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NULL CONTROLLABILITY OF A FOUR-STAGE AGE-STRUCTURED POPULATION DYNAMICS MODEL WITH SPATIAL DIFFUSION FOR DESERT LOCUSTS

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Abstract. This article focuses on the study of null controllability in a population dynamics model of the desert locust, structured by age and incorporating spatial diffusion as well as nonlocal boundary conditions. More specifically, we consider a four-stage model that includes second-order derivatives with respect to both age and space variables. Null controllability in this context is associated with the extinction of eggs, larvae, and the female subpopulation. We estimate a time T required to bring the density of these subpopulations to zero. Our approach combines the fixed-point theorem with Carleman estimates.

Keywords: Carleman estimates, desert locust, fixed-point.

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1 Introduction

We study a linear model describing the dynamics of a locust population structured by both age and space. Let $y_j(b,t,x)$, for $1 \leq j \leq 4$, be the density of individuals in the egg, larval, female, and male stages, respectively, with age $b \geq 0$ at time $t \geq 0$ and location $x \in \Omega$, where Ω is a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. We denote by B_j the life expectancy of an individual in stage j for $1 \leq j \leq 4$. Also, let T be a positive constant. Here, ∇ represents the spatial gradient, while Δ denotes the Laplace operator with respect to x. Moreover, one of our boundary conditions is given by $\partial_\eta y_j(b,t,x) = \nabla y_j(b,t,x) \cdot \eta$, where η represents the outward normal. The evolution of the distribution y_j is then governed by the following system (see [19])

$$
\begin{cases}
\partial_t y_1(b,t,x) + \partial_b\big[v_1(b,\tau(t),x)y_1(b,t,x)\big] - \beta\partial_b^2 y_1(b,t,x) - \mu_1\Delta y_1(b,t,x) \\
\qquad = -\big[\theta_1(b,\tau(t),x) + \gamma_1(b,\tau(t),x)\big]y_1(b,t,x) + \ \chi(b,x)\, u_1(b,t,x), \\
\partial_t y_2(b,t,x) + \partial_b\big[v_2(b,\tau(t),x)y_2(b,t,x)\big] - \beta\partial_b^2 y_2(b,t,x) - \mu_2\Delta y_2(b,t,x) \\
\qquad = -\big[\theta_2(b,\tau(t),x) + \gamma_2(b,\tau(t),x)\big]y_2(b,t,x) + \ \chi(b,x)\, u_2(b,t,x), \\
\partial_t y_3(b,t,x) + \partial_b\big[v_3(b,\tau(t),x)y_3(b,t,x)\big] - \beta\partial_b^2 y_3(b,t,x) - \mu_3\Delta y_3(b,t,x) \\
\qquad = -\theta_3(b,\tau(t),x)y_3(b,t,x) + \ \chi(b,x)\, u_3(b,t,x), \\
\partial_t y_4(b,t,x) + \partial_b\big[v_4(b,\tau(t),x)y_4(b,t,x)\big] - \beta\partial_b^2 y_4(b,t,x) - \mu_4\Delta y_4(b,t,x) \\
\qquad = -\theta_4(b,\tau(t),x)y_4(b,t,x),
\end{cases}
\tag{1.1}
$$

where

- $\theta_j(b,\tau(t),x)$, $1 \leq j \leq 4$, represents, respectively, the mortality rate of eggs, larvae, females and males of age b at temperature $\tau(t)$ in a geographical position x in the locust domain,
- $\gamma_j(b,\tau(t),x)$, $1 \leq j \leq 2$, represents the transition function of stage j; it is in fact the rate of individuals of age b at temperature $\tau(t)$ of stage j which pass to stage $j+1$ in a geographical position x of the domain,

- $\gamma_3(b, \tau(t), x)$, which will appear shortly in the boundary and initial conditions (1.2), is the egg-laying rate,
- $v_j(b, \tau(t), x)$, $1 \leq j \leq 4$, represents the growth rate of individuals of age b at temperature $\tau(t)$ (at time t) of stage j in a geographical position x of the domain,
- $\lambda > 0$, which will also appear in the boundary and initial conditions (1.2), represents sex ratio,
- the diffusion coefficient $\mu_j > 0$, $1 \leq j \leq 4$, represents the effects of dispersion of individuals or the different movements of individuals in space,
- the diffusion coefficient $\beta > 0$ represents the effects of dispersion or the various modifications in the organism of individuals during their development,
- $Q_j = (0, B_j) \times (0, T) \times \Omega$, $1 \leq j \leq 4$,
- the control functions for eggs, larvae, and females are denoted by u_1, u_2, and u_3, respectively.

We define the control domain as $\rho = (0, b) \times \omega$, where ω is a non-empty open subset of Ω. The characteristic function associated with this domain is denoted by $\chi(b, x)$, with $0 < b < \min\{B_1, B_2, B_3\}$. The boundary conditions and initial conditions are then expressed as follows

$$\begin{cases} \big[v_1(b, \tau(t), x)y_1(b, t, x) - \beta\partial_b y_1(b, t, x)\big]_{b=0} \\ \qquad\qquad = \displaystyle\int_0^{B_3} \gamma_3(b, \tau(t), x)y_3(b, t, x)\,\mathrm{d}b, \\ \big[v_2(b, \tau(t), x)y_2(b, t, x) - \beta\partial_b y_2(b, t, x)\big]_{b=0} \\ \qquad\qquad = \displaystyle\int_0^{B_1} \gamma_1(b, \tau(t), x)y_1(b, t, x)\,\mathrm{d}b, \\ \big[v_3(b, \tau(t), x)y_3(b, t, x) - \beta\partial_b y_3(b, t, x)\big]_{b=0} \\ \qquad\qquad = \displaystyle\int_0^{B_2} \lambda\gamma_2(b, \tau(t), x)y_2(b, t, x)\,\mathrm{d}b, \\ \big[v_4(b, \tau(t), x)y_4(b, t, x) - \beta\partial_b y_4(b, t, x)\big]_{b=0} \\ \qquad\qquad = \displaystyle\int_0^{B_2} (1-\lambda)\gamma_2(b, \tau(t), x)y_2(b, t, x)\,\mathrm{d}b, \\ \partial_\eta y_j(b, t, x) = 0 \text{ for } (b, t, x) \in (0, B_j) \times (0, T) \times \partial\Omega \text{ and } 1 \leq j \leq 4, \\ y_j(b, 0, x) = {y_j}^0(b, x) \text{ for } (b, x) \in (0, B_j) \times \Omega \text{ and } 1 \leq j \leq 4, \\ y_j(B_j, t, x) = 0 \ \text{ for } (t, x) \in (0, T) \times \Omega \text{ and } 1 \leq j \leq 4. \end{cases} \tag{1.2}$$

Here, the expressions

$$\int_0^{B_j} \lambda_j\gamma_j(b, \tau(t), x)y_j(b, t, x)\,\mathrm{d}b, \quad 1 \leq j \leq 3,$$

denote, respectively, the distribution of eggs, larvae and females of the newborns at time t and spatial position x, with $(\lambda_1, \lambda_2, \lambda_3) = (1, \lambda, 1)$. The quantity

$$\int_0^{B_2} (1-\lambda)\gamma_2(b, \tau(t), x)y_2(b, t, x)\,\mathrm{d}b$$

represents the distribution of male newborns with λ being the sex ratio.

Our objective is to control the states y_1, y_2 and y_3 governed by the system (1.1)–(1.2), by driving them to extinction at a fixed time $T > 0$. The control acts on a part $(0, T) \times \rho$ of the domain $Q = (0, B) \times (0, T) \times \Omega$, where $B = \max_{1 \leq j \leq 4} B_j$, through the controls u_1, u_2 and u_3. More precisely, we aim to show that for every ${y_j}^0 \in L^2(Q_j)$, $1 \leq j \leq 4$, and for every $T > 0$, there exist controls u_1, u_2 and u_3 in $L^2((0, T) \times \rho)$ such that the corresponding solution of the system (1.1)–(1.2) satisfies

$$y_1(b, T, x) = y_2(b, T, x) = y_3(b, T, x) = 0 \text{ a.e. in } (0, B) \times \Omega. \tag{1.3}$$

Desert locusts (*Schistocerca gregaria*) are among the most formidable pests in the arid and semi-arid regions of Africa, the Middle East, and Asia. Capable of devastating crops and pastures in record time, they cause significant agricultural losses and jeopardize the food security of millions of people. Their rapid proliferation and ability to migrate over vast areas make their control particularly challenging. In response to this threat, various control strategies have been implemented, ranging from the use of pesticides to biological and ecological approaches. However, the effectiveness of these interventions relies on a thorough understanding of locust population dynamics and the mechanisms that enable targeted control of their proliferation. In this context, control theory plays a key role by providing analytical tools to design strategies for regulating or eradicating locust populations. More specifically, the study of null controllability seeks to determine whether it is possible, through targeted action, to bring certain classes of individuals (eggs, larvae, adults) to total extinction within a finite time. This approach is based on dynamic models that integrate population evolution and their interactions with the environment while considering the effects of control interventions. The objective of our article is to analyze the null controllability of a model describing the dynamics of desert locusts, incorporating spatial diffusion terms and localized control actions. We focus particularly on the existence of strategies that allow for the elimination of certain population classes within a given time-frame. These findings are crucial for developing more effective and sustainable control methods against these pests, thereby contributing to crop protection and food security. The control method we consider here aims to eliminate or reduce the number of individuals, which can be achieved through the use of pesticides.

The desert locust, which is the focus of our study, belongs to the category of locusts known for exhibiting two distinct behaviors depending on population density and climatic conditions: solitary and gregarious behavior (see [26] and [16]). At low population densities, adult locusts adopt a solitary lifestyle. They briefly encounter each other to mate and then separate without undergoing a phase change. In this state, they are discreet, harmless, and often inconspicuous as they do not produce sound and remain scattered across their natural habitat, which spans the deserts of North and West Africa, around the Red Sea, and into Southwest Asia. Conversely, when locust numbers increase and reproduction occurs in concentrated areas, individuals tend to group together. If such favorable conditions persist, this can trigger a process known as gregarization, in which locusts transition from the solitary to the gregarious phase. In this phase, locusts form hopper bands or adult swarms that move together, causing severe damage to crops and pastures (see [15]).

It is important to note that desert locust reproduction generally occurs during the wet seasons, as rainfall creates ecological conditions favorable for sexual maturation, egg-laying, and the development of eggs and larvae. The choice of egg-laying sites and the degree of clustering are influenced by vegetation cover, the extent of bare soil, and soil structure. Dense vegetation is avoided, while bare, light, and loosely compacted soils are preferred (see [25]). Other factors such as temperature, light, soil moisture, and topography also play a significant role in determining where egg-laying occurs. Higher elevations often receive more rainfall than lowlands, and runoff can create breeding conditions tens of kilometers downstream from where the rain originally fell (see [25]).

In addition, flight behavior varies depending on the phase: solitary adults primarily fly at night, while gregarious adults fly during the day, moving with the prevailing winds (see [14]). Their flight performance is optimal when exposed to long or lengthening daylight hours (see [17]). According to Roffey and Magor (see [20]), during the first few days after fledging, both solitary and gregarious locusts walk and feed like larvae. As their exoskeleton hardens, they progressively become capable of sustained flight.

The optimal control and precise control of population dynamics structured by age and space have been the subject of numerous studies. Most of this research focuses primarily on optimal control problems or null controllability. For instance, B. Ainseba in [1] established the exact and approximate controllability of a linear age- and space-structured population dynamics model, based on a Carleman estimate for a nonlocal adjoint system in backward time. Similarly, O. Traoré in [29] proved a null controllability result for a nonlinear population dynamics model by combining a Carleman inequality with the Leray–Schauder fixed point theorem. He showed that an internal control acting on a small region of the domain can drive the population to extinction. Along the same lines, A. Traoré *et al.* in [27] investigated the null controllability of an age-structured model, considering nonlocal boundary conditions,without considering the spatial dimension. They specifically targeted the extinction of eggs, larvae, and the female population, and determined a minimal time T to reach this goal by combining a fixed point theorem and a Carleman estimate, supported by numerical simulations. In [4], B. Ainseba and M. Iannelli addressed the exact controllability of two nonlinear models: one structured solely by age and the other by both age and space. The controls are localized on small subdomains of the spatial domain or the age interval, and their approach relies on an observability result for the adjoint system along with Kakutani's fixed point theorem. Furthermore, O. Y. Imanuvilov and M. Yamamoto in [13] provided Carleman estimates for various types of equations. Y. He and B. Ainseba, in [11], demonstrated the exact null controllability of a Lobesia botrana model with nonlocal boundary conditions by acting on the eggs, larvae, and females via a control localized in a subdomain of the vineyard. Their method combines a Carleman estimate and a fixed point theorem. In a broader framework, Y. Simporé in [22] analyzed a linear infinite-dimensional control system describing population models structured by age, size, and space, using a method that does not rely on the Carleman inequality. He proved null controllability without explicitly using a Carleman estimate, instead relying on observability estimates and properties of the associated semigroup. In [24], Y. Simporé and O. Traoré established a null controllability result for a system derived from a nonlinear population dynamics model, using a method based on a Carleman inequality and the sentinel method. Additionally, B. Ainseba and M. Langlais in [5] addressed a control problem for a model structured by age and space, with a nonlocal birth process represented as a boundary condition. In [2], B. Ainseba and S. Anita studied the local exact controllability of a linear model with a nonlocal birth process, combining Carleman estimates, analytical results in L^2 spaces, and Banach's fixed point theorem. The authors of [3] examined an optimal harvesting problem in a nonlinear structured model with nonlocal boundary conditions and proved both the existence of an optimal control and the necessary optimality conditions. In [12], N. Hegoburu and S. Anita proposed an active control strategy over a specific age class, establishing null controllability under certain time constraints while preserving the non-negativity of the solution. They also showed the absence of null controllability in the linear Lotka–McKendrick equation with diffusion when the control is localized. In [9], the authors proved a new Carleman estimate adapted to weak solutions of the heat equation with nonhomogeneous Neumann boundary conditions, which allowed them to establish null controllability. In [23], Y. Simporé *et al.* adopted a dual formulation to address the controllability problem. It is also noted that in [21], Y. Simporé investigates the null controllability of a nonlinear population dynamics model. He shows that, under the action of a control localized both in the spatial variable and with respect to age, there exists a time T, depending on the constraints related to age, after which the system becomes null controllable. His proof relies in particular on the use of Schauder's fixed point theorem to establish the null controllability of the nonlinear system. Finally, A. Traoré *et al.* in [28] analyzed a nonlinear population dynamics model without spatial structure, where the nonlinearity is concentrated in the birth function. To the best of our knowledge, no existing result addresses the null controllability of a system structured simultaneously by age, space, and biological stages of desert locusts, involving second-order derivatives with respect to both age and space. These second-order terms represent small-scale fluctuations and model dispersion effects caused, for example, by temperature, humidity, or food quality. This paper is organized as follows. Section 2 presents the assumptions and the main result. Section 3 is devoted to the null controllability of an auxiliary model. Section 4 contains the proof of the main result, and Section 5 constitutes the conclusion.

This study builds upon the work of N. Ramdé *et al.* [19], with a particular focus on the issue of null controllability applied to desert locusts.

2 Assumptions and main result

For the remainder of the study, we make the following assumptions regarding the demographic parameters.

(F_1) Growth functions $v_j(b,\tau(t),x)$ are strictly positive and bounded, and $0 < \min v_j < v_j(b,\tau(t),x) < \max v_j$ for all $(b,t,x) \in Q_j = (0,B_j) \times (0,T) \times \Omega$ and $1 \leq j \leq 4$. In addition, $v_j(b,\tau(t),x) \in C^1([0,B])$ for any $t \in (0,T)$. Also, there are positive constants A_j such that $\|\partial_b v_j(b,\tau(t),x)\|_\infty \leq A_j$ for all $(b,t,x) \in Q_j = (0,B_j) \times (0,T) \times \Omega$ and $1 \leq j \leq 4$.

(F_2) The transition functions $\gamma_k(b,\tau(t),x)$, where $1 \leq k \leq 3$, are positive and $\gamma_k \in L^\infty((0,B_k) \times (0,T) \times \Omega)$.

(F_3) The mortality functions $\theta_k(b,\tau(t),x)$, where $1 \leq k \leq 3$, are positive, with $\theta_k \in L^\infty((0,B_k) \times (0,T) \times \Omega)$.

(F_4) The initial conditions ${y_j}^0(b,x) \geq 0$ for a.e. $(b,x) \in (0,B) \times \Omega$.

(F_5) The temperature function $\tau(t)$ is measurable.

These assumptions are biologically meaningful, as can be seen in the papers [6, 18, 30].

By setting $\hat{y}_j = e^{-\sigma t} y_j$ for $1 \leq j \leq 4$, the vector $(\hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4)$ then satisfies the following system

$$\begin{cases}\partial_t\hat y_1+\partial_b[v_1\hat y_1]-\beta\partial_b^2\hat y_1-\mu_1\Delta\hat y_1=-[\theta_1+\gamma_1+\sigma]\hat y_1+\chi(b,x)\,\hat u_1,\\ \partial_t\hat y_2+\partial_b[v_2\hat y_2]-\beta\partial_b^2\hat y_2-\mu_2\Delta\hat y_2=-[\theta_2+\gamma_2+\sigma]\hat y_2+\chi(b,x)\,\hat u_2,\\ \partial_t\hat y_3+\partial_b[v_3\hat y_3]-\beta\partial_b^2\hat y_3-\mu_3\Delta\hat y_3=-[\theta_3+\sigma]\hat y_3+\chi(b,x)\,\hat u_3,\\ \partial_t\hat y_4+\partial_b[v_4\hat y_4]-\beta\partial_b^2\hat y_4-\mu_4\Delta\hat y_4=-[\theta_4+\sigma]\hat y_4,\end{cases}$$

and

$$\begin{cases}\big[v_1(b,\tau(t),x)\hat y_1(b,t,x)-\beta\partial_b\hat y_1(b,t,x)\big]_{b=0}\\ \qquad\qquad=\displaystyle\int_0^{B_3}\gamma_3(b,\tau(t),x)\hat y_3(b,t,x)\,\mathrm{d}b,\\ \big[v_2(b,\tau(t),x)\hat y_2(b,t,x)-\beta\partial_b\hat y_2(b,t,x)\big]_{b=0}\\ \qquad\qquad=\displaystyle\int_0^{B_1}\gamma_1(b,\tau(t),x)\hat y_1(b,t,x)\,\mathrm{d}b,\\ \big[v_3(b,\tau(t),x)\hat y_3(b,t,x)-\beta\partial_b\hat y_3(b,t,x)\big]_{b=0}\\ \qquad\qquad=\displaystyle\int_0^{B_2}\lambda\gamma_2(b,\tau(t),x)\hat y_2(b,t,x)\,\mathrm{d}b,\\ \big[v_4(b,\tau(t),x)\hat y_4(b,t,x)-\beta\partial_b\hat y_4(b,t,x)\big]_{b=0}\\ \qquad\qquad=\displaystyle\int_0^{B_2}(1-\lambda)\gamma_2(b,\tau(t),x)\hat y_2(b,t,x)\,\mathrm{d}b,\\ \partial_\eta\hat y_j(b,t,x)=0\text{ for }(b,t,x)\in(0,B_j)\times(0,T)\times\partial\Omega\text{ and }1\le j\le4,\\ \hat y_j(b,0,x)=\hat y_j^0(b,x)\text{ for }(b,x)\in(0,B_j)\times\Omega\text{ and }1\le j\le4,\\ \hat y_j(B_j,t,x)=0\ \text{ for }(t,x)\in(0,T)\times\Omega\text{ and }1\le j\le4.\end{cases}$$

Now, consider the following auxiliary system

$$\begin{cases}\partial_t\hat y_1+\partial_b[v_1\hat y_1]-\beta\partial_b^2\hat y_1-\mu_1\Delta\hat y_1=-[\theta_1+\gamma_1+\sigma]\hat y_1+\chi(b,x)\,\hat u_1,\\ \partial_t\hat y_2+\partial_b[v_2\hat y_2]-\beta\partial_b^2\hat y_2-\mu_2\Delta\hat y_2=-[\theta_2+\gamma_2+\sigma]\hat y_2+\chi(b,x)\,\hat u_2,\\ \partial_t\hat y_3+\partial_b[v_3\hat y_3]-\beta\partial_b^2\hat y_3-\mu_3\Delta\hat y_3=-[\theta_3+\sigma]\hat y_3+\chi(b,x)\,\hat u_3,\\ \partial_t\hat y_4+\partial_b[v_4\hat y_4]-\beta\partial_b^2\hat y_4-\mu_4\Delta\hat y_4=-[\theta_4+\sigma]\hat y_4,\end{cases}\tag{2.1}$$

and

$$\begin{cases} \big[v_1(b,\tau(t),x)\hat{y}_1(b,t,x) - \beta\partial_b\hat{y}_1(b,t,x)\big]_{b=0} \\ \qquad\qquad = \displaystyle\int_0^{B_3} \gamma_3(b,\tau(t),x)\psi_3(b,t,x)\,\mathrm{d}b, \\ \big[v_2(b,\tau(t),x)\hat{y}_2(b,t,x) - \beta\partial_b\hat{y}_2(b,t,x)\big]_{b=0} \\ \qquad\qquad = \displaystyle\int_0^{B_1} \gamma_1(b,\tau(t),x)\psi_1(b,t,x)\,\mathrm{d}b, \\ \big[v_3(b,\tau(t),x)\hat{y}_3(b,t,x) - \beta\partial_b\hat{y}_3(b,t,x)\big]_{b=0} \\ \qquad\qquad = \displaystyle\int_0^{B_2} \lambda\gamma_2(b,\tau(t),x)\psi_2(b,t,x)\,\mathrm{d}b, \\ \big[v_4(b,\tau(t),x)\hat{y}_4(b,t,x) - \beta\partial_b\hat{y}_4(b,t,x)\big]_{b=0} \\ \qquad\qquad = \displaystyle\int_0^{B_2} (1-\lambda)\gamma_2(b,\tau(t),x)\psi_2(b,t,x)\,\mathrm{d}b, \\ \partial_\eta\hat{y}_j(b,t,x) = 0 \text{ for } (b,t,x) \in (0,B_j)\times(0,T)\times\partial\Omega \text{ and } 1 \le j \le 4, \\ \hat{y}_j(b,0,x) = \hat{y}_j^0(b,x) \text{ for } (b,x) \in (0,B_j)\times\Omega \text{ and } 1 \le j \le 4, \\ \hat{y}_j(B_j,t,x) = 0 \text{ for } (t,x) \in (0,T)\times\Omega \text{ and } 1 \le j \le 4. \end{cases} \tag{2.2}$$

The existence of solutions to problems similar to equations (2.1) and (2.2) can be found, for instance, in [19] as well as in the references cited therein.

Let $B = \max\{B_1, B_2, B_3, B_4\}$. We are now in a position to state the main result.

Theorem 2.1 *Assume that conditions* (F_1)–(F_5) *are satisfied. Let* $B > 0$ *and* $T > 0$ *be given. Then, for all* $(y_1^0, y_2^0, y_3^0) \in L^2((0,B)\times\Omega)^3$*, there exists a control* $u = (u_1,u_2,u_3) \in (L^2((0,T)\times\rho))^3$ *such that the associated solution of equations* (1.1) *and* (1.2) *satisfies* (1.3).

Summary of the main steps in the proof of Theorem 2.1 The proof proceeds along the following steps.

Step 1: *Formulation of an auxiliary problem.* We consider system (1.1)–(1.2) with the given initial conditions and introduce an auxiliary null-control problem.

Step 2: *Associated optimal control problem.* We define an optimal control problem aimed at minimizing a quadratic functional of the controls and the terminal states, thereby transforming the controllability problem into a convex optimization problem.

Step 3: *Existence and uniqueness of minimizers.* Using classical properties of continuity, convexity, and coercivity of the functional, and applying the direct method in the calculus of variations, we show that the problem admits a unique minimizer.

Step 4: *Adjoint equations and observability.* We introduce the adjoint variables associated with the optimal problem and establish an observability inequality linking the adjoint variables to the controls over the control domain.

Step 5: *Extraction of convergent sequences.* We construct sequences of approximate solutions and, using compactness and uniform bounds, prove their weak convergence to a solution of the original problem.

Step 6: *Application to null controllability.* Combining the previous results and the observability inequality, we conclude that for any initial state there exists a control $u = (u_1, u_2, u_3)$ that drives the system to zero at the final time T.

3 Null controllability of an auxiliary system

This section deals with the null controllability of the auxiliary system derived from equations (1.1) and (1.2):

$$\begin{cases} \partial_t y_1(b,t,x) + \partial_b\big[v_1(b,\tau(t),x)y_1(b,t,x)\big] - \beta\partial_b^2 y_1(b,t,x) - \mu_1\Delta y_1(b,t,x) \\ \qquad = -\big[\theta_1(b,\tau(t),x) + \gamma_1(b,\tau(t),x)\big]y_1(b,t,x) + \chi(b,x)\,u_1(b,t,x), \\ \partial_t y_2(b,t,x) + \partial_b\big[v_2(b,\tau(t),x)y_2(b,t,x)\big] - \beta\partial_b^2 y_2(b,t,x) - \mu_2\Delta y_2(b,t,x) \\ \qquad = -\big[\theta_2(b,\tau(t),x) + \gamma_2(b,\tau(t),x)\big]y_2(b,t,x) + \chi(b,x)\,u_2(b,t,x), \\ \partial_t y_3(b,t,x) + \partial_b\big[v_3(b,\tau(t),x)y_3(b,t,x)\big] - \beta\partial_b^2 y_3(b,t,x) - \mu_3\Delta y_3(b,t,x) \\ \qquad = -\theta_3(b,\tau(t),x)y_3(b,t,x) + \chi(b,x)\,u_3(b,t,x), \\ \partial_t y_4(b,t,x) + \partial_b\big[v_4(b,\tau(t),x)y_4(b,t,x)\big] - \beta\partial_b^2 y_4(b,t,x) - \mu_4\Delta y_4(b,t,x) \\ \qquad = -\theta_4(b,\tau(t),x)y_4(b,t,x), \end{cases} \tag{3.1}$$

and

$$\begin{cases} \big[v_1(b,\tau(t),x)y_1(b,t,x) - \beta\partial_b y_1(b,t,x)\big]_{b=0} = f_1(t,x), \\ \big[v_2(b,\tau(t),x)y_2(b,t,x) - \beta\partial_b y_2(b,t,x)\big]_{b=0} = f_2(t,x), \\ \big[v_3(b,\tau(t),x)y_3(b,t,x) - \beta\partial_b y_3(b,t,x)\big]_{b=0} = f_3(t,x), \\ \big[v_4(b,\tau(t),x)y_4(b,t,x) - \beta\partial_b y_4(b,t,x)\big]_{b=0} = f_4(t,x), \\ \partial_\eta y_j(b,t,x) = 0 \text{ for } (b,t,x) \in\ (0,B)\times(0,T)\times\partial\Omega \text{ and } 1 \le j \le 4, \\ y_j(b,0,x) = {y_j}^0(b,x) \text{ for } (b,x) \in (0,B)\times\Omega \text{ and } 1 \le j \le 4, \\ y_j(B_j,t,x) = 0\ \text{ for } (t,x) \in (0,T)\times\Omega \text{ and } 1 \le j \le 4, \end{cases} \tag{3.2}$$

where $(b,t,x) \in (0,B)\times(0,T)\times\Omega$ and

$$f_j = \int_0^B \int_\Omega \lambda_j \gamma_i(b,\tau(t),x)\ \psi_i(b,t,x)\,\mathrm{d}b,$$

with

$$(j,i,\lambda_j) \in \big\{(1,3,1),(2,1,1),(3,2,\lambda),(4,2,1-\lambda)\big\}.$$

First, we establish a Carleman-type inequality for the adjoint system associated with equations (3.1) and (3.2), given by

$$\begin{cases} -\partial_t p_1 - \partial_b(v_1p_1) - \beta\,\partial_b^2 p_1 - \mu_1\Delta p_1 + (\theta_1 + \gamma_1 + \partial_b v_1)p_1 = 0, \\ -\partial_t p_2 - \partial_b(v_2p_2) - \beta\,\partial_b^2 p_2 - \mu_2\Delta p_2 + (\theta_2 + \gamma_2 + \partial_b v_2)p_2 = 0, \\ -\partial_t p_3 - \partial_b(v_3p_3) - \beta\,\partial_b^2 p_3 - \mu_3\Delta p_3 + (\theta_3 + \partial_b v_3)p_3 = 0, \\ -\partial_t p_4 - \partial_b(v_4p_4) - \beta\,\partial_b^2 p_3 - \mu_4\Delta p_4 + (\theta_4 + \partial_b v_4)p_4 = 0 \end{cases} \tag{3.3}$$

for $(b,t,x) \in Q$, together with the boundary conditions:

$$\begin{cases} \partial_b p_j(0,t,x) = p_j(B,t,x) = 0 \text{ for } (t,x) \in (0,T)\times\Omega \text{ and } 1 \le j \le 3,\\ p_j(b,T,x) = -\varepsilon^{-1}\, y_j(b,T,x) \text{ for } (b,x) \in (0,B)\times\Omega \text{ and } 1 \le j \le 3,\\ p_4(b,T,x) = 0 \;\text{ for } (b,x) \in (0,B)\times\Omega,\\ \partial_\eta p_j(b,t,x) = 0 \;\text{ for } (b,t,x) \in (0,B)\times(0,T)\times\partial\Omega \text{ and } 1 \le j \le 3. \end{cases} \tag{3.4}$$

The system formed by equations (3.3) and (3.4) admits a unique solution.

For convenience, we set $X = (b,x) \in \varpi$.

Lemma 3.1 *Let $\omega_0 \subset\subset \rho = (0,b)\times\omega$ be a non-empty bounded subset of $\varpi = (0,B)\times\Omega$. Then, there exists a function $\xi \in C^2(\overline{\varpi})$ such that*

$$\begin{cases} \xi(X) > 0 \textit{ for any } X \in \varpi,\\ \xi(X) = 0 \textit{ for any } X \in \partial\varpi,\\ |\nabla_X \xi(X)| > 0 \textit{ for any } X \in \overline{\varpi\setminus\omega_0}. \end{cases}$$

Proof. The proof of Lemma 3.1 relies on arguments similar to those used in the proof of [10, Lemma 1.1], which we invite the reader to consult. □

Following [8, 10], we now introduce the weight functions constructed from the function $\xi(X)$:

$$\pi(b,t,x) = \frac{e^{\delta\xi(X)}}{t(T-t)}, \qquad \tilde{\pi}(b,t,x) = \frac{e^{-\delta\xi(X)}}{t(T-t)},$$
$$\phi(b,t,x) = \frac{e^{\delta\xi(X)} - e^{2\delta\|\xi\|_{C(\overline{\varpi})}}}{t(T-t)}, \qquad \tilde{\phi}(b,t,x) = \frac{e^{-\delta\xi(X)} - e^{2\delta\|\xi\|_{C(\overline{\varpi})}}}{t(T-t)}, \tag{3.5}$$

where δ represents a suitable positive constant.

Remark 3.2 *Starting from the definitions of π, $\tilde{\pi}$, ϕ, and $\tilde{\phi}$, we deduce the existence of a positive constant z such that: We note that the notation ∂_X is not standard when $X = (b,x)$ is a vector of variables. In some parts of this paper, it has been used interchangeably with ∇_X to denote the full gradient with respect to X. For clarity, we have:*

(1) $\partial_X\pi = \nabla_X\pi = \delta\pi\nabla_X\xi$, $\partial_X\phi = \nabla_X\phi = \delta\pi\nabla_X\xi$, $\partial_X\tilde{\pi} = \nabla_X\tilde{\pi} = -\delta\tilde{\pi}\nabla_X\xi$ *and* $\partial_X\tilde{\phi} = \nabla_X\tilde{\phi} = -\delta\tilde{\pi}\nabla_X\xi$,

(2) $\nabla\pi = \delta\nabla\xi\pi$, $\nabla\phi = \delta\nabla\xi\pi$, $\nabla\tilde{\pi} = -\delta\nabla\xi\tilde{\pi}$ *and* $\nabla\tilde{\phi} = -\delta\nabla\xi\tilde{\pi}$,

(3) $|\partial_t\pi| \le z\pi^2$, $|\partial_t\tilde{\pi}| \le z\tilde{\pi}^2$, $|\partial_t\phi| \le z\pi^2$ *and* $|\partial_t\tilde{\phi}| \le z\tilde{\pi}^2$,

(4) $|\partial_t^2\pi| \le z\pi^3$, $|\partial_t^2\tilde{\pi}| \le z\tilde{\pi}^3$, $|\partial_t^2\phi| \le z\pi^3$ *and* $|\partial_t^2\tilde{\phi}| \le z\tilde{\pi}^3$,

(5) $|\partial_t(\nabla_X\phi)| \le z\pi^2$ *and* $|\partial_t(\nabla_X\tilde{\phi})| \le z\tilde{\pi}^2$.

Lemma 3.3 *Let π, ϕ, $\tilde{\pi}$, and $\tilde{\phi}$ be defined as in* (3.5). *Then, there exists a constant $\hat{\delta} > 1$ such that for every $\delta > \hat{\delta}$ one can find a number $s_0(\delta)$ with the property that for all $s \geq s_0(\delta)$ and all solutions $p_j \in L^2(Q)$, $1 \leq j \leq 3$, of the adjoint problem* (3.3)–(3.4)*, the following inequalities hold*

$$\begin{aligned}\int_Q \Bigg[\frac{1}{s\pi}\left(|\partial_t p_j|^2 + \left|\partial_b^2 p_j\right|^2 + |\Delta p_j|^2\right) + s\pi\,|\partial_b p_j|^2 + s\pi\,|\nabla p_j|^2 \\ + s^3\pi^3 p_j^2\Bigg]\left(e^{2s\phi} + e^{2s\tilde{\phi}}\right)\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \\ \leq Z\int_{[0,T]\times\rho} s^3\pi^3 p_j^2\left(e^{2s\phi} + e^{2s\tilde{\phi}}\right)\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\int_Q \pi^3\left(p_1^2 + p_2^2 + p_3^2 + |\Delta p_1|^2 + |\Delta p_2|^2 + |\Delta p_3|^2\right)\left(e^{2s\phi} + e^{2s\tilde{\phi}}\right)\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \\ \leq Z\int_{[0,T]\times\rho} \pi^3\left(p_1^2 + p_2^2 + p_3^2\right)\left(e^{2s\phi} + e^{2s\tilde{\phi}}\right)\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x.\end{aligned} \tag{3.7}$$

Remark 3.4 *In our case, it is necessary to control the boundary conditions as in* [27]*, contrary to the situation studied in* [10].

Proof. Let us consider the operator $\widehat{M}$ defined as follows

$$\widehat{M}\,p_j = -\partial_t p_j - \beta\,\partial_b^2 p_j - \mu_j \Delta p_j, \quad 1 \leq j \leq 3. \tag{3.8}$$

Also, set

$$\tilde{h}_j(b,t,x) = -(\theta_j + \gamma_j)\,p_j + v_j \partial_b\, p_j, \quad (\gamma_3 = \gamma_4 = 0), \quad 1 \leq j \leq 3. \tag{3.9}$$

To avoid any confusion with the controls u_j introduced earlier, we denote here $u_j = e^{s\phi} p_j$ as the weighted adjoint variable. In other words, in this section, u_j is no longer the control but the adjoint variable modified by the exponential factor $e^{s\phi}$. Alternatively, one could use the notation $\hat{u}_j$ to explicitly distinguish this variable from the controls u_j.

Let $u_j = e^{s\phi} p_j$ and $\tilde{u}_j = e^{s\tilde{\phi}} p_j$, with $1 \leq j \leq 3$, where s is a positive constant. Let us set $H_0 = [0,T] \times \{B\} \times \Omega$, $H_1 = [0,T] \times \{0\} \times \Omega$, and $H = H_0 \cup H_1$. It follows from (3.5) and the boundary conditions satisfied by p_j that

$$u_j(b,T,x) = \tilde{u}_j(b,T,x) = u_j(b,0,x) = \tilde{u}_j(b,0,x) = 0 \;\text{ for } (b,x) \in [0,B] \times \Omega.$$

Let q and $\tilde{q}$ be the operators defined by the following expressions:

$$qu_j = e^{s\phi}\widehat{M}e^{-s\phi}u_j, \quad \tilde{q}u_j = e^{s\tilde{\phi}}\widehat{M}e^{-s\tilde{\phi}}u_j.$$

From equations (3.3), (3.8) and (3.9) it follows that

$$\begin{aligned} qu_j &= e^{s\phi}\widehat{M}e^{-s\phi}u_j = e^{s\phi}\tilde{h}_j \;\text{ on } Q, \\ \tilde{q}\tilde{u}_j &= e^{s\tilde{\phi}}\widehat{M}e^{-s\tilde{\phi}}\tilde{u}_j = e^{s\tilde{\phi}}\tilde{h}_j \;\text{ on } Q. \end{aligned}$$

Therefore, explicitly, we obtain

$$
\begin{aligned}
qu_j = & -\partial_t u_j - \beta\partial_b^2 u_j - \mu_1\Delta u_j + s\partial_t\phi u_j + 2\beta s\delta\partial_X\xi\pi\partial_b u_j \\
& + \beta s\delta\partial_X^2\xi\pi u_j + \beta s\delta^2(\partial_X\xi)^2\pi u_j - \beta s^2\delta^2(\partial_X\xi)^2\pi^2 u_j \\
& + 2\mu_j s\delta\pi\nabla\xi\nabla u_j + \mu_j s\delta\Delta\xi\pi u_j + \mu_j s\delta^2|\nabla\xi|^2\pi u_j - \mu_j s^2\delta^2|\nabla\xi|^2\pi^2 u_j,
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
\tilde{q}\tilde{u}_j = & -\partial_t\tilde{u}_j - \beta\partial_b^2\tilde{u}_j - \mu_1\Delta\tilde{u}_j + s\partial_t\phi\tilde{u}_j - 2\beta s\delta\partial_X\xi\tilde{\pi}\partial_b\tilde{u}_j \\
& - \beta s\delta\partial_X^2\xi\tilde{\pi}\tilde{u}_j + \beta s\delta^2(\partial_X\xi)^2\tilde{\pi}\tilde{u}_j - \beta s^2\delta^2(\partial_X\xi)^2\tilde{\pi}^2\tilde{u}_j \\
& - 2\mu_j s\delta\tilde{\pi}\nabla\xi\nabla\tilde{u}_j - \mu_j s\delta\Delta\xi\tilde{\pi}\tilde{u}_j + \mu_j s\delta^2|\nabla\xi|^2\tilde{\pi}\tilde{u}_j - \mu_j s^2\delta^2|\nabla\xi|^2\tilde{\pi}^2\tilde{u}_j.
\end{aligned} \tag{3.11}
$$

We will now introduce the operators $M_1, M_2, \tilde{M}_1, \tilde{M}_2$ as follows:

$$
\begin{aligned}
M_1 u_j = & -\beta\partial_b^2 u_j - \mu_j\Delta u_j - \beta s^2\delta^2(\partial_X\xi)^2\pi^2 u_j \\
& - \mu_j s^2\delta^2|\nabla\xi|^2\pi^2 u_j + s\partial_t\phi u_j, \\
M_2 u_j = & -\partial_t u_j + 2\beta s\delta\partial_X\xi\pi\partial_b u_j + 2\beta s\delta^2(\partial_X\xi)^2\pi u_j \\
& + 2\mu_j s\delta\pi\nabla\xi\nabla u_j + 2\mu_j s\delta^2|\nabla\xi|^2\pi u_j, \\
\tilde{M}_1\tilde{u}_j = & -\beta\partial_b^2\tilde{u}_j - \mu_j\Delta\tilde{u}_j - \beta s^2\delta^2(\partial_X\xi)^2\tilde{\pi}^2\tilde{u}_j \\
& - \mu_j s^2\delta^2|\nabla\xi|^2\tilde{\pi}^2\tilde{u}_j, + s\partial_t\phi\tilde{u}_j, \\
\tilde{M}_2\tilde{u}_j = & -\partial_t\tilde{u}_j - 2\beta s\delta\partial_X\xi\tilde{\pi}\partial_b\tilde{u}_j + 2\beta s\delta^2(\partial_X\xi)^2\tilde{\pi}u_j \\
& - 2\mu_j s\delta\tilde{\pi}\nabla\xi\nabla\tilde{u}_j + 2\mu_j s\delta^2|\nabla\xi|^2\tilde{\pi}\tilde{u}_j.
\end{aligned} \tag{3.12}
$$

From equations (3.9), (3.10), (3.11) and (3.12) we deduce that

$$
\begin{cases}
M_1 u_j + M_2 u_j = g_s \text{ in } Q, \\
\tilde{M}_1\tilde{u}_j + \tilde{M}_2\tilde{u}_j = \tilde{g}_s \text{ in } Q,
\end{cases} \tag{3.13}
$$

where

$$
\begin{aligned}
g_s &= \tilde{h}_j e^{s\phi} - \beta s\delta\pi\partial_X^2\xi u_j + \beta s\delta^2\pi(\partial_X\xi)^2 u_j - \mu_j s\delta\pi|\nabla\xi|^2 u_j + \mu_j s\delta^2\pi|\nabla\xi|^2 u_j, \\
\tilde{g}_s &= \tilde{h}_j e^{s\tilde{\phi}} + \beta s\delta\tilde{\pi}\partial_X^2\xi\tilde{u}_j + \beta s\delta^2\tilde{\pi}(\partial_X\xi)^2\tilde{u}_j + \mu_j s\delta\tilde{\pi}|\nabla\xi|^2\tilde{u}_j + \mu_j s\delta^2\tilde{\pi}|\nabla\xi|^2\tilde{u}_j.
\end{aligned}
$$

Next, by taking the $L^2(Q)$-norm in (3.13) we have

$$
\begin{aligned}
\|g_s\|^2_{L^2(Q)} &= \|M_1 u_j\|^2_{L^2(Q)} + \|M_2 u_j\|^2_{L^2(Q)} + 2\big(M_1 u_j, M_2 u_j\big)_{L^2(Q)}, \\
\|\tilde{g}_s\|^2_{L^2(Q)} &= \big\|\tilde{M}_1\tilde{u}_j\big\|^2_{L^2(Q)} + \big\|\tilde{M}_2\tilde{u}_j\big\|^2_{L^2(Q)} + 2\big(\tilde{M}_1\tilde{u}_j, \tilde{M}_2\tilde{u}_j\big)_{L^2(Q)}.
\end{aligned} \tag{3.14}
$$

From the definition of the function ξ, it is also known that for all $\Lambda \in \partial\Omega$ the following conditions are satisfied:

$$
\begin{aligned}
u_j(0,t,x) = u_j(B,t,x) &= \tilde{u}_j(0,t,x) = \tilde{u}_j(B,t,x) = 0 \\
&\qquad\qquad \text{in } (0,T)\times\Omega, \text{ where } 1 \leq j \leq 4, \\
u_j(b,t,\Lambda) &= \tilde{u}_j(b,t,\Lambda) = 0 \text{ in } \Gamma = (0,B)\times(0,T)\times\partial\Omega, \text{ where } 1 \leq j \leq 4, \\
\nabla\tilde{u}_j(b,t,\Lambda) &= \big(\nabla\tilde{u}_j(b,t,\Lambda).\eta(\Lambda)\big)\eta(\Lambda) \text{ for all } \Lambda\ \in \partial\Omega, \\
\nabla u_j(b,t,\Lambda) &= \big(\nabla u_j(b,t,\Lambda).\eta(\Lambda)\big)\eta(\Lambda) \text{ for all } \Lambda\ \in \partial\Omega, \\
\nabla\xi(\Lambda) &= \big(\nabla\xi(\Lambda).\eta(\Lambda)\big)\eta(\Lambda) \text{ for all } \Lambda\ \in \partial\Omega.
\end{aligned}
$$

Thus, from (3.12) and the various conditions, we obtain the following equalities:

$$
\begin{aligned}
2\big(M_1u_j, M_2u_j\big)_{L^2(Q)} = {}& 2\beta^2 s^3\delta^4 \int_Q (u_j)^2\pi^3\left(\partial_X\xi\right)^4 \,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \\
&+ 6\beta^2 s\delta^2 \int_Q \pi\left(\partial_X\xi\right)^2\left(\partial_b u_j\right)^2 \,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \\
&- 2\beta s\delta \int_0^T\int_\Omega \left[u_j\partial_t u_j\pi\left(\partial_X\xi\right)\right]_{b=0} \mathrm{d}t\,\mathrm{d}x \\
&+ 4\beta^2 s^2\delta^3 \int_0^T\int_\Omega \left[(u_j)^2\pi^2\left(\partial_X\xi\right)^3\right]_{b=0} \mathrm{d}t\,\mathrm{d}x \\
&+ 2\beta^2 s^3\delta^3 \int_0^T\int_\Omega \left[(u_j)^2\pi^3\left(\partial_X\xi\right)^3\right]_{b=0} \mathrm{d}t\,\mathrm{d}x \\
&- 2\beta^2 s\delta \int_0^T\int_\Omega \left[(\partial_b u_j)^2\pi\left(\partial_X\xi\right)\right]_0^B \mathrm{d}t\,\mathrm{d}x \qquad (3.15) \\
&- 2\beta s^2\delta \int_0^T\int_\Omega \left[(u_j)^2\pi\left(\partial_X\xi\right)\partial_t\phi\right]_{b=0} \mathrm{d}t\,\mathrm{d}x \\
&+ 2\mu_j^2 s^3\delta^4 \int_Q \pi^3 u_j^2|\nabla\xi|^4 \,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \\
&+ 2\mu_j^2 s\delta^2 \int_Q \pi\left|\nabla u_j\right|^2|\nabla\xi|^2 \,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \\
&+ 4\mu_j^2 s\delta^2 \int_Q \pi\left|\nabla u_j\cdot\nabla\xi\right|^2 \,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \\
&- 2\mu_j^2 s\delta \int_\Gamma \pi\,\frac{\partial\xi}{\partial\eta}\left|\frac{\partial u_j}{\partial\eta}\right|^2 \mathrm{d}b\,\mathrm{d}t\,\mathrm{d}\Lambda + X_1
\end{aligned}
$$

and

$$
\begin{aligned}
2\big(\tilde{M}_1\tilde{n}_j, \tilde{M}_2\tilde{n}_j\big)_{L^2(Q)} = {} & 2\beta^2 s^3\delta^4 \int_Q (\tilde{u}_j)^2 \tilde{\pi}^3 \left(\partial_X \xi\right)^4 \, \mathrm{d}b \, \mathrm{d}t \, \mathrm{d}x \\
& + 6\beta^2 s\delta^2 \int_Q \tilde{\pi} \left(\partial_X \xi\right)^2 \left(\partial_b \tilde{u}_j\right)^2 \, \mathrm{d}b \, \mathrm{d}t \, \mathrm{d}x \\
& + 2\beta s\delta \int_0^T \int_\Omega \left[\tilde{u}_j \partial_t \tilde{n}_j \tilde{\pi} \left(\partial_X \xi\right)\right]_{b=0} \, \mathrm{d}t \, \mathrm{d}x \\
& - 4\beta^2 s^2 \delta^3 \int_0^T \int_\Omega \big[(\tilde{u}_j)^2 \tilde{\pi}^2 \left(\partial_X \xi\right)^3\big]_{b=0} \mathrm{d}t \, \mathrm{d}x \\
& - 2\beta^2 s^3 \delta^3 \int_0^T \int_\Omega \big[(\tilde{u}_j)^2 \tilde{\pi}^3 \left(\partial_X \xi\right)^3\big]_{b=0} \mathrm{d}t \, \mathrm{d}x \\
& + 2\beta^2 s\delta \int_0^T \int_\Omega \big[(\partial_b \tilde{u}_j)^2 \tilde{\pi} \left(\partial_X \xi\right)\big]_0^B \, \mathrm{d}t \, \mathrm{d}x \\
& + 2\beta s^2 \delta \int_0^T \int_\Omega \big[(\tilde{u}_j)^2 \tilde{\pi} \left(\partial_X \xi\right) \partial_t \tilde{\phi}\big]_{b=0} \mathrm{d}t \, \mathrm{d}x \\
& + 2\mu_j^2 s^3 \delta^4 \int_Q \tilde{\pi}^3 \tilde{u}_j^2 |\nabla \xi|^4 \, \mathrm{d}b \, \mathrm{d}t \, \mathrm{d}x \\
& + 2\mu_j^2 s\delta^2 \int_Q \tilde{\pi} \left|\nabla \tilde{n}_j\right|^2 |\nabla \xi|^2 \, \mathrm{d}b \, \mathrm{d}t \, \mathrm{d}x \\
& - 4\mu_j^2 s\delta^2 \int_Q \tilde{\pi} \left|\nabla \tilde{u}_j \cdot \nabla \xi\right|^2 \, \mathrm{d}b \, \mathrm{d}t \, \mathrm{d}x \\
& + 2\mu_j^2 s\delta \int_\Gamma \tilde{\pi} \frac{\partial \xi}{\partial \eta} \left|\frac{\partial \tilde{u}_j}{\partial \eta}\right|^2 \, \mathrm{d}b \, \mathrm{d}t \, \mathrm{d}\Lambda + X_2
\end{aligned}
\tag{3.16}
$$

with

$$
\begin{aligned}
|X_1| \leq z_1 \int_Q \Big[& \left(s^3\delta^3\pi^3 + s^2\delta^4\pi^3\right) (u_j)^2 + (s\delta\pi + 1) \left|\partial_b u_j\right|^2 \\
& + \left(s\delta\pi + s\delta^2\pi\right) \left|\nabla u_j\right|^2 \Big] \mathrm{d}b \, \mathrm{d}t \, \mathrm{d}x \ \text{ for all } s \geq 1 \text{ and } \delta \geq 1,
\end{aligned}
$$

and

$$
\begin{aligned}
|X_2| \leq z_2 \int_Q \Big[& \left(s^3\delta^3\tilde{\pi}^3 + s^2\delta^4\tilde{\pi}^3\right) (\tilde{u}_j)^2 + (s\delta\tilde{\pi} + 1) \left|\partial_b \tilde{u}_j\right|^2 \\
& + \left(s\delta\tilde{\pi} + s\delta^2\tilde{\pi}\right) \left|\nabla \tilde{u}_j\right|^2 \Big] \mathrm{d}b \, \mathrm{d}t \, \mathrm{d}x \ \text{ for all } s \geq 1 \text{ and } \delta \geq 1.
\end{aligned}
$$

Taking into account equations (3.14), (3.15) and (3.16), and using the fact that $\xi(X) = 0$ on $\partial\varpi$, we deduce that

$$\begin{aligned}
&\|g_s\|^2_{L^2(Q)} + \|\tilde{g}_s\|^2_{L^2(Q)} \\
&\quad = \|M_1 u_j\|^2_{L^2(Q)} + \|M_2 u_j\|^2_{L^2(Q)} + \left\|\tilde{M}_1 \tilde{u}_j\right\|^2_{L^2(Q)} \\
&\qquad + \left\|\tilde{M}_2 \tilde{u}_j\right\|^2_{L^2(Q)} \ + \ 2X_1 \ + \ 2X_2 \\
&\qquad + 2\beta^2 s^3 \delta^4 \int_Q \left[(u_j)^2 \pi^3 (\partial_X \xi)^4 + (\tilde{u}_j)^2 \pi^3 (\partial_X \xi)^4\right] \,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \\
&\qquad + 6\beta^2 s \delta^2 \int_Q \left[\pi (\partial_X \xi)^2 (\partial_b u_j)^2 + \tilde{\pi} (\partial_X \xi)^2 (\partial_b \tilde{u}_j)^2\right] \,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \\
&\qquad + \ 2\mu_j^2 s^3 \delta^4 \int_Q \left[\pi^3 (u_j)^2 |\nabla \xi|^4 + \tilde{\pi}^3 (\tilde{u}_j)^2 |\nabla \xi|^4\right] \mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \\
&\qquad + 2\mu_j^2 s \delta^2 \int_Q \left[\pi |\nabla u_j|^2 |\nabla \xi|^2 + \tilde{\pi} |\nabla \tilde{u}_j|^2 |\nabla \xi|^2\right] \mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned} \tag{3.17}$$

Consequently, equality (3.17) leads to the following inequality

$$\begin{aligned}
&\|g_s\|^2_{L^2(Q)} + \|\tilde{g}_s\|^2_{L^2(Q)} \\
&\quad \geq \|M_1 u_j\|^2_{L^2(Q)} + \|M_2 u_j\|^2_{L^2(Q)} + \|\tilde{M}_1 \tilde{u}_j\|^2_{L^2(Q)} \\
&\qquad + \|\tilde{M}_2 \tilde{u}_j\|^2_{L^2(Q)} + X_1 + X_2 \\
&\qquad + 2\beta^2 s^3 \delta^4 \int_Q \left[(u_j)^2 \pi^3 (\partial_{b,x} \xi)^4 + (\tilde{u}_j)^2 \tilde{\pi}^3 (\partial_X \xi)^4\right] \,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \\
&\qquad + 2\beta^2 s \delta^2 \int_Q \left[\pi (\partial_X \xi)^2 (\partial_b u_j)^2 + \tilde{\pi} (\partial_X \xi)^2 (\partial_b \tilde{u}_j)^2\right] \,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \\
&\qquad + 2\mu_j^2 s^3 \delta^4 \int_Q \left[\pi^3 (u_j)^2 |\nabla \xi|^4 + \tilde{\pi}^3 (\tilde{u}_j)^2 |\nabla \xi|^4\right] \,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \\
&\qquad + 2\mu_j^2 s \delta^2 \int_Q \left[\pi |\nabla u_j|^2 |\nabla \xi|^2 + \tilde{\pi} |\nabla \tilde{u}_j|^2 |\nabla \xi|^2\right] \,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}$$

Thus, from the definition of ξ, there exists $\alpha > 0$ such that

$$|\nabla_X \xi(X)| > \alpha > 0 \ \text{ for all } X \in \overline{\varpi \backslash \omega_0}.$$

Following these calculations, estimate (3.6), which corresponds to the Carleman inequality established in this article, is obtained by arguments similar to those developed in [8, 10, 27]. Finally, summing over j yields inequality (3.7). □

We can now establish the observability inequality for the system defined by equations (3.3)–(3.4), and subsequently demonstrate the null controllability result for the system defined by equations (3.1)–(3.2). We begin by proving the observability inequality using the Carleman estimate.

Lemma 3.5 *For every solution* $p = (p_1, p_2, p_3) \in \left(L^2(Q)\right)^3$ *of the adjoint problem* (3.3)–(3.4)*, we have the following inequality*

$$\int_0^B \int_\Omega \sum_{j=1}^3 |p_j(b,0,x)|^2 \,\mathrm{d}x\,\mathrm{d}b + \int_0^T \int_\Omega \sum_{j=1}^3 |p_j(0,t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t \\ \leq C \int_{[0,T]\times\rho} \sum_{j=1}^3 |p_j(b,t,x)|^2 \,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x; \tag{3.18}$$

the constant $C > 0$ *depends on the final time* T*, the spatial domain* Ω*, the control region* ρ*, and the bounds of the coefficients of the system and the cut-off function* V.

Given a positive constant σ, we introduce the change of variables $\hat{p_j}(b,t,x) = e^{\sigma t} p_j(b,t,x)$, $1 \leq j \leq 4$. Under this transformation, the adjoint system (3.3)–(3.4) can be rewritten as follows:

$$\begin{cases} -\partial_t \hat{p_1} - \partial_b(v_1 \hat{p_1}) - \beta \partial_b^2 \hat{p_1} - \mu_1 \Delta \hat{p_1} + (\sigma + \theta_1 + \gamma_1 + \partial_b v_1)\hat{p_1} = 0, \\ -\partial_t \hat{p_2} - \partial_b(v_2 \hat{p_2}) - \beta \partial_b^2 \hat{p_2} - \mu_2 \Delta \hat{p_2} + (\sigma + \theta_2 + \gamma_2 + \partial_b v_2)\hat{p_2} = 0, \\ -\partial_t \hat{p_3} - \partial_b(v_3 \hat{p_3}) - \beta \partial_b^2 \hat{p_3} - \mu_3 \Delta \hat{p_3} + (\sigma + \theta_3 + \partial_b v_3)\hat{p_3} = 0, \\ -\partial_t \hat{p_4} - \partial_b(v_4 \hat{p_4}) - \beta \partial_b^2 \hat{p_4} - \mu_4 \Delta \hat{p_4} + (\sigma + \theta_4 + \partial_b v_4)\hat{p_4} = 0. \end{cases} \tag{3.19}$$

and

$$\begin{cases} \partial_b \hat{p_j}(0,t,x) = \hat{p_j}(B,t,x) = 0 \text{ for } (t,x) \in (0,T)\times\Omega \text{ and } 1 \leq j \leq 4, \\ \hat{p_j}(b,T,x) = -\varepsilon^{-1}\, e^{\sigma T} y_j(b,T,x) \text{ for } (b,x) \in (0,B)\times\Omega \text{ and } 1 \leq j \leq 3, \\ \hat{p_4}(b,T,x) = 0 \text{ for } (b,x) \in (0,B)\times\Omega, \\ \partial_\eta e^{-\sigma t} \hat{p_j}(b,t,x) = 0 \text{ for } (b,t,x) \in (0,B)\times(0,T)\times\partial\Omega \text{ and } 1 \leq j \leq 4. \end{cases} \tag{3.20}$$

In order to prove Lemma 3.5, it is necessary to use an auxiliary result, which is stated in the following proposition.

Proposition 3.6 *Let* $\hat{p} = (\hat{p}_1, \hat{p}_2, \hat{p}_3) \in \left(L^2(Q)\right)^3$ *be a solution of problem* (3.19)–(3.20)*. Then, we have*

$$\int_0^B \int_\Omega \left(|\hat{p}_1(b,0,x)|^2 + |\hat{p}_2(b,0,x)|^2 + |\hat{p}_3(b,0,x)|^2\right) \mathrm{d}b\,\mathrm{d}x \\ \leq C' \int_{[0,T]\times\rho} \left(|\hat{p}_1|^2 + |\hat{p}_2|^2 + |\hat{p}_3|^2\right) \mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x.$$

Proof of Proposition 3.6. Note that

$$\pi^3 e^{2s\tilde{\phi}} \geq \frac{e^{-2Z(1+(1/T)^2)}}{T^6}, \qquad t \in [(T/4), (3T/4)], \tag{3.21}$$

and

$$\pi^3 e^{2s\phi} \leq A_1 e^{3Z}, \qquad t \in [0,T]. \tag{3.22}$$

Using (3.21), (3.22) and (3.6), we obtain

$$\int_{\varpi\times[(T/4),(3T/4)]} (\hat{p}_j)^2 \, \mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \le C_1 \int_{[0,T]\times\rho} (\hat{p}_j)^2 \, \mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x. \tag{3.23}$$

Let $V \in C^2([0,T])$ be a cut-off function satisfying

$$\begin{cases} V(t) = 1 & \text{for } t \in [0,(T/4)], \\ 0 \le V(t) \le 1 & \text{for } t \in [(T/4),(3T/4)], \\ V(t) = 0 & \text{for } t > (3T/4). \end{cases}$$

Thus, by multiplying equation j of the system (3.19) by $V\hat{p}_j$ and integrating by parts over Q, we deduce that

$$\int_0^B \int_\Omega (\hat{p}_j(b,0,x))^2 \, \mathrm{d}b\,\mathrm{d}x + (2\sigma - Z_j) \int_Q (\hat{p}_j)^2 \, V \,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \\ \le \int_Q -\partial_t V \, (\hat{p}_j)^2 \, \mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x,$$

where $Z_j > 0$ is a constant depending on the bounds of the coefficients of the system.

From the definition of V, it follows that

$$\int_0^B \int_\Omega (\hat{p}_j(b,0,x))^2 \, \mathrm{d}b\,\mathrm{d}x \le C_2 \int_{\varpi\times[(T/4),(3T/4)]} (\hat{p}_j)^2 \, \mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x. \tag{3.24}$$

Next, from the inequalities (3.23) and (3.24), we obtain

$$\int_0^B \int_\Omega (\hat{p}_j(b,0,x))^2 \, \mathrm{d}b\,\mathrm{d}x \le C_3 \int_{[0,T]\times\rho} (\hat{p}_j)^2 \, \mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x. \tag{3.25}$$

Finally, by summing over j for $1 \le j \le 3$, equation (3.25) leads to the desired result. □

Proof of Lemma 3.5. We observe that, thanks to the regularity of p_j and standard energy arguments, we obtain an estimate of $p_j(0,t,x)$ in $L^2(Q)$. By applying the Cauchy–Schwarz and Young inequalities, and then integrating over the interval $[0,T]$, we obtain the inequality

$$\int_0^T |p_j(0,t,x)|^2 \, \mathrm{d}t \le \frac{B}{r(T/2)} \int_0^B \int_\Omega r(\partial_X p_j)^2 \,\mathrm{d}b\,\mathrm{d}x. \tag{3.26}$$

From equations (3.6), (3.22) and (3.26), we deduce that

$$\int_0^T |p_j(0,t,x)|^2 \, \mathrm{d}t \le C_s \int_{[0,T]\times\rho} p_j^2 \,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x, \tag{3.27}$$

with $C_s = s^3 A_1 e^{3Z}$. Now, since $\hat{p}_j(b,0,x) = p_j(b,0,x)$ and $\hat{p}_j^2 \le e^{2\sigma T} p_j^2$, using (3.27) and Proposition 3.6, we obtain (3.18). □

With our observability inequality, we can now prove the null controllability result.

Theorem 3.7 *Assume that the conditions* (F_1)–(F_5) *are satisfied. Let* $B > 0$ *and* $T > 0$ *be fixed. Then, for any initial data* $\left(y_1^0, y_2^0, y_3^0\right) \in \left(L^2((0,B)\times\Omega)\right)^3$*, there exists a control*

$$u = (u_1, u_2, u_3) \in \left(L^2((0,T)\times\rho)\right)^3$$

such that the associated solution of the system (3.1)–(3.2) *satisfies* (1.3).

Proof. Consider the systems (3.1) and (3.2) along with the following optimal control problem of minimizing

$$\frac{1}{2}\int_Q \left(|u_1(b,t,x)|^2 + |u_2(b,t,x)|^2 + |u_3(b,t,x)|^2\right) \mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \\ + \frac{1}{2\varepsilon}\int_0^B\int_\Omega \left(|y_1(b,T,x)|^2 + |y_2(b,T,x)|^2 + |y_3(b,T,x)|^2\right) \mathrm{d}x\,\mathrm{d}b,$$

where $Q = (0,B)\times(0,T)\times\Omega$, $u_j \in L^2(Q)$ and y_j, for $1 \le j \le 3$, are solutions of (3.1)–(3.2). For $u = (u_1, u_2, u_3)$ let

$$J_\varepsilon(u) = \frac{1}{2}\int_Q \left(|u_1(b,t,x)|^2 + |u_2(b,t,x)|^2 + |u_3(b,t,x)|^2\right) \mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \\ + \frac{1}{2\varepsilon}\int_0^B\int_\Omega \left(|y_1(b,T,x)|^2 + |y_2(b,T,x)|^2 + |y_3(b,T,x)|^2\right) \mathrm{d}x\,\mathrm{d}b.$$

The functional J_ε is continuous, strictly convex, and coercive. Consequently, it admits a unique minimizer, denoted by $u_\varepsilon = (u_{1,\varepsilon}, u_{2,\varepsilon}, u_{3,\varepsilon})$. After some elementary computations, we obtain

$$u_{j,\varepsilon}(b,t,x) = \chi(b,x)p_{j,\varepsilon}(b,t,x) \text{ for a.e. } (b,t,x) \in Q, \text{ where } 1 \le j \le 3. \tag{3.28}$$

Let $(p_{1,\varepsilon}, p_{2,\varepsilon}, p_{3,\varepsilon})$ be a solution of (3.3)–(3.4), and let $y_\varepsilon = (y_{1,\varepsilon}, y_{2,\varepsilon}, y_{3,\varepsilon})$ be the solution of (3.1)–(3.2) associated with u_ε. Multiplying the j-th equation of the system (3.3)–(3.4), for $j \in \{1,2,3\}$, by $y_{j,\varepsilon}$ and integrating over Q, we obtain

$$\int_Q p_{j,\varepsilon}\chi(b,x)u_{j,\varepsilon}\,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x + \frac{1}{\varepsilon}\int_0^B\int_\Omega y_{j,\varepsilon}^2(b,T,x)\,\mathrm{d}x\,\mathrm{d}b \\ + \int_0^B\int_\Omega p_{j,\varepsilon}(b,0,x)y_j^0(b,x)\,\mathrm{d}x\,\mathrm{d}b + \int_0^T p_{j,\varepsilon}(0,t,x)f_j(t,x)\,\mathrm{d}t = 0. \tag{3.29}$$

Considering equations (3.28) and (3.29), we obtain

$$\int_{[0,T]\times\rho} u_{j,\varepsilon}^2\,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x + \frac{1}{\varepsilon}\int_0^B\int_\Omega y_{j,\varepsilon}^2(b,T,x)\,\mathrm{d}x\,\mathrm{d}b \\ = -\int_0^B\int_\Omega p_{j,\varepsilon}(b,0,x)y_j^0(b,x)\,\mathrm{d}x\,\mathrm{d}b - \int_0^T p_{j,\varepsilon}(0,t,x)f_j(t,x)\,\mathrm{d}t. \tag{3.30}$$

Subsequently, by applying the Young inequality to (3.30), we have

$$\begin{aligned}
&\int_{[0,T]\times\rho} u_{j,\varepsilon}^2 \,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x + \frac{1}{\varepsilon}\int_0^B\int_\Omega y_{j,\varepsilon}^2(b,T,x)\,\mathrm{d}x\,\mathrm{d}b \\
&\quad \le \frac{1}{2C}\left(\int_0^B\int_\Omega p_{j,\varepsilon}^2(b,0,x)\,\mathrm{d}x\,\mathrm{d}b + \int_0^T p_{j,\varepsilon}^2(0,t,x)\,\mathrm{d}t\right) \\
&\qquad + 2C\left(\int_0^B\int_\Omega \left(y_j^0\right)^2(b,x)\,\mathrm{d}a + \int_0^T f_j^2(t,x)\,\mathrm{d}t\right).
\end{aligned}$$

So,

$$\begin{aligned}
&\int_{[0,T]\times\rho}\sum_{j=1}^3 u_{j,\varepsilon}^2 \,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x + \frac{1}{\varepsilon}\int_0^B\int_\Omega\sum_{j=1}^3 y_{j,\varepsilon}^2(b,T,x)\,\mathrm{d}b\,\mathrm{d}x \\
&\le \frac{1}{2\,C}\left(\int_0^B\int_\Omega\sum_{j=1}^3 p_{j,\varepsilon}^2(b,0,x)\,\mathrm{d}x\,\mathrm{d}b + \int_0^T\sum_{j=1}^3 p_{j,\varepsilon}^2(0,t,x)\,\mathrm{d}t\right) \qquad (3.31)\\
&\quad + 2C\left(\int_0^B\int_\Omega\sum_{j=1}^3 \left(y_j^0\right)^2(b,x)\,\mathrm{d}x\,\mathrm{d}b + \int_0^T\sum_{j=1}^3 f_j^2(t,x)\,\mathrm{d}t\right).
\end{aligned}$$

Starting from (3.18) and (3.31), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{[0,T]\times\rho}\sum_{j=1}^3 u_{j,\varepsilon}^2 \,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x + \frac{1}{\varepsilon}\int_0^B\int_\Omega\sum_{j=1}^3 y_{j,\varepsilon}^2(b,T,x)\,\mathrm{d}b\,\mathrm{d}x \\
&\quad \le 2C\left(\int_0^B\int_\Omega\sum_{j=1}^3 \left(y_j^0\right)^2(b,x)\,\mathrm{d}x\,\mathrm{d}b + \int_0^T\sum_{j=1}^3 f_j^2(t,x)\,\mathrm{d}t\right).
\end{aligned} \qquad (3.32)$$

Thus, from (3.32), we deduce that

$$\begin{aligned}
&\|u_{j,\varepsilon}\|^2_{L^2([0,T]\times\rho)} \\
&\quad \le 4C\left(\int_0^B\int_\Omega\sum_{j=1}^3 \left(y_j^0\right)^2(b,x)\,\mathrm{d}x\,\mathrm{d}b + \int_0^T\sum_{j=1}^3 f_j^2(t,x)\,\mathrm{d}t\right), \quad 1\le j\le 3, \\
&\int_0^B\int_\Omega y_{j,\varepsilon}^2(b,T,x)\,\mathrm{d}x\,\mathrm{d}b \\
&\quad \le 2\varepsilon C\left(\int_0^B\int_\Omega\sum_{j=1}^3 \left(y_j^0\right)^2(b,x)\,\mathrm{d}x\,\mathrm{d}b + \int_0^T\sum_{j=1}^3 f_j^2(t,x)\,\mathrm{d}t\right), \quad 1\le j\le 3.
\end{aligned} \qquad (3.33)$$

Hence, it is possible to extract subsequences, also denoted by $u_{j,\varepsilon}$, $y_{j,\varepsilon}$ $(1 \le j \le 3)$, such that $u_{j,\varepsilon} \longrightarrow u_j$ weakly in $L^2((0,T)\times\rho)$ and $y_{j,\varepsilon} \longrightarrow y_j$ weakly in $L^2(Q)$. Moreover, $y = (y_1, y_2, y_3)$ is the unique solution of the system (3.1) and (3.2) associated with $u = (u_1, u_2, u_3)$. Also, y satisfies (1.3). □

4 Proof of the main result

Let $\sigma_0 > 0$ be a real number, and for $1 \leq j \leq 4$ define $\tilde{y}_j = e^{-\sigma_0 t} y_j$. Then, $\tilde{y} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ is a solution of the system

$$\begin{cases} \partial_t \tilde{y}_1 + \partial_b[v_1 \tilde{y}_1] - \beta {\partial_b}^2 \tilde{y}_1 - \mu_1 \Delta \tilde{y}_1 + [\theta_1 + \gamma_1 + \sigma_0]\tilde{y}_1 = \chi(b,x)\tilde{u}_1, \\ \partial_t \tilde{y}_2 + \partial_b[v_2 \tilde{y}_2] - \beta {\partial_b}^2 \tilde{y}_2 - \mu_2 \Delta \tilde{y}_2 + [\theta_2 + \gamma_2 + \sigma_0]\tilde{y}_2 = \chi(b,x)\tilde{u}_2, \\ \partial_t \tilde{y}_3 + \partial_b[v_3 \tilde{y}_3] - \beta {\partial_b}^2 \tilde{y}_3 - \mu_3 \Delta \tilde{y}_3 + [\theta_3 + \sigma_0]\tilde{y}_3 = \chi(b,x)\tilde{u}_3, \\ \partial_t \tilde{y}_4 + \partial_b[v_4 \tilde{y}_4] - \beta {\partial_b}^2 \tilde{y}_4 - \mu_4 \Delta \tilde{y}_4 + [\theta_4 + \sigma_0]\tilde{y}_4 = 0. \end{cases} \tag{4.1}$$

It should be noted that no control term is introduced in the equation for $\tilde{y}_4$. The controls $\tilde{u}_j$ act only on the first three equations ($j = 1, 2, 3$), while the fourth component $\tilde{y}_4$ evolves without control. Moreover, B is the maximum age, given by $B = \max\{B_1, B_2, B_3, B_4\}$, so that all integrals can be taken over the same age interval.

For $(b, t, x) \in Q = (0, B) \times (0, T) \times \Omega$, and with the change of variables $\tilde{u}_j = e^{-\sigma_0 t} u_j$ for $1 \leq j \leq 4$, we obtain

$$\begin{cases} \left(v_1 \tilde{y}_1 - \beta \partial_b \tilde{y}_1\right)_{b=0} = \displaystyle\int_0^B \gamma_3 \tilde{y}_3(b,t,x)\,\mathrm{d}b, \\ \left(v_2 \tilde{y}_2 - \beta \partial_b \tilde{y}_2\right)_{b=0} = \displaystyle\int_0^B \gamma_1 \tilde{y}_1(b,t,x)\,\mathrm{d}b, \\ \left(v_3 \tilde{y}_3 - \beta \partial_b \tilde{y}_3\right)_{b=0} = \displaystyle\int_0^B \lambda\gamma_2 \tilde{y}_2(b,t,x)\,\mathrm{d}b, \\ \left(v_4 \tilde{y}_4 - \beta \partial_b \tilde{y}_4\right)_{b=0} = \displaystyle\int_0^B (1-\lambda)\gamma_2 \tilde{y}_2(b,t,x)\,\mathrm{d}b, \\ \partial_\eta \tilde{y}_j(b,t,x) = 0 \text{ for } (b,t,x) \in \ \Gamma = (0,B)\times(0,T)\times\partial\Omega \text{ and } 1 \leq j \leq 4, \\ \tilde{y}_j(b,0,x) = \tilde{y}_j^0(b,x) \text{ for } (b,x) \in \ Q_B = (0,B)\times\Omega \text{ and } 1 \leq j \leq 4, \\ \tilde{y}_j(B,t,x) = 0 \text{ for } (t,x) \in \ Q_T = (0,T)\times\Omega \text{ and } 1 \leq j \leq 4. \end{cases} \tag{4.2}$$

Now, let us consider the following system

$$\begin{cases} \partial_t \tilde{y}_1 + \partial_b(v_1 \tilde{y}_1) - \beta \partial_b^2 \tilde{y}_1 - \mu_1 \Delta \tilde{y}_1 + (\theta_1 + \gamma_1 + \sigma_0)\tilde{y}_1 = \chi(b,x)\tilde{u}_1, \\ \partial_t \tilde{y}_2 + \partial_b(v_2 \tilde{y}_2) - \beta \partial_b^2 \tilde{y}_2 - \mu_2 \Delta \tilde{y}_2 + (\theta_2 + \gamma_2 + \sigma_0)\tilde{y}_2 = \chi(b,x)\tilde{u}_2, \\ \partial_t \tilde{y}_3 + \partial_b(v_3 \tilde{y}_3) - \beta \partial_b^2 \tilde{y}_3 - \mu_3 \Delta \tilde{y}_3 + (\theta_3 + \sigma_0)\tilde{y}_3 = \chi(b,x)\tilde{u}_3, \\ \partial_t \tilde{y}_4 + \partial_b(v_4 \tilde{y}_4) - \beta \partial_b^2 \tilde{y}_4 - \mu_4 \Delta \tilde{y}_4 + (\theta_4 + \sigma_0)\tilde{y}_4 = 0, \end{cases} \tag{4.3}$$

and

$$\begin{cases} \left(v_1\tilde{y}_1 - \beta\partial_b\tilde{y}_1\right)_{b=0} = f_1(t,x), \\ \left(v_2\tilde{y}_2 - \beta\partial_b\tilde{y}_2\right)_{b=0} = f_2(t,x), \\ \left(v_3\tilde{y}_3 - \beta\partial_b\tilde{y}_3\right)_{b=0} = f_3(t,x), \\ \left(v_4\tilde{y}_4 - \beta\partial_b\tilde{y}_4\right)_{b=0} = f_4(t,x), \\ \partial_\eta\tilde{y}_j(b,t,x) = 0 \text{ for } (b,t,x) \in\ \Gamma = (0,B)\times(0,T)\times\partial\Omega \text{ and } 1\le j\le 4, \\ \tilde{y}_j(b,0,x) = \tilde{y}_j^0(b,x) \text{ for } (b,x)\in Q_B = (0,B)\times\Omega \text{ and } 1\le j\le 4, \\ \tilde{y}_j(B,t,x) = 0 \text{ for } (t,x)\in Q_T = (0,T)\times\Omega \text{ and } 1\le j\le 4, \end{cases} \tag{4.4}$$

with $f_1, f_2, f_3, f_4 \in L^2(Q_T)$ defined as in (4.2), and $Q_T = (0,T)\times\Omega$.

Note that, according to Section 3, the systems (4.3) and (4.4) are null controllable. We will now prove the null controllability of (4.1) and (4.2).

Let $\mathcal{H} = \prod_{j=1}^4 L^2(Q_T)$. We define the set $\mathcal{B}_\zeta(N_1,N_2,N_3,N_4)$ as follows:

$$\begin{aligned} &\mathcal{B}_\zeta(N_1,N_2,N_3,N_4) \\ &\quad = \Bigg\{ \tilde{u} = (\tilde{u}_1,\tilde{u}_2,\tilde{u}_3) \in \prod_{j=1}^3 L^2(p) \ :\ \tilde{y} \text{ is a solution of (4.3)–(4.4)}, \\ &\qquad\qquad \text{and } (\tilde{y}_1,\tilde{y}_2,\tilde{y}_3) \text{ satisfies (1.3); } f_j = N_j, \text{ and } \tilde{u} \text{ satisfies the bound (3.33)} \Bigg\}. \end{aligned}$$

Now, let us define the multivalued function G_ζ, which to every $(N_1,N_2,N_3,N_4)\in\mathcal{H}$ assigns the set $G_\zeta(N_1,N_2,N_3,N_4)\in 2^{\mathcal{H}}$ that consists of tuples

$$\left(\int_0^B \gamma_3\tilde{y}_3(b,t,x)\,\mathrm{d}b, \int_0^B \gamma_1\tilde{y}_1(b,t,x)\,\mathrm{d}b, \int_0^B \lambda\gamma_2\tilde{y}_2(b,t,x)\,\mathrm{d}b, \int_0^B (1-\lambda)\gamma_2\tilde{y}_2(b,t,x)\,\mathrm{d}b\right),$$

where $(\tilde{y}_{1,\tilde{u}},\tilde{y}_{2,\tilde{u}},\tilde{y}_{3,\tilde{u}})$ is a solution of (4.3)–(4.4) and $\tilde{u}\in\mathcal{B}_\zeta(N_1,N_2,N_3,N_4)$.

Our goal is to show that the multivalued function G_ζ has a fixed point. To achieve this, we will use a generalized version of the Leray–Schauder fixed point theorem. Or

$$\begin{aligned} \mathcal{R}_\zeta = \big\{ &(N_1,N_2,N_3,N_4)\in\mathcal{H} \ :\ \text{there exists } \Psi\in(0,1) \\ &\text{such that } (N_1,N_2,N_3,N_4)\in\Psi G_\zeta(N_1,N_2,N_3,N_4) \big\}. \end{aligned}$$

Subsequently, the existence of a fixed point for the multivalued function G_ζ directly follows from the following proposition.

Proposition 4.1 (a$_1$) *$\mathcal{R}_\zeta$ is bounded on $\mathcal{H}$.*

(a$_2$) *For all $(N_1,N_2,N_3,N_4)\in\mathcal{H}$, $G_\zeta(N_1,N_2,N_3,N_4)$ is a non-empty closed and convex subset of $\mathcal{H}$.*

(a$_3$) *$G_\zeta\colon \mathcal{H}\longrightarrow 2^{\mathcal{H}}$ is compact.*

(a$_4$) *The multivalued mapping G_ζ is upper semi-continuous on $\mathcal{H}$.*

Proof. Let us define $(\tilde{y}_{1,\tilde{u}}, \tilde{y}_{2,\tilde{u}}, \tilde{y}_{3,\tilde{u}}, \tilde{y}_{4,\tilde{u}}) = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$.

(a$_1$) Let $(N_1, N_2, N_3, N_4) \in \mathcal{R}_\zeta$. There exists $\Psi \in (0,1)$ such that

$$\frac{1}{\Psi}\left(N_1, N_2, N_3, N_4\right) \in G_\zeta\left(N_1, N_2, N_3, N_4\right).$$

Subsequently, there exists $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) \in \prod_{j=1}^3 L^2(\varpi)$ associated with $\tilde{u} \in \prod_{j=1}^3 L^2(p)$ such that

$$N_1 = \int_0^B \gamma_3 \tilde{y}_3 \, \mathrm{d}b, \qquad N_2 = \int_0^B \gamma_1 \tilde{y}_1 \, \mathrm{d}b,$$
$$N_3 = \int_0^B \lambda\gamma_2 \tilde{y}_2 \, \mathrm{d}b, \qquad N_4 = \int_0^B (1-\lambda)\gamma_2 \tilde{y}_2 \, \mathrm{d}b,$$

where $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ is the solution of the system (4.1)–(4.2) associated with $\tilde{u}$ satisfying the bound (3.33). We conclude that

$$\sum_{j=1}^4 \|N_j\|^2_{L^2(Q_T)} \le \sum_{j=1}^4 E_j \|\tilde{y}_j\|^2_{L^2(\varpi)} \tag{4.5}$$

with $E_j = B\,\|\gamma_j\|_{L^\infty([0,B]\times\Omega)}$ and $E_4 = 0$. Thus, by multiplying equation j of (4.1) and (4.2) by $\tilde{y}_j$, integrating by parts over ϖ, and then summing with respect to j, for $1 \le j \le 4$, we obtain

$$\begin{aligned}&\sum_{j=1}^4 (2\,\sigma_0 - Z_j - 1)\,\|\tilde{y}_j\|^2_{L^2(\varpi)} \\ &\quad\le \sum_{j=1}^4 \left\|\tilde{y}_j^0\right\|^2_{L^2([0,B]\times\Omega)} + \sum_{j=1}^4 \|N_j\|^2_{L^2(Q_T)} + \sum_{j=1}^4 \|\tilde{u}_j\|^2_{L^2(p)} \quad (u_4 \equiv 0 \text{ and } \tilde{u}_4 \equiv 0).\end{aligned} \tag{4.6}$$

Relying on relations (4.5), (4.6) and (3.33), we obtain

$$\begin{aligned}&\sum_{j=1}^4 (2\,\sigma_0 - Z_j - 1)\,\|\tilde{y}_j\|^2_{L^2(\varpi)} \\ &\quad\le (4C+1)\sum_{j=1}^4 \left\|\tilde{y}_j^0\right\|^2_{L^2([0,B]\times\Omega)} + (4C+1)\sum_{j=1}^4 E_j\,\|\tilde{y}_i\|^2_{L^2(\varpi)}.\end{aligned} \tag{4.7}$$

Consequently, inequality (4.7) leads to:

$$\sum_{j=1}^4 \big(2\,\sigma_0 - Z_j - 1 - (4C+1)E_j\big)\,\|\tilde{y}_j\|^2_{L^2(\varpi)} \le (4C+1)\sum_{j=1}^4 \left\|\tilde{y}_j^0\right\|^2_{L^2([0,B]\times\Omega)}. \tag{4.8}$$

Taking once again into account relations (4.5) and (4.8), we arrive at the desired result.

(a$_2$) Observe that for $(N_1, N_2, N_3, N_4) \in \mathcal{H}$ the set $G_\zeta(N_1, N_2, N_3, N_4)$ is non-empty. Indeed, systems (4.1) and (4.2) are equivalent to systems (3.1) and (3.2), which admit a solution. Moreover, $\tilde{u}$ satisfies (3.33).

The convexity of the set $G_\zeta(N_1, N_2, N_3, N_4)$ follows from the fact that the function $(N_1, N_2, N_3, N_4) \longmapsto (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ is affine.

To prove that the set $G_\zeta\left(N_1, N_2, N_3, N_4\right)$ is also closed let

$$\left(\omega_m^1, \omega_m^2, \omega_m^3, \omega_m^4\right) \in G_\zeta\left(N_1, N_2, N_3, N_4\right)$$

be such that

$$\left(\omega_m^1, \omega_m^2, \omega_m^3, \omega_m^4\right) \longrightarrow \left(\omega^1, \omega^2, \omega^3, \omega^4\right) \text{ in } \mathcal{H}.$$

Thus, for each m there exists $\tilde{u}_m$ satisfying (3.33) such that

$$\begin{aligned} \omega_m^1 &= \int_0^B \gamma_3 \tilde{y}_{3,m}\,\mathrm{d}b, & \omega_m^2 &= \int_0^B \gamma_1 \tilde{y}_{1,m}\,\mathrm{d}b, \\ \omega_m^3 &= \int_0^B \lambda\gamma_2 \tilde{y}_{2,m}\,\mathrm{d}b, & \omega_m^4 &= \int_0^B (1-\lambda)\gamma_2 \tilde{y}_{2,m}\,\mathrm{d}b, \end{aligned}$$

where $(\tilde{y}_{1,m}, \tilde{y}_{2,m}, \tilde{y}_{3,m})$ is the solution of the system (4.3)–(4.4) with (N_1, N_2, N_3, N_4) instead of (f_1, f_2, f_3, f_4), and $(\tilde{y}_{1,m}, \tilde{y}_{2,m}, \tilde{y}_{3,m})$ satisfies (4.8). From the inequalities (4.5), (4.8) and (3.33), it is possible to extract sequences, still denoted by $\left((\tilde{y}_{1,m}, \tilde{y}_{2,m}, \tilde{y}_{3,m})\right)_n$ and $(\tilde{u}_m)_n$, which converge weakly to $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ and $\tilde{u}$ in $\left(L^2(\varpi)\right)^3$ and in $\left(L^2(p)\right)^3$, respectively. Therefore,

$$\begin{aligned} \omega^1 &= \int_0^B \gamma_3 \tilde{y}_3\,\mathrm{d}b, & \omega^2 &= \int_0^B \gamma_1 \tilde{y}_1\,\mathrm{d}b, \\ \omega^3 &= \int_0^B \lambda\gamma_2 \tilde{y}_2\,\mathrm{d}b, & \omega^4 &= \int_0^B (1-\lambda)\gamma_2 \tilde{y}_2\,\mathrm{d}b. \end{aligned}$$

Moreover, $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ is the solution of the system (4.3) and (4.4), with $(N_1, N_2, N_3, N_4) = (f_1, f_2, f_3, f_4)$. Also, $\tilde{u}$ satisfies (3.33) and $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ satisfies (1.3). Therefore, $\left(\omega^1, \omega^2, \omega^3, \omega^4\right) \in G_\zeta\left(N_1, N_2, N_3, N_4\right)$.

(a$_3$) Let D be a bounded subset of $\mathcal{H}$, and let $(N_1, N_2, N_3, N_4) \in D$. Then, there exist $l_j > 0$, $1 \le j \le 4$, such that

$$\left\|N_j\right\|_{L^2(Q_T)} \le l_j, \quad 1 \le j \le 4. \tag{4.9}$$

Now, consider $\left(\left(\omega_m^1, \omega_m^2, \omega_m^3, \omega_m^4\right)\right)_n \in G_\zeta\left(N_1, N_2, N_3, N_4\right)$. For every $n \in \mathbb{N}$, there exist $(\tilde{y}_{1,m}, \tilde{y}_{2,m}, \tilde{y}_{3,m}) \in \left(L^2(\varpi)\right)^3$ and a control $\tilde{u}_m \in \left(L^2(p)\right)^3$ such that

$$\begin{aligned} \omega_m^1 &= \int_0^B \gamma_3 \tilde{y}_{3,m}\,\mathrm{d}b, & \omega_m^2 &= \int_0^B \gamma_1 \tilde{y}_{1,m}\,\mathrm{d}b, \\ \omega_m^3 &= \int_0^B \lambda\gamma_2 \tilde{y}_{2,m}\,\mathrm{d}b, & \omega_m^4 &= \int_0^B (1-\lambda)\gamma_2 \tilde{y}_{2,m}\,\mathrm{d}b, \end{aligned}$$

where $(\tilde{y}_{1,m}, \tilde{y}_{2,m}, \tilde{y}_{3,m})$ is a solution of the system (4.3) and (4.4), with $(N_{1,m}, N_{2,m}, N_{3,m}, N_{4,m})$ instead of (f_1, f_2, f_3, f_4), and $\tilde{u}_m$ satisfies (3.33). Thus, (3.33) allows us to obtain

$$\int_p \sum_{j=1}^3 \tilde{u}_{j,m}^2(b,t,x)\,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}x \le 4C\left(\int_0^B\!\!\int_\Omega \sum_{j=1}^4 \left(\tilde{y}_j^0\right)^2\,\mathrm{d}b\,\mathrm{d}x + \sum_{j=1}^4 l_j^2\right).$$

Thus, $(\tilde{u}_m)$ is bounded, Therefore, it is possible to extract a sequence, still denoted by $(\tilde{u}_m)$, which converges weakly to $\tilde{u}$ in $L^2(p)$.

First multiplying equation j of the system (4.3)–(4.4) by $\tilde{y}_{j,m}$, then integrating by parts over Q, and finally using (4.9), we obtain

$$\begin{aligned}\int_Q \sum_{j=1}^{4} \big(\beta \left((\tilde{y}_{j,m})_b\right)^2 (b,t,x) + \mu_j \left|\nabla_x \tilde{y}_{j,m}\right|^2 (b,t,x) + \tilde{y}_{j,m}^2(b,t,x)\big) \,\mathrm{d}b \,\mathrm{d}t \,\mathrm{d}x \\ \leq (4C+1)\left(\int_0^B \int_\Omega \sum_{j=1}^{4} \left(\tilde{y}_j^0\right)^2 \,\mathrm{d}b \,\mathrm{d}x + \sum_{j=1}^{4} l_j^2\right).\end{aligned} \tag{4.10}$$

Hence, the sequence $(\tilde{y}_{1,m}, \tilde{y}_{2,m}, \tilde{y}_{3,m})$ is bounded in $\left(L^2(Q)\right)^3$. Therefore, we can extract a sequence, denoted by $((\tilde{y}_{1,m_\beta}, \tilde{y}_{2,m_\beta}, \tilde{y}_{3,m_\beta}))$, which converges weakly to $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ in $\left(L^2(Q)\right)^3$.

Knowing that

$$\begin{aligned}\omega_{m_\beta}^1 &= \int_0^B \gamma_3 \tilde{y}_{3,m_\beta} \,\mathrm{d}b, & \omega_{m_\beta}^2 &= \int_0^B \gamma_1 \tilde{y}_{1,m_\beta} \,\mathrm{d}b, \\ \omega_{m_\beta}^3 &= \int_0^B \lambda \gamma_2 \tilde{y}_{2,m_\beta} \,\mathrm{d}b, & \omega_{m_\beta}^4 &= \int_0^B (1-\lambda) \gamma_2 \tilde{y}_{2,m_\beta} \,\mathrm{d}b,\end{aligned}$$

and referring to (4.10), we have

$$\sum_{j=1}^{4} \left\|\omega_{m_\beta}^j\right\|_{L^2(Q_T)} \leq C_3 \left(\int_0^B \int_\Omega \sum_{j=1}^{4} \left(\tilde{y}_j^0\right)^2 \,\mathrm{d}b \,\mathrm{d}x + \sum_{j=1}^{4} l_j^2\right). \tag{4.11}$$

Starting from (4.11), one can extract a sequence, denoted by

$$\left((\omega_{m_\beta}^1, \omega_{m_\beta}^2, \omega_{m_\beta}^3, \omega_{m_\beta}^4)\right),$$

which converges weakly to $\left(\omega^1, \omega^2, \omega^3, \omega^4\right)$ in $L^2\left(Q_T\right)$, that is,

$$\int_{Q_T} \Phi \omega_{m_i}^j \,\mathrm{d}t \,\mathrm{d}x \longrightarrow \int_{Q_T} \Phi \omega^i \,\mathrm{d}t \,\mathrm{d}x \;\text{ for all } \Phi \in L^2\left(Q_T\right) \text{ and } 1 \leq j \leq 4. \tag{4.12}$$

Moreover, the sequence

$$\left((\tilde{y}_{1,m_\beta}, \tilde{y}_{2,m_\beta}, \tilde{y}_{3,m_\beta}, \tilde{y}_{4,m_\beta})\right)$$

converges weakly to $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ in $\left(L^2(Q)\right)^4$. Consequently, the sequence

$$\left((\tilde{y}_{1,m_\beta}, \tilde{y}_{2,m_\beta}, \tilde{y}_{3,m_\beta}, \tilde{y}_{4,m_\beta})\right),$$

associated with

$$\left((\omega_{m_\beta}^1, \omega_{m_\beta}^2, \omega_{m_\beta}^3, \omega_{m_\beta}^4)\right),$$

also converges weakly to $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$. Furthermore, since $\Phi \gamma_\beta \in L^2\left(Q_T\right)$ for all $\Phi \in L^2\left(Q_T\right)$

and all $\beta \in \{1,2,3\}$, it follows that

$$\begin{cases} \displaystyle\int_{Q_T} \Phi\omega^1_{m_\beta}\,\mathrm{d}t\,\mathrm{d}x \longrightarrow \int_{Q_T} \Phi\left(\int_0^B \gamma_3\tilde{y}_3\,\mathrm{d}b\right)\,\mathrm{d}t\,\mathrm{d}x, \\ \displaystyle\int_{Q_T} \Phi\omega^2_{m_\beta}\,\mathrm{d}t\,\mathrm{d}x \longrightarrow \int_{Q_T} \Phi\left(\int_0^B \gamma_1\tilde{y}_1\,\mathrm{d}b\right)\,\mathrm{d}t\,\mathrm{d}x, \\ \displaystyle\int_{Q_T} \Phi\omega^3_{m_\beta}\,\mathrm{d}t\,\mathrm{d}x \longrightarrow \int_{Q_T} \Phi\left(\int_0^B \lambda\gamma_2\tilde{y}_2\,\mathrm{d}b\right)\,\mathrm{d}t\,\mathrm{d}x, \\ \displaystyle\int_{Q_T} \Phi\omega^4_{m_\beta}\,\mathrm{d}t\,\mathrm{d}x \longrightarrow \int_{Q_T} \Phi\left(\int_0^B (1-\lambda)\gamma_2\tilde{y}_2\,\mathrm{d}b\right)\,\mathrm{d}t\,\mathrm{d}x. \end{cases} \tag{4.13}$$

Consequently, equations (4.12) and (4.13) give

$$\begin{cases} \displaystyle\int_{Q_T} \Phi\left(\omega^1 - \int_0^B \gamma_3\tilde{y}_3\,\mathrm{d}b\right)\,\mathrm{d}t\,\mathrm{d}x = 0 \ \text{ for every } \Phi \in L^2(Q_T), \\ \displaystyle\int_{Q_T} \Phi\left(\omega^2 - \int_0^B \gamma_1\tilde{y}_1\,\mathrm{d}b\right)\,\mathrm{d}t\,\mathrm{d}x = 0 \ \text{ for every } \Phi \in L^2(Q_T), \\ \displaystyle\int_{Q_T} \Phi\left(\omega^3 - \int_0^B \lambda\gamma_2\tilde{y}_2\,\mathrm{d}b\right)\,\mathrm{d}t\,\mathrm{d}x = 0 \ \text{ for every } \Phi \in L^2(Q_T), \\ \displaystyle\int_{Q_T} \Phi\left(\omega^4 - \int_0^B (1-\lambda)\gamma_2\tilde{y}_2\,\mathrm{d}b\right)\,\mathrm{d}t\,\mathrm{d}x = 0 \ \text{ for every } \Phi \in L^2(Q_T). \end{cases}$$

We can then deduce that

$$\begin{cases} \omega^1 = \displaystyle\int_0^B \gamma_3\tilde{y}_3\,\mathrm{d}b \ \text{ for a.e. } t \in Q_T, \\ \omega^2 = \displaystyle\int_0^B \gamma_1\tilde{y}_1\,\mathrm{d}b \ \text{ for a.e. } t \in Q_T, \\ \omega^3 = \displaystyle\int_0^B \lambda\gamma_2\tilde{y}_2\,\mathrm{d}b \ \text{ for a.e. } t \in Q_T, \\ \omega^4 = \displaystyle\int_0^B (1-\lambda)\gamma_2\tilde{y}_2\,\mathrm{d}b \text{ for a.e. } t \in Q_T. \end{cases}$$

Thus, $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ is a solution of (4.3) and (4.4) with (N_1, N_2, N_3, N_4) instead of (f_1, f_2, f_3, f_4); $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ and $\tilde{u}$ satisfy (1.3) and (3.33), respectively.

(a$_4$) Let L be a closed subset of $\mathcal{H}$. We need to show that $G_\zeta^{-1}(L)$ is also a closed subset of $\mathcal{H}$. Observe that

$$G_\zeta^{-1}(L) = \big\{(N_1, N_2, N_3, N_4) \in \mathcal{H} \ : \ G_\zeta\,(N_1, N_2, N_3, N_4) \cap L \neq \emptyset\big\}.$$

Here, since G_ζ is multivalued, $G_\zeta^{-1}(L)$ denotes the set of quadruples (N_1, N_2, N_3, N_4) whose image under G_ζ intersects L.

Assume that the sequence $(N_{1,m}, N_{2,m}, N_{3,m}, N_{4,m})$ converges strongly to (N_1, N_2, N_3, N_4) in $\mathcal{H}$. Then, the sequence $(N_{1,m}, N_{2,m}, N_{3,m}, N_{4,m})$ is bounded. Moreover, there exists a sequence $\left(\omega_m^1, \omega_m^2, \omega_m^3, \omega_m^4\right)_n \in L$ such that

$$\left(\omega_m^1, \omega_m^2, \omega_m^3, \omega_m^4\right) \in G_\zeta\left(N_{1,m}, N_{2,m}, N_{3,m}, N_{4,m}\right) \quad \text{for all } n \geq 1.$$

Consequently, there exist $(\tilde{y}_{1,m}, \tilde{y}_{2,m}, \tilde{y}_{3,m}) \in \left(L^2(\varpi)\right)^3$ and a control $\tilde{u}_m \in \left(L^2(p)\right)^3$ such that

$$\begin{aligned} \omega_m^1 &= \int_0^B \gamma_3 \tilde{y}_{3,m}\,\mathrm{d}b, & \omega_m^2 &= \int_0^B \gamma_1 \tilde{y}_{1,m}\,\mathrm{d}b, \\ \omega_m^3 &= \int_0^B \lambda\gamma_2 \tilde{y}_{2,m}\,\mathrm{d}b, & \omega_m^4 &= \int_0^B (1-\lambda)\gamma_2 \tilde{y}_{2,m}\,\mathrm{d}b. \end{aligned}$$

Subsequently, $\tilde{u}_m$ and $(\tilde{y}_{1,m}, \tilde{y}_{2,m}, \tilde{y}_{3,m})$ satisfy (3.33), and $(\tilde{y}_{1,m}, \tilde{y}_{2,m}, \tilde{y}_{3,m}, \tilde{y}_{4,m})$ is a solution of the system (4.3)–(4.4) with $(N_{1,m}, N_{2,m}, N_{3,m}, N_{4,m})$ replacing (f_1, f_2, f_3, f_4); also, $(\tilde{y}_{1,m}, \tilde{y}_{2,m}, \tilde{y}_{3,m})$ satisfies (1.3). From (4.5), (4.8) and (3.33), we deduce that the sequences $(\tilde{u}_m)$ and $(\tilde{y}_{1,m}, \tilde{y}_{2,m}, \tilde{y}_{3,m})$ are bounded in $\left(L^2(p)\right)^3$ and $\left(L^2(\varpi)\right)^3$, respectively. Thus, there exist sequences denoted by $(\tilde{u}_m)_n$ and $((\tilde{y}_{1,m}, \tilde{y}_{2,m}, \tilde{y}_{3,m}))_n$ which converge weakly towards $\tilde{u}$ and $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$, respectively, in $\left(L^2(p)\right)^3$ and $\left(L^2(\varpi)\right)^3$. Moreover, $\tilde{u}$ satisfies (3.33) and

$$\begin{aligned} \omega^1 &= \int_0^B \gamma_3 \tilde{y}_3\,\mathrm{d}b, & \omega^2 &= \int_0^B \gamma_1 \tilde{y}_1\,\mathrm{d}b, \\ \omega^3 &= \int_0^B \lambda\gamma_2 \tilde{y}_2\,\mathrm{d}b, & \omega^4 &= \int_0^B (1-\lambda)\gamma_2 \tilde{y}_2\,\mathrm{d}b. \end{aligned}$$

In this context, $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ is a solution of (4.3)–(4.4) with (N_1, N_2, N_3, N_4) instead of (f_1, f_2, f_3, f_4), and $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ satisfies (1.3). Thus,

$$\left(\omega^1, \omega^2, \omega^3, \omega^4\right) \in G_\zeta\left(N_1, N_2, N_3, N_4\right). \tag{4.14}$$

Moreover, from (4.11), we can extract a sequence $\left(\left(\omega_m^1, \omega_m^2, \omega_m^3, \omega_m^4\right)\right)_n$ that converges strongly to $\left(\omega^1, \omega^2, \omega^3, \omega^4\right)$ in $\mathcal{H}$. Since L is a closed set, it follows that $\left(\omega^1, \omega^2, \omega^3, \omega^4\right) \in L$. Finally, by applying (4.14), we conclude that $(N_1, N_2, N_3, N_4) \in G_\zeta^{-1}(L)$. □

Proof of Theorem 2.1. The theorem is a direct consequence of [7, Theorem 3.1], since, by Proposition 4.1, all its assumptions are satisfied. Therefore, there exists $(N_1, N_2, N_3, N_4) \in \mathcal{H}$ such that $(N_1, N_2, N_3, N_4) \in G_\zeta\left(N_1, N_2, N_3, N_4\right)$. □

5 Conclusion

In this article, we have established the null controllability of an age- and space-structured model describing the population dynamics of desert locusts, consisting of four stages (see [19]). Three control actions are considered, targeting the eggs, larvae, and females, respectively. To further strengthen these results, we plan to investigate the controllability of the system with a reduced number of controls (two, or even one), and to develop algorithms that allow for the explicit computation of the control based on its characterization. We also plan to provide numerical simulations to support and validate the theoretical results.

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