alpha_1,\beta_1} + \lambda)(x(t) - g(t, x(t))) - I_{a^+}^{\alpha_2} f(t, x(t))\Big](a^+) = 0.$$

By Remark 2.12 and (3.2), we obtain

$$({}^{H}D_{a^+}^{\alpha_1,\beta_1}+\lambda)(x(t)-g(t,x(t)))-I_{a^+}^{\alpha_2}f(t,x(t))=0. \tag{3.3}$$

By Lemma 2.10, we have

$${}^{H}D_{a^+}^{\alpha_1,\beta_1}I_{a^+}^{\alpha_1}\big[(x(t)-g(t,x(t)))\big]=x(t)-g(t,x(t)),$$
$${}^{H}D_{a^+}^{\alpha_1,\beta_1}I_{a^+}^{\alpha_1+\alpha_2}f(t,x(t))=I_{a^+}^{\alpha_2}f(t,x(t)),$$

we can further write (3.3) as

$${}^{H}D_{a^+}^{\alpha_1,\beta_1}\Big\{x(t)-g(t,x(t))+\lambda I_{a^+}^{\alpha_1}\big[x(t)-g(t,x(t))\big]-I_{a^+}^{\alpha_1+\alpha_2}f(t,x(t))\Big\}=0.$$

From Lemma 2.11, we get

$$x(t)=g(t,x(t))-\lambda I_{a^+}^{\alpha_1}\big[x(t)-g(t,x(t))\big]+I_{a^+}^{\alpha_1+\alpha_2}f(t,x(t))+\frac{d}{\Gamma(\gamma_1)}(t-a)^{\gamma_1-1}. \tag{3.4}$$

Conversely, if $x(t)$ satisfies (3.4), then by Theorem 2.13,

$$x(t)-g(t,x(t))=-\lambda I_{a^+}^{\alpha_1}\big[x(t)-g(t,x(t))\big]+I_{a^+}^{\alpha_1+\alpha_2}f(t,x(t))+\frac{d}{\Gamma(\gamma_1)}(t-a)^{\gamma_1-1}\in C_{1-\gamma_1}^{\alpha_1,\beta_1},$$

and

$$[{}^{H}D_{a^+}^{\alpha_1,\beta_1}+\lambda](x(t)-g(t,x(t)))=I_{a^+}^{\alpha_2}f(t,x(t))\in C_{1-\gamma_1}^{\alpha_2,\beta_2}.$$

Applying ${}^{H}D_{a^+}^{\alpha_2,\beta_2}$ to the above equation, we get (1.1). Rearranging equation (3.4), we obtain

$$(I+\lambda I_{a^+}^{\alpha_1})(x(t)-g(t,x(t)))=I_{a^+}^{\alpha_1+\alpha_2}f(t,x(t))+\frac{d}{\Gamma(\gamma_1)}(t-a)^{\gamma_1-1}.$$

By Lemma 3.1 and Theorem 2.17, combing with

$$\begin{aligned}
&\sum_{k=0}^{\infty}(-\lambda)^k I_{a^+}^{k\alpha_1}(I_{a^+}^{\alpha_1+\alpha_2}f(\cdot,x(\cdot)))(t)\\
&=\int_a^t\sum_{k=0}^{\infty}\frac{(-\lambda)^k(t-s)^{(k+1)\alpha_1-1}}{\Gamma((k+1)\alpha_1)}(I_{a^+}^{\alpha_2}f(s,x(s)))\,\mathrm{d}s\\
&=\int_a^t(t-s)^{\alpha_1-1}\sum_{k=0}^{\infty}\frac{(-\lambda(t-s)^{\alpha_1})^k}{\Gamma(k\alpha_1+\alpha_1)}(I_{a^+}^{\alpha_2}f(s,x(s)))\,\mathrm{d}s\\
&=\int_a^t(t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(-\lambda(t-s)^{\alpha_1})(I_{a^+}^{\alpha_2}f(s,x(s)))\,\mathrm{d}s\\
&=(\mathbf{E}_{\alpha_1,\alpha_1+\alpha_2;\lambda}f)(t),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\Gamma(\gamma_1)}\sum_{k=0}^{\infty}(-\lambda)^k I_{a^+}^{k\alpha_1}((t-a)^{\gamma_1-1})\\
&=\frac{(t-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}+\frac{1}{\Gamma(\gamma_1)}\sum_{k=1}^{\infty}\frac{(-\lambda)^k}{\Gamma(k\alpha_1)}\int_a^t(t-s)^{k\alpha_1-1}(s-a)^{\gamma_1-1}\,\mathrm{d}s\\
&=\frac{(t-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}+\sum_{k=1}^{\infty}\frac{(-\lambda)^k(t-a)^{k\alpha_1+\gamma_1-1}}{\Gamma(k\alpha_1+\gamma_1)}\\
&=(t-a)^{\gamma_1-1}E_{\alpha_1,\gamma_1}(-\lambda(t-a)^{\alpha_1}),
\end{aligned}$$

we get

$$\begin{aligned} x(t) - g(t, x(t)) &= \sum_{k=0}^{\infty} (-\lambda)^k I_{a^+}^{k\alpha_1} \big[I_{a^+}^{\alpha_1+\alpha_2} f(t, x(t)) + \frac{d}{\Gamma(\gamma_1)} (t-a)^{\gamma_1 - 1} \big] \\ &= (\mathbf{E}_{\alpha_1, \alpha_1+\alpha_2; \lambda} f)(t) + d(t-a)^{\gamma_1-1} E_{\alpha_1, \gamma_1}(-\lambda(t-a)^{\alpha_1}). \end{aligned}$$

By (2.9), we have

$$I_{a^+}^{1-\gamma_1} [(t-a)^{\gamma_1-1} E_{\alpha_1, \gamma_1}(-\lambda(t-a)^{\alpha_1})] = E_{\alpha_1}(-\lambda(t-a)^{\alpha_1}).$$

Applying the boundary value condition (1.2) to (3.4), by Theorem 2.17 and the fact $E_{\alpha_1}(0) = 1$, we obtain

$$d = -\frac{(I_{a^+}^{1-\gamma_1} g)(T) + (\mathbf{E}_{\alpha_1, \alpha_1+\alpha_2-\gamma_1+1; \lambda} f)(T)}{1 + E_{\alpha_1}(-\lambda(T-a)^{\alpha_1})}.$$

Now we prove that (1.1) and (1.2) is equivalent to (3.1). □

Next, we will present an existence result of solutions to BVP (1.1), (1.2). To this end, we first make the following assumptions:

(H1) Let $f(t, \cdot) : \mathbb{R} \to \mathbb{R}$ be continuous for $t \in J$. For any $x(t) \in C_{1-\gamma_1}$, $f(t, x(t)) \in C_{1-\gamma_1}$, and there exists a non-negative function $l_f(t) \in \mathcal{C}$ such that

$$|f(t, x(t))| \le l_f(t) |x(t)|^{\theta}, \quad 0 < \theta < 1.$$

(H2) For any $x(t) \in C_{1-\gamma_1}$, $g(t, x(t)) \in C^{\alpha_1, \beta_1}$, and there exists a function $l_g(t) \in C([a, T], \mathbb{R}^+)$ such that

$$|g(t, x_1) - g(t, x_2)| \le l_g(t) \|x_1 - x_2\|_{1-\gamma_1}.$$

For convenience, we denote

$$\begin{aligned} \Lambda_1 &= (T-a)^{1-\gamma_1} \Big[1 + \frac{1}{\Gamma(\gamma_1)\Gamma(2-\gamma_1)[1 + E_{\alpha_1}(-\lambda(T-a)^{\alpha_1})]} \Big], \\ \Lambda_2 &= \|l_f\|_{\mathcal{C}} (T-a)^{\alpha_1+\alpha_2+(1-\theta)(1-\gamma_1)} \Gamma((\gamma_1-1)\theta+1) \Big[\frac{1}{\Gamma(\alpha_1+\alpha_2+(\gamma_1-1)\theta+1)} \\ &\quad + \frac{1}{\Gamma(\gamma_1)\Gamma(\alpha_1+\alpha_2+(1-\theta)(1-\gamma_1)+1)[1 + E_{\alpha_1}(-\lambda(T-a)^{\alpha_1})]} \Big]. \end{aligned}$$

Theorem 3.3 *Assume that (H1) and (H2) hold. If* $\Lambda_1 \|l_g\|_{\mathcal{C}} < 1$, *then* (1.1) *with* (1.2) *has at least one solution in* $C_{1-\gamma_1}^{\alpha_1, \beta_1}$.

Proof. We define an operator $\mathcal{T} : C_{1-\gamma_1} \to C_{1-\gamma_1}$ by

$$\begin{aligned} (\mathcal{T}x)(t) &= g(t, x(t)) + (\mathbf{E}_{\alpha_1, \alpha_1+\alpha_2; \lambda} f)(t) \\ &\quad - (t-a)^{\gamma_1-1} E_{\alpha_1, \gamma_1}(-\lambda(t-a)^{\alpha_1}) \Big[\frac{(I_{a^+}^{1-\gamma_1} g)(T) + (\mathbf{E}_{\alpha_1, \alpha_1+\alpha_2-\gamma_1+1; \lambda} f)(T)}{1 + E_{\alpha_1}(-\lambda(T-a)^{\alpha_1})} \Big]. \end{aligned}$$

It is evident that a fixed point of operator $\mathcal{T}$ is a solution of (3.1).

We decompose the operator $\mathcal{T}$ into two appropriate components, $\mathcal{T}_1$ and $\mathcal{T}_2$, defined on the closed ball $B_r = \{x \in C_{1-\gamma_1}; \|x\|_{C_{1-\gamma_1}} \leq r\}$ with some $r > 0$:

$$(\mathcal{T}_1 x)(t) = g(t, x(t)) - (t-a)^{\gamma_1 - 1} E_{\alpha_1,\gamma_1}(-\lambda(t-a)^{\alpha_1}) \Big[\frac{(I_{a^+}^{1-\gamma_1} g)(T)}{1 + E_{\alpha_1}(-\lambda(T-a)^{\alpha_1})} \Big],$$

$$(\mathcal{T}_2 x)(t) = (\mathbf{E}_{\alpha_1,\alpha_1+\alpha_2;\lambda} f)(t) - (t-a)^{\gamma_1 - 1} E_{\alpha_1,\gamma_1}(-\lambda(t-a)^{\alpha_1}) \Big[\frac{(\mathbf{E}_{\alpha_1,\alpha_1+\alpha_2-\gamma_1+1;\lambda} f)(T)}{1 + E_{\alpha_1}(-\lambda(T-a)^{\alpha_1})} \Big].$$

For $x \in B_r$, since

$$|g(t, x(t))| \leq |g(t, x(t)) - g(t, 0)| + |g(t, 0)| \leq r\|l_g\|_{\mathcal{C}} + M_g,$$

where $M_g := \sup_{t \in [a,T]} |g(t, 0)| < \infty$, we have

$$\begin{aligned} \int_a^T \frac{(T-s)^{-\gamma_1}}{\Gamma(1-\gamma_1)} |g(s, x(s))| \, \mathrm{d}s &\leq \int_a^T \frac{(T-s)^{-\gamma_1}}{\Gamma(1-\gamma_1)} \, \mathrm{d}s \cdot [r\|l_g\|_{\mathcal{C}} + M_g] \\ &= \frac{(T-a)^{1-\gamma_1}}{\Gamma(2-\gamma_1)} [r\|l_g\|_{\mathcal{C}} + M_g]. \end{aligned} \tag{3.5}$$

Moreover, we can obtain the following estimate:

$$\begin{aligned} &|(\mathbf{E}_{\alpha_1,\alpha_1+\alpha_2;\lambda} f)(t)| \\ &\leq \int_a^t (t-s)^{\alpha_1+\alpha_2-1} E_{\alpha_1,\alpha_1+\alpha_2} \Big[-\lambda(t-s)^{\alpha_1} \Big] |f(s, x(s))| \, \mathrm{d}s \\ &\leq r^\theta \|l_f\|_{\mathcal{C}} \int_a^t (t-s)^{\alpha_1+\alpha_2-1} E_{\alpha_1,\alpha_1+\alpha_2} \Big[-\lambda(t-s)^{\alpha_1} \Big] (s-a)^{(\gamma_1-1)\theta} \, \mathrm{d}s \\ &\leq \frac{r^\theta \|l_f\|_{\mathcal{C}}}{\Gamma(\alpha_1+\alpha_2)} \int_a^t (t-s)^{\alpha_1+\alpha_2-1} (s-a)^{(\gamma_1-1)\theta} \, \mathrm{d}s \\ &= r^\theta \frac{\|l_f\|_{\mathcal{C}} \Gamma((\gamma_1-1)\theta+1)(t-a)^{\alpha_1+\alpha_2+(\gamma_1-1)\theta}}{\Gamma(\alpha_1+\alpha_2+(\gamma_1-1)\theta+1)}. \end{aligned}$$

Similarly, we have

$$|(\mathbf{E}_{\alpha_1,\alpha_1+\alpha_2-\gamma_1+1;\lambda} f)(T)| \leq r^\theta \frac{\|l_f\|_{\mathcal{C}} \Gamma((\gamma_1-1)\theta+1)(T-a)^{\alpha_1+\alpha_2+(1-\theta)(1-\gamma_1)}}{\Gamma(\alpha_1+\alpha_2+(1-\theta)(1-\gamma_1)+1)}. \tag{3.6}$$

Thus

$$(t-a)^{1-\gamma_1} |(\mathcal{T}_1 x)(t) + (\mathcal{T}_2 x)(t)| \quad \leq \quad r\Lambda_1 \|l_g\|_{\mathcal{C}} + r^\theta \Lambda_2 + \Lambda_1 M_g,$$

which implies that $\|(\mathcal{T}_1 x) + (\mathcal{T}_2 x)\| \leq r$. If not, for each positive number $r > 0$, there exists $x(t) \in B_r$, for some $t_0 \in (a, T]$ such that

$$(t_0 - a)^{1-\gamma_1} |(\mathcal{T} x)(t_0)| > r,$$

this means, $\Lambda_1 \|l_g\|_{\mathcal{C}} + r^{\theta-1}\Lambda_2 + \frac{\Lambda_1 M_g}{r} > 1$. Let $r \to +\infty$, we get $\Lambda_1 \|l_g\|_{\mathcal{C}} \geq 1$, this contradicts $\Lambda_1 \|l_g\|_{\mathcal{C}} < 1$. Therefore, $\mathcal{T} x \in B_r$.

For $x, \tilde{x} \in C_{1-\gamma_1}$, we get

$$\begin{aligned}
&(t-a)^{1-\gamma_1}|(\mathcal{T}_1 x)(t) - (\mathcal{T}_1 \tilde{x})(t)| \\
&\le (t-a)^{1-\gamma_1}|g(t,x(t)) - g(t,\tilde{x}(t))| \\
&\quad + \frac{1}{\Gamma(\gamma_1)\Gamma(1-\gamma_1)[1+E_{\alpha_1}(-\lambda(T-a)^{\alpha_1})]} \int_a^T (T-s)^{-\gamma_1}|g(s,x(s)) - g(s,\tilde{x}(s))|\Big] \,\mathrm{d}s \\
&\le \Lambda_1 \|l_g\|_{\mathcal{C}} \|x - \widetilde{x}\|_{C_{1-\gamma_1}},
\end{aligned}$$

hence $\mathcal{T}_1$ is a contraction mapping.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ in B_r, then there exists $\varepsilon > 0$, such that $\|x_n - x\|_{C_{1-\gamma_1}} < \varepsilon$ for n sufficiently large. Thus

$$|f(t,x_n(t)) - f(t,x(t))| \le \|l_f\|_{\mathcal{C}} (t-a)^{(\gamma_1-1)\theta}\big(\varepsilon^\theta + 2r^\theta\big),$$

and

$$\begin{aligned}
&(t-a)^{1-\gamma_1}|(\mathcal{T}_2 x_n)(t) - (\mathcal{T}_2 x)(t)| \\
&\le (t-a)^{1-\gamma_1} \int_a^t (t-s)^{\alpha_1+\alpha_2-1} E_{\alpha_1,\alpha_1+\alpha_2}\Big[-\lambda(t-s)^{\alpha_1}\Big]|f(s,x_n(s)) - f(s,x(s))| \,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\gamma_1)[1+E_{\alpha_1}(-\lambda(T-a)^{\alpha_1})]} \\
&\quad \times \int_a^T (T-s)^{\alpha_1+\alpha_2-\gamma_1} E_{\alpha_1,\alpha_1+\alpha_2-\gamma_1+1}(-\lambda(T-s)^{\alpha_1})|f(s,x_n(s)) - f(s,x(s))| \,\mathrm{d}s \\
&\le \Lambda_2\big(\varepsilon^\theta + 2r^\theta\big).
\end{aligned}$$

Using (H1), for $t \in J$, we obtain $|f(s,x_n(s)) - f(s,x(s))| \to 0$ as $n \to \infty$, and hence $\|(\mathcal{T}_2 x_n) - (\mathcal{T}_2 x)\|_{C_{1-\gamma_1}} \to 0$ as $n \to \infty$. This yields the continuity of $\mathcal{T}_2$.

Next, we prove the equicontinuity of $\mathcal{T}_2$. For any $t_1, t_2 \in J$, $t_1 < t_2$ and $x \in B_r$, according to Theorem 2.18, we have

$$|(\mathbf{E}_{\alpha_1,\alpha_1+\alpha_2;\lambda} f)(t_1) - (\mathbf{E}_{\alpha_1,\alpha_1+\alpha_2;\lambda} f)(t_2)| \to 0, \quad \text{as} \quad t_1 \to t_2. \tag{3.7}$$

By Lemma 2.16 and (2.8), we get

$$\begin{aligned}
&(t_2-a)^{1-\gamma_1}|(t_1-a)^{\gamma_1-1}E_{\alpha_1,\gamma_1}(-\lambda(t_1-a)^{\alpha_1}) - (t_2-a)^{\gamma_1-1}E_{\alpha_1,\gamma_1}(-\lambda(t_2-a)^{\alpha_1})| \\
&\le (t_2-a)^{1-\gamma_1}\Bigg[|(t_1-a)^{\gamma_1-1} - (t_2-a)^{\gamma_1-1}| \cdot |E_{\alpha_1,\gamma_1}(-\lambda(t_1-a)^{\alpha_1})| \\
&\quad + (t_2-a)^{\gamma_1-1}|E_{\alpha_1,\gamma_1}(-\lambda(t_1-a)^{\alpha_1}) - E_{\alpha_1,\gamma_1}(-\lambda(t_2-a)^{\alpha_1})|\Bigg] \\
&\le \frac{1}{\Gamma(\gamma_1)}\Big|\Big(\frac{t_2-a}{t_1-a}\Big)^{1-\gamma_1} - 1\Big| + |E_{\alpha_1,\gamma_1}(-\lambda(t_1-a)^{\alpha_1}) - E_{\alpha_1,\gamma_1}(-\lambda(t_2-a)^{\alpha_1})| \\
&\to 0, \quad \text{as} \quad t_1 \to t_2.
\end{aligned} \tag{3.8}$$

Combining with (3.7), (3.8) and (3.6), we have

$$(t_2-a)^{1-\gamma_1}|(\mathcal{T}_2 x)(t_1) - (\mathcal{T}_2 x)(t_2)| \to 0, \quad \text{as} \quad t_1 \to t_2.$$

Thus, we have

$$\begin{aligned}
&|(t_1-a)^{1-\gamma_1}(\mathcal{T}_2x)(t_1)-(t_2-a)^{1-\gamma_1}(\mathcal{T}_2x)(t_2)|\\
&\le |1-\Big(\frac{t_2-a}{t_1-a}\Big)^{1-\gamma_1}||(t_1-a)^{1-\gamma_1}(\mathcal{T}_2x)(t_1)|+(t_2-a)^{1-\gamma_1}|(\mathcal{T}_2x)(t_1)-(\mathcal{T}_2x)(t_2)|\\
&\to 0, \quad \text{for} \quad t_1\to t_2.
\end{aligned}$$

Therefore, $\mathcal{T}_2$ is completely continuous which follows from the Arzelà-Ascoli theorem. Now, Theorem 2.19 shows that $\mathcal{T}$ has a fixed point in B_r which is a solution of (1.1) and (1.2). □

4 Application

Example 4.1 *We consider the following BVP:*

$$\begin{cases} {}^{H}D_{0^+}^{\frac{1}{3},\frac{1}{5}}({}^{H}D_{0^+}^{\frac{1}{2},\frac{1}{4}}+\frac{1}{6})\Big[x(t)-I_{0^+}^{\frac{1}{2}}(\frac{\sqrt{t}}{2}x(t))\Big]=2\sqrt[4]{t}x^{\frac{1}{3}}(t),\ t\in(0,1],\\ (I_{0^+}^{\frac{3}{8}}x)(0^+)+(I_{0^+}^{\frac{3}{8}}x)(1)=0. \end{cases} \tag{4.1}$$

When we set

$$\begin{aligned}
\alpha_1=\frac{1}{2},\ \alpha_2=\frac{1}{3},\ \beta_1=\frac{1}{4},\ \beta_2=\frac{1}{5},\ \lambda=\frac{1}{6},\ a=0,\ T=1,\\
g(t,x(t))=I_{0^+}^{\frac{1}{2}}(\frac{\sqrt{t}}{2}x(t)),\ f(t,x(t))=2\sqrt[4]{t}x^{\frac{1}{3}}(t),
\end{aligned}$$

we are able to rewrite (4.1) *as* (1.1) *with* (1.2) *and get* $\gamma_1=\frac{5}{8}$. *It is easy to verify that*

$$\begin{aligned}
|g(t,x(t))-g(t,\bar{x}(t))| &= I_{0^+}^{\frac{1}{2}}(\frac{\sqrt{t}}{2}|x(t)-\bar{x}(t)|)\\
&\le \frac{1}{2\Gamma(\frac{1}{2})}\int_0^t(t-s)^{-\frac{1}{2}}s^{\frac{1}{8}}ds\|x-\bar{x}\|_{C_{\frac{3}{8}}},\\
&= \frac{\Gamma(\frac{9}{8})t^{\frac{5}{8}}}{2\Gamma(\frac{13}{8})}\|x-\bar{x}\|_{C_{\frac{3}{8}}},\\
|f(t,x(t))| &\le 2|x(t)|^{\frac{1}{3}},
\end{aligned}$$

and $\|l_g\|_{\mathcal{C}}=\frac{\Gamma(\frac{9}{8})}{2\Gamma(\frac{13}{8})}$. *Moreover, we obtain*

$$\Lambda_1\|l_g\|_{\mathcal{C}}=\frac{\Gamma(\frac{9}{8})}{2\Gamma(\frac{13}{8})}\Big[1+\frac{1}{\Gamma(\frac{5}{8})\Gamma(\frac{11}{8})[1+E_{\frac{1}{2}}(-\frac{1}{6})]}\Big]\approx 0.75<1.$$

Now, all the assumptions of Theorem 3.3 are satisfied, and hence there exists at least a solution to BVP (4.1).

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