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APPROXIMATIONS TO SOLUTIONS OF STOCHASTIC PARTIAL INTEGRODIFFERENTIAL EQUATIONS IN HILBERT SPACES

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Abstract. This work investigates the existence and uniqueness of mild solutions for a specific class of stochastic partial integrodifferential equations in Hilbert spaces by developing Picard-type approximate sequences. We provide approximate results based on Taniguchi conditions. Additionally, we apply these abstract findings to the Heath-Jarrow-Morton-Musiela (HJMM) equation to reinforce the theoretical framework.

Keywords: Integrodifferential equations, Resolvent operator, mild solutions, Picard iterations.

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1 Introduction

Infinite dimensional systems are used to describe many phenomena in the real world. It is well known that heat conduction, properties of elastic-plastic material, fluid dynamics, diffusion-reaction processes, etc., all lie within this area. The object we are studying (temperature, displacement, concentration, velocity, etc.) is usually referred to as the state. We are interested in the case where the state satisfies proper integrodifferential equations that are derived from certain physical problems. By an infinite dimensional system we mean one whose corresponding state space is infinite dimensional. In particular, we are interested in the case where the state equations are stochastic partial integrodifferential equations (SPIDE).

Stochastic evolution equations have been used to model systems in control theory, chemistry, biology, economics and finance. They have been studied in many different aspects: qualitative and quantitative results. Indeed, there are a lots of results on the existence and uniqueness of solutions to stochastic differential equations in Hilbert spaces. See the monographs of Prato and Zabczyk [7], Liu [20] and Gawarecki and Mandrekar [12], the research papers [3, 4, 6, 15, 13, 14, 21, 24, 25, 26, 27, 28] and the references therein for more comments and citations.

Stochastic integrodifferential equations have received much attention because of their applications in many areas such as population dynamics, electrical engineering, ecology, sociology and other areas of science and engineering. Also the attention they received, is due to the fact that SPIDEs generalize several stochastic evolution equations. Qualitative properties such as existence, uniqueness, controllability and stability for various integrodifferential equations have been extensively studied by many researchers, see for instance [2, 9, 22]. However, the existence and uniqueness of solution is the only question in this work.

Concretely, we study Picard approximations of the mild solutions to the following SPIDE

$$\begin{cases} dx(t) = \left[Ax(t) + \displaystyle\int_0^t \Upsilon(t-s)x(s)\,\mathrm{d}s + F(t,x(t))\right]dt + B\left(t,x(t)\right)dw(t), \quad t \in \mathbf{J} = [0,T] \\ x(0) = \zeta : \Omega \to \mathsf{H} \text{ a measurable function,} \end{cases} \tag{1.1}$$

where A and $\Upsilon(t)$ are closed linear operators on a separable Hilbert space H with domain $\mathsf{Dom}(A) \subset \mathsf{Dom}(\Upsilon)$ which are independent of $t \in \mathbb{R}^+$. $w(t)$ is a Wiener process on the separable Hilbert space U with covariance operator $Q \in \mathcal{L}(\mathsf{U})$. $F : [0,T] \times \mathsf{H} \to \mathsf{H}$, $B : [0,T] \times \mathsf{H} \to \mathcal{L}(\mathsf{U},\mathsf{H})$ are measurable functions such that the involved integrals are well defined.

It is well known that if F and B are globally Lipschitz and satisfy the linear growth condition, we have unique strong solutions, using among other methods, the contraction mapping principle. However, this Lipschitz conditions seemed to be inappropriate when one discusses applications in real world. The nonlinear terms in equation (1.1) are not necessarily globally Lipschitz neither locally Lipschitz.

Here we assume the non-Lipschitz condition of Taniguchi type on the nonlinear terms. Thus, we investigate the existence and uniqueness of mild solution which is not a weak solution in the sense of general stochastic processes theory. Indeed, the mild solution terminology steps in from the type constant formula that we will use to define the so called mild solution. The constant formula here is the theory of resolvent operator in the sense of Grimmer [17]. Sufficient conditions for the existence are derived with the help of the resolvent operator.

On the other hand, recently in [10], the authors investigated deeply integrodifferential equations in Banach spaces with nonlocal and impulsive conditions. In this work by successive approximations, we consider a sufficient condition for the existence and uniqueness of mild solutions to the stochastic integrodifferential equation (1.1). The nonlinear terms are measurable, possibly unbounded and have linear growth in the sense of Taniguchi. It should be mentioned that this condition was introduced by Taniguchi [26] and was investigated by Cao, He and Zhang [5], Cao and He [6], Barbu and Bocşan [3] and others as non-Lipschitz condition of Taniguchi type.

The rest of the work is organized as follows. In Section 2, we recall some necessary preliminaries on stochastic integral and resolvent operator. In Section 3, we study the existence of the mild solutions of eqn. (1.1). Finally in Section 4, an example is presented which illustrates the main result for equation (1.1).

2 Preliminaries

This section is a presentation of an estimation and auxiliary results that will be used.

Stochastic integral

Let X and Y be Banach spaces, $\mathcal{L}(\mathsf{X},\mathsf{Y})$ denotes the space of bounded linear operator from X to Y, witch is simply denoted by $\mathcal{L}(\mathsf{X})$ when $\mathsf{X} = \mathsf{Y}$. We will assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$ is a complete filtered probability space. We are given a Q-Wiener process on the probability space which is having value in U a separable Hilbert space, one can construct $w(t)$ as follows,

$$w(t) := \sum_{n=1}^{+\infty} (\lambda_n)^{1/2} \beta_n(t) e_n \quad t \in \mathbb{R}^+,$$

where $\beta_n(t)(n = 1, 2, 3, \cdots)$ is a sequence of real valued standard Brownian motions mutually independent on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$, $\lambda_n \in \mathbb{R}^+ \quad (n = 1, 2, 3, \cdots)$ are positive real numbers such that $\sum_{n=1}^{+\infty} \lambda_n < +\infty$, $(e_n)_{n\geq 1}$ is a complete orthonormal basis in U, and $Q \in \mathcal{L}(\mathsf{U})$ is the incremental covariance operator of the w which is a symmetric non-negative trace class operator defined by $Qe_n = \lambda_n e_n \quad n = 1, 2, 3, \cdots$

For this analysis we recall the definition of H-valued stochastic integral with respect to the U-valued Q-Wiener process w. Let $\mathcal{L}_2^0 = L_2(\mathsf{U}_0, \mathsf{H})$ denote the space of all Hilbert-Schmidt operators from $\mathsf{U}_0 = Q^{1/2}(\mathsf{U})$ to H which is a separable Hilbert space, endowed with the norm

$$\|\varphi\|_2^2 = \mathsf{tr}(\varphi Q \varphi^*).$$

Let $\varphi : (0, +\infty) \to \mathcal{L}_2^0$ be a predictable $\mathcal{F}_t$-adapted process such

$$\int_0^t \mathbb{E}\|\varphi(s)\|_2^2 \, \mathrm{d}s < \infty \quad for\ t > 0$$

then we define the H-valued stochastic integral

$$\int_0^t \varphi(s) \, \mathrm{d}w(s)$$

which will be a continuous square integrable martingale. For more details on stochastic integral, we refer to [7].

Integrodifferential equation

Here we recall some fundamental results needed. Regarding the theory of resolvent operators, we refer the reader to [17]. Let Y be the Banach space $\mathsf{Dom}(A)$ equipped with the graph norm defined by $|y|_Y := |Ay| + |y|$ for $y \in Y$. The notation $\mathcal{C}(\mathbb{R}^+; Y)$ stands for the space of all continuous functions from $\mathbb{R}^+$ into Y.
We consider the following Cauchy problem :

$$\begin{cases} v'(t) = Av(t) + \displaystyle\int_0^t \Upsilon(t-s)v(s)\,\mathrm{d}s \quad for\ t \in \mathbb{R}^+ \\ v(0) = v_0 \in \mathsf{H}. \end{cases} \tag{2.1}$$

Definition 2.1 *[17] A resolvent operator for eqn.* (2.1) *is a bounded linear operator-valued function $R(t) \in \mathcal{L}(\mathsf{H})$ for $t \in \mathbb{R}^+$, having the following properties.*

1. *$R(0) = I$ (the identity map on H) and $\|R(t)\|_{\mathcal{L}(\mathsf{H})} \le Ne^{\gamma t}$ for some constants $N > 0$ and $\gamma \in \mathbb{R}$.*
2. *For each $x \in \mathsf{H}$, $R(t)x$ is strongly continuous for $t \in \mathbb{R}^+$.*
3. *$R(t) \in \mathcal{L}(Y)$ for $t \in \mathbb{R}^+$. For $x \in Y$, $R(.)x \in \mathcal{C}^1(\mathbb{R}^+; \mathsf{H}) \bigcap \mathcal{C}(\mathbb{R}^+; Y)$ and*

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t \Upsilon(t-s)R(s)x\,\mathrm{d}s \\ &= R(t)Ax + \int_0^t R(t-s)\Upsilon(s)x\,\mathrm{d}s \quad for\ t \in \mathbb{R}^+. \end{aligned}$$

From now on we make the following assumption.

(R)
1. The operator A is the infinitesimal generator of a C_0-semigroup $(S(t))_{t\in\mathbb{R}^+}$ on H.
2. For all $t \in \mathbb{R}^+$, $\Upsilon(t)$ is closed linear operator from $\mathsf{Dom}(A)$ to H and $\Upsilon(t) \in \mathcal{L}(Y, \mathsf{H})$. For any $y \in Y$, the map $t \mapsto \Upsilon(t)y$ is bounded, differentiable and the derivative $t \mapsto \Upsilon'(t)y$ is bounded uniformly continuous on $\mathbb{R}^+$.

The resolvent operator plays an important role to study the existence of solutions and to give a variation of constants formula for nonlinear systems. For more details on resolvent operators, we refer to [16, 17]. The following theorem gives a satisfactory answer to the problem of existence of solutions.

Theorem 2.2 *[17] Assume that* $(\mathbf{R})$ *holds. Then there exists a unique resolvent operator of the Cauchy problem* (2.1)*.*

In the following, we give some results for the existence of solutions for the following integrodifferential equation

$$\begin{cases} v'(t) = Av(t) + \displaystyle\int_0^t \Upsilon(t-s)v(s)\,\mathrm{d}s + q(t), \quad for\ t \in \mathbb{R}^+ \\ v(0) = v_0 \in \mathsf{H}, \end{cases} \tag{2.2}$$

where $q : \mathbb{R}^+ \to \mathsf{H}$ is a continuous function.

Definition 2.3 *[17] A continuous function $v : \mathbb{R}^+ \to \mathsf{H}$ is said to be a strict solution of eqn.* (2.2) *if $v \in \mathcal{C}^1(\mathbb{R}^+; \mathsf{H}) \bigcap \mathcal{C}(\mathbb{R}^+; Y)$, v satisfies* (2.2)*, for $t \geq 0$.*

Theorem 2.4 *[17] Assume that $(\mathbf{R})$ holds. If v is a strict solution of eqn.* (2.2)*, then*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)\,\mathrm{d}s, \quad for\ t \in \mathbb{R}^+.$$

Stochastic convolution

Let $\varphi : [0, +\infty) \to \mathcal{L}_2^0$ be a predictable, $\mathcal{F}_t$-adapted process. We define the stochastic convolution by

$$R \star \varphi(t) = \int_0^t R(t-s)\varphi(s)\,\mathrm{d}w(t) \quad t \in \mathbb{R}^+,$$

where $(R(t))_{t\in\mathbb{R}^+}$ is the resolvent operator of (2.2).
In case $\Upsilon \equiv 0$, it is well known that the resolvent operator $(R(t))_{t\in\mathbb{R}^+}$ corresponds to the semigroup $(S(t))_{t\in\mathbb{R}^+}$ for instance see [11]. Moreover, we have the following result.

Lemma 2.5 *[9] For arbitrary $p > 2$ there exists a constant $\kappa_p > 0$ such that the stochastic convolution $R \star \varphi$ satisfies the following inequality*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} \|R \star \varphi(t)\|\right) \leq \kappa_p \sup_{0\leq t\leq T} \|R(t)\|^p_{\mathcal{L}(\mathsf{H})} T^{p/2-1}\mathbb{E}\left(\int_0^T \|\varphi(s)\|_2^p\,\mathrm{d}s\right). \tag{2.3}$$

The process $R \star \varphi$ is stochastically continuous and therefore has a predictable version.
We end this section by giving the definition of solutions of (1.1).

Definition 2.6 *A mild solution of the integrodifferential equation* (1.1) *for $T > 0$ is a stochastic process $x : [0, T] \to \mathsf{H}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that x is $\mathcal{F}_t$-adapted predictable on $[0, T]$, satisfies with probability one $\int_0^T \|x(t)\|^2\,\mathrm{d}t < \infty$ and the following expression*

$$\begin{cases} x(t) = R(t)\zeta + \displaystyle\int_0^t R(t-s)F(s, x(s))\,\mathrm{d}s + \int_0^t R(t-s)B(s, x(s))\,\mathrm{d}w(s) \quad for\ t \in [0, T], \\ x(0) = \zeta. \end{cases} \tag{2.4}$$

It is of great importance to notice that this definition requires the existence of the resolvent and naturally, ζ should be $\mathcal{F}_0$ measurable and square integrable. To prove existence of a such process, we use Picard iterations in the next Section.

3 Approximations to solutions

From now on, we assume that the resolvent operator of the linear part of (2.2) exists. Let us fix a real number $p > 2$ and

$$L_p(\Omega, \mathcal{F}; \mathsf{H}) := \left\{x : \Omega \to \mathsf{H} : \int_\Omega \|x(\omega)\|^p \mathbb{P}(\,\mathrm{d}\omega) < \infty\right\}$$

denote the equivalence class of $\mathcal{F}$-measurable Lebesgue-Bochner p-th power summable functions.

Let $\Omega_T = [0,T] \times \Omega$. Let $\mathcal{P}_T$ be the space of all H-valued $\mathcal{F}_t$-adapted predictable processes x defined on Ω_T which are continuous in t for a.e. fixed $\omega \in \Omega$ and satisfy

$$\|x\|_{\mathcal{P}_T}^p = \mathbb{E}\left(\sup_{t\in\mathbf{J}} \|x(t)\|^p\right) < \infty.$$

By virtue of Borel-Cantelli lemma, the space $\mathcal{P}_T$ is a Banach space (see [1] for $p = 2$ the case $p > 2$ is has a similar proof). The following fact is of great importance in the sequel. For $x \in \mathcal{P}_T$, note that $x \in \mathcal{P}_t$ for all $t \in \mathbf{J}$ and

$$\mathbb{E}\|x(t)\|^p \leq \mathbb{E}\left(\sup_{s\in[0,t]} \|x(s)\|^p\right) = \|x\|_{\mathcal{P}_t}^p \leq \mathbb{E}\left(\sup_{s\in\mathbf{J}} \|x(s)\|^p\right) =: \|x\|_{\mathcal{P}_T}^p. \tag{3.1}$$

In order to set our problem, we make the following assumptions (called Taniguchi conditions) on F and B, which are:

($\mathbf{A1}$) The functions $F(t,x)$ and $B(t,x)$ are jointly measurable and there exists a function $L : \mathbf{J} \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\mathbb{E}(\|F(t,x)\|^p) + \mathbb{E}(\|B(t,x)\|_2^p) \leq L(t, \mathbb{E}\|x\|^p)$$

for all $t \in \mathbf{J}$ and all $x \in L_p(\Omega, \mathcal{F}, \mathsf{H})$.

($\mathbf{A2}$) $L(t,u)$ is integrable in t for each fixed $u \in \mathbb{R}^+$, it is continuous and nondecreasing in u for each fixed $t \in \mathbf{J}$ and for all $\lambda > 0, u_0 \geq 0$ the integral equation

$$u(t) = u_0 + \lambda \int_0^t L(s, u(s))\,\mathrm{d}s$$

has a global solution on $\mathbf{J}$.

($\mathbf{A3}$) There exists a function $K : \mathbf{J} \times \mathbb{R}^+ \to \mathbb{R}^+$ which is locally integrable in t for each fixed $u \in \mathbb{R}^+$ and continuous, monotone nondecreasing in u for each fixed $t \in \mathbf{J}$. Moreover, $K(t,0) \equiv 0$ and

$$\mathbb{E}\left(\|F(t,x) - F(t,y)\|^p\right) + \mathbb{E}\left(\|B(t,x) - B(t,y)\|_2^p\right) \leq K(t, \mathbb{E}(\|x-y\|^p)) \tag{3.2}$$

for all $t \in \mathbf{J}$ and all $x, y \in L_p(\Omega, \mathcal{F}, \mathsf{H})$.

($\mathbf{A4}$) If z is nonnegative integrable function satisfying

$$\begin{cases} z(t) \leq \lambda \displaystyle\int_0^t K(s, z(s))\,\mathrm{d}s, \quad t \in J, \\ z(0) = 0, \end{cases} \tag{3.3}$$

for some $\lambda > 0$, then $z(t) = 0$ for all $t \in \mathbf{J}$.

Now, we construct Picard type approximate sequences to (1.1):

$$\begin{cases} x_{n+1}(t) = R(t)\zeta + \int_0^t R(t-s)F(s,x_n(s))\,\mathrm{d}s + \int_0^t R(t-s)B(s,x_n(s))\,\mathrm{d}w(s), \\ x_0(t) = R(t)\zeta, \quad \text{for } t \in \mathbf{J}, n \geq 0. \end{cases} \tag{3.4}$$

Define the following operator $\mathcal{G}$ on $\mathcal{P}_T$ by

$$(\mathcal{G}(x))(t) = R(t)\zeta + \int_0^t R(t-s)F(s,x(s))\,\mathrm{d}s + \int_0^t R(t-s)B(s,x(s))\,\mathrm{d}w(s)$$

Obviously, eventual fixed points of $\mathcal{G}$ are mild solutions of equation (1.1). The next lemmas exhibit $\mathcal{G}$ fixed points.

Lemma 3.1 *Assume that the conditions* $(\mathbf{A1})$–$(\mathbf{A4})$ *hold and* $\zeta \in L_p(\Omega, \mathcal{F}_0, \mathsf{H})$ *with* $p > 2$. *Then the operator* $\mathcal{G} : \mathcal{P}_T \to \mathcal{P}_T$ *is well defined and continuous.*

Proof. Let $M_T = \sup\limits_{t\in[0,T]} \|R(t)\|_{\mathcal{L}(\mathsf{H})}$ which is finite by the uniform boundedness principle . From the strong continuity of the resolvent operator $(R(t))_{t\in\mathbb{R}^+}$ and the assumptions on F and B, we obtain that the process $\mathcal{G}x$ is $\mathcal{F}_t$-adapted and its paths are almost surely continuous. Let $x \in \mathcal{P}_T$. Using Hölder's inequality, we have the following estimations

$$\begin{aligned} \sup_{t\in\mathbf{J}} \|(\mathcal{G}x)(t)\|^p &\leq 3^p \sup_{t\in\mathbf{J}} \|R(t)\zeta\|^p + 3^p \sup_{t\in\mathbf{J}} \left\| \int_0^t R(t-s)F(s,x(s))\,\mathrm{d}s \right\|^p \\ &\quad +3^p \sup_{t\in\mathbf{J}} \left\| \int_0^t R(t-s)B(s,x(s))\,\mathrm{d}W(s) \right\|^p, \\ &\leq 3^p M_T^p \|\zeta\|^p + 3^p T^{p-1} M_T^p \int_0^T \|F(s,x(s))\|^p \ \mathrm{d}s \\ &\quad +3^p \sup_{t\in\mathbf{J}} \left\| \int_0^t R(t-s)B(s,x(s))\,\mathrm{d}W(s) \right\|^p, \quad \mathbb{P}-a.s.. \end{aligned}$$

Taking the expectation on both sides and applying Lemma 2.5 and Fubini theorem, we get

$$\begin{aligned} \mathbb{E}\left(\sup_{t\in\mathbf{J}} \|(\mathcal{G}x)(t)\|^p\right) &\leq 3^p M_T^p \mathbb{E}\|\zeta\|^p + 3^p T^{p-1} M_T^p \int_0^T \mathbb{E}\,\|F(s,x(s))\|^p \ \mathrm{d}s \\ &\quad +3^p \kappa_p M_T^p T^{p/2-1} \int_0^T \mathbb{E}\|B(s,x(s))\|_2^p \,\mathrm{d}s, \\ &\leq 3^p M^p \mathbb{E}\|\zeta\|^p + C_{T,p,M_T} \int_0^T L(s, \mathbb{E}\|x(s)\|^p)\,\mathrm{d}s, \qquad (3.5) \\ &\leq 3^p M^p \mathbb{E}\|\zeta\|^p + C_{T,p,M_T} \int_0^T L(s, \|x\|_{\mathcal{P}_T}^p)\,\mathrm{d}s < \infty, \end{aligned}$$

where $C_{T,p,M_T} = 2 \times 3^p M_T^p \max\left(T^{p-1}, \kappa_p T^{p/2-1}\right)$. Therefore, $\mathcal{G}x$ belongs to $\mathcal{P}_T$.

To show the continuity of the operator $\mathcal{G}$. Let $x \in \mathcal{P}_T$ and $(x_n)_{n\geq 0}$ a sequence in $\mathcal{P}_T$ converging to x. Using the above techniques, we have

$$\begin{aligned}
\|\mathcal{G}x-\mathcal{G}x_n\|^p_{\mathcal{P}_T} &= \mathbb{E}\left(\sup_{t\in\mathbf{J}}\|(\mathcal{G}x)(t)-(\mathcal{G}x_n)(t)\|^p\right)\\
&\le 2^pT^{p-1}M_T^p\int_0^T\mathbb{E}\|F(s,x(s))-F(s,x_n(s)\|^p\,\mathrm{d}s\\
&\quad +2^pM_T^p\kappa_pT^{p/2-1}\int_0^T\mathbb{E}\|B(s,x(s))-B(s,x_n(s)\|_2^p\,\mathrm{d}s \qquad (3.6)\\
&\le C'_{T,p,M_T}\int_0^T K(s,\mathbb{E}\|x(s)-x_n(s)\|^p)\,\mathrm{d}s\\
&\le C'_{T,p,M_T}\int_0^T K(s,\|x-x_n\|^p_{\mathcal{P}_T})\,\mathrm{d}s,
\end{aligned}$$

with $C'_{T,p,M_T}=2^{p+1}M_T^p\max\left(T^{p-1},\kappa_pT^{p/2-1}\right)$. By Lesbegue convergence theorem, the right hand side of the above inequality tend to zero as n goes to infinity. Hence $\mathcal{G}$ is continuous on $\mathcal{P}_T$. □

Lemma 3.2 *Assume that the conditions* $(\mathbf{A1})$–$(\mathbf{A4})$ *hold and* $\zeta\in L_p(\Omega,\mathcal{F}_0,\mathsf{H})$ *with* $p>2$. *Then there exists* $C'_{T,p,M_T}>0$ *such that*

$$\|\mathcal{G}x-\mathcal{G}y\|^p_{\mathcal{P}_t}\le C'_{T,p,M_T}\int_0^t K(s,\|x-y\|^p_{\mathcal{P}_s})\,\mathrm{d}s, \qquad (3.7)$$

for all $x,y\in\mathcal{P}_T$ *and* $t\in\mathbf{J}$.

Proof. Repeating the same arguments used above on $[0,t]$, we have

$$\begin{aligned}
\|\mathcal{G}x-\mathcal{G}y\|^p_{\mathcal{P}_t} &:= \mathbb{E}\left(\sup_{s\in[0,t]}\|(\mathcal{G}x)(s)-(\mathcal{G}y)(s)\|^p\right)\\
&\le 2^pt^{p-1}M_t^p\int_0^t\mathbb{E}\|F(s,x(s))-F(s,y(s)\|^p\,\mathrm{d}s\\
&\quad +2^pM_t^p\kappa_pt^{p/2-1}\int_0^t\mathbb{E}\|B(s,x(s))-B(s,y(s)\|_2^p\,\mathrm{d}s \qquad (3.8)\\
&\le C'_{t,p,M_t}\int_0^t K(s,\mathbb{E}\|x(s)-x_n(s)\|^p)\,\mathrm{d}s\\
&\le C'_{t,p,M_t}\int_0^t K(s,\|x-x_n\|^p_{\mathcal{P}_s})\,\mathrm{d}s.
\end{aligned}$$

The result follows using the fact that $C'_{t,p,M_t}\le C'_{T,p,M_T}$. □

Lemma 3.3 *Assume that the conditions* $(\mathbf{A1})$–$(\mathbf{A4})$ *hold and* $\zeta\in L_p(\Omega,\mathcal{F}_0,\mathsf{H})$ *with* $p>2$. *Then the sequence* $(x_n)_{n\ge0}$ *define by* (3.4) *is bounded sequence in* $\mathcal{P}_T$.

Proof. Mimicking the Proof of Lemma 3.1, precisely inequality (3.5), we get that

$$\begin{aligned}
\|x_{n+1}\|^p_{\mathcal{P}_t} &\le 3^pM_t^p\mathbb{E}\|\zeta\|^p+C_{t,p,M_t}\int_0^t L(s,\|x_n\|^p_{\mathcal{P}_t})\,\mathrm{d}s\\
&\le k_1+k_2\int_0^t L(s,\|x_n\|^p_{\mathcal{P}_s})\,\mathrm{d}s \qquad (3.9)
\end{aligned}$$

where $k_1 = 3^p M_T^p \mathbb{E}\|\zeta\|^p$, $k_2 = C_{T,p,M_T}$ and $t \in \mathbf{J}$.
Let $u(t), t \in \mathbf{J}$, be a global solution of the equation

$$u(t) = k_1 + k_2 \int_0^t L(s, u(s))\, \mathrm{d}s,$$

with an initial condition k_1. By induction, we have

$$\|x_{n+1}\|_{\mathcal{P}_t} \le u(t), \quad t \in \mathbf{J}.$$

Indeed, for $n = 0$, we have

$$\|x_0\|_{\mathcal{P}_t}^p = \sup_{s\in[0,t]} \mathbb{E}\left(\|R(s)\zeta\|^p\right) \le k_1 \le u(t), \quad t \in \mathbf{J}.$$

Now, assume that $\|x_n\|_{\mathcal{P}_t} \le u(t)$, $t \in \mathbf{J}$.
Then using (3.9), the induction assumption and the nondecreasing property of H, we have

$$\begin{aligned} u(t) - \|x_{n+1}\|_{\mathcal{P}_t} &\ge k_1 + k_2 \int_0^t L(s, u(s))\, \mathrm{d}s - k_1 - k_2 \int_0^t L(s, \|x_n\|_{\mathcal{P}_s}^p)\, \mathrm{d}s \\ &\ge k_2 \int_0^t \left\{ L(s, u(s))\|^p - L(s, \|x_n\|_{\mathcal{P}_s}^p) \right\} \mathrm{d}s \\ &\ge 0, \quad t \in \mathbf{J}. \end{aligned}$$

Hence the sequence $(x_n)_{n\ge0}$ is bounded in $\mathcal{P}_T$. □

Lemma 3.4 *Assume that the conditions* $(\mathbf{A1})$–$(\mathbf{A4})$ *hold and* $\zeta \in L_p(\Omega, \mathcal{F}_0, \mathsf{H})$ *with* $p > 2$. *Then the sequence* $(x_n)_{n\ge0}$ *is a Cauchy sequence in the space* $\mathcal{P}_T$ *and the limit is a mild solution for* (1.1).

Proof. Set $\gamma_n(t) := \sup_{m\ge n}\left(\|x_m - x_n\|_{\mathcal{P}_t}^p\right), \quad t \in \mathbf{J}, n \ge 0$. By Lemma 3.3, the functions $\gamma_n, n \ge 0$ are well defined and uniformly bounded. Moreover, $\gamma_n, n \ge 0$ are monotone nondecreasing. Since $\gamma_n, n \ge 0$ is a monotone nonincreasing sequence for each $t \in \mathbf{J}$, there exists a monotone nondecreasing function γ such that $\gamma_n(t) \longrightarrow \gamma(t)$, as $n \longrightarrow \infty$.
Now by inequality (3.7), we have

$$\|x_m - x_n\|_{\mathcal{P}_t}^p = \|\mathcal{G}x_{m-1} - \mathcal{G}x_{n-1}\|_{\mathcal{P}_t}^p \le k \int_0^t K(s, \|x_{m-1} - x_{n-1}\|_{\mathcal{P}_s}^p)\, \mathrm{d}s$$

where $k = C'_{T,p,M_T}$. Using the nondecreasing property of K in the above inequality, we get

$$\|x_m - x_n\|_{\mathcal{P}_t}^p \le k \int_0^t K(s, \gamma_{n-1}(s))\, \mathrm{d}s,$$

which implies that

$$\gamma_n(t) \le k \int_0^t K(s, \gamma_{n-1}(s))\, \mathrm{d}s.$$

Therefore, by using the Lebesgue convergence theorem, we get that

$$\gamma(t) \le k \int_0^t K(s, \gamma(s))\, \mathrm{d}s.$$

By assumption $(\mathbf{A4})$, we get that $\gamma \equiv 0$. Thus,

$$0 \le \|x_m - x_n\|^p_{\mathcal{P}_T} \le \gamma_n(T) \longrightarrow \gamma(T) = 0$$

as n goes to infinity. Hence $(x_n)_{n\ge 0}$ is a Cauchy sequence in the Banach space $\mathcal{P}_T$. Since $\mathcal{G}$ is continuous, the limit of $x_{n+1} = \mathcal{G}x_n$ is a mild solution for (1.1). □

Lemma 3.5 *Assume that the conditions* $(\mathbf{A1})$–$(\mathbf{A4})$ *hold and* $\zeta \in L_p(\Omega, \mathcal{F}_0, \mathsf{H})$ *with* $p > 2$*. Then Equation* (1.1) *has a unique mild solution in* $\mathcal{P}_T$*.*

Proof. Suppose x, y in $\mathcal{P}_T$ were two fixed points of $\mathcal{G}$, then by Lemma 3.2, we have

$$\|x - y\|^p_{\mathcal{P}_T} = \|\mathcal{G}x - \mathcal{G}y\|^p_{\mathcal{P}_T} \le C'_{T,p,M_T} \int_0^T K(s, \|x - y\|^p_{\mathcal{P}_s})\, \mathrm{d}s,$$

Once again, by assumption $(\mathbf{A4})$, $\|x - y\|^p_{\mathcal{P}_T} = 0$ that is $x = y$. □

By Lemmas 3.1-3.5, we have the following existence and uniqueness of mild solutions.

Theorem 3.6 *Suppose that the conditions* $(\mathbf{A1})$-$(\mathbf{A4})$ *hold. Assume that* $\zeta \in L_p(\Omega, \mathcal{F}_0, \mathsf{H})$ *and* $p > 2$*. Then the sequence* $(x_n)_{n\ge 0}$ *converges in* $\mathcal{P}_T$ *to the unique solution of* (1.1) *in* $\mathcal{P}_T$*.*

4 Application

Consider the following stochastic partial integrodifferential equation of the form

$$\begin{cases} d\,u(t)(z) &= \left(\dfrac{\partial}{\partial z} uz(t)(z) + \left\langle \mathbf{f}(t, u(t))(z), \displaystyle\int_0^z \mathbf{f}(t, u(t))(u)\ \mathrm{d}u \right\rangle_{\mathsf{V}} \right) dt \\ &\quad + \displaystyle\int_0^t \mathbf{g}(t-s) \left(\dfrac{\partial}{\partial z} u(s)(z)\right) \mathrm{d}s \\ &\quad + \langle \mathbf{f}(t, u(t))(z), dW(t)\rangle_{\mathsf{V}}, \quad t \in [0, T], \quad z \ge 0, \\ u(0)(z) &= u_0(z) \in \mathbb{R}, \quad z \ge 0, \end{cases} \tag{4.1}$$

where the family $\{u(t)(\cdot)\}_{t\in\mathbb{R}^+}$ is a stochastic process. $(\mathsf{V}, <\cdot,\cdot>_{\mathsf{V}})$ is a separable Hilbert space and W is an cylindrical Wiener process on V, and $\mathbf{g} : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded and C^1-function such that $\mathbf{g}'(\cdot)$ is bounded and uniformly continuous.

Observe that eqn. (4.1) degrades to the bond market's Heath-Jarrow-Morton-Musiela equation when $\mathbf{g} \equiv 0$. Then the families $\{u(t)(\cdot)\}_{t\in\mathbb{R}^+}$ and z are referred to as forward curve process and time maturity, respectively. The term $\mathbf{f}(t, u(t))(z)$ has a financial sense and is frequently used to allude to bond market volatility (for more details, see [18, 19, 23]). The forward curve process is defined by formula $P(t, T^*) = \exp\left(-\int_0^{T^*-t} u(t)(z)\,\mathrm{d}z\right)$ $0 \le t \le T^* \le T$ where $P(t, T^*)$ can be seen as the bond price at moment t that is now maturing at T^*.

For each $\mu \in \mathbb{R}$, let L^2_μ be the space of all (equivalence classes of) Lebesgue measurable functions $h : \mathbb{R}^+ \to \mathbb{R}$ such that

$$\int_0^\infty |h(x)|^2 e^{\mu x}\, \mathrm{d}x < \infty.$$

It is well know that for each $\mu \in \mathbb{R}$, H is a Hilbert space endowed with the norm

$$\|h\|_{\mu,2} = \left(\int_0^\infty |h(x)|^2 e^{\mu x}\, \mathrm{d}x \right)^{1/2}.$$

From there we set $\mathsf{H} = L^2_\mu$ and define the operator A by

$$\mathsf{Dom}(A) = \{h \in \mathsf{H} : Dh \in \mathsf{H}\}, \quad Ah = Dh,\ h \in \mathsf{Dom}(A),$$

where Dh is the first weak derivative of h. A generates a strongly continuous semigroup $S = \{S(t)\}_{t\in\mathbb{R}^+}$ on H defined by

$$S(t)h(z) = h(t+z), \quad h \in \mathsf{H}, \quad t, z \in \mathbb{R}^+.$$

Suppose that function $\mathbf{f}$ in equation (4.1) is defined by

$$\mathbf{f}(t,h)(z) = \mathbf{q}(t, z, h(t+z)), \quad h \in \mathsf{H}, \quad t, z \in \mathbb{R}^+,$$

where $\mathbf{q} : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathsf{V}$ is a given function. Define

$$B(t,h)[u](z) = \langle \mathbf{q}(t, x, h(t+z)), u\rangle_{\mathsf{V}}, \quad h \in \mathsf{H}, \quad t,\ z \in \mathbb{R}^+, \quad u \in \mathsf{V}, \tag{4.2}$$

and

$$F(t,h)(z) = \left\langle \mathbf{q}(t, z, h(t+z)), \int_0^z \mathbf{q}(t, u, h(u))\, \mathrm{d}u \right\rangle_{\mathsf{V}}, \quad h \in \mathsf{H}, \quad t,\ z \in \mathbb{R}^+. \tag{4.3}$$

For Volterra kernel, take $\Upsilon(t,s) \equiv \Upsilon(t-s) = \mathbf{g}(t-s)A$ for $t \geq s \geq 0$. Let $\mu \in \mathbb{R}$ such that

$$\int_0^\infty |u_0(z)|^2 e^{\mu z}\, \mathrm{d}x < \infty.$$

uniformly on $z \in \mathbb{R}^+$ and put $\zeta \equiv u_0$. Then eqn. (4.1) take the following form in the space H.

$$\begin{cases} d\,u(t) &= \left[\mathsf{A}(t)u(t) + \displaystyle\int_0^t \Upsilon(t,r)u(s)\, \mathrm{d}s\right] dt + F(t,u(t))dt + B(t,u(t))dW(t), \quad t \in [0,T] \\ u(0) &= \zeta. \end{cases} \tag{4.4}$$

Proposition 4.1 *Fix $T \in \mathbb{R}^+$. Let F and B be as given in* (4.3) *and* (4.2) *respectively. Assume that there exist functions $\lambda_1, \lambda_2 \in \mathsf{H}$ with λ_2 bounded such that for all $(t,z,x,y) \in (\mathbb{R}^+)^2 \times (\mathbb{R})^2$,*

$$|\mathbf{q}(t,z,x)|_{\mathsf{V}} \leq |\lambda_1(z)|,$$

and

$$|\mathbf{q}(t,z,x) - \mathbf{q}(t,z,y)|_{\mathsf{V}} \leq |\lambda_2(z)||x-y|.$$

Then for each $u_0 \in L^2_\mu(\Omega, \mathcal{F}_0, \mathbb{P}; \mathsf{H})$, there exists a unique L^2_μ-valued mild solution r to equation (4.4) *with respect to the initial data $u(0) = u_0$. .*

Proof. Since all the assumptions of Theorem 3.6 are fulfilled, the existence and uniqueness follow. □

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