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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A DISCRETE FOURTH-ORDER BOUNDARY VALUE PROBLEM

MAISAM BOROUN*, **SHAPOUR HEIDARKHANI**†
Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran

MUSTAFA AVCI‡
Department of Applied Mathematics, Faculty of Science and Technology,
Athabasca University, AB, Canada

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Abstract. The purpose of this paper is to establish some results on the existence of multiple solutions for a discrete fourth-order problem. The approach is based on variational methods. Some applications and examples are provided to illustrate the obtained results.

Keywords: Discrete equation, fourth-order boundary value problem, multiple solutions, variational methods.

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1 Introduction

Recently, some authors have studied problems of the type

$$\begin{cases} \Delta^4 u(t-2) + \eta\Delta^2 u(t-1) - \zeta u(t) \\ \qquad = \lambda f(t,u(t)) + \mu g(t,u(t)) + h(u(t)), \quad t \in [a+1, b+1]_{\mathbb{Z}}, \\ u(a) = \Delta^2 u(a-1) = 0, \quad u(b+2) = \Delta^2 u(b+1) = 0, \end{cases} \tag{$D^{f,g,h}_{\lambda,\mu}$}$$

*e-mail address: m.boroun.m.1989@gmail.com
†e-mail address: sh.heidarkhani@razi.ac.ir
‡e-mail address: mavci@athabascau.ca

where Δ denotes the forward difference operator, defined by

$$\Delta u(t) = u(t+1) - u(t),$$
$$\Delta^n u(t) = \Delta(\Delta^{n-1} u(t)),$$

a, b are two fixed integers, $[a+1, b+1]_{\mathbb{Z}}$ is the discrete interval $\{a+1, a+2, \cdots, b+1\}$, $f, g\colon [a+1, b+1]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ are two functions which are continuous in their second variable, and $h\colon \mathbb{R} \to \mathbb{R}$ is a strictly monotone Lipschitz continuous function with a Lipschitz constant $L > 0$ satisfying $h(0) = 0$. Moreover, the four real parameters η, ζ, λ and μ are such that $\lambda > 0$, $\mu \geq 0$, $\eta^2 + 4\zeta \geq 0$,

$$\eta < 8 \sin^2 \frac{\pi}{2(b-a+2)} \quad \text{and} \quad \zeta + 4\eta \sin^2 \frac{\pi}{2(b-a+2)} < 16 \sin^4 \frac{\pi}{2(b-a+2)}.$$

In the last decades, much attention has been concentrated on fourth-order difference equations which are derived from various discrete elastic beam problems. Some of the significant results on the existence and multiplicity of solutions for discrete fourth-order boundary value problems have been obtained using classical methods, such as fixed point theory and fixed point index theory [1, 2, 3, 9, 11, 12, 22], critical point theory [8, 13, 19], and the Krein–Rutman theorem and bifurcation theory [16, 17, 18].

In [3], Anderson *et al.*, using the Guo–Krasnosel'skii fixed point theorem and the Leggett–Williams theorem, studied the existence, multiplicity and non-existence of non-trivial solutions to the following problem

$$\begin{cases} \Delta^4 u(t-2) + \eta \Delta^2 u(t-1) = \lambda f(t, u(t)), & t \in [a+1, b+1]_{\mathbb{Z}}, \\ u(a) = \Delta^2 u(a-1) = 0, \quad u(b) = \Delta^2 u(b-1) = 0. \end{cases} \tag{1.1}$$

In [12], applying variational methods and critical point theory, some multiplicity results for discrete fourth-order boundary value problems with four parameters were established. Some closely related problems have been also recently studied in the papers [14, 15].

Let us consider the following condition.

(j_1) Let $h\colon \mathbb{R} \to \mathbb{R}$ be a strictly monotone function such that $h(0) = 0$, which is Lipschitz continuous with constant $L > 0$; that is, for all $x_1, x_2 \in \mathbb{R}$ we have

$$|h(x_1) - h(x_2)| \leq L|x_1 - x_2|.$$

We further assume that $0 < L < \frac{1}{2}\varepsilon_1$, where ε_1 will be introduced later.

Here, we deal with the problem $(D^{f,g,h}_{\lambda,\mu})$ when the continuous functions f and g have the subcritical growth and, via variational methods, we obtain the existence of at least one, two and three solutions whenever the parameters λ and μ belong to certain positive intervals. The main tools are critical point theorems established in [5, 6, 7]. Here, as an example, we present a special case of our results.

Theorem 1.1 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying*

$$\lim_{x\to 0^+} \frac{f(x)}{x^2} = +\infty. \tag{1.2}$$

Then, for each $\lambda \in (0, \lambda^*)$, where

$$\lambda^* = \frac{c^2\varepsilon_1}{2(b-a)\max_{|x|\le c}\int_0^x f(s)\,\mathrm{d}s}$$

for $c > 0$, the problem $(D^{f,g,0}_{\lambda,0})$ admits at least one non-trivial solution. (*The constant ε_1 will be specified later.*)

2 Preliminaries

We first introduce some basic notations. Let

$$E = \{u = \{u(t)\}_{t=a+1}^{b+1} : u(t) \in \mathbb{R}\}.$$

Then, E is a $(b-a+1)$-dimensional Hilbert space under the following inner product and norm

$$(u, v) = \sum_{t=a+1}^{b+1} u(t)v(t), \quad \|u\| = \left(\sum_{t=a+1}^{b+1} |u(t)|^2\right)^{\frac{1}{2}}, \quad u, v \in E.$$

Clearly, the following inequality is true

$$\max_{t\in[a+1,b+1]} |u(t)| \le \|u\|. \tag{2.1}$$

For $t \in [a+1, b+1]_{\mathbb{Z}}$ and $x \in \mathbb{R}$ set

$$F(t, x) = \int_0^x f(t, s)\,\mathrm{d}s, \quad G(t, x) = \int_0^x g(t, s)\,\mathrm{d}s, \quad H(x) = \int_0^x h(s)\,\mathrm{d}s.$$

We define the operators $K, T, A_\lambda\colon E \to E$, respectively, by

$$Ku(t) = \sum_{k=a+1}^{b+1} G(t, k)u(k), \quad Tu(t) = f(t, u(t)) + \frac{\mu}{\lambda} g(t, u(t)) + \frac{1}{\lambda} h(u(t)),$$

$$A_\lambda = \lambda KT, \quad u \in E,\ t \in [a+1, b+1]_{\mathbb{Z}}.$$

Since E is a $(b-a+1)$-dimensional Hilbert space and f, g are continuous in their second variable and $h\colon \mathbb{R} \to \mathbb{R}$ is a strictly monotone Lipschitz continuous function, the operators K and T are continuous. Consequently, $A_\lambda\colon E \to E$ is completely continuous.

Lemma 2.1 ([10, Lemma 2.1]) *Let $v \in E$ and $i \in \{1, 2\}$ be fixed. Also, let r_1, r_2 be the roots of the polynomial $P(r) = r^2 + \eta r - \zeta$. By the assumption on η and ζ, it follows that $r_1 \ge r_2 > -4\sin^2\frac{\pi}{2(b-a+2)}$. Then, the problem*

$$\begin{cases} -\Delta^2 u(t-1) + r_i u(t) = v(t), & t \in [a+1, b+1]_{\mathbb{Z}}, \\ u(a) = 0, \quad u(b+2) = 0, \end{cases}$$

has a unique solution

$$u(t) = \sum_{k=a+1}^{b+1} \mathbb{G}_i(t,k)v(k), \quad t \in [a, b+2]_{\mathbb{Z}},$$

where $\mathbb{G}_i(t,k)$ *is given by*

$$\mathbb{G}_i(t,k) = \frac{1}{\rho(1,0)\rho(b+2,a)} \begin{cases} \rho(t,a)\rho(b+2,k), & a \le t \le k \le b+1, \\ \rho(k,a)\rho(b+2,t), & a+1 \le k \le t \le b+2, \end{cases} \tag{2.2}$$

with

(i) $\rho(t,k) = \sin\varphi(t-k)$ *and* $\varphi = \arctan\dfrac{\sqrt{-r_i(r_i+4)}}{2+r_i}$, *when* $-4\sin^2\dfrac{\pi}{2(b-a+2)} < r_i < 0$,

(ii) $\rho(t,k) = t-k$, *when* $r_i = 0$,

(iii) $\rho(t,k) = \iota^{t-k} - \iota^{t-k}$ *and* $\iota = \dfrac{r_i + 2 + \sqrt{r_i(r_i+4)}r_i}{2}$, *when* $r_i > 0$.

Lemma 2.2 ([10, Lemma 2.2]) *Let* $w \in E$ *be fixed. Then, the following linear discrete fourth-order boundary value problem*

$$\begin{cases} \Delta^4 u(t-2) + \eta\Delta^2 u(t-1) - \zeta u(t) = w(t), & t \in [a+1, b+1]_{\mathbb{Z}}, \\ u(a) = \Delta^2 u(a-1) = 0, \quad u(b+2) = \Delta^2 u(b+1) = 0, \end{cases}$$

has a unique solution $u = \{u(t)\}_{t=a-1}^{b+3}$ *with*

$$\begin{aligned} u(t) &= \sum_{k=a+1}^{b+1} \left(\sum_{s=a+1}^{b+1} \mathbb{G}_1(t,s)\mathbb{G}_2(s,k) \right) w(k) \\ &= \sum_{k=a+1}^{b+1} \left(\sum_{s=a+1}^{b+1} \mathbb{G}_2(t,s)\mathbb{G}_1(s,k) \right) w(k), \quad t \in [a+1, b+1]_{\mathbb{Z}}. \end{aligned}$$

Lemma 2.3 ([10, Lemma 2.3]) *The linear continuous operator* $K\colon E \to E$ *is symmetric,* i.e., $(Ku, v) = (u, Kv)$ *for all* $u, v \in E$.

Lemma 2.4 ([10, Lemma 2.4]) *The linear continuous operator* $K\colon E \to E$ *has* $(b-a+1)$ *eigenvalues*

$$\frac{1}{\varepsilon_1}, \ \frac{1}{\varepsilon_2}, \ \ldots, \ \frac{1}{\varepsilon_{b-a+1}}$$

with the corresponding orthonormal eigenfunctions $e_1, e_1, \ldots, e_{b-a+1}$, *where* $e_k = \{e_k(t)\}_{t=a+1}^{b+1}$ *and*

$$e_k(t) = \sqrt{\frac{2}{b-a+2}} \sin\frac{t-a}{b-a+2}k\pi, \quad t \in [a+1, b+1]_{\mathbb{Z}},$$

for $k = 1, 2, \ldots, b-a+1$. *In addition, the algebraic multiplicity of each eigenvalue is* 1.

By Lemmas 2.1 and 2.2, we have $\mathbb{G}_i(t,k) > 0$ and $\mathbb{G}_i(t,k) = \mathbb{G}_i(k,t)$ for all $t,k \in [a+1,b+1]_{\mathbb{Z}}$ and $i = 1,2$. Let

$$\mathbb{G}(t,k) = \sum_{s=a+1}^{b+1} \mathbb{G}_1(t,s)\mathbb{G}_2(s,k), \quad t,k \in [a+1,b+1]_{\mathbb{Z}}.$$

Clearly, $\mathbb{G}(t,k) = \mathbb{G}(k,t)$ for all $t,k \in [a+1,b+1]_{\mathbb{Z}}$.

Remark 2.5 *From Lemma* 2.2*, it can easily be seen that the fixed point* $u = \{u(t)\}_{t=a+1}^{b+1} \in E$ *of* A_λ *is exactly the solution* $u = \{u(t)\}_{t=a-1}^{b+3}$ *of* $(D_{\lambda,\mu}^{f,g,h})$*, where* $u(a-1) = -u(a+1)$*,* $u(a) = 0, u(b+2) = 0$*,* $u(b+3) = -u(b+1)$*.*

Again from Lemmas 2.3 and 2.4, we obtain that $Ku(t) = \frac{1}{\varepsilon}u(t)$ for $u \in E, t \in [a+1,b+1]_{\mathbb{Z}}$, where $\frac{1}{\varepsilon} = \frac{1}{\varepsilon_k}$. From [10] we also know that

$$\varepsilon_k = 2\cos\frac{2k\pi}{b-a+2} + (2\eta - 8)\cos\frac{k\pi}{b-a+2} + 6 - 2\eta - \xi, \;\; k \in [1, b-a+1]_{\mathbb{Z}},$$

and

$$0 < \varepsilon_1 < \ldots < \varepsilon_{b-a+1}. \tag{2.3}$$

Moreover, the operator K has a unique inverse defined by $K^{-1}u = \varepsilon u$ for $u \in E$, where $\varepsilon = \varepsilon_k$, $k \in [1, b-a+1]_{\mathbb{Z}}$.

For $u \in E$ we define the functionals $\Phi, \Psi \colon E \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}(K^{-1}u, u) - \sum_{t=a+1}^{b+1} H(u(t)), \tag{2.4}$$

and

$$\Psi(u) = \sum_{t=a+1}^{b+1} \Big[F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))\Big]. \tag{2.5}$$

For every $\lambda > 0$ put $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$. Then, I_λ is continuously Fréchet differentiable and its derivative at the point $u \in E$ is

$$\begin{aligned} I'_\lambda(u) &= \Phi'(u) - \lambda\Psi'(u) \\ &= K^{-1}u - \sum_{t=a+1}^{b+1} h(u(t)) - \lambda \sum_{t=a+1}^{b+1} \Big[f(t,u(t)) + \frac{\mu}{\lambda}g(t,u(t))\Big] \\ &= K^{-1}u - \lambda Tu. \end{aligned} \tag{2.6}$$

Definition 2.6 *Let Φ and Ψ be two continuously differentiable functionals defined on a real Banach space E, and fix $r \in \mathbb{R}$. The functional $I = \Phi - \Psi$ is said to satisfy the Palais–Smale condition cut off above at r (in short $(P.S.)^r$) if any sequence $\{u_n\}_{n\in\mathbb{N}}$ in E such that*

(a) $\{I(u_n)\}$ *is bounded,*

(b) $\lim_{n\to\infty} \|I'(u_n)\|_{E^*} = 0$,

(c) $\Phi(u_n) < r$ *for each* $n \in \mathbb{N}$,

has a convergent subsequence.

Remark 2.7 *Since the operator equation $u = A_\lambda u$ is equivalent to the operator equation $K^{-1}u = \lambda T u$, we know that every critical point of the functional I_λ in E is a solution of $(D^{f,g,h}_{\lambda,\mu})$. Therefore, it will be enough only to find the critical points of the functional I_λ in E.*

The following theorem is a particular case of [4, Theorem 5.1], and it is the main tool we are going to use in the next section.

Theorem 2.8 (see [5, Theorem 2.3]) *Let E be a real Banach space, and let $\Phi, \Psi\colon E \to \mathbb{R}$ be two continuously differentiable functionals such that $\inf_{x\in E} \Phi(x) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{x} \in E$ with $0 < \Phi(\bar{x}) < r$ such that*

(a_1) $\dfrac{\sup_{\Phi(x)\le r} \Psi(x)}{r} < \dfrac{\Psi(\bar{x})}{\Phi(\bar{x})}$,

(a_2) *for each*

$$\lambda \in \left(\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)} \right)$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies $(P.S.)^r$ condition.

Then, for each

$$\lambda \in \left(\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)} \right)$$

there is $x_{0,\lambda} \in \Phi^{-1}((0,r))$ such that $I'_\lambda(x_{0,\lambda}) = 0$ and $I_\lambda(x_{0,\lambda}) \le I_\lambda(x)$ for all $x \in \Phi^{-1}((0,r))$.

Other next tool is the following abstract result.

Theorem 2.9 ([5, Theorem 3.2]) *Let E be a real Banach space, and let $\Phi, \Psi\colon E \to \mathbb{R}$ be two continuously differentiable functionals such that Φ is bounded from below and $\Phi(x) = \Phi(0) = \Psi(0) = 0$. Fix $r > 0$ and assume that for each*

$$\lambda \in \left(0, \frac{r}{\sup_{u\in\Phi^{-1}((-\infty,r))} \Psi(u)} \right)$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies $(P.S.)$ condition and is unbounded from below. Then, for each

$$\lambda \in \left(0, \frac{r}{\sup_{u\in\Phi^{-1}((-\infty,r))} \Psi(u)} \right)$$

the functional I_λ admits two distinct critical points.

Finally, we recall the following result of Bonanno and Marano (see [7]), stated here in a convenient form.

Theorem 2.10 ([7, Theorem 3.6]) *Let E be a reflexive real Banach space. Moreover, let $\Phi\colon E \to \mathbb{R}$ be a coercive, continuously differentiable and sequentially weakly lower semi-continuous functional whose continuous derivative admits a continuous inverse on E^*. Also, let $\Psi\colon E \to \mathbb{R}$ be a continuously differentiable functional with compact derivative such that*

$$\inf_{x\in E} \Phi(x) = \Phi(0) = \Psi(0).$$

Assume that there exist $r > 0$ and $\bar{x} \in E$ with $r < \Phi(\bar{x})$ such that

(b$_1$) $\dfrac{\sup_{\Phi(x)\le r} \Psi(u)}{r} < \dfrac{\Psi(\bar{x})}{\Phi(\bar{x})}$,

(b$_2$) *for each*

$$\lambda \in \Lambda := \left(\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(u)} \right)$$

the function $\Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in E.

Remark 2.11 *Throughout the abstract results, E denotes a real Banach space. In the present work, however, E specifically represents a finite-dimensional Hilbert space. This distinction causes no difficulty, since every finite-dimensional Hilbert space is naturally a Banach space under its induced norm. Consequently, all cited theoretical results remain valid in our framework.*

3 Main results

In this section, we deal with the existence of one solution for the problem $(D^{f,g,h}_{\lambda,\mu})$. For a real parameter $c > 0$ set

$$F_{\max} = \sum_{t=a+1}^{b+1} \max_{|x|\le c} F(t,x),$$

$$G_{\max} = \sum_{t=a+1}^{b+1} \max_{|x|\le c} G(t,x),$$

$$\sigma_1 = \left| \frac{c^2(\varepsilon_1 - L) - 2\lambda F_{\max}}{2G_{\max}} \right|,$$

$$\sigma_2 = \left| \frac{(\varepsilon_{b+a+1} + L)(b-a+1)d^2 - 2\lambda F_{\max}}{2\min\{0, G_{\max}\}} \right|,$$

$$\sigma_3 = \frac{\varepsilon_1}{\max\left\{0, 4\limsup_{|x|\to+\infty} \dfrac{G_{\max}}{x^2}\right\}},$$

$$\sigma = \min\{\sigma_1, \sigma_2, \sigma_3\}.$$

Now, we have the following theorem.

Theorem 3.1 *Let $f\colon [a+1,b+1]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Moreover, assume that*

(j$_2$) $\displaystyle\limsup_{x\to 0^+} \frac{\inf_{t\in[a+1,b+1]_{\mathbb{Z}}} F(t,x)}{x^2} = +\infty.$

Then, setting

$$\lambda^* = \frac{(\varepsilon_1 - L)c^2}{2F_{\max}},$$

where c is a positive constant and ε_1 is given in Lemma 2.4, for each parameter $\lambda \in (0,\lambda^*)$ and for each continuous function $g\colon [a+1,b+1]\times\mathbb{R}\to\mathbb{R}$ satisfying

$$\limsup_{|x|\to+\infty} \frac{\sum_{t=a+1}^{b+1} G(t,x)}{x^2} < +\infty, \tag{3.1}$$

and any $\mu \in [0,\sigma)$, the problem $(D_{\lambda,\mu}^{f,g,h})$ admits at least one non-trivial solution.

Proof. Our aim is to apply Theorem 2.8 to the space E, endowed with its standard norm, and to the functionals $\Phi, \Psi\colon E \to \mathbb{R}$ defined in (2.4) and (2.5), respectively. We prove that the functionals Φ and Ψ satisfy the assumptions of Theorem 2.8. By the definitions of Φ and Ψ, it is clear that both functionals continuously differentiable. Moreover, since h is strictly monotone, we get that the continuous derivative of Φ admits a continuous inverse on E^*. Now, we claim that the continuous derivative of Ψ is compact. Assume that there is a constant $M_1 > 0$ such that $\|u\| \le M_1$. Combining this with the continuity of f and g, and arguing as in [21], we conclude that $\Psi'(u)$ is uniformly bounded. Since f and g are continuous, they are uniformly continuous on $[a+1,b+1]_{\mathbb{Z}}\times[-M_1,M_1]$. In addition, since Ψ is continuous, it is also sequentially upper semi-continuous. We observe that Φ is sequentially lower semi-continuous. Moreover, since $\sum_{t=a+1}^{b+1} H(u(t))$ is continuous, putting $M(u) = \sum_{t=a+1}^{b+1} H(u(t))$ for $u \in E$ will lead to $M(u_n) \to M(u)$ as $u_n \to u$ in E. On the other hand, since E is a finite-dimensional Hilbert space and the inner product is continuous with respect to norm convergence in both variables, it follows that

$$\liminf_{n\to\infty} \Phi(u_n) = \liminf_{n\to\infty} \frac{(K^{-1}u_n,u_n)}{2} - \lim_{n\to\infty} M(u_n) \ge (K^{-1}u_n,u_n) - Mu = \Phi(u).$$

Also, as shown in [21], the functional Φ is coercive and

$$\frac{\varepsilon_1 - L}{2}\|u\|^2 \le \Phi(u) \le \frac{\varepsilon_{b-a+1} + L}{2}\|u\|^2.$$

Set

$$r = \frac{\varepsilon_1 - L}{2}c^2.$$

From (2.1), arguing as in [21], we have

$$\sup_{\Phi(u)\le 1} \Psi(u) \le \sum_{t=a+1}^{b+1} \max_{|x|\le c} F(t,x) + \frac{\mu}{\lambda}\sum_{t=a+1}^{b+1} \max_{|x|\le c} G(t,x) = F_{\max} + \frac{\mu}{\lambda}G_{\max}, \tag{3.2}$$

which implies that

$$\frac{\sup_{\Phi(u)\le 1} \Psi(u)}{r} \le \frac{2}{c^2(\varepsilon_1 - L)}\Big(F_{\max} + \frac{\mu}{\lambda}G_{\max}\Big).$$

If $G_{\max} \le 0$, by the fact that

$$\lambda < \frac{(\varepsilon_1 - L)c^2}{2F_{\max}},$$

we obtain

$$\frac{\sup_{\Phi(u)\le 1} \Psi(u)}{r} \le \frac{2F_{\max}}{(\varepsilon_1 - L)c^2} < \frac{1}{\lambda}. \tag{3.3}$$

If $G_{\max} > 0$, since $\lambda \in \Lambda$, we get

$$\frac{\sup_{\Phi(u)\le 1} \Psi(u)}{r} \le \frac{2F_{\max}}{(\varepsilon_1 - L)c^2} + \frac{2G_{\max}}{(\varepsilon_1 - L)c^2} \cdot \frac{\left|\frac{(\varepsilon_1 - L)c^2 - 2\lambda F_{\max}}{2G_{\max}}\right|}{\lambda} = \frac{1}{\lambda}. \tag{3.4}$$

By choosing $\bar{u} = d$ and using (j_1), together with the condition $c^2 < (b-a+1)d^2$, we obtain

$$\Phi(\bar{u}) \ge \frac{\varepsilon_1 - L}{2} c^2 = r.$$

Furthermore, we have

$$\frac{\Psi(\bar{u})}{\Phi(\bar{u})} \ge \frac{2\sum_{t=a+1}^{b+1} F(t,d)}{(\varepsilon_{b-a+1} + L)(b-a+1)d^2} + \frac{2\mu\sum_{t=a+1}^{b+1} G(t,d)}{\lambda(\varepsilon_{b-a+1} + L)(b-a+1)d^2}.$$

If $G_{\max} > 0$, from [21] we get

$$\frac{\Psi(\bar{u})}{\Phi(\bar{u})} \ge \frac{2\sum_{t=a+1}^{b+1} F(t,d)}{(\varepsilon_{b-a+1} + L)(b-a+1)d^2} > \frac{1}{\lambda}. \tag{3.5}$$

If $G_{\max} < 0$, since $\lambda \in \Lambda$, we obtain

$$\frac{\Psi(\bar{u})}{\Phi(\bar{u})} \ge \frac{1}{\lambda}. \tag{3.6}$$

Therefore, from (3.3)–(3.6), condition (a_1) of Theorem 2.8 holds. Also, from (2.1), (2.6) and the results in [21], we can see that for every $u \in E$ there exists a positive constant δ_μ such that

$$\sum_{t=a+1}^{b+1} G(t,u(t)) \le \frac{\varepsilon_1}{4\mu}\|u\|^2 + \delta_\mu. \tag{3.7}$$

Moreover, from condition (j_2), we have $\sum_{t=a+1}^{b+1} F(t,u(t)) \le z$ for $(t,x) \in [a+1,b+1] \times \mathbb{R}$, and hence

$$\lambda \sum_{t=a+1}^{b+1} F(t,u(t)) \le \frac{(\varepsilon_1 - L)c^2 z}{2F_{\max}}. \tag{3.8}$$

We also obtain

$$\Phi(u) - \lambda\Psi(u) \ge \frac{\varepsilon_1 - 2L}{4\chi}\|u\|^2 - \mu\delta_\mu - \frac{(\varepsilon_1 - L)c^2 z}{2F_{\max}},$$

which means that condition (a_2) of Theorem 2.8 holds. Hence, for each

$$\lambda \in \left(\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{1}{\sup_{\Phi(x)\le 1} \Psi(x)}\right)$$

Theorem 2.8 guarantees that u is a non-trivial solution of the problem $(D^{f,g,h}_{\lambda,\mu})$. □

Remark 3.2 *Theorem* 1.1 *in the introduction immediately follows from Theorem* 3.1.

Example 3.3 *Let* $\eta = \xi = 0$*,* $a = 0$ *and* $b = 4$*. Consider the following discrete fourth-order problem*

$$\begin{cases} \Delta^4 u(t-2) = \lambda f(t,u(t)) + \mu g(t,u(t)) + h(u(t)), \quad t \in [1,5]_{\mathbb{Z}}, \\ u(0) = \Delta^2 u(-1) = 0, \quad u(6) = \Delta^2 u(5) = 0, \end{cases} \tag{3.9}$$

where

$$f(t,x) = t\sqrt[3]{x^4}, \quad g(t,x) = 2tx, \quad h(x) = \arctan 0.5x, \quad t \in [1,5]_{\mathbb{Z}},\ x \in \mathbb{R}.$$

Taking into account that $\varepsilon_1 = 0.8$*,* $\varepsilon_5 = 1.2$*,* $L = 0.3$ *and* $c = 2$*, it is obvious that* f, g *are two continuous functions and* h *is a monotone Lipschitz continuous function. Moreover, we have*

$$\limsup_{x \to 0^+} \frac{\frac{3}{7} x^{\frac{7}{3}}}{x^2} = +\infty,$$

and

$$\lambda^* = \frac{1}{\frac{45}{7} \cdot 2^{\frac{7}{3}}}.$$

Also, since

$$\sigma_1 = \left| \frac{2 - \frac{90}{7} \cdot 2^{\frac{7}{3}}\,\lambda}{120} \right| \quad \textit{and} \quad \sigma_3 = \frac{0.8}{60} = 0.013\bar{3},$$

it follows that

$$\sigma = \min\{\sigma_1, \sigma_3\}.$$

Finally, we have

$$\limsup_{|x| \to +\infty} \frac{\sum_{t=a+1}^{b+1} G(t,x)}{x^2} = 15 < +\infty. \tag{3.10}$$

So, condition (j$_2$) *of Theorem* 3.1 *holds. Therefore, for each* $\lambda \in (0, 0.048)$ *and* $\mu \in [0, \sigma)$ *the problem* (3.9) *admits at least one non-trivial solution.*

4 Existence of two solutions

In this section, our goal is to obtain the existence of two distinct solutions for the problem $(D^{f,g,h}_{\lambda,\mu})$. The following result is obtained by applying Theorem 2.9.

Theorem 4.1 *Let* $f\colon [a+1, b+1]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ *be a continuous function. Assume that*

(j$_3$) *there exist constants* $\chi > 2$ *and* $r > 0$ *such that* $0 < \chi F(t,x) < x f(t,x)$ *for all* $t \in [a+1, b+1]_{\mathbb{Z}}$ *and* $|x| \geq r$*.*

Then, for each $\lambda \in (0, \lambda^*)$ *and for any continuous function* $g\colon [a+1, b+1] \times \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\limsup_{|x| \to +\infty} \frac{\sum_{t=a+1}^{b+1} G(t,x)}{x^2} < +\infty, \tag{4.1}$$

and for any $\mu \in [0, \sigma)$*, where* λ^* *and* σ *are as in Theorem* 3.1*, the problem* $(D^{f,g,h}_{\lambda,\mu})$ *admits at least two distinct solutions.*

Proof. We apply Theorem 2.9 to the space E, endowed with its standard norm, and to the functionals $\Phi, \Psi \colon E \to \mathbb{R}$ defined in (2.4) and (2.5). Integrating condition (j_3), there exist constants $\alpha_1, \beta_1 > 0$ such that

$$F(t,x) \geq \alpha_1 |x|^{\chi} - \beta_1$$

for all $(t,x) \in [a+1, b+1]_{\mathbb{Z}} \times \mathbb{R}$. For a fixed $\bar{u} \in E \setminus \{\theta\}$ and any $x > 1$, we have

$$I_\lambda(x\bar{u}) \leq \frac{1}{2}x^2 \|\bar{u}(t)\|^2 - \lambda\alpha_1 x^{\chi} \sum_{t=a+1}^{b+1} |\bar{u}(t)|^{\chi} + (b-a+1)\lambda\beta_1.$$

Since $\chi > 2$, this shows that I_λ is unbounded from below. By standard arguments, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies $(P.S.)$ condition (see, for instance, [20]). Therefore, all assumptions of Theorem 2.9 are satisfied. In conclusion, for each $\lambda \in (0, \lambda^*)$ the functional I_λ admits two distinct critical points, which are the solutions of the problem $(D^{f,g,h}_{\lambda,\mu})$. □

Remark 4.2 *We observe that, if $f(t,0) \neq 0$, then Theorem 4.1 ensures the existence of two non-trivial solutions for the problem $(D^{f,g,h}_{\lambda,\mu})$. Moreover, we point out that in most works concerning the existence of solutions for the problem $(D^{f,g,h}_{\lambda,\mu})$, the following condition is assumed*

$$\limsup_{x \to 0^+} \frac{\inf_{t \in [a+1,b+1]_{\mathbb{Z}}} F(t,x)}{x^2} = 0.$$

It is easily seen that this condition is incompatible with the assumption $f(t,0) \neq 0$.

Example 4.3 *Consider the following discrete fourth-order problem*

$$\begin{cases} \Delta^4 u(t-2) = \lambda f(t,u(t)) + \mu g(t,u(t)) + h(u(t)), & t \in [2,7]_{\mathbb{Z}}, \\ u(1) = \Delta^2 u(0) = 0, \quad u(8) = \Delta^2 u(7) = 0, \end{cases} \tag{4.2}$$

where

$$f(t,x) = a + b\sqrt[5]{x^6}, \quad g(t,x) = \frac{x}{t}, \quad h(x) = \arctan 0.5x, \quad t \in [2,7]_{\mathbb{Z}}, \, |x| \geq r.$$

Clearly, f and g are two continuous functions and h is a monotone Lipschitz continuous function. We see that $F(t,x) = ax + \frac{5b}{11}x^{\frac{11}{5}}$ and $G(t,x) = 7$. Then, for a fixed r and $x \leq -r$ we have

$$\begin{aligned} xf(t,x) - \chi F(t,x) &= ax + bx\sqrt[5]{x^6} - \chi\left(ax + \frac{5b}{11}x^{\frac{11}{5}}\right) \\ &= ax(1-\chi) + bx^{\frac{11}{5}}\left(1 - \frac{5}{11}\chi\right) \\ &= |x|(a(\chi - 1) + b|x|^{\frac{6}{5}}\left(\frac{5}{11}\chi - 1\right) > 0. \end{aligned}$$

In addition, if $x \geq r$ for

$$r^{\frac{5}{6}} > \frac{a(\chi-1)}{b(1-\frac{5}{11}\chi)},$$

we have

$$\begin{aligned} xf(t,x) - \chi F(t,x) &= ax + bx\sqrt[5]{x^6} - \chi\left(ax + \frac{5b}{11}x^{\frac{11}{5}}\right) \\ &= ax(1-\chi) + bx^{\frac{11}{5}}\left(1 - \frac{5}{11}\chi\right) \\ &\geq r\left((a(1-\chi) + br^{\frac{6}{5}}\left(1 - \frac{5}{11}\chi\right)\right) > 0. \end{aligned}$$

Then, condition (j_3) *is satisfied. Hence, the problem* (4.2) *admits at least two distinct solutions.*

5 Existence of three solutions

In this section, we deal with the existence of at least three solutions for the problem $(D^{f,g,h}_{\lambda,\mu})$.

Theorem 5.1 *Let* $f\colon [a+1,b+1]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ *be a continuous function. Assume that*

- (j_4) *there exist* $c_1 \in [0,+\infty)$ *and* $1 < \alpha < 2$ *such that* $F(t,x) \leq c_1(1+|x|^{\alpha})$ *for all* $(t,x) \in [a+1,b+1]_{\mathbb{Z}} \times \mathbb{R}$,
- (j_5) $F(t,x) \geq 0$ *for each* $(t,x) \in [a+1,b+1]_{\mathbb{Z}} \times \mathbb{R}^+$,
- (j_6) *there exist* $r > 0$ *and* $\iota > 0$ *with* $r < \frac{1}{2}(\varepsilon_1 - L)(b-a+1)\iota^2$ *such that*

$$\omega := \frac{2F_{\max}}{r(\varepsilon_1 - L)c^2} < \frac{\inf_{t\in[a+1,b+1]_{\mathbb{Z}}} F(t,\iota)}{\frac{1}{2}(\varepsilon_1 - L)(b-a+1)\iota^2}.$$

Also, let $g\colon [a+1,b+1]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$. *Then, for every*

$$\lambda \in \left(\frac{\frac{1}{2}(\varepsilon_1 - L)(b-a+1)\iota^2}{\inf_{t\in[a+1,b+1]_{\mathbb{Z}}} F(t,\iota)}, \frac{1}{\omega}\right)$$

and for any $\mu \in [0,\sigma)$, *where* σ *is as in Theorem* 3.1, *the problem* $(D^{f,g,h}_{\lambda,\mu})$ *admits at least three solutions.*

Proof. Our aim is to apply Theorem 2.10 to the space E, endowed with its standard norm, and to the functionals $\Phi, \Psi\colon E \to \mathbb{R}$ defined in (2.4) and (2.5), respectively. As shown before, the functionals Φ and Ψ satisfy the regularity assumptions required by Theorem 2.10. Now, let $\nu \in E$ be defined by

$$\nu(t) = \iota. \tag{5.1}$$

We have

$$\frac{\varepsilon_1 - L}{2}(b-a+1)\iota^2 < \Phi(\nu) < \frac{\varepsilon_{b-a+1} + L}{2}(b-a+1)\iota^2.$$

By (j_5),

$$\Psi(\nu) \geq \sum_{t=1}^{b+1}\left[F(t,\nu(t)) + \frac{\mu}{\lambda}G(t,\nu(t))\right] \geq \inf_{t\in[a+1,b+1]_{\mathbb{Z}}}\left[F(t,\iota) + \frac{\mu}{\lambda}G(t,\iota)\right],$$

so that

$$\frac{\Psi(\nu)}{\Phi(\nu)} \geq \frac{\inf_{t\in[a+1,b+1]} \Psi(\iota)}{\frac{1}{2}(\varepsilon_1 - L)(b-a+1)\iota^2}. \tag{5.2}$$

As $r < \frac{1}{2}(\varepsilon_1 - L)(b-a+1)\iota^2$, we have $r < \Phi(\nu)$. Also, from (2.1) it follows that

$$\sum_{t=a+1}^{b+1} |u(t)|^\alpha < \sum_{t=a+1}^{b+1} \max_{t\in[a+1,b+1]_{\mathbb{Z}}} |u(t)|^\alpha \leq \|u\|^\alpha \tag{5.3}$$

for every $u \in E$. So, (5.3) implies that for each $u \in \Phi^{-1}((-\infty, r))$ we have

$$\begin{aligned}
\Psi(u) &= \sum_{t=a+1}^{b+1} \Big[F(t,u(t)) + \frac{\mu}{\lambda} G(t,u(t))\Big] \\
&\leq \sum_{t=a+1}^{b+1} c_1(1 + |u(t)|^\alpha) + \sum_{t=a+1}^{b+1} \frac{\mu}{\lambda} G(t,u(t)) \\
&\leq c_1(\|u\|^\alpha + (b-a+2)) + \sum_{t=a+1}^{b+1} \frac{\mu}{\lambda} G(t,u(t)).
\end{aligned}$$

From (j_6) and (5.2) we obtain

$$\frac{\sup_{\Phi(u)\leq r} \Psi(u)}{r} < \frac{\Psi(\nu)}{\Phi(\nu)}.$$

Hence, condition (b_1) of Theorem 2.10 is satisfied. Define $I_\lambda := \Phi - \lambda\Psi$. Then,

$$I_\lambda(u) \geq \frac{\|u\|^2}{2} - \lambda c_1(\|u\|^\alpha + (b-a+2)) + \sum_{t=a+1}^{b+1} \mu G(t,u(t)),$$

which shows that I_λ is coercive. By above calculation, we have

$$\Lambda \subseteq \left(\frac{\Phi(v)}{\Psi(v)}, \frac{r}{\sup_{\Phi(u)\leq r} \Psi(u)}\right).$$

Theorem 2.10 ensures that for each $\lambda \in \Lambda$, the functional I_λ admits at least three critical points in E that are the solutions of the problem $(D^{f,g,h}_{\lambda,\mu})$. □

Remark 5.2 *We observe that, if $f(t,0) \neq 0$, then by Theorem* 5.1*, we obtain the existence of at least three non-zero solutions.*

Example 5.3 *We set $a = 1$, $b = 5$, $c = 4$, and $c_1 = 6$. Since*

$$F(t,x) = \frac{x}{t} \geq 0 \quad \textit{for } (t,x) \in [2,6] \times [0,+\infty),$$

condition (j_5) *of Theorem* 5.1 *is satisfied. Moreover, the inequality*

$$\frac{x}{t} \leq 6(1 + |x|^\alpha)$$

ensures that condition (j_4) *holds. Next, we take* $L = 0.2$, $r = 10$, $\iota = 4$, $\varepsilon_1 = 0.5$, *and* $G(t,x) = x^2 + 3t$. *Then, simple calculations give*

$$\omega := \frac{1}{360} < \frac{2}{12}.$$

Also, since

$$\sigma_1 = \left| \frac{1.2 - \frac{2}{5}\lambda}{212} \right| \quad \textit{and} \quad \sigma_3 = 0.5,$$

we obtain

$$\sigma = \min\{\sigma_1, \sigma_3\}.$$

Hence, for $\lambda \in (6, 360)$ *and* $\mu \in [0, \sigma)$, *the problem* $(D^{f,g,h}_{\lambda,\mu})$ *admits at least three non-zero solutions.*

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