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COMPLETIONS OF LOCALLY CONVEX SPACES AND ALMOST PERIODICITY

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Abstract. In this paper, we consider the completions of locally convex spaces, the completions of metric linear spaces and present several new structural results concerning the (Stepanov) ρ-almost periodic functions with values in locally convex spaces. We formulate and prove the Bochner criterion for the (Stepanov) almost periodic functions with values in non-complete locally convex spaces.

Keywords: ρ-almost periodic functions, Stepanov ρ-almost periodic functions, locally convex spaces, metric linear spaces, completions.

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1 Introduction and preliminaries

If Y is a locally convex space and $\mathcal{W}$ is a base of neighbourhoods of zero in Y, then a continuous function $F : \mathbb{R}^n \to Y$ is said to be almost periodic if, for every $W \in \mathcal{W}$, there exists a number $l > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists a point $\tau \in B(\mathbf{t}_0, l)$ such that $F(t+\tau) - F(t) \in W$ for all $\mathbf{t} \in \mathbb{R}^n$. If this is the case, the range of $F(\cdot)$ is totally bounded in Y and $F(\cdot)$ is uniformly continuous; the space of all almost periodic functions, denoted by $AP(\mathbb{R}^n : Y)$, is translation invariant, closed under uniform convergence and closed under reflexions at zero (cf. [7] and [8] for some pioneering results established by G. M. N'Guérékata in this direction, the research article [2] by D. Bugajewski and G. M. N'Guérékata for more details about almost periodic functions with values in Fréchet space, and the research article [5] by S. Gal and G. M. N'Guérékata for more details about almost periodic functions with values in p-Fréchet spaces, where $0 < p < 1$). We have

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recently proved that $AP(\mathbb{R}^n : Y)$ is the vector space with the usual operations if Y is an arbitrary locally convex space ([12]); in the same paper, we have provided an illustrative application of the above-mentioned result to the integral equations in sequentially complete locally convex spaces. For more details concerning the almost periodic functions and their applications, we refer the reader to the research monographs [3, 6, 10, 14, 15] and references cited therein; *cf.* also the excellent survey article [1] by J. Andres, A. M. Bersani, and R. F. Grande.

The main aim of this research article is to continue the analysis raised in [12]. We provide several useful remarks concerning the completions of locally convex spaces and extend the well-known result of F. Y. Sayas [17], who considered the completions of normed spaces, to the metric linear spaces. We also provide several new structural results concerning the (Stepanov) ρ-almost periodic functions with values in locally convex spaces; in all established results, we connect the (Stepanov) ρ-almost periodic functions with values in non-complete locally convex spaces with the corresponding (Stepanov) ρ-almost periodic functions with values in the completions of these spaces. Concerning some results which have not been clarified in [12], we would like to note that all established results concerning the completions of locally convex spaces and the completions of metric linear spaces are new and not considered in [12], where we have only used the construction of a completion of a normed space given by F. Y. Sayas [17] for the corresponding local Banach spaces of Y; cf. [12, Theorem 2.1]. Also, all established results for Stepanov ρ-almost periodic type functions are completely new as well as the Bochner Criterion clarified in Theorem 2.7.

The organization of this paper is very simple. After recalling the necessary definitions of Bohr $(\mathcal{B}, I', \rho)$-almost periodic functions and the Stepanov-$(\Omega, p, \rho, \mathcal{B}, \Lambda', \nu)$-almost periodic functions, we briefly explain the notation used throughout the paper. Our main results concerning the completions of locally convex spaces, the completions of metric linear spaces and the (Stepanov) ρ-almost periodic functions with values in locally convex spaces are given in Section 2. In particular, we formulate and prove the Bochner criterion for the (Stepanov) almost periodic functions with values in non-complete locally convex spaces.

In our recent research article [4], we have considered several new classes of (metrically) ρ-almost periodic type functions $F : I \times X \to Y$, where $\emptyset \neq I \subseteq \mathbb{R}^n$, X is an arbitrary non-empty set and Y is a locally convex space. We need the following notion:

Definition 1.1 *([4]) Suppose that (C1) holds, where:*

(C1) $\emptyset \neq I \subseteq \mathbb{R}^n$, $\emptyset \neq I' \subseteq \mathbb{R}^n$, $I' + I \subseteq I$, *Y is a locally convex space over the field of complex numbers, the topology on Y is induced by the fundamental system $\circledast_Y$ of seminorms, ρ is a binary relation on Y, X is an arbitrary non-empty set and $\mathcal{B}$ is a collection of certain non-empty subsets of X such that for each $x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$.*

Let $F : I \times X \to Y$. Then we say that $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, I', \rho)$-almost periodic if, for every $\epsilon > 0$, $\kappa \in \circledast_Y$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap I'$ such that, for every $\mathbf{t} \in I$ and $x \in B$, there exists an element $y_{\mathbf{t};x} \in \rho(F(\mathbf{t}; x))$ such that

$$\kappa\big(F(\mathbf{t} + \tau; x) - y_{\mathbf{t};x}\big) \leq \epsilon, \quad \mathbf{t} \in I,\ x \in B.$$

If X is a topological space, then we say that $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, I', \rho)$-almost periodic if $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, I', \rho)$-almost periodic and continuous; furthermore, if $\rho = c\mathrm{I}$, where $c \in \mathbb{C} \setminus \{0\}$, then we say that $F(\cdot;\cdot)$ is (pre-)Bohr $(\mathcal{B}, I', c)$-almost periodic.

We omit the term "$\mathcal{B}$" from the notation if $X = \{0\}$, the term "ρ" from the notation if $\rho = \mathrm{I}$ and term "I'" from the notation if $I' = I$.

In our recent analysis of Stepanov ρ-almost periodic functions with values in locally convex spaces ([11]), we have introduced the following notion:

Definition 1.2 *Suppose that the following condition holds true:*

(SM): $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a Lebesgue measurable set such that $m(\Omega) > 0$, $\Lambda' + \Lambda \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$, $p > 0$, Y is a locally convex space over the field of complex numbers, the topology on Y is induced by the fundamental system $\circledast_Y$ of seminorms, ρ is a binary relation on Y, X is an arbitrary non-empty set, $\mathcal{B}$ is a collection of certain non-empty subsets of X such that for each $x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$, and $\nu \in L^p_{loc}(\Lambda : [0, \infty))$.

Then we say that a function $F : \Lambda \times X \to Y$ is Stepanov-$(\Omega, p, \rho, \mathcal{B}, \Lambda', \nu)$-almost periodic if, for every $\epsilon > 0$, $\kappa \in \circledast_Y$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $\mathbf{t} \in \Lambda$ and $x \in B$, the mapping $\mathbf{s} \mapsto G_x(\mathbf{s}) \in \rho(F(\mathbf{s}; x))$, $\mathbf{s} \in \mathbf{t} + \Omega$ is well defined, and

$$\int_{\mathbf{t}+\Omega} \Big[\kappa\Big(F(\mathbf{s} + \tau; x) - G_x(\mathbf{s})\Big)\Big]^p \nu^p(\mathbf{s})\, d\mathbf{s} \leq \epsilon, \quad \mathbf{t} \in \Lambda. \tag{1.1}$$

Suppose now that Y is a Fréchet space equipped with the calibration $\circledast_Y = (\|\cdot\|_k)_{k \in \mathbb{N}}$ of increasing seminorms, $1 \leq p \leq +\infty$ and $\nu(\mathbf{u}) > 0$ for a.e. $\mathbf{u} \in \Omega$. Then

$$L^p_\nu(\Omega : Y) := \Big\{ f : \Omega \to Y : \|f(\cdot)\|_k \in L^p_\nu(\Omega) \text{ for all } k \in \mathbb{N} \Big\}$$

is a vector space with the usual operations. In [11], we have proved that $L^p_\nu(\Omega : Y)$ is a Fréchet space with the family of seminorms $(k_{p,\nu})_{k \in \mathbb{N}}$, where $k_{p,\nu}(f) := \| \|f(\cdot)\|_k \|_{L^p_\nu(\Omega)}$ for all $f \in L^p_\nu(\Omega : Y)$ and $k \in \mathbb{N}$.

In the remainder of this paper, we will use the following notation:

Notation and preliminaries. We will always assume henceforth that Y is a Hausdorff locally convex space over the field of complex numbers. The abbreviation $\circledast_Y$ stands for the fundamental system of seminorms which defines the topology of Y and I denotes the identity operator on Y. If X is also a Hausdorff locally convex spaces over the field of complex numbers, then $L(X, Y)$ denotes the space consisting of all continuous linear mappings from X into Y, $L(X) \equiv L(X, X)$. By Y^* and Y' we denote the space of all linear functions $y : Y \to \mathbb{C}$ and the space of all continuous linear functions $y : Y \to \mathbb{C}$, respectively; we equip Y' with the strong topology. We refer the reader to [16, Definition, p. 32] for the notion of a metric linear space. The interested reader may consult [16], [18] and [9] for more details about topological vector spaces, locally convex spaces and abstract Volterra integro-differential equations in locally convex spaces.

2 Completions of locally convex spaces and connections with almost periodicity

If Y is a locally convex space, then a completion of Y is a pair $(\hat{Y}, K)$, where $\hat{Y}$ is a complete locally convex space, $K : Y \to \hat{Y}$ is a continuous linear injection, $K(Y)$ is dense in $\hat{Y}$ and

$K^{-1} : K(Y) \to Y$ is continuous; see [16, Definition, p. 258]. Every two completions of a locally convex space Y is linearly and topologically isomorphic; see *e.g.*, [16, Lemma 22.20].

If Y is a locally convex space, then a completion of Y always exists; the pair $(\hat{Y}, K)$, where $\hat{Y}$ is a complete locally convex space and $K(\cdot)$ satisfies the requirements given in the definition of completion, can be constructed as follows (see the proof of [16, Proposition 22.21]): Suppose that $\mathcal{U}$ is a fixed base of absolutely convex neighbourhoods of zero in Y. Define

$$Y'^{\times} := \Big\{x \in \big(Y'\big)^{*} : x_{|U^{\circ}} \text{ is bounded for all } U \in \mathcal{U}\Big\};$$

here, U° denotes the polar of U, which is defined through $U^{\circ} := \{y \in Y' : |y(x)| \leq 1 \text{ for all } x \in U\}$. If we define $p_U(x) := \sup_{y \in U^{\circ}} |x(y)|$, $x \in Y'^{\times}$, then $(p_U)_{U \in \mathcal{U}}$ is a fundamental system of seminorms which defines a topology on $Y'^{\times}$. Equipped with this calibration of seminorms, the locally convex space $Y'^{\times}$ is complete; furthermore, the first component of completion of Y is defined by $\hat{Y} := \overline{K(Y)}$, where the closure is taken in the space $Y'^{\times}$, and $K : Y \to Y'^{\times}$ is given by $[K(x)](y) := y(x)$, $x \in Y$, $y \in Y'$.

Suppose now that $T \in L(Y)$ is given. Then we define $T^{\times} : Y'^{\times} \to Y'^{\times}$ by $[T^{\times}A](y) := A(y \circ T)$, $y \in Y'$, $A \in Y'^{\times}$. It is clear that $T^{\times}(\cdot)$ is linear and well-defined; furthermore, if $U \in \mathcal{U}$, then we have

$$p_U\big(T^{\times}A\big) := \sup_{y \in U^{\circ}} |A(y \circ T)| \leq \sup_{z \in V^{\circ}} |A(z)|,$$

where $V := T^{-1}(U)$ is an absolutely convex neighbourhood of zero in Y, which simply implies that $T^{\times} \in L(Y'^{\times})$. Now we will prove that $T^{\times}(\hat{Y}) \subseteq \hat{Y}$. If $A \in \hat{Y}$ is given, then there exists a net $(A_\alpha)_{\alpha \in I}$ in $K(Y)$ such that $\lim_{\alpha \in I} A_\alpha = A$ in $Y'^{\times}$. Since $A_\alpha \in K(Y)$ for all $\alpha \in I$, we have the existence of a net $(x_\alpha)_{\alpha \in I}$ in Y such that $A_\alpha y = y(x_\alpha)$ for all $\alpha \in I$ and $y \in Y'$. This clearly implies that $[T^{\times}A_\alpha](y) = A_\alpha(y \circ T) = (y \circ T)(x_\alpha) = y(T(x_\alpha))$ for all $\alpha \in I$ and $y \in Y'$; in particular, $T^{\times}(K(y)) = K(Ty)$, $y \in Y$. It suffices to show that $\lim_{\alpha \in I} p_U(A_\alpha - A) = 0$ for all $U \in \mathcal{U}$, *i.e.*, that $\lim_{\alpha \in I} \sup_{y \in U^{\circ}} |(A_\alpha - A)(y \circ T)| = 0$. Let V be defined by $V := T^{-1}(U)$. Then we have $\sup_{y \in U^{\circ}} |(A_\alpha - A)(y \circ T)| \leq \sup_{z \in V^{\circ}} |(A_\alpha - A)(z)|$, which simply implies $T^{\times}(\hat{Y}) \subseteq \hat{Y}$ since $\lim_{\alpha \in I} A_\alpha = A$ in $Y'^{\times}$. Define $\hat{T} := (T^{\times})_{|\hat{Y}}$; by the foregoing, we have:

$$\hat{T} \in L\big(\hat{Y}\big) \text{ and } \hat{T}(Ky) = K(Ty), \quad y \in Y. \tag{2.1}$$

Remark 2.1 *(i) If Y_κ denotes the local Banach space for the seminorm $\kappa \in \circledast_Y$, then the mapping $K : Y \to \prod_{\kappa \in \circledast_Y} Y_\kappa$, given by $K(y) := (y + \{x \in Y : \kappa(x) = 0\})_{\kappa \in \circledast_Y}$, is linear, continuous and injective, and the inverse mapping $K^{-1} : K(Y) \to Y$ is continuous. The completion $\hat{Y}$ can be also described as the closure of $K(Y)$ in $\prod_{\kappa \in \circledast_Y} Y_\kappa$; see [16, pp. 279-280] for more details.*

(ii) Suppose that Y is a locally convex space, $(A, \leq)$ is a fixed directed set, $c(Y)$ is the vector space of all Cauchy nets $(x_\alpha)_{\alpha \in A}$ in Y and $c_0(Y)$ is the vector space of all nets $(x_\alpha)_{\alpha \in A}$ in Y such that $\lim_{\alpha \in A} x_\alpha = 0$. Following F. J. Sayas [17], we introduce the following vector space $Z := \{(x_\alpha)_{\alpha \in A} + c_0(Y) : (x_\alpha)_{\alpha \in A} \in c(Y)\}$, where we identify the classes $(x_\alpha)_{\alpha \in A} + c_0(Y)$ and $(y_\alpha)_{\alpha \in A} + c_0(Y)$ if $\lim_{\alpha \in A}(x_\alpha - y_\alpha) = 0$. It can be simply proved that the operations of addition

$$\Big[(x_\alpha)_{\alpha \in A} + c_0(Y)\Big] + \Big[(y_\alpha)_{\alpha \in A} + c_0(Y)\Big] := \Big[(x_\alpha + y_\alpha)_{\alpha \in A} + c_0(Y)\Big],$$

for any $(x_\alpha)_{\alpha\in A} \in c(Y)$ *and* $(y_\alpha)_{\alpha\in A} \in c(Y)$, *and the multiplication with scalars*

$$\lambda \cdot \Big[(x_\alpha)_{\alpha\in A} + c_0(Y)\Big] := \Big[(\lambda \cdot x_\alpha)_{\alpha\in A} + c_0(Y)\Big], \quad \lambda \in \mathbb{C},\ (x_\alpha)_{\alpha\in A} \in c(Y),$$

are well-defined as well as that Z *is a vector space when equipped with these operations. Set*

$$\kappa_Z\Big((x_\alpha)_{\alpha\in A} + c_0(Y)\Big) := \lim_{\alpha\in A} \kappa\big(x_\alpha\big), \quad (x_\alpha)_{\alpha\in A} \in c(Y),\ \kappa \in \circledast_Y.$$

Then it can be simply proved that for each $\kappa \in \circledast_Y$ *the mapping* $\kappa_Z(\cdot)$ *is a well-defined seminorm on* Z *as well as that the family* $(\kappa_Z)_{\kappa\in\circledast_Y}$ *is a fundamental system of seminorms on* Z*. Define* $K : Y \to Z$ *by* $K(y) := (y, y, ..., y, ...) + c_0(Y)$, $y \in Y$*. Then* $K(\cdot)$ *is linear, continuous, injective,* $K^{-1} : K(Y) \to Y$ *is linear, continuous, injective and* $\kappa_Z(Ky) = \kappa(y)$ *for all* $y \in Y$ *and* $\kappa \in \circledast_Y$.

Now we will prove that $K(Y)$ *is dense in* Z*; the proof is a little bit different from the corresponding proof of F. J. Sayas given in the case that* Y *is a normed space, when* Z *is the Banach space and* (Z, K) *is the completion of* Y*. So, let* $(x_\alpha)_{\alpha\in A} \in c(Y)$*. We will prove that* $\lim_{\alpha\in A} K(x_\alpha) = (x_\alpha)_{\alpha\in A} + c_0(Y)$ *in* Z*, which implies that* $K(Y)$ *is dense in* Z*. In order to see that, fix* $\kappa \in \circledast_Y$*; then we need to prove that* $\lim_{\alpha\in A}\lim_{\beta\in A} \kappa(x_\beta - x_\alpha) = 0$*. Let* $\epsilon > 0$ *be fixed. We know that there exists* $\alpha_\epsilon \in A$ *such that for each* $\alpha \in A$ *and* $\beta \in A$ *such that* $\alpha_\epsilon \le \alpha$ *and* $\alpha_\epsilon \le \beta$ *one has* $\kappa(x_\beta - x_\alpha) \le \epsilon$*. This clearly implies* $\lim_{\beta\in A} \kappa(x_\beta - x_\alpha) \le \epsilon$ *for* $\alpha_\epsilon \le \alpha$ *and proves the claimed assertion.*

Therefore, if $(\hat{Z}, K_Z)$ *is a completion of* Z*, then* $(\hat{Z}, K_Z \circ K)$ *is the completion of* Y*, as easily approved.*

(iii) By [16, Remark 22.18(d)], every sequentially complete metric linear space is complete; furthermore, we know that for each locally convex space Y*, the following statements are equivalent (see [16, Exercise 1, p. 260]):*

(a) Y *has a countable zero neighbourhood basis.*

(b) Y *has a countable fundamental system of seminorms.*

(c) There exists a metric d *on* Y *which induces the topology on* Y *and for which* (Y, d) *is a metric linear space.*

Using this fact and the construction given in the proof of [16, Proposition 22.21], it readily follows that the completion of any locally convex space Y *which has a countable fundamental system of seminorms is a Fréchet space.*

Further on, we would like to emphasize that the construction of a completion of Y *can be established following the ideas of F. J. Sayas, without the use of any result from the dual theory of locally convex spaces and any result concerning the local Banach spaces for the seminorms of* Y*. Let* $(A, \le) = (\mathbb{N}, \le)$*. By (ii), it suffices to show that the space* Z *considered above is sequentially complete. In order to do that, we may assume without loss of generality that the topology of* Y *is given by the increasing family of seminorms* $(\|\cdot\|_k)_{k\in\mathbb{N}}$*; see also [16, p. 294]. Let* $((x_l^N)_{l\in\mathbb{N}} + c(Y))_{N\in\mathbb{N}}$ *be a Cauchy sequence in* Z*. If* $N \in \mathbb{N}$ *is given, then* $(x_l^N)_{l\in\mathbb{N}}$ *is a Cauchy sequence in* Y *and therefore there exists an integer* $a_N \in \mathbb{N}$ *such that* $a_N \ge N$ *and, for every* $m \ge a_N$ *and* $s \ge a_N$*, we have* $\|x_m^N - x_s^N\|_N \le 1/N$*. Define* $y_N := x_{a_N}^N$, $N \in \mathbb{N}$*. We claim that* $(y_N)_{N\in\mathbb{N}}$ *is a Cauchy sequence in* Y*. To show this, let a number* $\epsilon > 0$ *and an integer* $k \in \mathbb{N}$ *be fixed. We know that there exists* $N_0 \in \mathbb{N}$ *such that, for* *every $N \geq N_0$ and $M \geq N_0$, we have $\lim_{s\to+\infty} \|x_s^N - x_s^M\|_k \leq \epsilon/6$. Therefore, there exists a sufficiently large integer $s = s(M,N) \geq \max(a_N, a_M)$ such that $\|x_s^N - x_s^M\|_k \leq \epsilon/3$. If $1/N \leq \epsilon/3$, $1/M \leq \epsilon/3$ and $\min(M,N) \geq \max(k, N_0)$, then we have:*

$$\begin{aligned}
\|y_N - y_M\|_k &\leq \|x_{a_N}^N - x_{s(M,N)}^N\|_k + \|x_{s(M,N)}^N - x_{s(M,N)}^M\|_k + \|x_{s(M,N)}^M - x_{a_M}^M\|_k \\
&\leq \|x_{a_N}^N - x_{s(M,N)}^N\|_N + \|x_{s(M,N)}^N - x_{s(M,N)}^M\|_k + \|x_{s(M,N)}^M - x_{a_M}^M\|_M \\
&\leq \frac{1}{N} + \|x_{s(M,N)}^N - x_{s(M,N)}^M\|_k + \frac{1}{M} \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Now we will prove that $\lim_{N\to+\infty}[(x_s^N)_{s\in\mathbb{N}} + c(Y)] = (y_s)_{s\in\mathbb{N}} + c(Y)$ in Z. Fix again a number $\epsilon > 0$ and an integer $k \in \mathbb{N}$. We need to prove that there exists $N_1 \in \mathbb{N}$ such that, for every $N \geq N_1$, we have

$$\lim_{s\to+\infty} \|x_s^N - y_s\|_k \leq \epsilon. \tag{2.2}$$

Towards this end, let us first observe that there exists $w = w(\epsilon, k) \in \mathbb{N}$ such that, for every $m \geq w$ and $s \geq w$, we have $\|y_s - y_m\|_k \leq \epsilon/2$. Furthermore, if $s \geq \max(a(N), w)$, $N \geq \max(k, w)$ and $1/N \leq \epsilon/2$, then we have:

$$\begin{aligned}
\|x_s^N - y_s\|_k = \|x_s^N - x_{a_s}^s\|_k &\leq \|x_s^N - x_{a_N}^N\|_k + \|x_{a_N}^N - x_{a_s}^s\|_k \\
&\leq \|x_s^N - x_{a_N}^N\|_N + \|x_{a_N}^N - x_{a_s}^s\|_k \leq \frac{1}{N} + \frac{\epsilon}{2} \leq \epsilon.
\end{aligned}$$

This implies (2.2) *for all $N \geq N_1 \equiv \max(k, w, 2/\epsilon)$ and completes the proof.*

Now we are able to formulate the following result:

Theorem 2.2 *Suppose that* (C1) *holds, $F : I \times X \to Y$ and the completion $(\hat{Y}, K)$ is defined as above. Set $\hat{\rho}_K := \{(Kx, Ky) : (x, y) \in \rho\}$. Then the following holds:*

(i) The function $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, I', \rho)$-almost periodic if and only if the function $G : I \times X \to \hat{Y}$, given by $G(\mathbf{t}; x) := K(F(\mathbf{t}; x))$, $\mathbf{t} \in I$, $x \in X$ is pre-$(\mathcal{B}, I', \hat{\rho}_K)$-almost periodic. Moreover, if X is a topological space, then $F(\cdot;\cdot)$ is continuous if and only if $G(\cdot;\cdot)$ is continuous.

(ii) Suppose that $\rho = T \in L(Y)$. Then the function $F : I \times X \to Y$ is pre-$(\mathcal{B}, I', T)$-almost periodic [Bohr $(\mathcal{B}, I', T)$-almost periodic, if X is a topological space] if and only if the function $G : I \times X \to \hat{Y}$, given by $G(\mathbf{t}; x) := K(F(\mathbf{t}; x))$, $\mathbf{t} \in I$, $x \in X$ is pre-$(\mathcal{B}, I', \hat{T})$-almost periodic [Bohr $(\mathcal{B}, I', \hat{T})$-almost periodic, if X is a topological space].

Proof. Suppose that the function $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, I', \rho)$-almost periodic. Let W be a zero neighbourhood in $\hat{Y}$ and let $B \in \mathcal{B}$ be given. Then $U = K^{-1}(W)$ is a zero neighbourhood in Y, which implies that there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap I'$ such that, for every $\mathbf{t} \in I$ and $x \in B$, there exists an element $y_{\mathbf{t};x} \in \rho(F(\mathbf{t}; x))$ such that $F(\mathbf{t} + \tau; x) - y_{\mathbf{t};x} \in U$, $\mathbf{t} \in I$, $x \in B$. Since $K(y_{\mathbf{t};x}) \in \hat{\rho}_K(K(F(\mathbf{t}; x)))$, $\mathbf{t} \in I$, $x \in B$, the above implies $(K \circ F)(\mathbf{t}+\tau;x) - K(y_{\mathbf{t};x}) \in W$, $\mathbf{t} \in I,\ x \in B$; therefore, $G(\cdot;\cdot)$ is pre-$(\mathcal{B}, I', \hat{\rho}_K)$-almost periodic. To prove the converse statement, let a zero neighbourhood U in Y and a set $B \in \mathcal{B}$ be fixed. Since the mapping $K^{-1} : K(Y) \to Y$ is linear and continuous, it follows there exists a zero neighbourhood W in $\hat{Y}$ such that $K(U) = W \cap K(Y)$. Therefore, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap I'$ such that, for every $\mathbf{t} \in I$ and $x \in B$, there exists an element $y_{\mathbf{t};x;K} \in \hat{\rho}_K(K(F(\mathbf{t};x)))$ such that $K(F(\mathbf{t}+\tau;x)) - y_{\mathbf{t};x;K} \in W \cap K(Y)$, $\mathbf{t} \in I,\ x \in B$. Since $K(\cdot)$ is injective, the above yields that for each $\mathbf{t} \in I$ and $x \in B$ there exists an element $G_x(\mathbf{t}) \in \rho(F(\mathbf{t};x))$ such that $y_{\mathbf{t};x;K} = K(G_x(\mathbf{t}))$ and $F(\mathbf{t}+\tau;x) - G_x(\mathbf{t}) \in K^{-1}(W \cap K(Y)) = U$. This readily implies that $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, I', \rho)$-almost periodic. Clearly, if X is a topological space, then the continuity of $F(\cdot;\cdot)$ is equivalent to the continuity of $(K \circ F)(\cdot;\cdot)$ since $K : Y \to \hat{Y}$ is linear and continuous as well as $K^{-1} : K(Y) \to Y$ is linear and continuous.

Assume now that $\rho = T \in L(Y)$. Then $\hat{\rho}_K$ is equal to the part of the operator $\hat{T}$, defined as above, in $K(Y)$. Due to (i), it suffices to show that the assumption that the function $G(\cdot;\cdot)$ is pre-$(\mathcal{B}, I', \hat{T})$-almost periodic implies that the function $F(\cdot;\cdot)$ is pre-$(\mathcal{B}, I', T)$-almost periodic. In order to see that, we can repeat verbatim the corresponding part of proof of (i) and apply the second equality in (2.1). $\square$

As an immediate consequence, we have the following result:

Corollary 2.3 *Suppose that* (C1) *holds,* $F : I \times X \to Y$*,* $c \in \mathbb{C} \setminus \{0\}$ *and* $\rho = c\mathrm{I} \in L(Y)$*. Then* $F(\cdot;\cdot)$ *is pre-*$(\mathcal{B}, I', c)$*-almost periodic [Bohr* $(\mathcal{B}, I', c)$*-almost periodic, if* X *is a topological space] if and only if the function* $G : I \times X \to \hat{Y}$*, given by* $G(\mathbf{t};x) := K(F(\mathbf{t};x))$*,* $\mathbf{t} \in I$*,* $x \in X$*, is pre-*$(\mathcal{B}, I', c)$*-almost periodic [Bohr* $(\mathcal{B}, I', c)$*-almost periodic, if* X *is a topological space].*

Now we will prove the following result for the Stepanov ρ-almost periodic type functions with values in locally convex spaces:

Theorem 2.4 *Suppose that* (SM) *holds,* $F : \Lambda \times X \to Y$*, the completion* $(\hat{Y}, K)$ *is defined as above and* $\hat{\rho}_K = \{(Kx, Ky) : (x,y) \in \rho\}$*. Then the following holds:*

(i) The function $F(\cdot;\cdot)$ *is Stepanov-*$(\Omega, p, \rho, \mathcal{B}, \Lambda', \nu)$*-almost periodic if and only if the function* $G : I \times X \to \hat{Y}$*, given by* $G(\mathbf{t};x) := K(F(\mathbf{t};x))$*,* $\mathbf{t} \in I$*,* $x \in X$ *is Stepanov-*$(\Omega, p, \hat{\rho}_K, \mathcal{B}, \Lambda', \nu)$*-almost periodic. Moreover, if* X *is a topological space, then* $F(\cdot;\cdot)$ *is continuous if and only if* $G(\cdot;\cdot)$ *is continuous.*

(ii) Suppose that $\rho = T \in L(Y)$*. Then the function* $F : I \times X \to Y$ *is Stepanov-*$(\Omega, p, T, \mathcal{B}, \Lambda', \nu)$*-almost periodic if and only if the function* $G : I \times X \to \hat{Y}$*, given by* $G(\mathbf{t};x) := K(F(\mathbf{t};x))$*,* $\mathbf{t} \in I$*,* $x \in X$ *is Stepanov-*$(\Omega, p, \hat{T}, \mathcal{B}, \Lambda', \nu)$*-almost periodic.*

Proof. We will only prove (i) since (ii) can be deduced similarly as in the corresponding part of Theorem 2.2. Let $B \in \mathcal{B}$, let U be an absolutely convex zero neighbourhood in Y, and let $\kappa \in \circledast_Y$ and $\delta > 0$ be such that $\{y \in Y : \kappa(y) \leq \delta\} \subseteq U$. Since the function $F : \Lambda \times X \to Y$ is Stepanov-$(\Omega, p, \rho, \mathcal{B}, \Lambda', \nu)$-almost periodic, for every $\epsilon > 0$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $\mathbf{t} \in \Lambda$ and $x \in B$, the mapping

$\mathbf{s} \mapsto G_x(\mathbf{s}) \in \rho(F(\mathbf{s}; x))$, $\mathbf{s} \in \mathbf{t} + \Omega$ is well defined, and (1.1) holds with the number ϵ replaced with the number $\epsilon \cdot \delta^p$ therein. Due to the definition of $\hat{\rho}_K$, it suffices to prove that:

$$\int_{\mathbf{t}+\Omega} \Big[p_U\Big(K\big(F(\mathbf{s}+\tau; x)\big) - K\big(G_x(\mathbf{s})\big)\Big)\Big]^p \nu^p(\mathbf{s})\, d\mathbf{s} \leq \epsilon, \quad \mathbf{t} \in \Lambda,\ x \in B,$$

i.e.,

$$\int_{\mathbf{t}+\Omega} \left[\sup_{y \in U^\circ} \Big|\big\langle K\big(F(\mathbf{s}+\tau; x)\big) - K\big(G_x(\mathbf{s})\big), y\big\rangle\Big|\right]^p \nu^p(\mathbf{s})\, d\mathbf{s} \leq \epsilon, \quad \mathbf{t} \in \Lambda,\ x \in B. \tag{2.3}$$

Due to the definition of $K(\cdot)$, it is clear that (2.3) is equivalent to

$$\int_{\mathbf{t}+\Omega} \left[\sup_{y \in U^\circ} \Big|\big\langle y, F(\mathbf{s}+\tau; x) - G_x(\mathbf{s})\big\rangle\Big|\right]^p \nu^p(\mathbf{s})\, d\mathbf{s} \leq \epsilon, \quad \mathbf{t} \in \Lambda,\ x \in B. \tag{2.4}$$

Using the generalization of the norm formula given in [16, Proposition 22.14], we have that (2.4) is equivalent to

$$\int_{\mathbf{t}+\Omega} \Big[\inf\big\{t > 0 : F(\mathbf{s}+\tau; x) - G_x(\mathbf{s}) \in tU\big\}\Big]^p \nu^p(\mathbf{s})\, d\mathbf{s} \leq \epsilon, \quad \mathbf{t} \in \Lambda,\ x \in B. \tag{2.5}$$

If $\mathbf{s} \in \mathbf{t} + \Omega$ is given, then there exist two possibilities: $\kappa(F(\mathbf{s}+\tau; x) - G_x(\mathbf{s})) = 0$ or $\kappa(F(\mathbf{s}+\tau; x) - G_x(\mathbf{s})) > 0$. In the first case, we have $F(\mathbf{s}+\tau; x) - G_x(\mathbf{s}) \in tU$ for all $t > 0$ and therefore $\inf\{t > 0 : F(\mathbf{s}+\tau; x) - G_x(\mathbf{s}) \in tU\} = 0$. In the second case, we have

$$F(\mathbf{s}+\tau; x) - G_x(\mathbf{s}) \in \frac{\kappa(F(\mathbf{s}+\tau; x) - G_x(\mathbf{s}))}{\delta} U,$$

so that

$$\inf\big\{t > 0 : F(\mathbf{s}+\tau; x) - G_x(\mathbf{s}) \in tU\big\} \leq \frac{\kappa(F(\mathbf{s}+\tau; x) - G_x(\mathbf{s}))}{\delta},$$

which also holds in the first case. Since (1.1) holds with the number ϵ replaced with the number $\epsilon \cdot \delta^p$ therein, the previous inequality yields that (2.5) holds true. In order to prove the converse statement, let us assume that the function $G(\cdot; \cdot)$ is Stepanov-$(\Omega, p, \hat{\rho}_K, \mathcal{B}, \Lambda', \nu)$-almost periodic. Let $\epsilon > 0$, $\kappa \in \circledast_Y$ and $B \in \mathcal{B}$ be given. Set $U := \{y \in Y : \kappa(y) \leq \delta\}$. Using the properties of mapping $K(\cdot)$ and the arguments contained in the proof of Theorem 2.2, it follows that, for every $\epsilon > 0$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $\mathbf{t} \in \Lambda$ and $x \in B$, the mapping $\mathbf{s} \mapsto G_x(\mathbf{s}) \in \rho(F(\mathbf{s}; x))$, $\mathbf{s} \in \mathbf{t} + \Omega$ is well defined, and (2.5) holds. But, for each $y \in Y$ we have

$$\inf\big\{t > 0 : y \in tU\big\} = \frac{\kappa(y)}{\delta},$$

which is easily proved. Keeping in mind this equality and (2.5), we simply get that the estimate (1.1) holds true. The proof of the theorem is thereby complete. □

Now we will state the following simple consequence of Theorem 2.4:

Corollary 2.5 *Suppose that* (SM) *holds,* $c \in \mathbb{C} \setminus \{0\}$ *and* $\rho = c\mathrm{I} \in L(Y)$. *Then the function* $F : \Lambda \times X \to \hat{Y}$ *is Stepanov-*$(\Omega, p, c, \mathcal{B}, \Lambda', \nu)$*-almost periodic if and only if the function* $G : \Lambda \times X \to \hat{Y}$*, given by* $G(\mathbf{t}; x) := K(F(\mathbf{t}; x))$, $\mathbf{t} \in I$, $x \in X$ *is Stepanov-*$(\Omega, p, c, \mathcal{B}, \Lambda', \nu)$*-almost periodic.*

It is clear that many structural results for the (Stepanov) ρ-almost periodic type functions with values in locally convex spaces can be proved directly, without using the completeness of the underlying space Y (observe, however, that M. A. Latif and M. I. Bhatti have assumed a priori the completeness of the underlying space Y in [13, Definition 5], where the authors have introduced the notion of almost periodicity for a continuous function $F : \mathbb{R}^n \to Y$). The results established in this section are essentially important in the analysis of some important criteria for the almost periodicity of functions with values in non-complete locally convex spaces. For the sake of brevity, we will only formulate here the Bochner criterion for the almost periodic functions with values in non-complete locally convex spaces.

We need the following preliminaries. Suppose that X is a topological space, Y is a Fréchet space and the function $F : \mathbb{R}^n \times X \to Y$ is Bohr $\mathcal{B}$-almost periodic, where $\mathcal{B}$ is any family of compact subsets of X. Then we know that the set $\{F(\mathbf{t}; x) : \mathbf{t} \in \mathbb{R}^n,\ x \in B\}$ is bounded in Y; see [11]. Fix now a set $B \in \mathcal{B}$ and define $l_\infty(B : Y)$ as the space of all bounded functions from B into Y. If we endow $l_\infty(B : Y)$ with the family of seminorms $(\kappa_\infty)_{\kappa \in \circledast_Y}$, where $\kappa_\infty(f) := \sup_{x \in B} \kappa(f(x))$ for all $f \in l_\infty(B : Y)$ and $\kappa \in \circledast_Y$, then $l_\infty(B : Y)$ is a Fréchet space. We can simply prove that the function $F : \mathbb{R}^n \times X \to Y$ is Bohr $\mathcal{B}$-almost periodic if and only if for each set $B \in \mathcal{B}$ the function $F_B : \mathbb{R}^n \to l_\infty(B : Y)$, given by $[F_B(\mathbf{t})](x) := F(\mathbf{t}; x)$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in B$, is Bohr almost periodic.

In order to formulate our next result, we need the following lemma:

Lemma 2.6 *([11])*

(i) Suppose that X is a topological space, $\mathcal{B}$ is any family of compact subsets of X, Y is a Fréchet space and $F : \mathbb{R}^n \times X \to Y$ is continuous. Then $F(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic if and only if for each set $B \in \mathcal{B}$ and for each sequence $(\mathbf{b}_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ there exists a subsequence $(\mathbf{a}_k)_{k\in\mathbb{N}}$ of $(\mathbf{b}_k)_{k\in\mathbb{N}}$ such that the sequence of functions $(F_B(\cdot+\mathbf{a}_k))_{k\in\mathbb{N}}$ converges in $l_\infty(B : Y)$.

(ii) Suppose that $1 \le p < +\infty$, X is a topological space, $\mathcal{B}$ is any family of compact subsets of X, Y is a Fréchet space equipped with the calibration $\circledast_Y = (\|\cdot\|_k)_{k\in\mathbb{N}}$ of increasing seminorms and $F : \mathbb{R}^n \times X \to Y$ satisfies condition (C)*, where:*

(C) For every $x \in X$, the function $F(\cdot; x)$ is p-locally integrable and, for every $x \in X$, $\epsilon > 0$, $k \in \mathbb{N}$ and $\mathbf{t}_0 \in \mathbb{R}^n$, there exists an open set O in X such that $x \in O$ and $\|F(\mathbf{t}_0 + \mathbf{u}; x) - F(\mathbf{t}_0 + \mathbf{u}; y)\|_k \le \epsilon$ for a.e. $\mathbf{u} \in \Omega$ and $y \in O$.

Then $F(\cdot;\cdot)$ is Stepanov-$(\Omega, p, \mathcal{B})$-almost periodic if and only if for each set $B \in \mathcal{B}$ and for each sequence $(\mathbf{b}_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ there exists a subsequence $(\mathbf{a}_k)_{k\in\mathbb{N}}$ of $(\mathbf{b}_k)_{k\in\mathbb{N}}$ such that the sequence of functions $(\hat{F}_{\Omega;B}(\cdot + \mathbf{a}_k))_{k\in\mathbb{N}}$, defined by

$$\Big\{\Big[\hat{F}_{\Omega;B}(\mathbf{t})\Big](x)\Big\}(\mathbf{u}) := F(\mathbf{t} + \mathbf{u}; x), \quad \mathbf{t} \in \mathbb{R}^n,\ \mathbf{u} \in \Omega,\ x \in B,$$

converges in $l_\infty(B : L^p(\Omega : Y))$.

Keeping in mind Corollary 2.3, Corollary 2.5 and Lemma 2.6, we can clarify the following result:

Theorem 2.7 *(The Bochner Criterion)*

(i) *Suppose that X is a topological space, $\mathcal{B}$ is any family of compact subsets of X, Y is a locally convex space equipped with a countable fundamental system of seminorms, and $F : \mathbb{R}^n \times X \to Y$ is continuous. Let $(\hat{Y}, K)$ be the completion of Y. Then $F(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic if and only if for each set $B \in \mathcal{B}$ and for each sequence $(\mathbf{b}_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$, there exists a subsequence $(\mathbf{a}_k)_{k\in\mathbb{N}}$ of $(\mathbf{b}_k)_{k\in\mathbb{N}}$ such that the sequence of functions $((K \circ F)_B(\cdot + \mathbf{a}_k))_{k\in\mathbb{N}}$ converges in $l_\infty(B : \hat{Y})$.*

(ii) *Suppose that $1 \leq p < +\infty$, X is a topological space, $\mathcal{B}$ is any family of compact subsets of X, Y is a locally convex space equipped with the calibration $\circledast_Y = (\|\cdot\|_k)_{k\in\mathbb{N}}$ of increasing seminorms and $F : \mathbb{R}^n \times X \to Y$ satisfies condition* (C)*. Let $(\hat{Y}, K)$ be the completion of Y. Then $F(\cdot;\cdot)$ is Stepanov-$(\Omega, p, \mathcal{B})$-almost periodic if and only if for each set $B \in \mathcal{B}$ and for each sequence $(\mathbf{b}_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$, there exists a subsequence $(\mathbf{a}_k)_{k\in\mathbb{N}}$ of $(\mathbf{b}_k)_{k\in\mathbb{N}}$ such that the sequence of functions $((K \hat{\circ} F)_{\Omega;B}(\cdot + \mathbf{a}_k))_{k\in\mathbb{N}}$, defined in the obvious way, converges in $l_\infty(B : L^p(\Omega : \hat{Y}))$.*

Proof. The proof of (i) is trivial and we will present here the main details of the proof of (ii), only. Let us consider the completion $(\hat{Y}, K)$ of Y constructed in Remark 2.1(ii)-(iii). Then $\hat{Y}$ is a Fréchet space and the topology of $\hat{Y}$ is induced by the increasing fundamental system of seminorms $(\|\cdot\|_{k,\hat{Y}})_{k\in\mathbb{N}}$ satisfying $\|Ky\|_{k,\hat{Y}} = \|y\|_k$ for all $k \in \mathbb{N}$ and $y \in Y$. This simply implies that condition (C) holds for the function $K \circ F : \mathbb{R}^n \times X \to \hat{Y}$. Then the result immediately follows by applying Corollary 2.5 and Lemma 2.6(ii). $\square$

Finally, let us note that the analogues of Theorem 2.4 and Corollary 2.5 can be deduced for the Weyl ρ-almost periodic type functions with values in locally convex spaces, as well; more details will be given somewhere else. Our structural results can be also formulated for the corresponding classes of uniformly recurrent functions (cf. [10] for further information in this direction).

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