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ON A VARIABLE-ORDER FRACTIONAL PARABOLIC PROBLEM WITH L^1 DATA

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Abstract. In this paper, we discuss the existence and uniqueness of renormalized solution to the variable-order fractional parabolic problems. The functional setting involves Lebesgue and variable-order Sobolev spaces. Some a-priori estimates are used to obtain our results.

Keywords: Existence and uniqueness of renormalized solution, parabolic problem, variable-order fractional p-Laplacian.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N, N > 2$, be a bounded smooth domain. In this article, we would like to establish the existence and uniqueness of the renormalized solution to the following problem involving variable-order fractional Laplacian

$$(P_T)\begin{cases} u_t + (-\Delta)_p^{s(\cdot)} u + |u|^{q-2}u = f & \text{in } Q_T := \Omega \times (0,T), \\ u = 0 & \text{in } \left(\mathbb{R}^N \setminus \Omega\right) \times (0,T), \\ u(.,0) = u_0(.) & \text{in } \mathbb{R}^N, \end{cases}$$

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on the assumptions:

$i)$ $\quad f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$

$ii)$ $\quad 0 < s^- = \min_{(z,\xi)\in\mathbb{R}^N\times\mathbb{R}^N} s(z,\xi) \leq s(z,\xi) \leq s^+ = \max_{(z,\xi)\in\mathbb{R}^N\times\mathbb{R}^N} s(z,\xi) < 1$

$iii)$ $\quad N > ps(\cdot)$ and $1 < q < \frac{Np}{N-ps(\cdot)}$.

The operator $(-\Delta)_p^{s(\cdot)}$ is a variable-order fractional p-Laplacian defined by:

$$(-\Delta)_p^{s(\cdot)}u(z) = 2P.V \int_{\mathbb{R}^N} \frac{|u(z)-u(\xi)|^{p-2}(u(z)-u(\xi))}{|z-\xi|^{N+p.s(z,\xi)}}\,\mathrm{d}\xi,\ z \in \mathbb{R}^N,$$

where P.V denotes the Cauchy principle value. When $s(\cdot) = s(constant) \in (0,1)$, the operator $(-\Delta)_p^{s(\cdot)}$ reduces to the usual fractional p-Laplacian.
The fractional variable order derivatives suggested by Lorenzo and Hartley in [10] appeared in nonlinear diffusion processes. In particular, some diffusion processes reacting to temperature changes may be better described by using variable order derivatives in a nonlocal integro-differential operator, see for example [11]. There is growing interest in partial differential equations involving nonlocal operators [6, 7, 8, 9, 20, 22] due to their various applications in various scientific and technical fields for modeling complex phenomena. Here are some areas of application where these partial differential equations can be used: finance, biology, physics, geophysics, materials science, image processing, etc, see [1, 2, 5]. More specifically, partial differential equations involving variable-order fractional Laplacian are useful for modeling phenomena where non-local behaviors and complex spatial dependencies play a crucial role. They allow for the consideration of more realistic and complex phenomena than traditional partial differential equations.
Very interesting works addressing the variable-order fractional Laplacian exist in the literature. Injection results have been established by M. Xiang *et al.* [20] with the aim of demonstrating the existence of multiple solutions distinct from the problem (P_1) using the mountain pass theorem and Ekeland's variational principle.

$$(P_1)\ \begin{cases} (-\Delta)^{s(\cdot)} + \lambda V(x)u = \alpha|u|^{q(.)-2}u + \beta|u|^{q(.)-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases} \tag{1.1}$$

The works of Sabri [16] and Sabri *et al.* [15] present very interesting results. In [16], the study considers a more general operator; it is the variable-order fractional $p(\cdot)$–Laplacian, as indicated by the following problem:

$$(P_2)\begin{cases} u_t + (-\Delta)_{p(\cdot)}^{s(\cdot)}u = f & \text{in } Q_T := \Omega \times (0,T), \\ u = 0 & \text{in } \Sigma_T := \Omega \times (0,T), \\ u(.,0) = u_0(.) & \text{in } \mathbb{R}^N. \end{cases}$$

With L^∞-data, Sabri demonstrated the existence and uniqueness of a weak solution using the Rothe's time-discretization method.
However, it should be noted that the case of problems involving the variable-order fractional Laplacian with L^1–data seems to be less addressed in the literature. Most of the works we have encountered that deal with problems involving the fractional Laplacian $(-\Delta)_{p(\cdot)}^s$, with a constant-order s.
Problem (P_T) can be viewed as generalization of the following equation studied in [8]:

$$(P_3)\begin{cases} u_t + (-\Delta)_p^{s(\cdot)}u + |u|^{q-2}u = \lambda\frac{\partial F}{\partial u} & \text{in } Q_T := \Omega \times (0,T), \\ u = 0 & \text{in } \left(\mathbb{R}^N \setminus \Omega\right) \times (0,T), \\ u(.,0) = u_0(.) & \text{in } \mathbb{R}^N, \end{cases}$$

where $F \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is locally Lipschitz uniformly in second variable and λ is a positive parameter. The authors established the existence and uniqueness of the weak solution of problem (P_3), for regular data. As it has been shown in [14, 17], the distributional solution is in general not unique. In order to get well-posedness for L^1-data, the notion of a renormalized solution was introduced by DiPerna and Lions in [3]. Thus, the present paper represents a significant advancement in the study of fractional parabolic problems of variable order due to the right-hand side belonging to L^1.

The rest of the paper is organized as follows: In section 2 we recall some basic proprieties of Lebesgue and Sobolev spaces with variable exponent and in section 3 we state and prove our main result.

2 Preliminary

Let's define the function space as

$$W = W^{s(\cdot),p}(\Omega) := \left\{h : \mathbb{R}^N \longrightarrow \mathbb{R} \text{ is measurable}, h \in L^p(\Omega) \text{ and } [h]_{s(\cdot),\Omega} < \infty\right\},$$

where

$$[h]_{s(\cdot),\Omega} = \Big(\int_\Omega \int_\Omega \frac{|h(z) - h(\xi)|^p}{|z-\xi|^{N+ps(z,\xi)}} \,\mathrm{d}z\,\mathrm{d}\xi \Big)^{1/p}. \tag{2.1}$$

$W^{s(\cdot),p}(\Omega)$ is a Banach space equipped with the norm $||\cdot||_{s(\cdot),\Omega}$ defined as follows:

$$||h||_{s(\cdot),\Omega} = \left(||h||^p_{L^p(\Omega)} + [h]^p_{s(\cdot),\Omega}\right)^{\frac{1}{p}}.$$

Let us recall the following functional space,

$$W_0 = W_0^{s(\cdot),p}(\Omega) := \left\{ h \in W^{s(\cdot),p}(\Omega) : h = 0 \text{ a.e in } \mathbb{R}^N \setminus \Omega \right\}.$$

Equip $W_0^{s(\cdot),p}(\Omega)$ with the norm

$$||h||_{W_0^{s(\cdot),p}(\Omega)} = [h]_{s(\cdot),\Omega}.$$

Let consider a measurable function $s(\cdot)$ and two constants s_1 and s_2 such that $0 < s_1 < s(z,\xi) < s_2 < 1$, for all $(z,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$. The following lemma shows that $[\cdot]_{s(\cdot),(\Omega)}$ is an equivalent norm of $W_0^{s(\cdot),p}(\Omega)$.

Lemma 2.1 *[20] The embedding $W_0^{s_2,p}(\Omega) \hookrightarrow W_0^{s(\cdot),p}(\Omega) \hookrightarrow W_0^{s_1,p}(\Omega)$ are continuous. Moreover, if $N > ps_1$, for any fixed constant $\alpha \in [1, \frac{Np}{N-ps_1}]$, $W_0^{s_1,p}(\Omega)$ can be continuously embedded into $L^\alpha(\Omega)$.*

Lemma 2.2 *[8] Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Assume that $s : \mathbb{R}^N \times \mathbb{R}^N \to (0,1)$ is continuous and p satisfies $1 < q < \frac{Np}{N-ps(\cdot)}$. Then, there exists a positive constant $C_q = C(N,q,s^+,s^-)$ such that for any $h \in W^{s(\cdot),p}(\Omega)$, $||h||_{L^q(\Omega)} \le C_q||h||_{s(\cdot),\Omega}$. That is, the embedding $W^{s(\cdot),p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous. Furthermore, this embedding is compact. If $h \in W_0^{s(\cdot),p}(\Omega)$, then there exists $C_q = C(N,q,s^+,s^-)$ such that*

$$||h||_{L^q(\Omega)} \le C_q[h]_{s(\cdot),\Omega}.$$

Lemma 2.3 *[4] Let $l^+ = \max(l, 0)$. If $h \in W_0^{s(\cdot),p}(\Omega)$, then*

$$|h(z) - h(\xi)|^{p-2}(h^+(z) - h^+(\xi))(h(z) - h(\xi)) \geq |h^+(z) - h^+(\xi)|.$$

Lemma 2.4 *[4] Let τ be a positive constant. For all $\varsigma, \upsilon \in \mathbb{R}^N$,*

$$\langle |\varsigma|^{p-2}\varsigma - |\upsilon|^{p-2}\upsilon, \varsigma - \upsilon\rangle \geq \begin{cases} \tau|\varsigma - \upsilon|^p & \text{if } p \geq 2, \\ \tau\dfrac{|\varsigma - \upsilon|^2}{(|\varsigma| + |\upsilon|)^{2-p}} & \text{if } p < 2. \end{cases} \tag{2.2}$$

Remark 2.5 *Given real numbers a, b, c, and d, we have:*

$$ab - cd = \frac{(a-c)(b+d)}{2} + \frac{(a+c)(b-d)}{2}.$$

Let us define the truncating function $T_k(u)$, for any $u \in \mathbb{R}$, as

$$T_k(u) = \max\{-k, \min\{k, u\}\} = \begin{cases} k & \text{if } u \geq k, \\ u & \text{if } |u| < k, \\ -k & \text{if } u \leq -k. \end{cases} \tag{2.3}$$

and denote $\mathcal{T}_0^{s(\cdot),p}(Q_T) = \Big\{u(\text{measurable}) : Q_T \to \mathbb{R}, T_k(u) \in L^p(0, T; W_0^{s(\cdot),p}(\Omega)), k > 0\Big\}$.
We denote by $\Theta_k(u) : \mathbb{R} \to \mathbb{R}^+$ the primitive of $T_k(u)$ defined by

$$\Theta_k(u) = \int_0^u T_k(s)\,\mathrm{d}s = \begin{cases} \dfrac{u^2}{2} & \text{if } |u| < k, \\ k|u| - \dfrac{k^2}{2} & \text{if } |u| \geq k. \end{cases} \tag{2.4}$$

Note that

$$0 \leq \Theta_k(u) \leq k|u|.$$

Moreover, let us note that: $\lim\limits_{k \to 0} \dfrac{T_k(u)}{k} = \text{sign}_0(u)$, with

$$\text{sign}_0(u) = \begin{cases} 1 & \text{if } u > 0, \\ 0 & \text{if } u = 0, \\ -1 & \text{if } u < 0. \end{cases} \tag{2.5}$$

Lemma 2.6 *[21] Let $a, b \in \mathbb{R}$ and $p \geq 1$, then*

$$|a - b|^{p-2}(a - b)(T_k(a) - T_k(b)) \geq |T_k(a) - T_k(b)|^p. \tag{2.6}$$

Let define, for all $\epsilon > 0$, the function S_ϵ, which is useful in the proof of uniqueness.

$$S_\epsilon(h) = \begin{cases} h & \text{if } |h| < \epsilon, \\ (\epsilon + \frac{1}{2}) \pm \frac{1}{2}(h \pm (\epsilon + 1))^2 & \text{if } \epsilon \leq \pm h \leq \epsilon + 1 \\ \pm(\epsilon + \frac{1}{2}) & \text{if } \pm h > \epsilon + 1, \end{cases} \tag{2.7}$$

which gives

$$S'_\epsilon(h) = \begin{cases} 1 & \text{if} \quad |h| < \epsilon, \\ \epsilon + 1 + |h| & \text{if} \quad \epsilon \leq \pm h \leq \epsilon + 1, \\ 0 & \text{if} \quad |h| > \epsilon + 1. \end{cases} \tag{2.8}$$

3 Mains results

Definition 3.1 *A function $u \in C\big(0, T; L^1(\Omega)\big)$ is said to be a renormalized solution to problem (P_T) if $u \in \mathcal{T}_0^{s(\cdot),p}(Q_T)$ and the following conditions are satisfied:*

$$(i) \lim_{a\to\infty} \int_{\{t\in(0,T),(z,\xi)\in\Omega\times\Omega:(u(z,t),u(\xi,t))\in R_a\}} \frac{|u(z,t) - u(\xi,t)|^{p-1}}{|z-\xi|^{N+s(z,\xi)p}} \,\mathrm{d}z\,\mathrm{d}\xi dt = 0 \tag{3.1}$$

where

$$R_a = \{(v, w) \in \mathbb{R}^2 : \max\{|v|, |w|\} \geq a + 1 \textit{ and } \min\{|v|, |w|\} < a \textit{ or } vw < 0\}. \tag{3.2}$$

(ii) *For any $\varphi \in C_0^\infty(Q_T)$ and $S \in W^{1,\infty}(\mathbb{R})$ piecewise $C^1(\mathbb{R})$ satisfying that S' has a compact support,*

$$\int_0^T \int_{\Omega\times\Omega} \frac{|u(z,t) - u(\xi,t)|^{p-2}(u(z,t) - u(\xi,t))(S'(u)\varphi(z,t) - S'(u)\varphi(\xi,t))}{|z-\xi|^{N+s(z,\xi)p}} \,\mathrm{d}z\,\mathrm{d}\xi\,\mathrm{d}t$$
$$+ \int_{Q_T} \frac{\partial S(u)}{\partial t}\varphi(z,t)\,\mathrm{d}z\,\mathrm{d}t + \int_{Q_T} |u|^{q-2}uS'(u)\varphi(z,t)\,\mathrm{d}zdt = \int_{Q_T} fS'(u)\varphi(z,t)\,\mathrm{d}zdt. \tag{3.3}$$

Theorem 3.2 *Assume that* $(\mathbf{i}) - (\mathbf{iii})$ *hold, then problem (P_T) admits a unique renormalized solution $u \in \mathcal{T}_0^{s(\cdot),p}(Q_T) \cap L^1(Q_T)$.*

We give the proof of Theorem 3.2 in two subsections.

3.1 Existence of a renormalized solution

For the proof of the existence, we proceed in three steps as follows.

Step 1. The semi-discrete problem

Let $M \in \mathbb{N}^*$, $T > 0$ and $\delta = \dfrac{T}{M}$ and define $t_n = n\delta$, $u^n = u(t_n, .)$.
Let us introduce the approximate problem

$$(P_n) \quad \begin{cases} u^n + \delta(-\Delta)_p^{s(\cdot)}u^n + \delta|u^n|^{q-2}u^n = u^{n-1} + \delta f^n & \text{in } \Omega, \\ u^n = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u^0 = u_0 & \text{in } \mathbb{R}^N, \end{cases}$$

where $f^n(z) = \dfrac{1}{\delta}\int_{t_{n-1}}^{t_n} f(\varsigma, .)\,\mathrm{d}\varsigma$ in Ω, $u_0 \in L^\infty(\Omega)$ and $f^n \in L^\infty(\Omega)$.
Due to the density of $L^\infty(\Omega)$ in $L^1(\Omega)$, we have:

$$f^n \to f \text{ strongly in } L^1(\Omega).$$

Lemma 3.3 *Let's assume that the discretized problem* (P_n) *has a weak solution* $(u^n)_{0\leq n\leq M}$*. Then* $u^n \in L^\infty(\Omega)$ *for all* $0 \leq n \leq M$.

Proof. For $n = 1$, problem (P_1) is given as follows

$$(P_1)\begin{cases} u^1 + \delta(-\Delta)_p^{s(\cdot)}u^1 + \delta|u^1|^{q-2}u^1 = g^1 & \text{in } \Omega,\\ u^1 = 0 & \text{in } \mathbb{R}^N \setminus \Omega,\\ u^0 = u_0 & \text{in } \mathbb{R}^N, \end{cases}$$

We will demonstrate that if problem (P_1) has a solution u^1, then u^1 belongs to set $L^\infty(\Omega)$.
Let $\varphi \in W_0^{s(\cdot),p}(\Omega)$, if u^1 is a solution of (P_1), then the following equation is satisfied

$$\int_\Omega u^1\varphi\,\mathrm{d}z + \delta\langle(-\Delta)_p^{s(\cdot)}u^1, \varphi\rangle + \delta\langle|u^1|^{q-2}u^1, \varphi\rangle = \int_\Omega g^1\varphi\,\mathrm{d}z, \tag{3.4}$$

with $g^1 = u^0 + \delta f^1 \in L^\infty(\Omega)$.
Let $\epsilon \in \mathbb{R}^+$. Then $u^1 - \epsilon \in W_0^{s(\cdot),p}(\Omega)$ and $(u^1 - \epsilon)^+ \in W_0^{s(\cdot),p}(\Omega)$. We can take $\varphi = (u^1 - \epsilon)^+$ in (3.4) to get

$$\int_\Omega u^1(u^1-\epsilon)^+\,\mathrm{d}z + \delta\langle(-\Delta)_p^{s(\cdot)}u^1, (u^1-\epsilon)^+\rangle + \delta\langle|u^1|^{q-2}u^1, (u^1-\epsilon)^+\rangle = \int_\Omega g^1(u^1-\epsilon)^+\,\mathrm{d}z. \tag{3.5}$$

Let us define Ξ by

$$\Xi = \{z \in \Omega, u^1 \geq \epsilon\}. \tag{3.6}$$

We obtain

$$\langle|u^1|^{q-2}u^1, (u^1-\epsilon)^+\rangle = \int_\Xi |u^1|^{q-2}u^1(u^1-\epsilon)\,\mathrm{d}z \geq 0, \tag{3.7}$$

$$\langle(-\Delta)_p^{s(\cdot)}u^1, (u^1-\epsilon)^+\rangle \geq \int_{\Xi\times\Xi} \frac{|u^1(z) - u^1(\xi)|^p}{|z-\xi|^{N+s(z,\xi)p}}\,\mathrm{d}z\,\mathrm{d}\xi \geq 0. \tag{3.8}$$

It follows that

$$\int_\Omega u^1(u^1-\epsilon)^+\,\mathrm{d}z \leq \int_\Omega g^1(u^1-\epsilon)^+\,\mathrm{d}z. \tag{3.9}$$

That to say

$$0 \leq \int_\Omega [(u^1-\epsilon)^+]^2\,\mathrm{d}z \leq \int_\Omega (g^1-\epsilon)(u^1-\epsilon)^+\,\mathrm{d}z. \tag{3.10}$$

By setting $\epsilon = ||g^1||_\infty$, we have $g^1 - \epsilon \leq 0$ a.e in Ω.
Consequently $(u^1-\epsilon)^+ = 0$ a.e in Ω for all $\epsilon = ||g^1||_\infty$, which amounts to $u^1 \leq ||g^1||_\infty$ a.e in Ω.
In the same way, by taking $(u^1+\epsilon)^-$ as a test function in (3.4), we demonstrate that $-||g^1||_\infty \leq u^1$.
From all that precedes, we have $u^1 \in L^\infty(\Omega)$.
By induction, we deduce that $u^n \in L^\infty(\Omega)$ for all $n = \overline{1, M}$. □

Lemma 3.4 *Let* $u_0 \in L^\infty(\Omega)$ *and* $f^n \in L^\infty(\Omega)$*. For* $n = \overline{1, M}$*, the problem* (P_n) *has a unique weak solution* $u^n \in L^\infty(\Omega) \cap W_0^{s(\cdot),p}(\Omega)$*, that is, for any* $\varphi \in W_0^{s(\cdot),p}(\Omega)$*, one has*

$$\int_\Omega u^n\varphi\,\mathrm{d}z + \delta\langle(-\Delta)_p^{s(\cdot)}u^n, \varphi\rangle + \delta\langle|u^n|^{q-2}u^n, \varphi\rangle = \int_\Omega g^n\varphi\,\mathrm{d}z, \tag{3.11}$$

with $g^n = u^{n-1} + \delta f^n$.

Proof. For $n = 1$, the problem is given as follows:

$$(P_1)\begin{cases} u^1 + \delta(-\Delta)_p^{s(\cdot)} u^1 + \delta|u^1|^{q-2}u^1 = g^1 & \text{in } \Omega,\\ u^1 = 0 & \text{in } \mathbb{R}^N \setminus \Omega,\\ u^0 = u_0 & \text{in } \mathbb{R}^N, \end{cases}$$

For simplicity, we write $u = u^1$. Define the functional

$$\psi(u) = \frac{1}{2}\int_\Omega u^2(z)\,\mathrm{d}z + \delta\int_{\Omega\times\Omega}\frac{1}{p}\frac{|u(z)-u(\xi)|^p}{|z-\xi|^{N+s(z,\xi)p}}\,\mathrm{d}z\,\mathrm{d}\xi + \delta\int_\Omega \frac{1}{q}|u(z)|^q\,\mathrm{d}z - \int_\Omega g^1(z)u(z)\,\mathrm{d}z.$$

Note that ψ is well-defined and Gateaux differentiable on $W_0^{s(\cdot),p}(\Omega)$. Standard arguments allow us to establish an equivalence between the minimizer of ψ and the weak solution of problem (P_1). Indeed, let $u \in W_0^{s(\cdot),p}(\Omega)$ be a minimizer of ψ. We will show that u satisfies problem (P_1). We have:

$$\begin{aligned} 0 &= \frac{d}{dt}\psi(u+tv)\big|_{t=0}\\ &= \Big[\frac{1}{2}\int_\Omega \frac{d}{dt}(u(z)+tv(z))^2\,\mathrm{d}z + \delta\int_\Omega\int_\Omega \frac{d}{dt}\frac{|u(z)-u(\xi)+t\big(v(z)-v(\xi)\big)|^p}{p|z-\xi|^{N+ps(z,\xi)}}\,\mathrm{d}z\,\mathrm{d}\xi\\ &\quad - \int_\Omega \frac{d}{dt}g^1(u(z)+tv(z))\,\mathrm{d}z\Big]_{t=0}\\ &= \int_\Omega u(z)v(z)\,\mathrm{d}z + \delta\int_\Omega\int_\Omega \frac{|u(z)-u(\xi)|^{p-2}(u(z)-u(\xi))(v(z)-v(\xi))}{|z-\xi|^{N+ps(z,\xi)}}\,\mathrm{d}z\,\mathrm{d}\xi\\ &\quad - \int_\Omega g^1(z)v(z)\,\mathrm{d}z. \end{aligned} \tag{3.12}$$

Thus, u is a weak solution of (P_1), with v being a test function. Conversely, let us consider a weak solution u of problem (P_1) and show that it minimizes ψ. Let $v \in C_0^\infty(\overline{\Omega})$, then we have the following weak formulation:

$$\begin{aligned} \int_\Omega u(z)v(z)\,\mathrm{d}z \quad &+ \quad \delta\int_\Omega\int_\Omega \frac{|u(z)-u(\xi)|^{p-2}(u(z)-u(\xi))(v(z)-v(\xi))}{|z-\xi|^{N+s(z,\xi)p}}\,\mathrm{d}z\,\mathrm{d}\xi\\ &= \int_\Omega g^1(z)v(z)\,\mathrm{d}z = 0, \end{aligned} \tag{3.13}$$

which is nothing other than $\psi'(u) = 0$, thus u minimizes ψ.
To achieve this and demonstrate the existence of a weak solution, we will show that ψ possesses a global minimizer $u \in W_0^{s(\cdot),p}(\Omega) \cap L^\infty(\Omega)$. Indeed,

$$\begin{aligned} \psi(u) \;=\; &\frac{1}{2}\int_\Omega u^2(z)\,\mathrm{d}z + \delta\int_{\Omega\times\Omega}\frac{1}{p}\frac{|u(z)-u(\xi)|^p}{|z-\xi|^{N+s(z,\xi)p}}\,\mathrm{d}z\,\mathrm{d}\xi\\ &+\delta\int_\Omega\frac{1}{q}|u(z)|^q\,\mathrm{d}z - \int_\Omega g^1(z)u(z)\,\mathrm{d}z \;\geq\; \frac{\delta}{p}[u]^p_{s(\cdot),\Omega} - c[u]_{s(\cdot),\Omega}\,, \end{aligned}$$

that is to say,

$$\frac{\psi(u)}{[u]_{s(\cdot),\Omega}} \geq \frac{\delta}{p}[u]^{p-1}_{s(\cdot),\Omega} - c\,.$$

Since $p > 1$, it follows that ψ is coercive. Furthermore, ψ is lower bounded and weakly lower semicontinuous. Therefore, ψ has a global minimizer $u \in W_0^{s(\cdot),p}(\Omega)$.
We will now prove that the minimizer of ψ is unique. Suppose that ψ has two minimizers u and v. By using the test functions $u - v$ and $v - u$ for u and v respectively, we have

$$\int_\Omega u(u-v)\,\mathrm{d}z + \delta \int_{\Omega\times\Omega} \frac{|u(z)-u(\xi)|^{p-2}(u(z)-u(\xi))\big(u(z)-u(\xi)-\big(v(z)-v(\xi)\big)\big)}{|z-\xi|^{N+s(z,\xi)p}}\,\mathrm{d}z\,\mathrm{d}\xi + \delta\int_\Omega |u|^{q-2}u(u-v)\,\mathrm{d}z \;=\; \int_\Omega g^1(u-v)\,\mathrm{d}z \tag{3.14}$$

and

$$\int_\Omega v(v-u)\,\mathrm{d}z + \delta \int_{\Omega\times\Omega} \frac{|v(z)-v(\xi)|^{p-2}(v(z)-v(\xi))\big(v(z)-v(\xi)-\big(u(z)-u(\xi)\big)\big)}{|z-\xi|^{N+s(z,\xi)p}}\,\mathrm{d}z\,\mathrm{d}\xi + \delta\int_\Omega |v|^{q-2}v(v-u)\,\mathrm{d}z \;=\; \int_\Omega g^1(v-u)\,\mathrm{d}z. \tag{3.15}$$

By summing the two equations, we obtain:

$$\int_\Omega (u-v)^2\,\mathrm{d}z + \delta\langle(-\Delta)_p^{s(\cdot)}u - (-\Delta)_p^{s(\cdot)}v, u-v\rangle + \delta\int_\Omega \big(|u|^{q-2}u - |v|^{q-2}v\big)(u-v)\,\mathrm{d}z = 0. \tag{3.16}$$

Since $(-\Delta)_p^{s(\cdot)}$ and $h \longmapsto |h|^{q-2}h$ are monotone. Then, we deduce from (3.16) that $u = v$. Hence (P_1) has a unique weak solution $u^1 \in L^\infty(\Omega) \cap W_0^{s(.),p}(\Omega)$.
Using the same manner as the case $n = 1$, we deduce that the problem (P_n) has a unique weak solution $u^n \in L^\infty(\Omega) \cap W_0^{s(.),p}(\Omega)$ for all $n = 1, \cdots, M$. □

The following result establishes the renormalization condition for further work.

Proposition 3.5 *Let $(u^n)_{0\le n\le M}$ be a weak solution of (P_n), then we have:*

$$\lim_{a\to\infty} \int_{\{(z,\xi)\in\Omega\times\Omega:(u^n(z),u^n(\xi))\in R_a\}} \frac{|u^n(z)-u^n(\xi)|^{p-1}}{|z-\xi|^{N+s(z,\xi)p}}\,\mathrm{d}z\,\mathrm{d}\xi = 0 \tag{3.17}$$

where

$$R_a = \{(v,w)\in\mathbb{R}^2 : \max\{|v|,|w|\} \ge a+1 \textit{ and } \min\{|v|,|w|\} < a \textit{ or } vw < 0\}. \tag{3.18}$$

Proof. In order to prove (3.17), we take $\varphi = T_1(G_a(u^n))$ as test function in (3.11), where $G_a(u^n)$ is given by $G_a(u^n) = u^n - T_a(u^n) = \begin{cases} u^n \pm a & \text{if } |u^n| > a\,, \\ 0 & \text{if } |u| \le a\,. \end{cases}$
We have

$$\int_\Omega u^n T_1(G_a(u^n))\,\mathrm{d}z + \delta\langle(-\Delta)_p^{s(z,\xi)}u^n, T_1(G_a(u^n))\rangle + \delta\int_\Omega |u^n|^{q-2}u^n T_1(G_a(u^n))\,\mathrm{d}z = \int_\Omega g^n T_1(G_a(u^n))\,\mathrm{d}z. \tag{3.19}$$

Equation (3.19) becomes

$$\int_{\{|u^n|>a\}} u^n T_1(u_n \pm a)\,\mathrm{d}z + \delta\langle(-\Delta)_p^{s(z,\xi)} u^n, T_1(u_n \pm a)\rangle \\ + \delta \int_{\{|u^n|>a\}} |u^n|^{q-2} u^n T_1(u_n \pm a)\,\mathrm{d}z = \int_{\{|u^n|>a\}} g^n T_1(u_n \pm a)\,\mathrm{d}z. \tag{3.20}$$

It is evident that the first term on the left-hand side is positive. Note that the third term of left hand side of (3.20) is also positive. Indeed,

$$\begin{aligned} &\int_{\{|u^n|>a\}} |u^n|^{q-2} u^n T_1(u_n \pm a)\,\mathrm{d}z \\ &= \int_{\{u^n>a\}} |u^n|^{q-2} u^n T_1(u^n - a)\,\mathrm{d}z + \int_{\{u^n<-a\}} |u^n|^{q-2} u^n T_1(u^n + a)\,\mathrm{d}z \\ &= \int_{\{u^n>a+1\}} |u^n|^{q-2} u^n\,\mathrm{d}z + \int_{\{a<u^n<a+1\}} |u^n|^{q-2} u^n (u^n - a)\,\mathrm{d}z \\ &\quad + \int_{\{-a-1<u^n<-a\}} |u^n|^{q-2} u^n (u^n + a)\,\mathrm{d}z + \int_{\{u^n<-a-1\}} |u^n|^{q-2} (-u^n)\,\mathrm{d}z. \end{aligned} \tag{3.21}$$

This gives $\int_\Omega |u^n|^{q-2} u^n T_1(G_a(u^n)) \geq 0$, and (3.20) becomes

$$\delta\langle(-\Delta)_p^{s(z,\xi)} u^n, T_1(G_a(u^n))\rangle \leq \int_\Omega g^n T_1(G_a(u^n))\,\mathrm{d}z. \tag{3.22}$$

since for all $(u^n(z), u^n(\xi)) \in R_a$,

$$|u^n(z) - u^n(\xi)|^{p-2}(u^n(z) - u^n(\xi))\big(T_1(G_a(u^n))(z) - T_1(G_a(u^n))(\xi)\big) \geq |u^n(z) - u^n(\xi)|^{p-1}.$$

Substituting g^n with its expression, we get

$$\lim_{a\to\infty} \int_{\{(z,\xi)\in\Omega\times\Omega:(u^n(z),u^n(\xi))\in R_a\}} \frac{|u^n(z) - u^n(\xi)|^{p-1}}{|z-\xi|^{N+s(z,\xi)p}}\,\mathrm{d}z\,\mathrm{d}\xi \\ \leq \lim_{a\to\infty} \big(\delta \int_{\{|u^n|\geq a\}} |f^n| + \int_{\{|u^n|\geq a\}} |u^{n-1}|\big). \tag{3.23}$$

Given that $\lim_{a\to\infty} meas\{|u^n| > a\} = 0$ uniformly in n, it follows that

$$\lim_{a\to\infty} \int_{\{(z,\xi)\in\Omega\times\Omega:(u^n(z),u^n(\xi))\in R_a\}} \frac{|u^n(z) - u^n(\xi)|^{p-1}}{|z-\xi|^{N+s(z,\xi)p}}\,\mathrm{d}z\,\mathrm{d}\xi = 0. \tag{3.24}$$

□

Step 2. Stability results

The following result is essentials for the study of the convergence of the Euler forward scheme.

Lemma 3.6 *Let $(u^n)_{0\leq n\leq M}$ be a renormalized solution of (P_n), then there exists a positive constant $C(u^0, f)$ independent on M, such that for all $n = 1, \cdots, M$, we have*

(i) $\|u^n\|_1 \leq C(u^0, f)$;

(ii) $\sum_{i=1}^{n} \|u^i - u^{i-1}\|_1 \leq C(u^0, f)$;

(iii) $\sum_{i=1}^{n} \|u^i\|^p_{W_0^{s(\cdot),p}(\Omega)} \leq C(u^0, f)$.

Proof. To prove (i), we choose $\varphi = T_k(u^i)$ as test function in (3.11) to obtain

$$\int_\Omega u^i T_k(u^i)\,\mathrm{d}z + \delta\langle(-\Delta)_p^{s(\cdot)}u^i, T_k(u^i)\rangle + \delta\int_\Omega |u^i|^{q-2}u^i T_k(u^i) = \int_\Omega g^i T_k(u^i)\,\mathrm{d}z. \tag{3.25}$$

Since u^i and $T_k(u^i)$ have the same sign, so $|u^i|^{q-2}u^i T_k(u^i) \geq 0$. Therefore, it follows that

$$\int_\Omega u^i T_k(u^i)\,\mathrm{d}z \leq \delta\int_\Omega k|f^i| + \int_\Omega k|u^{i-1}|. \tag{3.26}$$

Dividing (3.26) by $k > 0$ leads to

$$\int_\Omega u^i \frac{T_k(u^i)}{k}\,\mathrm{d}z \leq \delta\int_\Omega |f^i| + \int_\Omega |u^{i-1}|. \tag{3.27}$$

Note that $\lim\limits_{k\to 0} \dfrac{T_k(u^i)}{k} = sign_0(u^i)$. Then, passing to limit in (3.27) and using Fatou's lemma, we obtain:

$$\int_\Omega |u^i|\,\mathrm{d}z \leq \delta\int_\Omega |f^i| + \int_\Omega |u^{i-1}|. \tag{3.28}$$

This gives $\|u^i\|_1 \leq \delta\|f^i\|_1 + \|u^{i-1}\|_1$ and summing from $i = 1$ to n yields

$$\|u^n\|_1 \leq \|f\|_1 + \|u_0\|_1.$$

Next, testing (P_n) by $\varphi = T_\delta(u^i - T_k(u^{i-1}))$ leads to

$$\begin{aligned}&\int_\Omega (u^i - u^{i-1})T_\delta(u^i - T_k(u^{i-1}))\,\mathrm{d}z + \delta\langle(-\Delta)_p^{s(\cdot)}u^i, T_\delta(u^i - T_k(u^{i-1}))\rangle\\ &+\int_\Omega \delta|u^i|^{q-2}u^i T_\delta(u^i - T_k(u^{i-1}))\,\mathrm{d}z = \int_\Omega \delta f^i T_\delta(u^i - T_k(u^{i-1}))\,\mathrm{d}z.\end{aligned} \tag{3.29}$$

For the sequel, let's divide (3.29) by $\delta > 0$. We have

$$\begin{aligned}&\int_\Omega (u^i - u^{i-1})\frac{T_\delta(u^i - T_k(u^{i-1}))}{\delta}\,\mathrm{d}z + \int_\Omega \langle(-\Delta)_p^{s(z,\xi)}u^i, T_\delta(u^i - T_k(u^{i-1}))\rangle\\ &+\int_\Omega |u^i|^{q-2}u^i T_\delta(u^i - T_k(u^{i-1}))\,\mathrm{d}z \leq ||f^i||_{L^1(\Omega)}.\end{aligned} \tag{3.30}$$

Let's consider the subdivision of Ω as follows:

$$\begin{aligned}\Omega_1^i &= \{|u^i - u^{i-1}| \leq \delta; |u^{i-1}| \leq k\}\\ \Omega_2^i &= \{|u^i - k sign_0(u^{i-1})| \leq \delta; |u^{i-1}| > k\}.\end{aligned}$$

Let $Y^{(i)}$ and $Z^{(i)}$ denote the second and third terms of the left-hand side of equation (3.30). Then $Y^{(i)}$ is written as follows:

$$\begin{aligned}Y^{(i)} =& \int_{\Omega_1^i\times\Omega_1^i} \frac{|u^i(z) - u^i(\xi)|^{p-2}(u^i(z) - u^i(\xi))}{|z-\xi|^{N+s(z,\xi)p}}\\ &\qquad\times\left[(u^i - u^{i-1})(z) - (u^i - u^{i-1})(\xi)\right]\mathrm{d}z\,\mathrm{d}\xi\\ &+\int_{\Omega_2^i\times\Omega_2^i} \frac{|u^i(z) - u^i(\xi)|^p}{|z-\xi|^{N+s(z,\xi)p}}\,\mathrm{d}z\,\mathrm{d}\xi.\end{aligned}$$

Using the convexity property, we have :

$$\frac{1}{p}||u^i||^p_{W_0^{s(\cdot),p}(\Omega_1^i)} - \frac{1}{p}||u^{i-1}||^p_{W_0^{s(\cdot),p}(\Omega_1^i)} \leq Y^{(i)}.$$

We have $Z^{(i)} = \int_\Omega \delta |u^i|^{q-2}u^i T_\delta(u^i - T_k(u^{i-1}))\,\mathrm{d}z$. In $\Omega \setminus \Omega_1^i$, by using the same manner as in (3.21) leads to $Z^{(i)} \geq 0$.
In Ω_1^i, using the convexity property, we obtain

$$\frac{1}{q}||u^i||^q_{L^q(\Omega_1^i)} - \frac{1}{q}||u^{i-1}||^q_{L^q(\Omega_1^i)} \leq Z^{(i)}.$$

Now, summing to $i = 1, ...n$ in (3.30), we obtain:

$$\sum_{i=1}^n \int_\Omega (u^i - u^{i-1}) \frac{T_\delta(u^i - T_k(u^{i-1}))}{\delta}\,\mathrm{d}z \leq \sum_{i=1}^n \left(\|f\|_{L^1(Q_T)} + \|u^0\|_{W_0^{s(\cdot),p}(\Omega_1^i)} + \|u^0\|^q_{L^q(\Omega_1^i)} \right). \tag{3.31}$$

We have (ii) by passing to the limit for $k \to \infty$ and $\delta \to 0$ in (3.31).
To end with (iii), we take $\varphi = T_k(u^i)$ as test function in (P_n) to have

$$\begin{aligned} \delta\langle(-\Delta)_p^{s(\cdot)}u^i, (T_k(u^i))\rangle \quad &+ \quad \delta \int_\Omega |u^i|^{q-2}u^i T_\delta(T_k(u^i))\,\mathrm{d}z \\ &= \quad \delta \int_\Omega f^i(T_k(u^i))\,\mathrm{d}z + \int_\Omega (u^{i-1} - u^i)(T_k(u^i))\,\mathrm{d}z. \end{aligned} \tag{3.32}$$

Since $T_k(u^i)$ has the same sign as u^i, the second term on the left-hand side is positive. Adding for $i = 1$ to n and using the Lemma 2.6, inequality (3.32) yields

$$\frac{\delta}{k} \sum_{i=1}^n \int_{\Omega\times\Omega} \frac{|T_k(u^i(z)) - T_k(u^i(\xi))|^p}{|z-\xi|^{N+s(z,\xi)p}}\,\mathrm{d}z\,\mathrm{d}\xi \leq \|f\|_{L^1(Q_T)} + \|u^i - u^{i-1}\|_{L^1(Q_T)}.$$

and we deduce (iii). □

Step 3. Renormalized solution
Let define Rothe's functions u_δ and $\tilde{u}_\delta$ for $n = 1, ..., M$ and $t \in [t_{n-1}, t_n]$ by:

$$u_\delta(t) = u^n \text{ and } \tilde{u}_\delta(t) = \frac{t - t_{n-1}}{\delta}(u^n - u^{n-1}) + u^{n-1}, \tag{3.33}$$

with $u_\delta(0) = \tilde{u}_\delta(0) = u_0$. We also define $f_\delta(t,z) = f^n(z)$. Note that $\dfrac{\partial \tilde{u}_\delta}{\partial t} = \dfrac{u^n - u^{n-1}}{\delta}$.
The discretized problem (P_n) can thus be rewritten as follows:

$$(P_\delta) \begin{cases} \dfrac{\partial \tilde{u}_\delta}{\partial t} + (-\Delta)_p^{s(\cdot)}u_\delta + |u_\delta|^{q-2}u_\delta = & f_\delta \quad \text{in } Q_T, \\ u_\delta = \tilde{u}_\delta = & 0 \quad \text{in } \left(\mathbb{R}^N \setminus \Omega\right) \times (0,T), \\ u_\delta(.,0) = \tilde{u}_\delta(.,0) = & u_{0\delta} \quad \text{in } \mathbb{R}^N. \end{cases} \tag{3.34}$$

Taking $S'(u_\delta)\varphi$ as test function with $\varphi \in C_0^\infty(Q_T)$ and $S \in W^{1,\infty}(\mathbb{R})$ piecewise C^1 satisfying that S' has a compact support, the solution of (P_δ) satisfy the following equation:

$$\begin{aligned} \int_0^T \int_\Omega \frac{\partial \tilde{u}_\delta}{\partial t} S'(u_\delta)\varphi(z,t)\,\mathrm{d}z\,\mathrm{d}t + \int_0^T \langle(-\Delta)_p^{s(z,\xi)}u_\delta, S'(u_\delta)\varphi(z,t))\rangle\,\mathrm{d}t \\ + \int_0^T \langle|u_\delta|^{q-2}u_\delta, S'(u_\delta)\varphi(z,t)\rangle\,\mathrm{d}t = \int_0^t \int_\Omega f_\delta S'(u_\delta)\varphi(z,t)\,\mathrm{d}z\,\mathrm{d}t. \end{aligned} \tag{3.35}$$

According to the stability results, we deduce the following a priori estimates.

Lemma 3.7 *Let $n = 1, ..., M$, $t \in [t_{n-1}, t_n]$ and $\delta = \frac{T}{M}$. Then, the function u_δ and $\tilde{u}_\delta$ satisfy the following estimations:*

(i) $\|\tilde{u}_\delta - u_\delta\|_{L^1(Q_T)} \leq \frac{1}{\delta} C(T, u^0, f)$;

(ii) $\left\| \dfrac{\partial \tilde{u}_\delta}{\partial t} \right\|_{L^1(Q_T)} \leq C(T, u^0, f)$;

(iii) $\|u_\delta\|_{L^1(Q_T)} \leq C(T, u^0, f)$;

(iv) $\|\tilde{u}_\delta\|_{L^1(Q_T)} \leq C(T, u^0, f)$;

(v) $\|T_k(u_\delta)\|_{L^p(0,T;W_0^{s(\cdot),p}(\Omega))} \leq C(T, u^0, f, k)$,

where $C(T, u^0, f)$ and $C(T, u^0, f, k)$ are positive constants independents of M.

Proof. See [15, Lemma 5] □

Lemma 3.8 *The sequence $(\tilde{u}_\delta)$ converges to u in $C([0, T]; L^1(\Omega))$.*

Proof. We consider two integer η and ϱ, then the weak form of (P_δ) is written as follows:

$$\int_0^T \langle (\tilde{u}_\varrho - \tilde{u}_\eta)_t, \phi \rangle \, \mathrm{d}t + \int_0^T \langle (-\Delta)_p^{s(z,\xi)} u_\varrho - (-\Delta)_p^{s(\cdot)} u_\eta, \phi \rangle \, \mathrm{d}t + \int_0^T \langle |u_\varrho|^{q-2} u_\varrho - |u_\eta|^{q-2} u_\eta, \phi \rangle \, \mathrm{d}t = \int_{Q_T} (f_\varrho - f_\eta) \phi \, \mathrm{d}z \, \mathrm{d}t \tag{3.36}$$

for all $\phi \in L^p(0, T; W_0) \cap L^\infty(Q_T)$.
Let $t \leq T$ and considering $\phi = T_1(u_\varrho - u_\eta)_{\chi(0,t)}$, then

$$\begin{aligned} \int_0^t \langle (\tilde{u}_\varrho - \tilde{u}_\eta)_t, T_1(u_\varrho - u_\eta) \rangle \, \mathrm{d}\tau \quad &+ \quad \int_0^t \langle (-\Delta)_p^{s(\cdot)} u_\varrho - (-\Delta)_p^{s(\cdot)} u_\eta, T_1(u_\varrho - u_\eta) \rangle \, \mathrm{d}\tau \\ &+ \quad \int_0^t \langle |u_\varrho|^{q-2} u_\varrho - |u_\eta|^{q-2} u_\eta, T_1(u_\varrho - u_\eta) \rangle \, \mathrm{d}\tau \\ &= \quad \int_\Omega \int_0^t (f_\varrho - f_\eta) T_1(u_\varrho - u_\eta)_{\chi(0,t)} \, \mathrm{d}z \, \mathrm{d}\tau. \end{aligned} \tag{3.37}$$

Since $u \longmapsto (-\Delta)_p^{s(\cdot)} u$ and $u \longmapsto |u|^{q-2} u$ are monotone, we deduce that

$$\langle (-\Delta)_p^{s(\cdot)} u_\varrho - (-\Delta)_p^{s(\cdot)} u_\eta, T_1(u_\varrho - u_\eta) \rangle \geq 0 \tag{3.38}$$

and

$$\langle |u_\varrho|^{q-2} u_\varrho - |u_\eta|^{q-2} u_\eta, T_1(u_\varrho - u_\eta) \rangle \geq 0. \tag{3.39}$$

Hence, we obtain the following inequality:

$$\int_0^t \langle (\tilde{u}_\varrho - \tilde{u}_\eta)_t, T_1(u_\varrho - u_\eta) \rangle \, \mathrm{d}\tau \leq \int_{Q_T} (f_\varrho - f_\eta) T_1(u_\varrho - u_\eta)_{\chi(0,t)} \, \mathrm{d}z \, \mathrm{d}t \tag{3.40}$$

which give

$$\int_\Omega \Theta_1(\tilde{u}_\varrho - \tilde{u}_\eta)(t)\,\mathrm{d}z \le \int_\Omega \Theta_1(u_{0\varrho} - u_{0\eta})(t)\,\mathrm{d}z + ||f_\varrho - f_\eta||_{L^1(Q_T)} := \beta_{\varrho,\eta}. \tag{3.41}$$

Therefore,

$$\begin{aligned}\int_{\{|\tilde{u}_\varrho - \tilde{u}_\eta|<1\}} \frac{|\tilde{u}_\varrho - \tilde{u}_\eta|^2}{2}\,\mathrm{d}z + \int_{\{|\tilde{u}_\varrho - \tilde{u}_\eta|\ge 1\}} \frac{|\tilde{u}_\varrho - \tilde{u}_\eta|}{2}\,\mathrm{d}z &\le \int_\Omega \Theta_1(\tilde{u}_\varrho - \tilde{u}_\eta)(t)\,\mathrm{d}z \\ &\le \beta_{\varrho,\eta}. \end{aligned} \tag{3.42}$$

It follows that

$$\begin{aligned}\int_\Omega |\tilde{u}_\varrho - \tilde{u}_\eta|(t)\,\mathrm{d}z &= \int_{\{|\tilde{u}_\varrho - \tilde{u}_\eta|<1\}} |\tilde{u}_\varrho - \tilde{u}_\eta|(t)\,\mathrm{d}z + \int_{\{|\tilde{u}_\varrho - \tilde{u}_\eta|\ge 1\}} |\tilde{u}_\varrho - \tilde{u}_\eta|(t)\,\mathrm{d}z \\ &\le \left(\int_{\{|\tilde{u}_\varrho - \tilde{u}_\eta|<1\}} |\tilde{u}_\varrho - \tilde{u}_\eta|^2(t)\,\mathrm{d}z\right)^{\frac{1}{2}} meas(\Omega)^{\frac{1}{2}} + 2\beta_{\varrho,\eta} \\ &\le \big(2meas(\Omega)\beta_{n,\eta}\big)^{\frac{1}{2}} + 2\beta_{\varrho,\eta}. \end{aligned} \tag{3.43}$$

Since f_δ and $u_{0\delta}$ converge respectively in $L^1(Q_T)$ and $L^1(\Omega)$, we have $\beta_{\varrho,\eta} \longrightarrow 0$ as ϱ and η tend towards 0. Therefore $\tilde{u}_\delta$ is a Cauchy sequence in $C([0,T];L^1(\Omega))$. This imply that $\tilde{u}_\delta$ converges to u in $C([0,T];L^1(\Omega))$. □

The following lemma is obtained by using Lemma 3.7.

Lemma 3.9 *Let $\tilde{u}_\delta$ and u_δ satisfy problem (P_δ). Then there exists a function u in $L^1(Q_T)$ such that $T_k(u) \in L^p(0,T;W_0^{s(\cdot),p}(\Omega))$ for all $k>0$, and we have:*
(i) $\tilde{u}_\delta$, u_δ converges to u in $L^1(Q_T)$,
(ii) $T_k(u_\delta)$ converges to $T_k(u)$ weakly in $L^p(0,T,W_0^{s(\cdot),p}(\Omega))$,
(iii) $T_k(u_\delta)$ converges to $T_k(u)$ strongly in $L^p(0,T,W_0^{s(\cdot),p}(\Omega))$.

To achieve strong convergence of $T_k(u_\delta)$ to $T_k(u)$, Landes regularization in time is employed. To do this, let's consider the subsequence $(T_k(u))_\mu$ defined by:

$$(T_k(u))_\mu(z,t) := \mu\int_0^t e^{\mu(s-t)}T_k(u(z,s))\,\mathrm{d}s,$$

$(T_k(u))_\mu$ satisfies the following lemma, whose proof relies on standard arguments:

Lemma 3.10 *For all $\mu>0$, we have:*

(i) $|(T_k(u))_\mu(z,t)| \le k(1-e^{-\mu t}) \le k$;

(ii) $(T_k(u))_\mu(z,t) \in L^p(0,T,W_0^{s(\cdot),p}(\Omega)) \cap L^\infty(Q_T)$;

(iii) $\dfrac{\partial (T_k(u))_\mu(z,t)}{\partial t} = \mu(T_k(u) - (T_k(u))_\mu(z,t))$.

Proof. For (i), we have

$$\begin{aligned} |(T_k(u))_\mu(z,t)| &= \left|\mu \int_0^t e^{\mu(s-t)} T_k(u(z,s))\,\mathrm{d}s\right| \\ &\le \mu \int_0^t e^{\mu(s-t)} |T_k(u(z,s))|\,\mathrm{d}s. \end{aligned} \tag{3.44}$$

Since $|T_k(u(z,s)| \le k$, we have

$$\begin{aligned} |(T_k(u))_\mu(z,t)| &\le \mu \int_0^t e^{\mu(s-t)} k\,\mathrm{d}s \\ &\le k\left(1 - e^{-\mu t}\right). \end{aligned} \tag{3.45}$$

For (ii), let's first show that $(T_k(u))_\mu(z,t)$ belongs to $L^p(0,T;W_0)$. We have

$$||(T_k(u))_\mu(z,t)||_{L^p(0,T;W_0)} = \left(\int_0^T ||(T_k(u))_\mu(z,t)||^p_{W_0}\,\mathrm{d}t\right)^{\frac{1}{p}}. \tag{3.46}$$

Let's estimate $||(T_k(u))_\mu(z,t)||^p_{W_0}$:

$$\begin{aligned} ||(T_k(u))_\mu(z,t)||^p_{W_0} &= \int_{\Omega\times\Omega} \frac{|(T_k(u))_\mu(z,t) - (T_k(u))_\mu(\xi,t)|^p}{|z-\xi|^{N+s(z,\xi)p}}\,\mathrm{d}z\,\mathrm{d}\xi \\ &= \mu \int_0^t e^{\mu(y-t)} \left(\int_{\Omega\times\Omega} \frac{|(T_k(u))(z,y) - (T_k(u))(\xi,y)|^p}{|z-\xi|^{N+s(z,\xi)p}}\,\mathrm{d}z\,\mathrm{d}\xi\right)\mathrm{d}y \\ &\le \mu \int_0^t e^{\mu(y-t)} ||T_k(u)||^p_{W_0}\,\mathrm{d}y. \end{aligned} \tag{3.47}$$

Starting from relation (3.47) , using the integral Minkowski inequality, and (i), equation (3.46) becomes:

$$\begin{aligned} ||(T_k(u))_\mu(z,t)||_{L^p(0,T;W_0)} &\le \mu \int_0^T \left(\int_0^t e^{-\mu(t-y)}\,\mathrm{d}y\right)^{\dfrac{1}{p}} ||T_k(u)||^p_{W_0}\,\mathrm{d}t \\ &\le \int_0^T \left(1 - e^{-\mu(t)}\right)^{\dfrac{1}{p}} ||T_k(u)||^p_{W_0}\,\mathrm{d}t \\ &\le \int_0^T ||T_k(u)||^p_{W_0}\,\mathrm{d}t \end{aligned} \tag{3.48}$$

and we deduce that $(T_k(u))_\mu(z,t)$ belongs to $L^p(0,T;W_0)$.
Next, we show that $(T_k(u))_\mu(z,t)$ belongs to $L^\infty(Q_T)$. We have:

$$\begin{aligned} ||(T_k(u))_\mu||_{L^\infty(Q_T)} &= ess - sup_{t\in[0,T]} ||(T_k(u))_\mu||_{L^\infty(\Omega)} \\ &= ess - sup_{t\in[0,T]} \left\|\mu \int_0^t e^{\mu(s-t)} T_k(u(z,s))\,\mathrm{d}s\right\|_{L^\infty(\Omega)} \\ &\le \mu\, ess - sup_{t\in[0,T]} \int_0^t e^{\mu(s-t)} \|T_k(u)\|_{L^\infty(\Omega)}\,\mathrm{d}s \\ &\le \|T_k(u)\|_{L^\infty(\Omega)}. \end{aligned}$$

Hence, $(T_k(u))_\mu(z,t) \in L^\infty(Q_T)$.
To prove (iii), let us set $b(t) = t$ and $Q(s,t) = \mu e^{\mu(s-t)} T_k(u(z,s))$, we have:

$$\begin{aligned}
\frac{\partial (T_k(u))_\mu(z,t)}{\partial t} &= \frac{\partial}{\partial t}\left(\int_0^{b(t)} Q(s,t)\,\mathrm{d}s\right) \\
&= \int_0^{b(t)} \frac{\partial Q(s,t)}{\partial t}\,\mathrm{d}s + Q(b(t),t)\frac{db(t)}{dt} \qquad (3.49)\\
&= \mu\left(T_k(u(z,t)) - (T_k(u))_\mu(z,t)\right). \qquad (3.50)
\end{aligned}$$

This concludes the proof of the lemma. □

Following Lemma 3.10 and according to [18, Step 2 of the proof of Theorem 1.1], we have:

$$(T_k(u))_\mu \longrightarrow T_k(u) \text{ strongly in } L^p(0,T;W_0^{s(\cdot),p}(\Omega)),$$

and from the weak convergence of $T_k(u_n)$ to $T_k(u)$ in $L^p(0,T;W_0^{s(\cdot),p}(\Omega))$, we conclude:

$$T_k(u_n) \longrightarrow T_k(u) \text{ strongly in } L^p(0,T;W_0^{s(\cdot),p}(\Omega)).$$

Now, let $\varphi \in C^1(Q_T)$ with $\varphi = 0$ in $C\Omega \times (0,T)$, $\varphi(.,T) = 0$ in Ω and $S \in W^{1,\infty}(\mathbb{R})$ be such that $suppS' \subset [-M,M]$ for some $M > 0$. Using remark 2.5, (3.35) can be written as follows:

$$\begin{aligned}
&\int_0^t \int_\Omega \frac{\partial \tilde{u}_\delta}{\partial t} S'(u_\delta)\varphi\,\mathrm{d}z\,\mathrm{d}\tau + \int_0^T \int_\Omega |u_\delta|^{q-2}u_\delta S'(u_\delta)\varphi\,\mathrm{d}z\,\mathrm{d}t \\
+ &\int_0^T \int_{\Omega\times\Omega} \frac{|U_\delta^{z,\xi,t}|^{p-2}(U_\delta^{z,\xi,t})(\varphi(z,t)-\varphi(\xi,t))}{|z-\xi|^{N+s(z,\xi)p}} \times \frac{S'(u_\delta)(z,t)+S'(u_\delta)(\xi,t)}{2}\,\mathrm{d}z\,\mathrm{d}\xi\,\mathrm{d}t \\
+ &\int_0^T \int_{\Omega\times\Omega} \frac{|U_\delta^{z,\xi,t}|^{p-2}(U_\delta^{z,\xi,t})(S'(u_\delta)(z,t)-S'(u_v)(y,t))}{|z-\xi|^{N+s(z,\xi)p}} \times \frac{(\varphi(z,t)+\varphi(y,t))}{2}\,\mathrm{d}z\,\mathrm{d}\xi\,\mathrm{d}t \\
= &\int_0^T \int_\Omega f_\delta S'(u_\delta)\varphi\,\mathrm{d}z\,\mathrm{d}t,
\end{aligned}$$

where $U^{z,\xi,t} = u(z,t) - u(\xi,t)$. We denote

$$\begin{aligned}
I_1^\delta &= \int_0^t \int_\Omega \frac{\partial \tilde{u}_\delta}{\partial t} S'(u_\delta)\varphi\,\mathrm{d}z\,\mathrm{d}\tau, \\
I_2^\delta &= \int_0^T \int_\Omega |u_\delta|^{q-2}u_\delta S'(u_\delta)\varphi\,\mathrm{d}z\,\mathrm{d}t, \\
I_3^\delta &= \int_0^T \int_{\Omega\times\Omega} \frac{|U_\delta^{z,\xi,t}|^{p-2}(U_\delta^{z,\xi,t})(\varphi(z,t)-\varphi(\xi,t))}{|z-\xi|^{N+s(z,\xi)p}} \times \frac{S'(u_\delta)(z,t)+S'(u_\delta)(y,t)}{2}\,\mathrm{d}z\,\mathrm{d}\xi\,\mathrm{d}t, \\
I_4^\delta &= \int_0^T \int_{\Omega\times\Omega} \frac{|U_\delta^{z,\xi,t}|^{p-2}(U_\delta^{z,\xi,t})(S'(u_\delta)(z,t)-S'(u_v)(\xi,t))}{|z-\xi|^{N+s(z,\xi)p}} \times \frac{(\varphi(z,t)+\varphi(\xi,t))}{2}\,\mathrm{d}z\,\mathrm{d}\xi\,\mathrm{d}t, \\
I_5^\delta &= \int_0^T \int_\Omega f_\delta S'(u_\delta)\varphi\,\mathrm{d}z\,\mathrm{d}t.
\end{aligned}$$

The transition to the limit for the sequences I_1^δ, I_3^δ, I_4^δ, I_5^δ is carried out in the same way as in [18, Step 3 of the proof of Theorem 1.1]. Thus, we obtain:

$$\begin{aligned}
I_1^\delta &\longrightarrow \int_0^t\int_\Omega \frac{\partial\tilde{u}}{\partial t}S'(u)\varphi\,\mathrm{d}z\,\mathrm{d}\tau,\\
I_3^\delta &\longrightarrow \int_0^T\int_{\Omega\times\Omega}\frac{|U^{z,\xi,t}|^{p-2}(U^{z,\xi,t})(\varphi(z,t)-\varphi(\xi,t))}{|z-\xi|^{N+s(z,\xi)p}}\times\frac{S'(u)(z,t)+S'(u)(\xi,t)}{2}\,\mathrm{d}z\,\mathrm{d}\xi\,\mathrm{d}t,\\
I_4^\delta &\longrightarrow \int_0^T\int_{\Omega\times\Omega}\frac{|U^{z,\xi,t}|^{p-2}(U^{z,\xi,t})(S'(u)(z,t)-S'(u)(\xi,t))}{|z-\xi|^{N+s(z,\xi)p}}\times\frac{(\varphi(z,t)+\varphi(\xi,t))}{2}\,\mathrm{d}z\,\mathrm{d}\xi\,\mathrm{d}t,\\
I_5^\delta &\longrightarrow \int_0^T\int_\Omega fS'(u)\varphi\,\mathrm{d}z\,\mathrm{d}t.
\end{aligned}$$

Let's now focus on I_2^δ. Taking $T_k(u_\delta)$ as test function in (P_δ), we have:

$$\begin{aligned}
\int_0^t\int_\Omega\frac{\partial\tilde{u}_\delta}{\partial t}T_k(u_\delta)\,\mathrm{d}z\,\mathrm{d}\tau+\int_0^t\langle(-\Delta)_p^{s(z,\xi)}u_\delta,T_k(u_\delta)\rangle\,\mathrm{d}\tau\\
+\int_0^t\langle|u_\delta|^{q-2}u_\delta,T_k(u_\delta)\rangle\,\mathrm{d}\tau=\int_0^t\int_\Omega f_\delta T_k(u_\delta)\,\mathrm{d}z\,\mathrm{d}t. \quad (3.51)
\end{aligned}$$

Since $\int_0^t\langle(-\Delta)_p^{s(z,\xi)}u_\delta,T_k(u_\delta)\rangle\,\mathrm{d}\tau\geq 0$, we have:

$$\int_0^t\int_\Omega\frac{\partial\tilde{u}_\delta}{\partial t}T_k(u_\delta)\,\mathrm{d}z\,\mathrm{d}\tau+\int_0^t\langle|u_\delta|^{q-2}u_\delta,T_k(u_\delta)\rangle\,\mathrm{d}\tau\leq\int_0^t\int_\Omega f_\delta T_k(u_\delta)\,\mathrm{d}z\,\mathrm{d}t. \quad (3.52)$$

For the rest we need $\int_0^t\langle|u_\delta|^{q-2}u_\delta,T_k(u_\delta)\rangle\,\mathrm{d}\tau\leq C$. To check this, we will go back to the semi-discretized form. Thus, (3.52) is equivalent to:

$$\begin{aligned}
\int_\Omega u^nT_k(u^n)\,\mathrm{d}z+\delta\int_\Omega|u^n|^{q-2}u^nT_k(u^n) &\leq \delta\int_\Omega f^nT_k(u^n)+\int_\Omega u^{n-1}T_k(u^n)\\
&\leq k\big(\delta\int_\Omega|f^n|+\int_\Omega|u^{n-1}|\big), \quad (3.53)
\end{aligned}$$

and since u^n and $T_k(u^n)$ have the same sign, we obtain

$$\delta\int_\Omega|u^n|^{q-2}u^nT_k(u^n)\leq Ck.$$

Hence, $\int_0^t\langle|u_\delta|^{q-2}u_\delta,T_k(u_\delta)\rangle\,\mathrm{d}\tau\leq C$.
In addition,

$$\begin{aligned}
\int_{Q_T}|u_\delta|^{q-1}\,\mathrm{d}z\,\mathrm{d}t &= \int_{\{|u_\delta|<k\}}|u_\delta|^{q-1}\,\mathrm{d}z\,\mathrm{d}t+\int_{\{|u_\delta|\geq k\}}|u_\delta|^{q-1}\,\mathrm{d}z\,\mathrm{d}t\\
&\leq k^{q-2}\int_{\{|u_\delta|<k\}}|u_\delta|\,\mathrm{d}z\,\mathrm{d}t+\frac{1}{k}\int_{\{|u_\delta|\geq k\}}|u_\delta|^{q-1}|T_k(u_\delta)|\,\mathrm{d}z\,\mathrm{d}t\\
&\leq C. \quad (3.54)
\end{aligned}$$

Since u_δ converges to u in Q_T,we have

$$u_\delta\rightharpoonup u \text{ weakly in } L^{q-1}(Q_T).$$

Now, we take $T_k(u_\varrho - u_\eta)$ as a test function in (P_δ) we have:

$$\begin{aligned}\int_0^T \langle (\tilde{u}_\varrho - \tilde{u}_\eta)_t, T_k(u_\varrho - u_\eta)\rangle \,\mathrm{d}t + \int_0^t \langle (-\Delta)_p^{s(z,\xi)} u_\varrho - (-\Delta)_p^{s(z,\xi)} u_\eta, T_k u_\varrho - u_\eta)\rangle \,\mathrm{d}t \\ + \int_0^t \langle |u_\varrho|^{q-2}u_\varrho - |u_\eta|^{q-2}u_\eta, T_k(u_\varrho - u_\eta)\rangle \,\mathrm{d}t \\ = \int_\Omega \int_0^t (f_\varrho - f_\eta) T_k(u_\varrho - u_\eta)\,\mathrm{d}z\,\mathrm{d}t. \qquad (3.55)\end{aligned}$$

Let's consider u^n and u^m as the semi-discretized forms of u_ϱ and u_η respectively. As before, we obtain:

$$\begin{aligned}\int_\Omega \big(|u^n|^{q-2}u^n - |u^m|^{q-2}u^m\big) T_k(u^n - u^m)\,\mathrm{d}z\,\mathrm{d}t \le k\,\Big(\int_\Omega |f^n - f^m|\,\mathrm{d}z\,\mathrm{d}t \\ + \int_\Omega |u^{n-1} - u^{m-1}|\,\mathrm{d}z\Big), \qquad (3.56)\end{aligned}$$

which gives

$$\int_\Omega \big(|u^n|^{q-2}u^n - |u^m|^{q-2}u^m\big) T_k(u^n - u^m)\,\mathrm{d}z\,\mathrm{d}t \longrightarrow 0 \text{ when } n,m \longrightarrow \infty. \qquad (3.57)$$

In other words,

$$\int_\Omega \big(|u_\varrho|^{q-2}u_\varrho - |u_\eta|^{q-2}u_\eta\big) T_k(u_\varrho - u_\eta)\,\mathrm{d}z\,\mathrm{d}t \longrightarrow 0. \qquad (3.58)$$

Furthermore,

$$\begin{aligned}\int_{Q_T} |u_\varrho - u_\eta|^{q-1}\,\mathrm{d}z\,\mathrm{d}t &= \int_{\{|u_\varrho - u_\eta|<k\}} |u_\varrho - u_\eta|^{q-1}\,\mathrm{d}z\,\mathrm{d}t + \int_{\{u_\varrho - u_\eta \ge k\}} |u_\varrho - u_\eta|^{q-1}\,\mathrm{d}z\,\mathrm{d}t \\ &\le k^{q-2}\int_{\{|u_\varrho - u_\eta|<k\}} |u_\varrho - u_\eta|\,\mathrm{d}z\,\mathrm{d}t \\ &\quad + \frac{1}{k}\int_{\{|u_\varrho - u_\eta|\ge k\}} |u_\varrho - u_\eta|^{q-1}|T_k(u_\varrho - u_\eta)|\,\mathrm{d}z\,\mathrm{d}t. \qquad (3.59)\end{aligned}$$

Using Lemma 2.4, we have

$$\begin{aligned}\frac{1}{k}\int_{\{|u_\varrho - u_\eta|\ge k\}} |u_\varrho - u_\eta|^{q-1}|T_k(u_\varrho - u_\eta)|\,\mathrm{d}z\,\mathrm{d}t \;\le\; \frac{1}{k}\int_{\{|u_\varrho - u_\eta|\ge k\}} \big(|u_\varrho|^{q-2}u_\varrho - |u_\eta|^{q-2}u_\eta\big) \\ \times \operatorname{sign}_0(u_\varrho - u_\eta)|T_k(u_\varrho - u_\eta)|\,\mathrm{d}z\,\mathrm{d}t \\ \longrightarrow 0 \text{ as } \varrho, \eta \longrightarrow \infty.\end{aligned}$$

Since $\{u_\delta\}$ is a Cauchy sequence in $C([0,T];L^1(\Omega))$, according to the previous equation, we have

$$\int_{Q_T} |u_\varrho - u_\eta|^{q-1}\,\mathrm{d}z\,\mathrm{d}t \longrightarrow 0 \text{ as } \varrho, \eta \longrightarrow \infty.$$

Therefore, $\{u_\delta\}$ is a Cauchy sequence in $L^{q-1}(Q_T)$. Consequently, from the convergence of u_δ in $L^{q-1}(Q_T)$, it follows that

$$|u_\delta|^{q-2}u_\delta \longrightarrow |u|^{q-2}u \text{ strongly in } L^1(Q_T)$$

The convergence of I_2^δ is deduced from the previous strong convergence result and the starred weak convergence of $S(u_\delta)$ to $S(u)$.
From all the preceding, taking the limit yields:

$$\lim_{\delta\longrightarrow\infty} I_2^\delta = \int_0^T \int_\Omega |u|^{q-2} u S'(u)\varphi \,\mathrm{d}z\,\mathrm{d}t.$$

By putting together the convergence results of $I_1^\delta - I_5^\delta$, we obtain

$$\begin{aligned}&\int_0^T \int_{\Omega\times\Omega} \frac{|u(z,t)-u(\xi,t)|^{p-2}(u(z,t)-u(\xi,t))(S'(u)\varphi(z,t)-S'(u)\varphi(\xi,t))}{|z-\xi|^{N+s(z,\xi)p}}\,\mathrm{d}z\,\mathrm{d}\xi\,\mathrm{d}t\\ &+\int_{Q_T}\frac{\partial S(u)}{\partial t}\varphi(z,t)\,\mathrm{d}z\,\mathrm{d}t+\int_{Q_T}|u|^{q-2}uS'(u)\varphi(z,t)\,\mathrm{d}z\,\mathrm{d}t=\int_{Q_T} fS'(u)\varphi(z,t)\,\mathrm{d}z\,\mathrm{d}t.\end{aligned} \tag{3.60}$$

Then, in the sense of Definition 3.1, problem (P_T) admits a renormalized solution.

3.2 Uniqueness of renormalized solution

Let's establish the uniqueness of the renormalized solution for the problem (P_T). Consider two renormalized solutions, u and v, for the problem. Note that $S_\epsilon \in W^{1,\infty}(\mathbb{R})$ with $\operatorname{supp} S'_\epsilon \subset [-\epsilon-1,\epsilon+1]$.
For the renormalized solution u, we take $S=S_\epsilon$ and $\varphi(z,t)=S'_\epsilon(v)T_\epsilon(S_\epsilon(u)-S_\epsilon(v))$ to obtain

$$\begin{aligned}&\int_{Q_T}\frac{\partial u}{\partial t}S'_\epsilon(u)S'_\epsilon(v)T_\epsilon(S_\epsilon(u)-S_\epsilon(v))\,\mathrm{d}z\,\mathrm{d}t\\ &+\int_0^T\langle(-\Delta)_p^{s(z,\xi)}u-(-\Delta)_p^{s(z,\xi)}u, S'_\epsilon(u)S'_\epsilon(v)T_\epsilon(S_\epsilon(u)-S_\epsilon(v))\rangle\,\mathrm{d}t\\ &+\int_{Q_T}|u|^{q-2}uS'_\epsilon(u)S'_\epsilon(v)T_\epsilon(S_\epsilon(u)-S_\epsilon(v))\,\mathrm{d}z\,\mathrm{d}t\\ &=\int_{Q_T} fS'_\epsilon(u)S'_\epsilon(v)T_\epsilon(S_\epsilon(u)-S_\epsilon(v))\,\mathrm{d}z\,\mathrm{d}t.\end{aligned}\tag{3.61}$$

Similarly, for the renormalized solution v, we take $S=S_\epsilon$ and $\varphi(z,t)=S'_\epsilon(u)T_\epsilon(S_\epsilon(u)-S_\epsilon(v))$ and we also have

$$\begin{aligned}&\int_{Q_T}\frac{\partial v}{\partial t}S'_\epsilon(u)S'_\epsilon(v)T_\epsilon(S_\epsilon(u)-S_\epsilon(v))\,\mathrm{d}z\,\mathrm{d}t\\ &+\int_0^T\langle(-\Delta)_p^{s(z,\xi)}u, S'_\epsilon(u)S'_\epsilon(v)T_\epsilon(S_\epsilon(u)-S_\epsilon(v))\rangle\,\mathrm{d}t\\ &+\int_{Q_T}|v|^{q-2}vS'_\epsilon(u)S'_\epsilon(v)T_\epsilon(S_\epsilon(u)-S_\epsilon(v))\,\mathrm{d}z\,\mathrm{d}t\\ &=\int_{Q_T} fS'_\epsilon(u)S'_\epsilon(v)T_\epsilon(S_\epsilon(u)-S_\epsilon(v))\,\mathrm{d}z\,\mathrm{d}t.\end{aligned}\tag{3.62}$$

We subtract the above equality to have

$$\begin{aligned}&\int_{Q_T}(\frac{\partial u}{\partial t}-\frac{\partial v}{\partial t})S'_\epsilon(u)S'_\epsilon(v)T_\epsilon(S_\epsilon(u)-S_\epsilon(v))\,\mathrm{d}z\,\mathrm{d}t\\&+\int_0^T\langle(-\Delta)_p^{s(z,\xi)}u-(-\Delta)_p^{s(z,\xi)}v,S'_\epsilon(u)S'_\epsilon(v)T_\epsilon(S_\epsilon(u)-S_\epsilon(v))\rangle\,\mathrm{d}t\\&+\int_{Q_T}(|u|^{q-2}u-|v|^{q-2}v)S'_\epsilon(u)S'_\epsilon(v)T_\epsilon(S_\epsilon(u)-S_\epsilon(v))\,\mathrm{d}z\,\mathrm{d}t\;=\;0.\end{aligned}\tag{3.63}$$

We will show that the final two terms on the left-hand side of the previous equation are both positive. This amounts to showing that the term $T_\epsilon(S_\epsilon(u)-S_\epsilon(v))$ has the same sign as $u-v$ over the support of S_ϵ; which is obtained by using the definition of S_ϵ. Therefore, employing monotonicity, we infer that all these terms are positive. The equation transforms into

$$\int_{Q_T}(\frac{\partial u}{\partial t}-\frac{\partial v}{\partial t})S'_\epsilon(u)S'_\epsilon(v)T_\epsilon(S_\epsilon(u)-S_\epsilon(v))\,\mathrm{d}z\,\mathrm{d}t\le 0.\tag{3.64}$$

Since S'_ϵ is bounded and positive, the expressions $\int_{Q_T}(\frac{\partial u}{\partial t}-\frac{\partial v}{\partial t})S'_\epsilon(u)S'_\epsilon(v)T_\epsilon(S_\epsilon(u)-S_\epsilon(v))\,\mathrm{d}z\,\mathrm{d}t$ and $\int_{Q_T}(\frac{\partial u}{\partial t}-\frac{\partial v}{\partial t})T_\epsilon(S_\epsilon(u)-S_\epsilon(v))\,\mathrm{d}z\,\mathrm{d}t$ have the same sign.
For the remainder, we reason according to the support of S_ϵ. Let us take arbitrary $\tau\in(0,T)$, with $u(.,0)=v(.,0)$.

$$\begin{aligned}&\int_{\{(z,\tau)\in\Omega\times(0,t);u(z,\tau),v(z,\tau)\in(-\epsilon,\epsilon)\}}(\frac{\partial u}{\partial\tau}-\frac{\partial v}{\partial\tau})T_\epsilon(S_\epsilon(u)-S_\epsilon(v))\,\mathrm{d}z\,\mathrm{d}\tau\\&=\int_{\{(z,\tau)\in\Omega\times(0,t);u(z,\tau),v(z,\tau)\in(-\epsilon,\epsilon)\}}(\frac{\partial u}{\partial\tau}-\frac{\partial v}{\partial\tau})(u-v)\,\mathrm{d}z\,\mathrm{d}\tau\\&=\frac{1}{2}\int_{\{z\in\Omega;u(z),v(z)\in(-\epsilon,\epsilon)\}}(u(z,t)-v(z,t))^2\,\mathrm{d}z\le 0,\end{aligned}\tag{3.65}$$

which gives $u=v$.
After calculation, outside $(-\epsilon,\epsilon)$, the expression of $\int_{Q_T}(\frac{\partial u}{\partial t}-\frac{\partial v}{\partial t})T_\epsilon(S_\epsilon(u)-S_\epsilon(v))\,\mathrm{d}z\,\mathrm{d}\tau$ reduces to

$$\pm\epsilon\int_{Q_T}(\frac{\partial u}{\partial t}-\frac{\partial v}{\partial t})\,\mathrm{d}z\,\mathrm{d}\tau=\pm\epsilon\int_{\{z\in\Omega,u(z),v(z)\in(-\epsilon-1,-\epsilon)\cup(\epsilon,\epsilon+1)\}}(u(z,t)-v(z,t))\,\mathrm{d}z.\tag{3.66}$$

with $\pm\epsilon(u(z,t)-v(z,t))\ge 0$. Hence, from (3.64) it follows that $u(z,t)=v(z,t)$.

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References

[1] D. Applebaum, *Processes and Stochastic Calculus*, Camb. Stud. Adv. Math. 116, Second Ed., Cambridge University Press, Cambridge, 2009.

[2] R. Cont and P. Tankov, *Financial Modelling with jump processes*, Chapman and Hall/CRC Financial Mathematics Series, Boca Raton, FL, 2004.

[3] R. J. DiPerna, P.-L. Lions, *On the Cauchy problem for Boltzmann equations: Global existence and weak stability,* Annals of Mathematics **130** (2) (1989), pp. 321–366.

[4] J. Giacomoni, S. Tiwari, *Existence and global behavior of solutions to fractional p-Laplacian parabolic problems*, Electronic Journal of Differential Equations **2018** (2018), no. 44, pp. 1–20. `http://ejde.math.txstate.edu` or `http://ejde.math.unt.edu`.

[5] T. C. Halsey, *Electrorheological fluids*, Science **258** (1992), pp. 761–766.

[6] M. Hsini, K. Kefi and N. Irzi, *On a fractional problem with variable exponent*, Proceedings of the Romanian Academy, Series A, Volume 21, Number 2/2020, pp. 105–114.

[7] S. Korbéogo , F. D. Y. Zongo and A. Ouédraogo, *On a fractional parabolic problems with variable exponent*, Linear and Nonlinear Analysis Journal **9** (2023) no. 1, pp. 65–81.

[8] S. Korbéogo , F. D. Y. Zongo and A. Ouédraogo, *On a variable-order fractional parabolic problems*, Proceeding of the 2nd International Conference on Nonlinear Dynamics and Applications (ICNDA 2024), Vol. 2: Complex Systems, Fractals and Nonlinear Flows (2024), pp. 204–219, doi:10.1007/978-3-031-69134-8-14.

[9] S. Liang, G. M. Bisci, B. Zhang *Sign-changing solutions for Kirchhoff-type problems involving variable-order fractional Laplacian and critical exponents*, Nonlinear Analysis: Modelling and Control **27** (2022) no. 3, pp. 556–575, doi:10.15388/namc.2022.27.26575.

[10] C. F. Lorenzo, T. T. Hartley, *Initialized fractional calculus,* International Journal of Applied Mathematics **3** (2000), pp. 249–265.

[11] C. F. Lorenzo, T. T. Hartley, *Variable order and distributed order fractional operators*, Nonlinear Dynamics **29** (2002), pp. 57–98.

[12] S. Ouaro, A. Ouédraogo, *Nonlinear parabolic problems with variable exponent and L^1 data*, Electronic Journal of Differential Equations **2017** (2017) no. 32, pp. 1–32.

[13] S. Ouaro, N. Rabo and U. Traoré, *Numerical analysis of nonlinear parabolic problems with variable exponent and L^1 data*, CUBO, A Mathematical Journal **24** (2022) no. 02, pp. 187–209, doi:10.56754/0719-0646.2402.0187.

[14] A. Prignet, *Remarks on existence and uniqueness of solutions of elliptic problems with right hand side measures*, Rendiconti di Matematica e delle sue Applicazioni (7) **15** (3) (1995), pp. 321–337.

[15] A. Sabri, A. Jamea, H. T. Alaoui, *Existence of entropy solutions to nonlinear degenerate parabolic problems with variable exponent and L^1–data*, Communications in Mathematics **28** (2020), no. 1, pp. 67–88, doi:10.2478/cm-2020-0006.

[16] A. Sabri, *Weak Solution for Nonlinear Fractional $p(\cdot)-$Laplacian Problem with Variable Order via Rothe's Time-Discretization Method*, Mathematical Modelling and Analysis **27** (2022), Issue 4, pp. 533–546, doi:10.3846/mma.2022.15740.

[17] J. Serrin, *Pathological solutions of elliptic differential equations,* Annali della Scuola Normale Superiore di Pisa-Classe di Scienze (3) **18** (1964), pp. 385–387.

[18] K. Teng, C. Zhang, Shulin Zhou, *Renormalized and entropy solutions for the fractional p-Laplacian evolution equations*, Journal of Evolution Equations **19** (2019), pp. 599–584.

[19] L. Wang and B. Zhang *Infinitely many solutions for Kirchhoff-type variable-order fractional Laplacian problems involving variable exponents*, Applicable Analysis **100** (2021), doi:10.1080/00036811.2019.1688790.

[20] M. Xiang, B. Zhang, D. Yang, *Multiplicity results for variable-order fractional Laplacian equations with variable growth*, Nonlinear Analysis **178** (2019), pp. 190–204, doi:10.1016/j.na.2018.07.016.

[21] C. Zhang and X. Zhang, *Renormalized solutions for the fractional $p(x)$-Laplacian equation with L^1 data*, arXiv:1708.04481v1 (2017).

[22] F. D. Y. Zongo, S. Korbéogo and A. Ouédraogo, *On some fractional magnetic problems*, Pan-American Journal of Mathematics **2** (2023) no. 17, doi:10.28919/cpr-pajm/2-17.