Journal of Nonlinear Evolution Equations and Applications
Volume 2026, Number 2, pp. 31–50 (June 2026)
ISSN 2161-3680
http://www.jneea.com

NEW RESULTS ON THE CONTROLLABILITY FOR FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS VIA MEASURE OF NONCOMPACTNESS

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Received January 9, 2026

Accepted January 13, 2026

Communicated by Gaston M. N'Guérékata

Abstract. In this paper, we consider a nonlinear fractional integrodifferential equation with the Caputo fractional derivative and discuss the controllability of its solutions. By applying Mönch's fixed point theorem, the resolvent operator in the Grimmer sense, and the measure of noncompactness, we obtain sufficient conditions for the controllability of solutions. To verify the accuracy and efficiency of the analytical results, we provide an example to show how the solution behaves with respect to different fractional orders, nonlinearities and boundary conditions.

Keywords: fractional integrodifferential equations, controllability, resolvent operator, measures of non-compactness, fixed point theorems, nonlocal conditions.

2010 Mathematics Subject Classification: 34K06, 34A12, 37L05, 93B05.

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1 Introduction

The study of fractional differential equations and the principles of fractional calculus has recently garnered significant attention within the mathematical community. Various sectors, such as engineering, biology, control theory, social sciences, signal processing, image processing, and physical sciences, are increasingly investigating the concept of fractional calculus. Fractional differential equations provide effective models for numerous scenarios. In many practical applications, models based on fractional orders demonstrate superior performance compared to those based on integer orders. Differential inclusions (practical extensions of differential equations and inequalities commonly associated with control theory) serve multiple purposes and have diverse applications. Their utility facilitates the analysis of real-world dynamic systems, particularly when velocities cannot be solely attributed to the state of the system itself. Research has extensively focused on bounded value problems. Additionally, there has been considerable investigation into the existence of solutions for fractional differential inclusions and related systems. To substantiate the theory and its applications concerning fractional calculus, we reference several research publications [22, 27, 29], along with papers such as [1, 3, 7, 14].

The notion of controllability is essential in mathematical control theory and plays a significant role in both theoretical and practical mathematics. In the realm of contemporary fractional calculus, the importance of controllability is even more pronounced. As a result, a group of researchers is actively involved in this field, collaborating to connect control theory with fractional differential systems. This joint endeavor has fostered the emergence of innovative ideas and concepts, which have enriched our comprehension of dynamical systems. For a comprehensive overview of recent findings, please consult the references [4, 13, 23–25, 28, 31, 32].

We emphasize that controllability is founded on two essential principles. There are two distinct forms of controllability: exact controllability, often referred to as complete controllability, and approximate controllability. In infinite-dimensional spaces, the ability to exercise precise control enables the system to reach any desired final state, while the capacity for approximate control allows it to be directed toward a specified local vicinity of that ultimate state. According to [18], a prevalent method for addressing exact controllability involves transforming the controllability issue into a fixed-point problem, predicated on the assumption that the controllability operator is invertible. Additionally, approximate controllability has been thoroughly examined through the lens of fixed-point theorem and semigroup theory, with the assumption that the linear component of the relevant nonlinear system is approximately controllable (refer to [5] for further information). It is important to note that in finite-dimensional spaces, the concepts of exact and approximate control are mutually exclusive. A significant body of research has been undertaken by various authors, such as [6, 33, 34] regarding the theory surrounding the existence of solutions for fractional differential equations.

Partial differential equations offer a valuable framework for understanding various processes in fields such as engineering, biology, and physics. However, these equations are limited to depicting the behavior of a system at a specific point in time. Their failure to incorporate the cumulative impacts of prior events can result in an inaccurate representation of the system's dynamics. This limitation becomes particularly apparent when dealing with models related to heat conduction, porous elastic materials, and nuclear reactor operations, where the significance of memory effects is crucial (refer to [19] for further details). A viable approach to address this issue is to integrate an integral component into the conventional partial differential equation. This modification effectively resolves the limitation, leading to an integro-differential equation. This enhanced formulation not only serves as a solution but also proves versatile by accommodating factors such as memory effects and past influences, thereby increasing the adaptability of the model. An integro-differential equation of this nature could be expressed as:

$$\begin{cases} x(t) = Ax(t) + \displaystyle\int_0^t \Upsilon(t-s)x(s)\,\mathrm{d}s + \mathcal{I}(t), \quad \text{for } t \geq 0, \\ x(0) = x_0 \in \mathcal{X}, \end{cases} \tag{1.1}$$

where $\mathcal{I} \ : \mathbb{R}^+ \to \mathcal{X}$ is the input function, $(\mathcal{X}, \|\cdot\|)$ is a Banach space, A is a family of linear operators (not necessarily bounded) on $\mathcal{X}$, and $(\Upsilon(t))_{t\geq 0}$, is a family of closed linear operators with domain $D(\Upsilon(\cdot)) \supset D(A)$ time-independent.

Equation (1.1) has been extensively discussed in the current literature. For example, Grimmer *et al.* [8, 15, 16] thoroughly investigated this equation, seeking mild solutions and examining their regularity in various contexts. They utilized resolvent operator theory, an extension of standard semigroup theory. In contrast to semigroups, resolvent operators can exhibit growth rates exceeding exponential. The existence, controllability and other qualitative and quantitative properties of integrodifferential equations have been considered by many authors in the last two decades, for instance, see [11, 12, 17] and the references therein.

Motivated by all above mentioned works, in the present paper, we investigate the nonlocal controllability of the following fractional integrodifferential equations:

$$\begin{cases} {}^cD^q x(t) = Ax(t) + Cu(t) + \displaystyle\int_0^t \Upsilon(t-s)x(s)\,\mathrm{d}s + F(t, x(t)),\ t \in J = [0, a], \\ x(0) + h(x) = x_0, \end{cases} \tag{1.2}$$

where $x(\cdot)$ is the state variable with values in the Banach space $\mathcal{X}$ with norm $\|\cdot\|$; $u(\cdot) \in L^2(J; \mathbb{U})$ is the variable control, and $\mathbb{U}$ is another Hilbert space; $C : \mathbb{U} \to \mathcal{X}$ is a bounded linear operator; $F : J \times \mathcal{X} \to \mathcal{X}$ is an $\mathcal{X}$-valued function and nonlocal term $h : \mathcal{C}(J, \mathcal{X}) \to \mathcal{X}$.

To the best of our knowledge, the study of nonlocal controllability for integrodifferential equations of the form (1.2) is an untreated topic in the literature. The main contributions of this work are listed in the following order:

- A new class of fractional integrodifferential systems with non-local conditions in the form of system (1.2) is constructed.
- A novel set of sufficient conditions is established to ensure the nonlocal controllability results for system (1.2).
- We express the mild solution of (1.2) by applying the theory of resolvent operator in Grimmer's sense.
- By using measure of noncompactness method and Mönch fixed point theorem, we prove the nonlocal controllability of system (1.2).
- An example is provided at the end to demonstrate the practical value of the analytical findings.

The rest of this paper is organized as follows. First, in Section 2, we present some basic ideas and findings that will be applied throughout the work. In Section 3, we derive the controllability results for system (1.2) by means of the Mönch fixed point theorem. In the end, an example is developed to confirm the main findings in Section 4.

2 Preliminaries

In this segment, we present some essential concepts and tools, which will be used throughout the current work.

2.1 Caputo fractional derivative

Definition 2.1 *[20] The fractional integral of order r with the lower limit zero for a function g is defined as*

$$\mathscr{I}^r g(t) = \frac{1}{\Gamma(r)} \int_0^t \frac{g(s)}{(t-s)^{1-r}} \,\mathrm{d}s, \quad t > 0,\, r > 0,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 *[20] The Riemann-Liouville derivative of order r with the lower limit zero for a function $g : [0, +\infty) \to \mathbb{R}$ can be written as*

$${}^{L}D^r g(t) = \frac{1}{\Gamma(n-r)} \left(\frac{\mathrm{d}^n}{\mathrm{d}t^n}\right) \int_0^t (t-s)^{n-r-1} g(s) \,\mathrm{d}s, \quad n-1 < r < n,\, t > 0.$$

Definition 2.3 *[20, 21] The Caputo derivative of order r for a function $g : [0, +\infty) \to \mathbb{R}$ can be written as*

$${}^{C}D^r g(t) = {}^{L}D^r \left(g(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} g^{(k)}(0) \right), \quad t > 0,\, n-1 < r < n.$$

Remark 2.1 *[20]*

(1) If $g(t) \in C^m[0, +\infty)$, then

$${}^{C}D^r g(t) = \frac{1}{\Gamma(n-r)} \int_0^t (t-s)^{n-r-1} g^{(n)}(s) \,\mathrm{d}s = \mathscr{I}^{n-r} g^{(n)}(t), \quad t > 0.$$

(2) The Caputo derivative of a constant is equal to zero.

(3) If g is an abstract function with values in $\mathbb{X}$, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

2.2 Integrodifferential equations in Banach spaces

Now, let us review some of the fundamental aspects of the concept of the resolvent operator. Consider the following integrodifferential equation

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t \Upsilon(t-s)x(s) \,\mathrm{d}s, & \text{for } t \geq 0, \\ x(0) = x_0 \in \mathcal{X}, \end{cases} \tag{2.1}$$

where A and $\Upsilon(t)$ are closed operators on the Banach space $\mathcal{X}$. Let $\mathbb{Y}$ be the Banach space formed from $D(A)$ with the graph norm

$$\|x\|_{\mathbb{Y}} = \|Ax\| + \|x\|, \text{ for } x \in D(A).$$

In what follows, we suppose that A and $\Big(\Upsilon(t)\Big)_{t\geq 0}$ satisfy the following conditions:

(R_1) A is the infinitesimal generator of C_0-semigroup $\big(S(t)\big)_{t\geq 0}$ in $\mathcal{X}$.

(R_2) For all $t \geq 0$, $\Upsilon(t) \in \mathcal{L}(\mathbb{Y}, \mathcal{X})$, and for each $x \in \mathbb{Y}$, the function $\Upsilon(\cdot)x$ is bounded, differentiable and the derivative $\Upsilon'(\cdot)x$ is bounded and uniformly continuous on $[0, +\infty)$.

Definition 2.4 [15] *A family of bounded linear operators $\big(\mathcal{Q}(t)\big)_{t\geq 0} \in \mathcal{L}(\mathcal{X})$ is said to be a resolvent operator for Eq.* (2.1) *if $\mathcal{Q}(t)$ satisfies the following properties for all $t \geq 0$:*

(i) *$\mathcal{Q}(0) = I$ (identity operator on $\mathcal{X}$) and $\|\mathcal{Q}(t)\| \leq Me^{\omega t}$ for some constants $\omega \in \mathbb{R}$ and $M \geq 1$.*

(ii) *For each $x \in \mathcal{X}$, $\mathcal{Q}(t)x$ is strongly continuous for $t \geq 0$.*

(iii) *For $x \in \mathbb{Y}$, $\mathcal{Q}(\cdot)x \in \mathcal{C}^1([0, +\infty), \mathcal{X}) \cap \mathcal{C}([0, +\infty), \mathbb{Y})$ and*

$$\begin{aligned}\mathcal{Q}'(t)x &= A\mathcal{Q}(t)x + \int_0^t \Upsilon(t-s)\mathcal{Q}(s)x \, \mathrm{d}s \\ &= \mathcal{Q}(t)Ax + \int_0^t \mathcal{Q}(t-s)\Upsilon(s)x \, \mathrm{d}s \quad \textit{for} \;\; t \geq 0.\end{aligned}$$

Theorem 2.1 [15] *Assume that (R_1) and (R_2) hold. Then, Eq.* (2.1) *has a unique resolvent operator $\big(\mathcal{Q}(t)\big)_{t\geq 0}$.*

Theorem 2.2 [8] *Assume that (R_1) and (R_2) hold. Let $\big(\mathcal{Q}(t)\big)_{t\geq 0}$ be the resolvent operator of equation* (2.1)*. Then, $\mathcal{Q}(t)$ for $t > 0$ is operator-norm continuous (or continuous in the uniform operator topology) if and only if $S(t)$ is operator-norm continuous for $t > 0$.*

Let $\big(R(t)\big)_{t\geq 0}$ be the resolvent operator of equation (2.1). We define $\big(\mathcal{Q}_q(t)\big)_{t\geq 0}$ and $\big(\mathcal{P}_q(t)\big)_{t\geq 0}$ on $\mathcal{X}$ by

$$\mathcal{Q}_q(t)x = \int_0^{+\infty} \Upsilon_q(s)Q(st^q)x \, \mathrm{d}s, \; t \geq 0, x \in \mathcal{X},$$

and

$$\mathcal{P}_q(t)x = \int_0^{+\infty} qs\Upsilon_q(s)Q(st^q)x \, \mathrm{d}s, \; t \geq 0, x \in \mathcal{X},$$

where

$$\Upsilon_q(s) = \frac{1}{\pi q}\sum_{n=1}^{+\infty}(-s)^{n-1}\frac{\Gamma(1+qn)}{n!}\sin(n\pi q), \; q \in (0,1),$$

is the Wright type function defined on $(0, \infty)$ in which $0 \leq \Upsilon_q(s)$ for $s \in (0, \infty)$,

$$\int_0^{+\infty} \Upsilon_q(s) \, \mathrm{d}s = 1, \quad \int_0^{+\infty} s^\lambda \Upsilon_q(s) \, \mathrm{d}s = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda q)}, \; \lambda \in [0,1].$$

Assume that there exists $M \geq 1$ such that $\|R(t)\|_{\mathcal{L}(\mathcal{X})} \leq M$ for each $t \in [0, a]$.

Lemma 2.3 *[10] The families $\big(\mathcal{Q}_q(t)\big)_{t\geq 0}$ and $\big(\mathcal{P}_q(t)\big)_{t\geq 0}$ satisfy the following properties:*

(i) For each $t \geq 0$, $\mathcal{Q}_q(t)$ and $\mathcal{P}_q(t)$ are linear and bounded on $\mathcal{X}$. Moreover,

$$\|\mathcal{Q}_q(t)x\| \leq M\|x\|, \text{ and } \|\mathcal{P}_q(t)x\| \leq \frac{Mq}{\Gamma(1+q)}\|x\|, \text{ for } x \in \mathcal{X}.$$

(ii) $\mathcal{Q}_q(t)$ and $\mathcal{P}_q(t)$ are strongly continuous on $\mathcal{X}$.

(iii) If $R(t)$ is norm-continuous for $t > 0$, then $\mathcal{Q}_q(t)$ and $\mathcal{P}_q(t)$ are norm-continuous for $t > 0$.

Let us consider the following linear equation:

$$\begin{cases} {}^cD_t^q x(t) = Ax(t) + \displaystyle\int_0^t \Upsilon(t-s)x(s)\,\mathrm{d}s + \mathcal{I}(t), \quad \text{for } t \geq 0, \\ x(0) = x_0 \in \mathcal{X}, \end{cases} \tag{2.2}$$

where $\mathcal{I} \in L^p_{loc}(\mathbb{R}^+, \mathcal{X})$. By the definition of the Gamma function $\Gamma(\cdot)$, and the Caputo fractional derivative, we can rewrite the Cauchy problem (2.2) in the following equivalent form:

$$\begin{cases} x(t) = x_0 + \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t (t-s)^{q-1}\Big[Ax(s) + \int_0^s \Upsilon(s-\tau)x(\tau)\ \mathrm{d}\tau + \mathcal{I}(s)\Big]\,\mathrm{d}s, \quad \text{for } t \geq 0, \\ x(0) = x_0 \in \mathcal{X}, \end{cases} \tag{2.3}$$

provided that this integral exists.

Theorem 2.4 *[10] If x is a solution of equation* (2.3)*, then*

$$x(t) = \mathcal{Q}_q(t)x_0 + \int_0^t (t-s)^{q-1}\mathcal{P}_q(t-s)\mathcal{I}(s)\,\mathrm{d}s, \text{ for } t \geq 0. \tag{2.4}$$

Definition 2.5 *A function $x \in \mathcal{C}([0,a];\mathcal{X})$ satisfying* (2.4) *for $t \in [0,a]$ is said a mild solution of equation* (2.2) *on $[0,a]$.*

We end this section by the following useful lemma introduced by Ezzinbi *et al.* [9].

Lemma 2.5 *Let $\mathcal{X}$ be a Banach space and $(T_n)_{n\geq 1}$ be a sequence of bounded linear maps on $\mathcal{X}$. Assume that $T_n\zeta \to T\zeta$ for all $\zeta \in \mathcal{X}$ and for some $T \in \mathcal{L}(\mathcal{X})$. Then, for any compact set K in $\mathcal{X}$, T_n converges to T uniformly in K, namely,*

$$\sup_{\zeta\in K}\|T_n(\zeta) - T(\zeta)\| \to 0, \quad as \quad n \to +\infty.$$

2.3 Measure of noncompactness

Now let us recall some fundamental facts of the notion of Hausdorff measure of noncompactness.

Definition 2.6 *Let K be a bounded subset of $\mathcal{H}$. The Hausdorff measure of non-compactness is defined by*

$$\alpha(K) = \inf\{\epsilon > 0 | K \subset \cup_{i=1}^{n} B(k_i, \tau_i),\ k_i \in K,\ \tau_i < \epsilon,\ 1 \leq i \leq n\},$$

and $B(k_i, \tau_i)$ represents the open ball of radius τ_i with center at k_i.

Lemma 2.6 ([2]) *Let α denote Kuratowski measure of noncompactness on the real Banach spaces $\mathcal{H}$ and $K_1, K_2 \subseteq \mathcal{H}$ be bounded. The following properties are satisfied:*

(i) *If $K_1 \subseteq K_2$, then $\alpha(K_1) \leq \alpha(K_2)$ (Monotonicity).*

(ii) $\alpha(K_1) = \alpha(\bar{K_1}) = \alpha(co(K_1))$, *where $\bar{K_1}$ and $co(K_1)$ mean the closure and convex hull of K_1, respectively.*

(iii) *K_1 is pre-compact if and only if $\alpha(K_1) = 0$ (Regularity).*

(iv) $\alpha(\lambda K_1) = |\lambda| \alpha(K_1)$ *for any $\lambda \in \mathbb{R}$.*

(v) $\alpha(K_1 \cup K_2) \leq \max\{\alpha(K_1), \alpha(K_2)\}$.

(vi) $\alpha(K_1 + \nu) = \alpha(K_1)$ *for all $\nu \in \mathcal{H}$.*

(vii) $\alpha(K_1 + K_2) \leq \alpha(K_1) + \alpha(K_2)$ *where* $K_1 + K_2 = \{x_1 + x_2 : x_1 \in K_1,\ x_2 \in K_2\}$.

(viii) *If the map $\vartheta : D(\vartheta) \subseteq \mathcal{H} \to \mathcal{U}$ is Lipschitz continuous with constant κ, then*

$$\alpha(\vartheta K_1) \leq \kappa\, \alpha(K_1)$$

for any bounded subset $K_1 \subseteq D(\vartheta)$.

Theorem 2.7 *[30] If $\{x_k\}_{k=1}^{\infty}$ is a sequence of Bochner integrable functions from J into $\mathcal{X}$ with the estimation $\|x_k(t)\| \leq \nu(t)$ for almost all $t \in J$ and every $k \geq 1$, where $\nu \in L^1(J, \mathbb{R})$, then the function $\psi(t) = \beta\big(\{x_k(t) : k \geq 1\}\big)$ is in $L^1(J, \mathbb{R})$ and satisfies*

$$\beta\Big(\Big\{ \int_0^t x_k(s)\, \mathrm{d}s : k \geq 1 \Big\}\Big) \leq 2 \int_0^t \psi(s)\, \mathrm{d}s. \tag{2.5}$$

Now, we define the mild solution of the nonlocal fractional system (1.2).

Definition 2.7 *The mild solution of system* (1.2) *is a function $x \in \mathcal{C}(J, \mathcal{X})$ satisfying the following integral equation*

$$\begin{aligned} x(t) &= \mathcal{Q}_q(t)[x_0 - h(x)] \\ &\quad + \int_0^t (t-s)^{q-1} \mathcal{P}_q(t-s) B u(s)\, \mathrm{d}s + \int_0^t (t-s)^{q-1} \mathcal{P}_q(t-s) F(s, x(s))\, \mathrm{d}s,\ t \in J. \end{aligned} \tag{2.6}$$

Definition 2.8 *The nonlocal system* (1.2) *is said to be controllable on J if and only if, for every $x_0, x_1 \in \mathcal{X}$, there exists a control $u \in L(J, \mathbb{U})$ such that the mild solution $x(\cdot)$ of the system* (1.2) *satisfies $x(a) + h(x) = x_1$.*

For ending this section, we present the following Mönch fixed point theorem.

Theorem 2.8 (Mönch fixed point) *[26]* Let $\mathcal{X}$ be a Banach space, $\mathcal{F} \subset \mathcal{X}$ open, $0 \in \mathcal{F}$, and $\mathcal{G} : \overline{\mathcal{F}} \to \mathcal{X}$ continuous. Moreover, $\mathcal{G}$ satisfies Mönch's conditions:

(i) $N \subset \overline{\mathcal{F}}$ is countable, $N \subset \overline{co}(0 \cup \mathcal{G}(N))$ implies N is relatively compact.

(ii) Boundary condition: $\vartheta \in \overline{\mathcal{F}}$ and $\gamma \in (0,1)$ and $\vartheta = \gamma \mathcal{G}(\vartheta)$ imply $\vartheta \notin \partial\mathcal{F}$.

Then, there exists an element $w \in \overline{\mathcal{F}}$ such that $\mathcal{G}(w) = w$.

3 Controllability results

In this section, we present and prove the controllability results for system (1.2). To establish the main results, we require the following assumptions:

$(C.1)$ The resolvent operator $\mathcal{R}(t)$, is norm-continuous for $t > 0$.

$(C.2)$ The function $F : J \times \mathcal{X} \to \mathcal{X}$ satisfies:

- **(i)** $F(\cdot, x)$ is strongly measurable for every $x \in \mathcal{X}$, and $F(t, \cdot)$ is continuous for a.e. $t \in J$.
- **(ii)** There exist a constant $0 < \eta_1 \leq \eta$, a function $m \in L^{\frac{1}{\eta_1}}(J, \mathbb{R}^+)$ and a continuous increasing function $f : [0, \infty) \to [0, \infty)$ such that $\|F(t,x)\| \leq m(t) f(\|x\|), x \in \mathcal{X}, t \in J$, where f satisfies $\liminf_{\varsigma \to +\infty} \frac{f(\varsigma)}{\varsigma} = 0$.
- **(iii)** There exist a constant $0 < \eta_2 \leq \eta$ and a function $g \in L^{\frac{1}{\eta_2}}(J, \mathbb{R}^+)$ such that, for every bounded subset $\mathcal{Z} \subset \mathcal{X}$, $\beta(F(t, \mathcal{Z})) \leq g(t)\beta(\mathcal{Z})$ for a.e. $t \in J$.

$(C.3)$ The function $h : \mathcal{C}(J; \mathcal{X}) \to \mathcal{X}$ is continuous, compact operator and there exists $b > 0$ such that $\|h(x) - h(y)\| \leq b\|x - y\|$.

$(C.4)$ **(i)** The linear operator $C : L^2(J; \mathbb{U}) \to L^1(J; \mathcal{X})$ is bounded, $\mathcal{W} : L^2(J; \mathbb{U}) \to \mathcal{X}$ defined by

$$\mathcal{W}u = \int_0^a (a-s)^{1-q} \mathcal{P}_q(a-s) C u(s)\, \mathrm{d}s$$

has an inverse operator $\mathcal{W}^{-1}$ which take the values in $L^2(J; \mathbb{U})/ker\mathcal{W}$ and there exist two positive values $\mathcal{T}_2, \mathcal{T}_3$ such that $\|C\|_{L_a(\mathcal{U},\mathcal{X})} \leq \mathcal{T}_2$, $\|\mathcal{W}^{-1}\|_{L_a(\mathcal{X},\mathcal{U}/ker\mathcal{W})} \leq \mathcal{T}_3$.

(ii) There exist a constant $0 < \eta_0 \leq \eta$ and a function $\mathcal{M}_{\mathcal{W}} \in L^{\frac{1}{\eta_0}}(J, \mathbb{R}^+)$ such that for every bounded set $Q \subset \mathcal{X}$, $\beta\big(\mathcal{W}^{-1}Q\big)(t) \leq \mathcal{M}_{\mathcal{W}}(t)\beta(Q)$.

$(C.5)$ $\Omega = \Big(1 + \dfrac{qM\mathcal{T}_2}{\Gamma(1+q)} a^{q-\eta_0} \|K_W\|_{L^{\frac{1}{\eta_0}}}\Big) \dfrac{2qM}{\Gamma(1+q)} a^{q-\eta_2} \|g\|_{L^{\frac{1}{\eta_2}}} < 1.$

For each positive number ρ, let us define $\mathcal{E}_\rho = \{x \in \mathcal{X} : \|x\| \leq \rho\}$. Using the assumption $(C.4)$, for an arbitrary function $x(\cdot) \in \mathcal{C}(J; \mathcal{X})$, we define the control

$$u_x(t) = \mathcal{W}^{-1}\Big(x_1 - \mathcal{Q}_q(a)[x_0 - h(x)] - \int_0^a (a-s)^{q-1} \mathcal{P}_q(a-s) F(s, x(s))\, \mathrm{d}s\Big)(t). \qquad (3.1)$$

First, we shall prove that $\|u_x(t)\| \le \Omega_\rho$, where

$$\Omega_\rho := \mathcal{T}_3\Big(\|x_1\| + M(\|x_0\| + b\rho + \|h(0)\|) + \frac{qM}{\Gamma(1+q)} f(\rho)\Big[\int_0^a ((a-s)^{q-1})^{\frac{1}{1-\eta_1}}\,\mathrm{d}s\Big]^{1-\eta_1} \times \Big[\int_0^a \|m(s)\|^{\frac{1}{\eta_1}}\,\mathrm{d}s\Big]^{\eta_1}\Big]\Big).$$

For each $t \in J$ and $x \in \mathcal{E}_\rho$, we have

$$\begin{aligned}
\|u_x(t)\| &\le \|\mathcal{W}^{-1}\|\Big(\|x_1\| + \|\mathcal{Q}_q(a)\|(\|x_0\| + \|h(x) - h(0)\| + \|h(0)\|) \\
&\quad + \frac{qM}{\Gamma(1+q)} \int_0^b (b-s)^{q-1} f(\|x(s)\|) m(s)\,\mathrm{d}s\Big) \\
&\le \|\mathcal{W}^{-1}\|\Big(\|x_1\| + \|\mathcal{Q}_q(a)\|(\|x_0\| + b\|x\| + \|h(0)\|) \\
&\quad + \frac{qM}{\Gamma(1+q)} f(\|x\|)\Big[\int_0^a ((a-s)^{q-1})^{\frac{1}{1-\eta_1}}\,\mathrm{d}s\Big]^{1-\eta_1} \times \Big[\int_0^a \|m(s)\|^{\frac{1}{\eta_1}}\,\mathrm{d}s\Big]^{\eta_1}\Big) \\
&\le \mathcal{T}_3\Big(\|x_1\| + M(\|x_0\| + b\rho + \|h(0)\|) \\
&\quad + \frac{qM}{\Gamma(1+q)} f(\rho)\Big[\int_0^a ((a-s)^{q-1})^{\frac{1}{1-\eta_1}}\,\mathrm{d}s\Big]^{1-\eta_1} \times \Big[\int_0^a \|m(s)\|^{\frac{1}{\eta_1}}\,\mathrm{d}s\Big]^{\eta_1}\Big) \\
&= \Omega_\rho.
\end{aligned}$$

Hence, the desired result follows.

It is convenient, in the sequel, to use the constant $0 < \Omega < 1$, where

$$\Omega = \Big(1 + \frac{qM\mathcal{T}_2}{\Gamma(1+q)} a^{q-\eta_0} \|K_W\|_{L^{\frac{1}{\eta_0}}}\Big) \frac{2qM}{\Gamma(1+q)} a^{q-\eta_2} \|g\|_{L^{\frac{1}{\eta_2}}}.$$

Remark 3.1 *It is easy to verify that $x(a) = x_1$. Substituting $t = a$ in (2.6) with the above control (3.1) and using the definition of $\mathcal{W}$ in assumption $(C.4)$, we get*

$$\begin{aligned}
x(t) &= \mathcal{Q}_q(a)[x_0 - h(x)] + \int_0^a (a-s)^{q-1}\mathcal{P}_q(a-s)Bu(s)\,\mathrm{d}s \\
&\quad + \int_0^a (a-s)^{q-1}\mathcal{P}_q(t-s)F(s,x(s))\,\mathrm{d}s \\
&= \mathcal{Q}_q(a)[x_0 - h(x)] \\
&\quad + \mathcal{W}\mathcal{W}^{-1}\Big[\Big(x_1 - \mathcal{Q}_q(a)[x_0 - h(x)] - \int_0^a (a-s)^{q-1}\mathcal{P}_q(t-s)F(s,x(s))\,\mathrm{d}s\Big)\Big] \\
&\quad + \int_0^a (a-s)^{q-1}\mathcal{P}_q(t-s)F(s,x(s))\,\mathrm{d}s \\
&= x_1.
\end{aligned}$$

Theorem 3.1 Assume that $(C.1)$-$(C.4)$ hold. Then, the fractional system (1.2) has a mild solution on J provided that

$$\Delta := b\big(1 + a^q \frac{\mathcal{T}_2\mathcal{T}_3}{\Gamma(1+q)}\big)M < 1. \tag{3.2}$$

Proof. Using the control (3.1), we define the operator $\mathcal{N} : \mathcal{C}(J, \mathcal{X}) \to \mathcal{C}(J, \mathcal{X})$ by

$$(\mathcal{N}x)(t) = \mathcal{Q}_q(t)[x_0 - h(x)] + \int_0^t (t-s)^{q-1}\mathcal{P}_q(t-s)Cu_x(s)\,\mathrm{d}s + \int_0^t (t-s)^{q-1}\mathcal{P}_q(t-s)F(s, x(s))\,\mathrm{d}s. \quad (3.3)$$

By virtue of Definition 2.7, it is easy to show that the existence of a mild solution of the nonlocal fractional system (1.2) is equivalent to the existence of a fixed point of the operator $\mathcal{N}$. We shall prove the above theorem through the following steps.

Step 1: There is a constant $\rho > 0$ such that $\mathcal{N}(\mathcal{E}_\rho) \subset \mathcal{E}_\rho$, where $\mathcal{E}_\rho = \{x \in \mathcal{X} : \|x\| \leq \rho\}$.

Suppose on the contrary that this is not true. Then for each positive number ρ, there exists a function $x^\rho(\cdot) \in \mathcal{E}_\rho$ such that $\mathcal{N}(x^\rho) \notin \mathcal{E}_\rho$ *i.e.*, $\|\mathcal{N}x_\rho(\tau)\| > \rho$, for some $\tau = \tau(\rho) \in J$, that is,

$$\begin{aligned}
\rho \leq \|\mathcal{N}x_\rho(\tau)\| &= \|\mathcal{Q}_q(\tau)[x_0 - h(x_\rho)]\| + \Big\| \int_0^\tau (\tau-s)^{q-1}\mathcal{P}_q(t-s)Bu_{x_\rho}(s)\,\mathrm{d}s \Big\| \\
&\quad + \Big\| \int_0^\tau (\tau-s)^{q-1}\mathcal{P}_q(t-s)F(s, x_\rho(s))\,\mathrm{d}s \Big\| \\
&\leq M(\|x_0\| + b\|x_\rho\| + \|h(0)\|) + \frac{qM}{\Gamma(1+q)} \int_0^\tau (\tau-s)^{q-1}\|u_{x_\rho}(s)\|\,\mathrm{d}s \\
&\quad + \frac{qM}{\Gamma(1+q)} \int_0^\tau (\tau-s)^{q-1} f(\|x(s)\|)m(s)\,\mathrm{d}s \\
&\leq M(\|x_0\| + b\rho + \|h(0)\|) + \frac{a^q M \mathcal{T}_2\mathcal{T}_3}{\Gamma(1+q)} \Big[\|x_1\| + M(\|x_0\| + b\rho + \|h(0)\|) \\
&\quad + \frac{qM}{\Gamma(1+q)} f(\rho) \Big[\int_0^a ((a-l)^{q-1})^{\frac{1}{1-\eta_1}}\,\mathrm{d}l \Big]^{1-\eta_1} \times \\
&\quad \Big[\int_0^a \|m(l)\|^{\frac{1}{\eta_1}}\,\mathrm{d}l \Big]^{\eta_1} \Big] + \frac{qM}{\Gamma(1+q)} \frac{a^{(\alpha_1+1)(q-1)}}{(\alpha_1+1)^{(q-1)}} f(\rho)\|m\|_{L^{\frac{1}{\eta_1}}}, \qquad (3.4)
\end{aligned}$$

where $\alpha_1 = \dfrac{q-1}{\eta_1 - 1} + 1$. Dividing both sides by ρ and taking the lower limit as $\rho \to \infty$, we get

$$1 \leq b\big(1 + a^q \frac{\mathcal{T}_2\mathcal{T}_3}{\Gamma(1+q)}\big)M,$$

which is a contradiction to (3.2). Hence, there exists $\rho > 0$ such that $\mathcal{N}(\mathcal{E}_\rho) \subset \mathcal{E}_\rho$ and $\mathcal{E}_\rho$ is bounded.

Step 2: The operator $\mathcal{N}$ is continuous on $\mathcal{E}_\rho$.
To this end, let $\{x_n\}_{n\geq 1} \subset \mathcal{E}_\rho$ such that $x_n \to x$ in $\mathcal{E}_\rho$. Set

$$\begin{aligned}
\mathcal{N}_1 &= \mathcal{Q}_q(t)[x_0 - h(x)], \\
\mathcal{N}_2 &= \int_0^t (t-s)^{q-1}\mathcal{P}_q(t-s)F(s, x(s))\,\mathrm{d}s \\
&\quad + \int_0^t (t-s)^{q-1}\mathcal{P}_q(t-s)Cu_x(s)\,\mathrm{d}s \ \text{ for } \ t \in J,
\end{aligned}$$

then $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2$. Therefore, since h is continuous, we have that

$$\begin{aligned}
\|\mathcal{N}_1(x_n) - \mathcal{N}_1(x)\| &= \|\mathcal{Q}_q(t)[h(x_n) - h(x)]\| \\
&\leq \|\mathcal{Q}_q(t)\|\|h(x_n) - h(x)\| \\
&\leq M\|h(x_n) - h(x)\| \to 0 \text{ as } n \to \infty. \qquad (3.5)
\end{aligned}$$

To see the continuity of $\mathcal{N}_2$, we first set $\mathbb{F}_n(s) = F(s, x_n(s))$ for every n and a.e.s, and $\mathbb{F}(s) = F(s, x(s))$ a.e. s. Thus by $(C.2)$-(i), $\mathbb{F}_n(s) \to \mathbb{F}(s)$ and by $(C.2)$,

$$\|\mathbb{F}_n(s)\| = \|F(s, x_n(s))\| \leq m(s) f(\|x_n\|) \leq m(s) f(\rho)$$

for every n and a.e. s. It follows from Lebesgue dominated convergence theorem that

$$\int_0^t \|\mathbb{F}_n(s) - \mathbb{F}(s)\| \, \mathrm{d}s \to 0 \text{ as } n \to \infty.$$

Moreover, we have that

$$\begin{aligned} \|\mathcal{N}_2(x_n) - \mathcal{N}_2(x)\| \leq \frac{Mq}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \|F(s, x_n(s)) - F(s, x(s))\| \, \mathrm{d}s \\ + \frac{Mq}{\Gamma(1+q)} \times \mathcal{T}_3 \sqrt{\frac{a^{2q-1}}{2q-1}} \|u_{x_n}(s) - u_x(s)\|_{L^2(J,\mathbb{U})}, \end{aligned}$$

where

$$\begin{aligned} \|u_{x_n}(s) - u_x(s)\|_{L^2(J,\mathbb{U})} \leq \mathcal{T}_3 \Big(M \|h(x_n) - h(x)\| \\ + \frac{qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \|F(s, x_n(s)) - F(s, x(s))\| \, \mathrm{d}s \Big). \end{aligned}$$

Accordingly, it follows that $\|\mathcal{N}_2(x_n) - \mathcal{N}_2(x)\| \to 0$ as $n \to \infty$, showing that $\mathcal{N}_2$ is continuous on $\mathcal{E}_\rho$. Hence $\mathcal{N}$ is continuous on $\mathcal{E}_\rho$.

Step 3: The Mönch condition holds.

Now consider $\mathcal{M} \subset \mathcal{E}_\rho$ is countable and $\mathcal{M} \subset conv(\{0\} \cup \mathcal{N}(\mathcal{M}))$. We have to show that $\mathcal{M}$ is relatively compact. To this end, it suffices to show that $\beta(\mathcal{M}) = 0$, where β is the Hausdorff MNC. Since $\mathcal{M}$ is countable, we can describe it as $\mathcal{M} = \{x_n\}_{n=1}^\infty$ and its relative compactness implies that $\mathcal{M}$ is also relatively compact. So we have to prove that $\mathcal{N}(\mathcal{M})$ is equibounded and equicontinuous on J in order to use Ascoli-Arzela's theorem. We show that $\mathcal{N}(\mathcal{M})$ is equicontinuous. For this purpose, let $y \in \mathcal{N}(\mathcal{M})$, and $0 \leq t_1 < t_2 \leq a$, there is $x \in \mathcal{M}$ such that $y = \mathcal{N}x$ and then

$$\begin{aligned} \|y(t_2) - y(t_1)\| &= \Big\| \mathcal{Q}_q(t)[x(0) - h(x)] + \int_0^{t_2} (t_2 - s)^{q-1} \mathcal{P}_q(t_2 - s) C u_x(s) \, \mathrm{d}s \\ &\quad + \int_0^{t_2} (t_2 - s)^{q-1} \mathcal{P}_q(t_2 - s) F(s, x(s)) \, \mathrm{d}s \\ &\quad - \mathcal{Q}_q(t)[x(0) - h(x)] - \int_0^{t_1} (t_1 - s)^{q-1} \mathcal{P}_q(t_1 - s) C u_x(s) \, \mathrm{d}s \\ &\quad - \int_0^{t_1} (t_1 - s)^{q-1} \mathcal{P}_q(t_1 - s) F(s, x(s)) \, \mathrm{d}s \Big\| \\ &= \Big\| (\mathcal{Q}_q(t_2) - \mathcal{Q}_q(t_1))[x(0) - h(x)] + \int_0^{t_2} (t_2 - s)^{q-1} \mathcal{P}_q(t_2 - s) C u_x(s) \, \mathrm{d}s \\ &\quad + \int_0^{t_2} (t_2 - s)^{q-1} \mathcal{P}_q(t_2 - s) F(s, x(s)) \, \mathrm{d}s \\ &\quad - \int_0^{t_1} (t_1 - s)^{q-1} \mathcal{P}_q(t_1 - s) C u_x(s) \, \mathrm{d}s \\ &\quad - \int_0^{t_1} (t_1 - s)^{q-1} \mathcal{P}_q(t_1 - s) F(s, x(s)) \, \mathrm{d}s \Big\| \end{aligned}$$

$$
\begin{aligned}
&\le \Big\|(\mathcal{Q}_q(t_2)-\mathcal{Q}_q(t_1))x_0\Big\| + \Big\|(\mathcal{Q}_q(t_2)-\mathcal{Q}_q(t_1))h(x)\Big\| \\
&\quad + \Big\| \int_0^{t_1} \Big[(t_2-s)^{q-1}\mathcal{P}_q(t_2-s)-(t_1-s)^{q-1}\mathcal{P}_q(t_1-s)\Big] Cu_x(s)\,\mathrm{d}s\Big\| \\
&\quad + \Big\| \int_{t_1}^{t_2} (t_2-s)^{q-1}\mathcal{P}_q(t_2-s)Cu_x(s)\,\mathrm{d}s\Big\| \\
&\quad + \Big\| \int_0^{t_1} \Big[(t_2-s)^{q-1}\mathcal{P}_q(t_2-s)-(t_1-s)^{q-1}\mathcal{P}_q(t_1-s)\Big] F(s,x(s))\,\mathrm{d}s\Big\| \\
&\quad + \Big\| \int_{t_1}^{t_2} (t_2-s)^{q-1}\mathcal{P}_q(t_2-s)F(s,x(s))\,\mathrm{d}s\Big\| \\
&:= \sum_{k=1}^{6} A_k,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= \Big\|(\mathcal{Q}_q(t_2)-\mathcal{Q}_q(t_1))x_0\Big\|, \\
A_2 &= \Big\|(\mathcal{Q}_q(t_2)-\mathcal{Q}_q(t_1))h(x)\Big\|, \\
A_3 &= \Big\| \int_0^{t_1} \Big[(t_2-s)^{q-1}\mathcal{P}_q(t_2-s)-(t_1-s)^{q-1}\mathcal{P}_q(t_1-s)\Big] Cu_x(s)\,\mathrm{d}s\Big\|, \\
A_4 &= \Big\| \int_{t_1}^{t_2} (t_2-s)^{q-1}\mathcal{P}_q(t_2-s)Cu_x(s)\,\mathrm{d}s\Big\|, \\
A_5 &= \Big\| \int_0^{t_1} \Big[(t_2-s)^{q-1}\mathcal{P}_q(t_2-s)-(t_1-s)^{q-1}\mathcal{P}_q(t_1-s)\Big] F(s,x(s))\,\mathrm{d}s\Big\|, \\
A_6 &= \Big\| \int_{t_1}^{t_2} (t_2-s)^{q-1}\mathcal{P}_q(t_2-s)F(s,x(s))\,\mathrm{d}(s)\Big\|.
\end{aligned}
$$

Firstly for $t_1 > 0$, by the norm-continuity of $\mathcal{Q}_q(t)$, we get A_1 tends to zero as $t_2 \to t_1$. Moreover, we have that

$$
A_2 = \Big\|(\mathcal{Q}_q(t_2)-\mathcal{Q}_q(t_1))h(x)\Big\| \le \Big\|(\mathcal{Q}_q(t_2)-\mathcal{Q}_q(t_1))\Big\| h_r,
$$

where $h_r = \sup\{\|h(x)\| : \|x\| \le r\}$. Now for $t_1 = 0$, we have that

$$
\| \mathcal{Q}_q(z)h(x)-h(x)\| \le \sup_{\omega\in\overline{h(\mathcal{E}_\rho)}} \|\mathcal{Q}_q(z)\omega-\omega\| \to 0, \text{as } z \to 0^+,
$$

by Lemma 2.5, since $\overline{h(\mathcal{E}_\rho)}$ is compact. Therefore A_2 tends to 0 as $t_2 \to t_1$.

$$
\begin{aligned}
A_3 \le &\mathcal{T}_2 \int_0^{t_1} (t_2-s)^{q-1}\|\mathcal{P}_q(t_2-s)-\mathcal{P}_q(t_1-s)\|\|u(s)\|\,\mathrm{d}s \\
&+ \frac{qM}{\Gamma(1+q)}\mathcal{T}_2 \int_{t_1}^{t_2} |\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]| \|u_x(s)\|\,\mathrm{d}s \\
\le &\left[\int_0^{t_1} [(t_2-s)^{q-1}\|\mathcal{P}_q(t_1-s)-\mathcal{P}_q(t_2-s)\|_{\mathcal{L}(\mathcal{X})}]^2\,\mathrm{d}s\right]^{\frac{1}{2}} \|u_x\|_{\mathbb{L}^2} \\
&+ \frac{qM}{\Gamma(1+q)}\left(\int_{t_1}^{t_2} \left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]^2\mathrm{d}s\right)^{\frac{1}{2}} \times \|u\|_{\mathbb{L}^2}
\end{aligned}
$$

$$= \left[\int_0^{t_1} \left[s^{q-1}\|\mathcal{P}_q(s)-\mathcal{P}_q(t_2-t_1+s)\|\right]^2 \mathrm{d}s\right]^{\frac{1}{2}} \times \|u_x\|_{\mathbb{L}^2}$$
$$+ \frac{qM}{\Gamma(1+q)}\left(\int_{t_1}^{t_2}\left[s^{q-1}-(t_2-t_1+s)^{q-1}\right]^2 \mathrm{d}s\right)^{\frac{1}{2}} \times \|u_x\|_{\mathbb{L}^2}.$$

Since

$$\lim_{t_2\to t_1} s^{q-1}\|\mathcal{P}_q(s)-\mathcal{P}_q(t_2-t_1+s)\|_{\mathcal{L}(\mathcal{X})} = 0, \quad \text{a.e} \quad s\in[0,t_1],$$
$$\lim_{t_2\to t_1} |s^{q-1}-(t_2-t_1+s)^{q-1}| = 0, \quad \text{a.e} \quad s\in[0,t_1],$$
$$s^{q-1}\|\mathcal{P}_q(s)-\mathcal{P}_q(t_2-t_1+s)\|_{\mathcal{L}(\mathcal{X})} \le \frac{2qM}{\Gamma(1+q)}\frac{1}{s^{1-q}},$$

and

$$|s^{q-1}-(t_2-t_1+s)^{q-1}| \le \frac{2}{s^{1-q}},$$

using the Lebesgue dominated convergence theorem, we can affirm that $\lim\limits_{t_2\to t_1} A_3 = 0$ uniformly for $g\in\Omega$.

Now, regarding to A_4 we have

$$A_4 \le \frac{qM}{\Gamma(1+q)}\left(\int_{t_1}^{t_2}|t_2-s|^{2(q-1)}\right)^{\frac{1}{2}}\|u_x\|_{\mathbb{L}^2}$$
$$= \frac{qM}{\Gamma(1+q)}\sqrt[p]{\left(\frac{1}{2q-1}\right)}(t_2-t_1)^{(2q-1)/2}\|u_x\|_{\mathbb{L}^2},$$

which implies that $\lim\limits_{t_2\to t_1} A_4 = 0$ uniformly for $v\in\Omega$.

Similar to the proofs for A_3 and A_4, we get A_5 and A_6 tend to zero as $t_2\to t_1$. The proof is complete. Hence, the set $\mathcal{N}(\mathcal{M})$ is equicontinuous for $t\in J$.

Now consider $\mathcal{M}\subset\mathcal{E}_\rho$ is countable and $\mathcal{M}\subset conv(\{0\}\cup\mathcal{N}(\mathcal{M}))$. We shall demonstrate that $\beta(\mathcal{M})=0$, β is the Hausdorff MNC. We can assume, without loss of generality, $\mathcal{M}=\{x_n\}_{n=1}^{\infty}$. We now show that $\mathcal{N}(\mathcal{M})$ is equibounded. To do this, we show that for $t\in[0,a]$, the set $\{\mathcal{N}(x)(t) : x\in\mathcal{M}\}$ is relatively compact. We achieve this by using the measure of noncompactness. We have that for $t=0$, the set $\{\mathcal{N}(x)(0) : x\in\mathcal{M}\} = \{x_0+h(x) : x\in\mathcal{M}\}$ is relatively compact in $\mathcal{X}$. Since h is compact, $\overline{h(\mathcal{M})}$ is compact also.

For $t\in(0,a]$, we have that

$$\beta\big(\{(\mathcal{N}_1x_n)(t)\}_{n=1}^{\infty}\big) \le \mathcal{N}_1\beta\big(\{\mathcal{Q}_q(t)[x(0)-h(x_n)]\}_{n=1}^{\infty}\big) = 0,$$

thanks to the compactness of h. By Lemma 2.3 and Theorem 2.7, we have that

$$\begin{aligned}\beta\big(\{(\mathcal{N}_2x_n)(t)\}_{n=1}^{\infty}\big) &\leq \beta\big(\{\int_0^t (t-s)^{q-1}\mathcal{P}_q(t-s)F(s,x_n(s))\,\mathrm{d}s\}_{n=1}^{\infty}\big)\\ &\quad+\beta\big(\{\int_0^t (t-s)^{q-1}\mathcal{P}_q(t-s)Cu_{x_n}(s)\,\mathrm{d}s\}_{n=1}^{\infty}\big)\\ &\leq \frac{2qM}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}g(s)\,\mathrm{d}s\cdot\beta(\mathcal{M}(t))\\ &\leq \frac{2qM\mathcal{T}_2}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\,\mathrm{d}s\cdot\beta(u_{x_n})\\ &= K_1+K_2,\end{aligned}$$

where

$$\begin{aligned}K_1 &\leq \frac{2qM}{\Gamma(1+q)}\beta(\mathcal{M}(t))\Big[\int_0^t \big[(t-s)^{q-1}\big]^{\frac{1}{1-\eta_2}}\,\mathrm{d}s\Big]^{1-\eta_2}\Big[\int_0^t \big[\|g(s)\|\big]^{\frac{1}{\eta_2}}\,\mathrm{d}s\Big]^{\eta_2}\\ &\leq \frac{2qM}{\Gamma(1+q)}a^{q-\eta_2}\Big(\frac{1-\eta_2}{\Gamma(q-\eta_2)}\Big)^{1-\eta_2}\|g\|_{L^{\frac{1}{\eta_2}}},\end{aligned}$$

and

$$\begin{aligned}K_2 &= \frac{2qM\mathcal{T}_2}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\,\mathrm{d}s\cdot\beta(u_{x_n})\\ &\leq \frac{2qM\mathcal{T}_2}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\,\mathrm{d}s\cdot\beta\Big(\mathcal{W}^{-1}\Big(\{x_1-h(x_n)-\mathcal{Q}_q(t)[x(0)-h(x_n)]\\ &\quad-\int_0^a (a-r)^{q-1}\mathcal{P}_q(a-r)F(r,x_n(r))\,\mathrm{d}r\}_{n=1}^{\infty}\Big)(s)\Big)\\ &\leq \frac{2qM\mathcal{T}_2}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}K_W(s)\,\mathrm{d}s\cdot\frac{qM}{\Gamma(1+q)}\int_0^a (a-r)^{q-1}g(r)\,\mathrm{d}r\cdot\beta(\mathcal{M}(s))\\ &\leq \frac{2q^2M^2\mathcal{T}_2}{(\Gamma(1+q))^2}a^{2q-\eta_0-\eta_2}\|g\|_{L^{\frac{1}{\eta_2}}}\|K_W\|_{L^{\frac{1}{\eta_0}}}.\end{aligned}$$

It follows that,

$$\begin{aligned}\beta\big(\mathcal{N}(\mathcal{M})(t)\big) &\leq \beta\big(\mathcal{N}_1(\mathcal{M})(t)\big)+\beta\big(\mathcal{N}_2(\mathcal{M})(t)\big)\\ &\leq \Big(1+\frac{qM\mathcal{T}_2}{\Gamma(1+q)}a^{q-\eta_0}\|K_W\|_{L^{\frac{1}{\eta_0}}}\Big)\frac{2qM}{\Gamma(1+q)}a^{q-\eta_2}\|g\|_{L^{\frac{1}{\eta_2}}}\beta\big(\mathcal{M}(t)\big)\\ &= \Omega\beta\big(\mathcal{M}(t)\big),\end{aligned}$$

where $\Omega=\Big(1+\frac{qM\mathcal{T}_2}{\Gamma(1+q)}a^{q-\eta_0}\|K_W\|_{L^{\frac{1}{\eta_0}}}\Big)\frac{2qM}{\Gamma(1+q)}a^{q-\eta_2}\|g\|_{L^{\frac{1}{\eta_2}}}<1$. But from Mönch's condition, we have

$$\beta\big(\mathcal{M}(t)\big)\leq\beta\big(conv(\{0\}\cup\mathcal{N}(\mathcal{M}(t)))\big)=\beta\big(\mathcal{N}(\mathcal{M}(t))\big)\leq\Omega\beta\big(\mathcal{M}(t)\big).$$

This implies that $\beta\big(\mathcal{M}(t)\big)=0$, since $\Omega<1$, which implies that $\beta\big(\mathcal{N}(\mathcal{M})(t)\big)=0$. This shows that $\overline{\mathcal{N}(\mathcal{M})(t)}$ is compact that is $\overline{\{\mathcal{N}(x)(t);x\in\mathcal{M}\}}$ is compact as desired. As a consequence

$\mathcal{N}(\mathcal{M})$ is equicontinuous and equibounded and therefore by Ascoli-Arzela's theorem, we have that $\mathcal{N}(\mathcal{M})$ is relatively compact. But

$$\beta\big(\mathcal{M}(t)\big) \leq \beta\big(conv(\{0\} \cup \mathcal{N}(\mathcal{M}(t)))\big) = \beta\big(\mathcal{N}(\mathcal{M}(t))\big) \leq \Omega\beta\big(\mathcal{N}(\mathcal{M}(t))\big) = 0.$$

This implies that $\beta\big(\mathcal{M}\big) = 0$, showing that $\bar{\mathcal{M}}$ is compact in $\mathcal{X}$ as desired. Thus $\mathcal{M}$ is relatively compact and the Mönch's condition is satisfied. Consequently, $\mathcal{N}$ has a fixed point x on $\mathcal{E}_\rho$, which is the mild solution of the system (1.2) satisfying $x(a) - h(x) = x_1$. □

4 An example

The following fractional integrodifferential system is taken under consideration to apply our results:

$$\begin{cases} {}^cD^q\xi(t,\theta) = \dfrac{\partial^2}{\partial\theta^2}\xi(t,\theta) + \displaystyle\int_0^t l(t-s)A\xi(s,\theta)\,\mathrm{d}s \\ \qquad\qquad\qquad + \omega\xi(t,\theta) + \dfrac{e^{-t}}{\lambda + e^t}\sin(\xi(t,\theta)),\ t \in J = [0,\pi], \\ \xi(t,0) = \xi(t,\pi) = 0, \\ \xi(0,\theta) = \xi_0(\theta) + \displaystyle\int_0^\pi \int_0^a \Pi(t,\theta)\ln(1 + |\xi(t,\theta)|^{\frac{1}{2}})\,\mathrm{d}t\,\mathrm{d}\theta,\ 0 < \theta < \pi, \end{cases} \tag{4.1}$$

where $\frac{1}{2} < q < 1$, $l(t) = e^{-t}$, for $t \geq 0$, $\mathcal{X} = L^2(0,\pi)$, $\mathbb{U} = L^2(0,\pi)$ is a separable Hilbert space, $\Pi \in C(J \times [0,\pi])$ and $\Pi(t,0) = \Pi(t,\pi) = 0$. We denote by $\widehat{P}(\lambda)$ the Laplace transform of a given function $P(t)$. The following theorem will be needed to give an explicit form to $R(t)$.

Theorem 4.1 *[8, Theorem 3.1] Assume that the following conditions are satisfied:*

a) A generates an analytic semigroup $(T(t))_{t\geq 0}$ and satisfies the following estimate:

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(\mathcal{X})} \leq \frac{M'}{|\lambda|},\ Re(\lambda) > 0,\ \textit{and}\ M' \geq 1.$$

b) $\|\Upsilon(t)\|_{\mathcal{L}(Y,\mathcal{X})} \leq b(t)$ for some $b(\cdot) \in \mathbb{L}^1_{loc}(\mathbb{R}^+)$ and $\Upsilon(t)x$ is strongly measurable for each $x \in Y$.

c) For $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$, $\widehat{\Upsilon}(\lambda)$ exists as an element of $\mathcal{L}(Y,\mathcal{X})$ and

$$\|\widehat{\Upsilon}(\lambda)\|_{\mathcal{L}(Y,\mathcal{X})} \leq \frac{N}{|\lambda|^\beta},\ \textit{for}\ \beta > 0,\ \textit{and}\ N \geq 1.$$

Then, equation (2.1) *has a unique resolvent operator defined on $\mathcal{X}$ by*

$$R(t)x = \begin{cases} T(t)x + R_1(t)x, & \textit{if}\ \ t > 0, \\ x,\ \textit{if}\ \ t = 0, \end{cases}$$

where

$$R_1(t)x = \frac{1}{2\pi i}\int_{\zeta-i\infty}^{\zeta+i\infty} e^{\lambda t}\sum_{j=1}^{+\infty}\left[(\lambda I - A)^{-1}\widehat{G}(\lambda)\right]^j (\lambda I - A)^{-1}x \,\mathrm{d}\lambda,$$

in which $\zeta^\beta > 2(2M' + 1)N$.

Let us define A by $Az = z''$ for $z \in H^2(0,\pi) \cap H_0^1(0,\pi)$. Here, $\Upsilon(t) = \delta(t)A$ for $t \geq 0$. The operators A and $\Upsilon(t)$, $t \geq 0$ satisfy the assumptions $(\mathbf{H_1})$ and $(\mathbf{H_2})$, respectively. Moreover, they satisfy conditions a), b) and c) in Theorem 4.1. Indeed, it is easily seen that condition a) holds with $M' = 1$. For condition b), let $x \in D(A)$, then $\Upsilon(t)x = e^{-t}Ax$ is strongly measurable. Moreover,

$$\|\Upsilon(t)x\| \leq e^{-t}(\|x\| + \|Ax\|),$$

which implies that condition b) holds with $b(t) = e^{-t}$. For condition c), let $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$, then $\widehat{G}(\lambda) = \dfrac{1}{\lambda+1}A$ exists as an element of $\mathcal{L}(D(A),\mathcal{X})$. Moreover, $\|\widehat{\Upsilon}(\lambda)x\| \leq \dfrac{1}{|\lambda|}(\|x\| + \|Ax\|)$, which implies that condition c) holds with $N = \beta = 1$.

It is well known that A generates an analytic C_0-semigroup $(T(t))_{t\geq 0}$ given by

$$T(t)w = \sum_{n=1}^{+\infty} e^{-n^2 t}\langle w, e_n\rangle e_n, \quad \text{for} \quad w = \sum_{n=1}^{+\infty}\langle w, e_n\rangle e_n \in \mathcal{X},$$

where $e_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$ for each $n \geq 1$. Let $w = \sum_{n=1}^{+\infty} w_n e_n \in \mathcal{X}$, then

$$(\lambda I - A)^{-1}w = \sum_{n=1}^{+\infty}\frac{1}{\lambda + n^2}w_n e_n,$$

and

$$(\lambda I - A)^{-1}Aw = \sum_{n=1}^{+\infty}\frac{-n^2}{\lambda + n^2}w_n e_n.$$

Hence,

$$R(t)w = \sum_{n=1}^{+\infty} b_n(t) w_n e_n,$$

where

$$\begin{aligned}
b_n(t) &= e^{-n^2 t} + \frac{1}{2\pi i}\int_{\zeta-i\infty}^{\zeta+i\infty} e^{\lambda t}\sum_{j=1}^{+\infty}\left[\frac{-n^2}{(\lambda+n^2)(\lambda+1)}\right]^j \frac{1}{(\lambda+n^2)}\,\mathrm{d}\lambda \\
&= e^{-n^2 t} + \frac{1}{2\pi i}\int_{\zeta-i\infty}^{\zeta+i\infty}\frac{-n^2 e^{\lambda t}}{(\lambda+n^2)[n^2 + (\lambda+n^2)(\lambda+1)]}\,\mathrm{d}\lambda \\
&= e^{-n^2 t} + \frac{1}{2\pi i}\int_{\zeta-i\infty}^{\zeta+i\infty} F_n(\lambda)e^{\lambda t}\,\mathrm{d}\lambda,
\end{aligned}$$

for

$$F_n(\lambda) = \frac{-n^2}{(\lambda+n^2)[n^2+(\lambda+n^2)(\lambda+1)]}, \quad n \geq 1.$$

Set

$$\begin{aligned} x(t)(\theta) &= \xi(t,\theta),\ 0<\theta<\pi, \\ F(t,x(t))(\theta) &= \frac{e^{-t}}{\lambda+e^t}\sin(\xi(t,\theta)),\ 0<\theta<\pi, \\ (Cu)(t)(\theta) &= \omega\zeta(t,\theta),\ 0<\theta<\pi, \\ \mathrm{h}(\vartheta)(\xi) &= \int_0^1\int_0^b \Pi(t,\xi)\log(1+|\vartheta(t,r)|^{\frac{1}{2}})\ \mathrm{d}t\,\mathrm{d}r, \quad 0<\theta<\pi. \end{aligned}$$

Equation (4.1) is then transformed into the following form

$$\begin{cases} {}^cD^q x(t) = Ax(t) + Cu(t) + \displaystyle\int_0^t \Upsilon(t-s)x(s)\,\mathrm{d}s + F(t,x(t)),\ t\in J=[0,\pi], \\ x(0)+h(x) = x_0. \end{cases} \tag{4.2}$$

We have that F is Lipschitz continuous with respect to the second variable and moreover we have that

$$\|F(t,x)\| \leq \frac{e^{-t}}{N+e^{-t}}.$$

Consequently F satisfies $(C.2)$-(i), $(C.2)$-(ii) and $(C.2)$-(iii) with $f:\mathbb{R}^+\to\mathbb{R}^+$ defined by $f(x) = 1$.

Lemma 4.2 *The map* $\mathrm{h}:\mathcal{C}(J,\mathcal{X})\to\mathcal{X}$ *defined by*

$$\mathrm{h}(\vartheta)(\xi) = \int_0^\pi\int_0^b \Pi(t,\xi)\log(1+|\vartheta(t,r)|^{\frac{1}{2}})\ \mathrm{d}t\,\mathrm{d}r$$

is compact.

Proof. Let $E\subset\mathcal{C}([0,\pi],\mathcal{X})$ be bounded. Then

$$\|\mathrm{h}(x)\|_{\mathcal{C}([0,\pi],\mathcal{X})} \leq \pi \max_{t\in[0,\pi],\xi\in[0,1]} |\Pi(t,\xi)|[\|x\|_{\mathcal{C}(J,\mathcal{X})}+1], \quad \text{for all } x\in E.$$

So $\mathrm{h}(E)$ is bounded. Now since Π is uniformly continuous on $[0,1]\times[0,\pi]$, it follows that $\mathrm{h}(E)$ is equicontinuous on $[0,1]$. Therefore, by Ascoli-Alzela's theorem, $\mathrm{h}(E)$ is relatively compact. Hence h is compact and $(C.3)$ is satisfied. It is clear that with $h_r = \sup\{\|\mathsf{h}(x)\| : \|x\|\leq r\}$, we get $\lim_{r\to\infty}\inf\frac{h_r}{r} = 0$. □

We assume that the operator $\mathcal{W}$ defined as

$$(\mathcal{W}u)(\cdot) = \int_0^\pi (\pi-s)^{-q}\mathcal{P}_q(1-s)\omega\zeta(s,\cdot)\,\mathrm{d}s$$

satisfies (C_4). Accordingly, all conditions of Theorem 3.1 are fulfilled and we conclude that system (4.1) is nonlocal controllable on J.

5 Conclusion

The manuscript presents findings on the controllability of fractional integrodifferential equations with nonlocal initial conditions within Banach spaces. This is achieved through the application of the Hausdorff Measure of Noncompactness and the Mönch fixed point theorem. The findings indicate that even in the absence of compactness in the resolvent operator related to the linear homogeneous component, the Mönch fixed point theorem remains a viable tool for deriving controllability results when certain sufficient conditions are met. Future research could investigate a variety of promising avenues. One potential direction is to utilize the existing framework for systems characterized by more complex dynamics, including those that feature nonlinear or time-dependent coefficients. Furthermore, the suggested methodology might be adapted to represent more sophisticated systems in disciplines such as engineering, economics, or biology, where fractional stochastic models are gaining relevance. Implementing numerical simulations to evaluate the practical application of fractional integrodifferential systems would additionally support the theoretical findings and aid in assessing their efficacy in real-world settings.

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