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ALMOST PERIODIC SOLUTIONS OF STOCHASTIC SINGULAR DIFFERENCE EQUATIONS

PAUL H. BEZANDRY *
Department of Mathematics, Howard University, Washington, DC 20059

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Abstract. In this paper we study and obtain the existence of almost periodic solutions of the following class of stochastic singular difference equations of the form:

$$AX(k+1) + BX(k) = f(k, X(k))\,\xi(k+1),\ k \in \mathbb{Z}\,,$$

where A and B are singular $N \times N$ random matrices $(\det A = \det B = 0)$, $f : \mathbb{Z} \times L^1(\Omega; \mathbb{R}^N) \to L^1(\Omega; \mathbb{R}^N)$ is almost periodic in the first variable uniformly in the second one, and $\xi = \{\xi(k), k \in \mathbb{Z}\}$ is an almost periodic random sequence and stochastically independent of f. These results are, subsequently, applied to find the existence of almost periodic solutions of the second-order stochastic singular difference equations.

Keywords: almost periodic sequence, stochastic difference equation, singular random matrix.

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1 Introduction

The study of almost periodicity which generalizes the notion of periodicity is an area of interest in its own right and has sundry applications in fields like Physics. For a study of almost periodic sequences we refer the reader to ([2], [3], [4], [5], [7], [8]) and references therein. Almost periodicity is also of importance in the study of stochastic processes. On the other hand, a first-order difference equation may be used to describe phenomena that evolve in discrete time when the size of each generation is a function of the preceding one. However, the real world often refuses to conform to such a neat mathematical representation. Unpredictable effects may be included in the form of a sequence of random variables and the result is a stochastic difference equation.

*e-mail address: pbezandry@howard.edu

In Bezandry *et al.* [3], the notion of mean almost periodicity was introduced and used to study the existence and uniqueness of mean almost periodic solutions to the stochastic Beverton-Holt equation.

In this paper, we study the existence of almost periodic solutions to the following class of systems of stochastic singular difference equations of the form:

$$AX(k+1) + BX(k) = f(k, X(k))\,\xi(k+1),\ k \in Z, \tag{1.1}$$

on $\mathbb{R}^N$, where A and B are singular $N \times N$ random matrices $(\det A = \det B = 0)$, $f : \mathbb{Z} \times L^1(\Omega; \mathbb{R}^N) \to L^1(\Omega; \mathbb{R}^N)$ is a function to be specified later. We assume that $\xi = \{\xi(k), k \in \mathbb{Z}\}$ is an almost periodic random sequence in $\mathbb{R}$ and stochastically independent of f, and $\mathbf{E}\,|\xi(k)| < \infty$ for all $k \in \mathbb{Z}$. We also assume that the random matrices A and B are stochastically independent and independent of $X(0)$ and $\xi(0)$. This assumption together with eq. (1.1) implies that (A, B) is stochastically independent of the sequences $\{X(k)\}_{k\in\mathbb{Z}}$ and $\{\xi(k)\}_{k\in\mathbb{Z}}$.

To the best of our knowledge, the existence of almost periodic solutions to stochastic singular difference equation in the form (1.1) is an untreated question which constitutes the main motivation of this paper. Recently, there has been an increasing interest in extending certain classical results to stochastic cases. This is due to the fact that almost all problems in a real life situation to which mathematical models are applicable are basically stochastic rather than deterministic. The deterministic case of eq. (1.1) has been discussed in Diagana-Pennequin [6].

The paper is organized as follows. In Section 2, we recall a basic theory of mean almost periodic random sequences on $\mathbb{Z}$. In Section 3, we review some results for the existence of mean almost periodic solutions to some systems of first-order stochastic linear nonsingular difference equations. In Section 4, we apply the techniques developed in Section 2 and use the results reviewed in Section 3 to find some sufficient conditions for the existence of mean almost periodic solutions to some class of systems of stochastic singular difference equations. Section 5 is devoted to application of existence results obtained in Section 4 to find mean almost periodic solutions of some second-order (and higher-order) stochastic singular difference equation. In the final section, we draw the conclusions and indicate the future directions of the work.

2 Preliminaries

In this section we recall the basic properties of almost periodic random sequences. To facilitate our task, we first introduce the notations needed in the sequel.

Let $(\mathbb{R}^N, \|\cdot\|)$ be the N-dimensional Euclidean space and let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space. Throughout the rest of the paper, $\mathbb{Z}$ denotes the set of all integers. Define $L^1(\Omega; \mathbb{R}^N)$ to be the space of all $\mathbb{R}^N$-valued random variables V such that

$$\mathbf{E}\,\|V\| := \Big(\int_\Omega \|V(\omega)\| d\mathbf{P}(\omega)\Big) < \infty. \tag{2.1}$$

It is then routine to check that $L^1(\Omega; \mathbb{R}^N)$ is a Banach space when it is equipped with its natural norm $\|\cdot\|_1$ defined by $\|V\|_1 := \mathbf{E}\,\|V\|$ for each $V \in L^1(\Omega, \mathbb{R}^N)$.

Let $X = \{X(k)\}_{k\in\mathbb{Z}}$ be a sequence of $\mathbb{R}^N$-valued random variables satisfying $\mathbf{E}\,\|X(k)\| < \infty$ for each $k \in \mathbb{Z}$. Thus, interchangeably we can, and do, speak of such a sequence as a function, which goes from $\mathbb{Z}$ into $L^1(\Omega; \mathbb{R}^N)$.

Definition 2.1 *$X = \{X(k)\}_{k\in\mathbb{Z}}$ is bounded if there exists an $M > 0$ such that $\mathbf{E}\,\|X(k)\| \le M$ for all $k \in \mathbb{Z}$.*

Let $UB(\mathbb{Z};\ L^1(\Omega;\mathbb{R}^N))$ denote the collection of all uniformly bounded $L^1(\Omega;\mathbb{R}^N)$-valued random sequences $X = \{X(k)\}_{k\in\mathbb{Z}}$. It is then easy to check that the space $UB(\mathbb{Z};\ L^1(\Omega;\mathbb{R}^N))$ is a Banach space when it is equipped with the norm:

$$\|X\|_\infty = \sup_{k\in\mathbb{Z}} \mathbf{E}\,\|X(k)\|.$$

This setting requires the following preliminary definitions.

Definition 2.2 *A $L^1(\Omega;\mathbb{R}^N)$-valued random sequence $X = \{X(k)\}_{k\in\mathbb{Z}}$ is said to be Bohr almost periodic in mean if for each $\varepsilon > 0$ there exists $N_0(\varepsilon) > 0$ such that among any N_0 consecutive integers there exists at least an integer $p > 0$ for which*

$$\mathbf{E}\,\|X(k+p) - X(k)\| < \varepsilon,\ \forall\, k \in \mathbb{Z}.$$

An integer $p > 0$ with the above-mentioned property is called an ε-almost period for X. The collection of all $\mathbb{R}^N$-valued random sequences $X = \{X(k)\}_{k\in\mathbb{Z}}$ which are Bohr almost periodic in mean is then denoted by $AP(\mathbb{Z}; L^1(\Omega;\mathbb{R}^N))$.

Definition 2.3 *A $L^1(\Omega;\mathbb{R}^N)$-valued random sequence $X = \{X(k)\}_{k\in\mathbb{Z}}$ is said to be almost periodic in probability if for each $\varepsilon > 0$, and $\eta > 0$ there exists $N_0(\varepsilon,\eta) > 0$ such that among any N_0 consecutive integers there exists at least an integer $p > 0$ for which*

$$\mathbf{P}\big\{\omega \in \Omega :\ \|X(\omega,k+p) - X(\omega,k)\| > \varepsilon\big\} < \eta,\ \forall\, k \in \mathbb{Z}.$$

Theorem 2.4 *If X is almost periodic in mean, then it is almost periodic in probability and there also exists a constant $M > 0$ such that $\mathbf{E}\|X(k)\| \le M$ for all $k \in \mathbb{Z}$. Conversely, if X is almost periodic in probability and the sequence $\big\{\|X(k)\|,\ k \in \mathbb{Z}\big\}$ is uniformly integrable, then X is almost periodic in mean.*

Proof. The proof is straightforward and omitted. □

Definition 2.5 *A function $F : \mathbb{Z} \times L^1(\Omega;\mathbb{R}^N) \mapsto L^1(\Omega;\mathbb{R}^N)$, $(k,U) \mapsto F(k,U)$ is said to be almost periodic in mean in $k \in \mathbb{Z}$ uniformly in $U \in L^1(\Omega;\mathbb{R}^N)$ if for any $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that among any $n_0(\varepsilon)$ consecutive integers there exists at least an integer p with the following property*

$$\mathbf{E}\,\|F(k+p,U) - F(k,U)\| < \varepsilon$$

for each random variable $U \in L^1(\Omega;\mathbb{R}^N)$ and $k \in \mathbb{Z}$.

We now state the following composition result.

Theorem 2.6 [2] Let $F : \mathbb{Z} \times L^1(\Omega; \mathbb{R}^N) \mapsto L^1(\Omega; \mathbb{R}^N)$, $(k, U) \mapsto F(k, U)$ be almost periodic in mean in $k \in \mathbb{Z}$ uniformly in $U \in L^1(\Omega; \mathbb{R}^N)$. If in addition, F is Lipschitz in $U \in L^1(\Omega; \mathbb{R}^N)$ (that is, there exists $L > 0$ such that

$$\mathbf{E}\,\|F(k,U) - F(k,V)\| \leq L\ \mathbf{E}\,\|U - V\| \ \ \forall\, U, V\ \in L^1(\Omega; \mathbb{R}^N),\ k \in \mathbb{Z})$$

then for any almost periodic random sequence $X = \{X(k)\}_{k \in Z}$, then the $L^1(\Omega; \mathbb{R}^N)$-valued random sequence $Y(k) = F(k, X(k))$ is almost periodic in mean.

The following results will play a key role in the study of almost periodic solutions of linear and nonlinear stochastic difference equations.

Theorem 2.7 *Let A be an $N \times N$ random matrix and $X : \mathbb{Z} \to L^1(\Omega; \mathbb{R}^N)$ be almost periodic random sequence. Assume that A and X are independent. Then $\{AX(k), k \in \mathbb{Z}\}$ is almost periodic in mean.*

Remark 2.8 *If A is a random matrix, then $\mathbf{E}\,\|A\| < \infty$. To see this, note that since A is bounded almost surely, it implies that it is essentially bounded. That is, there exists an $M > 0$ such that $\mathbf{P}\,(\|A\| > M) = 0$. The latter implies that $\mathbf{P}(\|A\| > x) = 0$ for all $x > M$. We then have*

$$\begin{aligned}
\mathbf{E}\,\|A\| &= \int_0^\infty \mathbf{P}\,(\|A\| > x)\ dx \\
&= \int_0^M \mathbf{P}\,(\|A\| > x)\ dx + \int_M^\infty \mathbf{P}\,(\|A\| > x)\ dx \\
&= \int_0^M \mathbf{P}\,(\|A\| > x)\ dx < \infty.
\end{aligned}$$

Proof. (Theorem 2.7) First, note that A is bounded almost surely. It follows from Remark 2.8 that $\mathbf{E}\,\|A\| < \infty$. Now, fix $\varepsilon > 0$. By assumption, we can then choose an $N_0(\varepsilon) > 0$ such that among any N_0 consecutive integers there exists at least an integer $p > 0$ for which

$$\mathbf{E}\,\|X(k+p) - X(k)\| < \frac{\varepsilon}{\mathbf{E}\,\|A\|}\ .$$

Using the independence, we have

$$\mathbf{E}\,\|AX(k+p) - AX(k)\| \leq \mathbf{E}\,\|A\|\ \mathbf{E}\,\|X(k+p) - X(k)\| < \varepsilon\,,$$

for each $k \in \mathbb{Z}$. Therefore AX is almost periodic in mean. □

Theorem 2.9 *Let $\alpha : \mathbb{Z} \to \mathbb{C}$ be an almost periodic deterministic sequence and $X : \mathbb{Z} \to L^1(\Omega; \mathbb{R}^N)$ be a mean almost periodic random sequence. Then $\alpha X : \mathbb{Z} \to L^1(\Omega; \mathbb{R}^N)$ defined by $(\alpha X)(k) = \alpha(k)X(k),\ k \in \mathbb{Z}$ is almost periodic in mean.*

Proof. Fix $\varepsilon > 0$. By assumption, we can then choose an $N_0(\varepsilon) > 0$ such that among any N_0 consecutive integers there exists at least an integer $p > 0$ for which

(i) $\mathbf{E}\,\|X(k+p) - X(k)\| < \frac{\varepsilon}{2M_1}$,

(ii) $|\alpha(k+p)-\alpha(k)| < \frac{\varepsilon}{2M_2}$,

for each $k \in \mathbb{Z}$ and where $M_1 = \sup_{n\in\mathbb{Z}} |\alpha(n)|$ and $M_2 = \sup_{n\in\mathbb{Z}} \mathbf{E}\,\|X(n)\|$.

We then have

$$\begin{aligned}
&\mathbf{E}\,\|\alpha(k+p)X(k+p)-\alpha(k)X(k)\| \\
&\leq |\alpha(k+p)|\,\mathbf{E}\,\|X(k+p)-X(k)\| + |\alpha(k+p)-\alpha(k)|\,\mathbf{E}\,\|X(k)\| \\
&\leq M_1\,\mathbf{E}\,\|X(k+p)-X(k)\| + M_2|\alpha(k+p)-\alpha(k)| < \varepsilon\,.
\end{aligned}$$

Therefore αX is almost periodic in mean. □

More generally, we have

Theorem 2.10 *Let $X : \mathbb{Z} \to L^1(\Omega;\mathbb{C})$ be mean almost periodic random sequence and $Y : \mathbb{Z} \to L^1(\Omega;\mathbb{R}^N)$ be mean almost periodic random sequence. Assume that X and Y are stochastically independent. Then $XY : \mathbb{Z} \to L^1(\Omega;\mathbb{R}^N)$ defined by $(XY)(k) = X(k)Y(k),\ k \in \mathbb{Z}$ is almost periodic in mean.*

Proof. Fix $\varepsilon > 0$. By assumption, we can then choose an $N_0(\varepsilon) > 0$ such that among any N_0 consecutive integers there exists at least an integer $p > 0$ for which

(i) $\mathbf{E}\,\|Y(k+p)-Y(k)\| < \frac{\varepsilon}{2M_1}$,

(ii) $\mathbf{E}\,\|X(k+p)-X(k)\| < \frac{\varepsilon}{2M_2}$,

for each $k \in \mathbb{Z}$ and where $M_1 = \sup_{n\in\mathbb{Z}} \mathbf{E}\,\|X(n)\|$ and $M_2 = \sup_{n\in\mathbb{Z}} \mathbf{E}\,\|Y(n)|$.

Using that fact that X and Y are stochastically independent,we then have

$$\begin{aligned}
&\mathbf{E}\,\|X(k+p)Y(k+p)-X(k)Y(k)\| \\
&\leq \mathbf{E}\,\|X(k+p)\|\,\mathbf{E}\,\|Y(k+p)-Y(k)\| + \mathbf{E}\,\|X(k+p)-X(k)\|\,\mathbf{E}\,\|Y(k)\| \\
&\leq M_1\,\mathbf{E}\,\|Y(k+p)-Y(k)\| + M_2\,\mathbf{E}\,\|X(k+p)-X(k)\| < \varepsilon\,.
\end{aligned}$$

Therefore, XY is almost periodic in mean. □

Theorem 2.11 *Let $b : \mathbb{Z}_+ \to L^1(\Omega;\mathbb{C})$ be a summable random sequence, i.e $\sum_{l=0}^{\infty} \mathbf{E}\,|b(l)| < \infty$. Then for any almost periodic random sequence $X : \mathbb{Z} \to L^1(\Omega;\mathbb{R}^N)$ and independent of b, the random sequence $W(\cdot)$ defined by*

$$W(k) = \sum_{l=-\infty}^{k} b(k-l)X(l),\ \ k \in \mathbb{Z}$$

is also almost periodic in mean.

Proof. Fix $\varepsilon > 0$. By assumption, we can then choose an $N_0(\varepsilon) > 0$ such that among any N_0 consecutive integers there exists at least an integer $p > 0$ for which

$$\mathbf{E}\,\|X(k+p)-X(k)\| < \varepsilon/M$$

with $M = \sum_{l=0}^{\infty} \mathbf{E}\,|b(l)| < \infty$.

Define $W(k) = \sum_{l=-\infty}^{k} b(k-l)X(l)$. Using the independence of b and X, the summability of b, and the periodicity of X, we then have

$$\begin{aligned}
\mathbf{E}\,\|W(k+p) - W(k)\| &= \mathbf{E}\Big\| \sum_{l=-\infty}^{k+p} b(k+p-l)X(l) - \sum_{l=-\infty}^{k} b(k-l)X(l) \Big\| \\
&= \mathbf{E}\Big\| \sum_{l=-\infty}^{k} b(k-l)[X(l+p) - X(l)] \Big\| \\
&\le \sum_{l=-\infty}^{k} \mathbf{E}\,|b(k-l)|\, \mathbf{E}\,\|X(l+p) - X(l)\| \\
&\le \Big(\sum_{j=0}^{\infty} \mathbf{E}\,|b(j)| \Big) \cdot \sup_{l\in\mathbb{Z}} \mathbf{E}\,\|X(l+p) - X(l)\| < M \cdot \frac{\varepsilon}{M} = \varepsilon\,.
\end{aligned}$$

Hence,

$$\sup_{k\in\mathbb{Z}} \mathbf{E}\,\|W(k+p) - W(k)\| < \varepsilon\,.$$

□

3 Almost Periodic Solutions of Nonsingular Stochastic Difference Equations

In this section we review the existence of almost periodic solutions of the following systems of nonsingular first-order stochastic linear difference equations of type

$$X(\omega, k+1) = A(\omega)X(\omega, k) + g(\omega, k)\,\xi(\omega, k+1)\,, \tag{3.1}$$

where A is an $N \times N$ square random matrix, $g : \Omega \times \mathbb{Z} \to L^1(\Omega, \mathbb{R}^N)$, and $\xi = \{\xi(k), k \in \mathbb{Z}\}$ is an almost periodic random sequence in $\mathbb{R}$. We assume that ξ is stochastically independent of g and $\mathbf{E}\,|\xi(k)| < \infty$ for all $k \in \mathbb{Z}$. We also assume that the random matrix A is stochastically independent of $X(0)$ and $\xi(0)$.

We begin with the scalar case. We denote by $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Theorem 3.1 [1] Suppose that $A := \lambda \in \mathbb{C} \setminus \mathbb{S}^1$. If $g : \Omega \times \mathbb{Z} \to L^1(\Omega; \mathbb{R}^N)$ is almost periodic in mean, then there is a mean almost periodic solution of eq. (3.1) given by

(i) $X(k) = \displaystyle\sum_{l=-\infty}^{k} \lambda^{k-l} g(l-1)\,\xi(l)$ in case $|\lambda| < 1$;

(ii) $X(k) = -\displaystyle\sum_{l=k}^{\infty} \lambda^{k-l-1} g(l)\,\xi(l+1)$ in case $|\lambda| > 1$.

Proof. Our proof follows closely that of ([1]). Since we are in stochastic case and for sake of clarity, we reproduce it here with slight modifications.

To prove (i), define $Y(k) = \lambda^k \xi(k)$ and since $|\lambda| < 1$ almost surely and using independence, we have

$$\mathbf{E}\sum_{k=1}^{\infty} |Y(k)| = \sup_{j\in\mathbb{Z}} \mathbf{E}\,|\xi(j)|\ \mathbf{E}\left|\frac{1}{1-|\lambda|}\right| < \infty\,.$$

It follows from Theorem 2.11 that X is almost periodic in mean.

To prove (ii), define $Y(k) = \lambda^{-k}\xi(k)$ and since $|\lambda| > 1$ almost surely, we have

$$\mathbf{E}\sum_{l=0}^{\infty} |Y(l)| = \sup_{j\in\mathbb{Z}} \mathbf{E}\,|\xi(j)|\,\mathbf{E}\left|\frac{1}{|\lambda|-1}\right| < \infty\,.$$

It follows from Theorem 2.11 that X is almost periodic in mean. □

As a consequence of the previous theorem, we obtain the following result in case of a nonsingular random matrix A.

Theorem 3.2 [1] Suppose A is a constant $N \times N$ nonsingular random matrix with eigenvalues $\lambda \notin \mathbb{S}^1$. Then, if g is almost periodic in mean, there is a mean almost periodic solution of eq. (3.1).

Proof. It is well known that there exists a nonsingular matrix S such that $S^{-1}A\,S = B$ is an upper triangular matrix. Setting $X(k) = S\,Y(k)$, eq. (3.1) becomes

$$Y(k+1) = B\,Y(k) + S^{-1}\,g(k)\,\xi(k+1),\ k\in\mathbb{Z}. \tag{3.2}$$

Obviously, eq. (3.2) is the same type as eq. (3.1). One can easily show using Theorem 2.7 and Theorem 2.10 that $S^{-1}\,g(k)\,\xi(k+1)$ is almost periodic in mean. The general case of an arbitrary random matrix A can now be reduced to the scalar case. Indeed, the last equation of eq. (3.2) is of the form

$$Z(k+1) = \lambda\,Z(k) + C(k)\,\xi(k+1),\ k\in\mathbb{Z} \tag{3.3}$$

where λ is a random element of $\mathbb{C}$ and $C(k)\,\xi(k+1)$ is a mean almost periodic random sequence. Hence, all we need to show is that any solution $Z(k)$ of eq. (3.3) is almost periodic in mean. But this is the content of Theorem 3.1. It then implies that the N-th component $Y_N(k)$ of the solution $Y(k)$ of eq. (3.2) is almost periodic in mean. Then substituting $Y_N(k)$ in the $(N-1)$th equation of eq. (3.2) we obtain again an equation of the form eq. (3.3) for $Y_{N-1}(k)$; and so on. The proof is complete. □

4 Almost Periodic Solutions of Stochastic Singular Difference Equations

4.1 Linear case

We are first interested in the case when the forcing term f does not depend on X. Namely, we study the existence of mean almost periodic solutions for the singular stochastic difference equation

$$AX(k+1) + BX(k) = C(k)\,\xi(k+1),\quad k\in\mathbb{Z} \tag{4.1}$$

where A, B are $N \times N$ square random matrices satisfying $\det A = \det B = 0$ and $C : \mathbb{Z} \to L^1(\Omega; \mathbb{R}^N)$ is a mean almost periodic sequence and ξ is as before.

Define

$$\rho(A, B) = \{\lambda \in \mathbb{C} : \lambda A + B \text{ is invertible}\} .$$

We now state one of our main results.

Theorem 4.1 *If* $\mathbb{S}^1 \subset \rho(A, B)$*, then eq.* (4.1) *has a unique mean almost periodic solution.*

Proof. We borrow Diagana-Pennequin's proof [6] and adapt it in stochastic case.

We set

$$\widehat{A} = (A + B)^{-1} A, \ \ \widehat{B} = (A + B)^{-1} B, \ \text{ and } \ \widehat{C}(k) = (A + B)^{-1} C(k), \ k \in \mathbb{Z} .$$

We can easily show that eq. (4.1) is equivalent to

$$\widehat{A}\, X(k + 1) + \widehat{B}\, X(k) = \widehat{C}(k)\, \xi(k + 1), \ k \in \mathbb{Z} . \tag{4.2}$$

Using the identity $\widehat{A} + \widehat{B} = I_N$, we deduce that $\widehat{A}\widehat{B} = \widehat{B}\widehat{A}$ and consequently, one can find common basis of trigonalization for $\widehat{A}$ and $\widehat{B}$. Thus, there exists an invertible matrix T such that

$$\widehat{A} = T^{-1} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} T$$

and

$$\widehat{B} = T^{-1} \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} T$$

where A_1, B_2 are invertible and A_2, B_1 are nilpotent.

Noting $A_i + B_i$ is the identity matrix of the same size as A_i and letting

$$TX(k) = \begin{bmatrix} W(k) \\ V(k) \end{bmatrix}$$

and

$$T\widehat{C}(k) = \begin{bmatrix} \alpha(k) \\ \beta(k) \end{bmatrix}$$

where $\{\alpha(k)\}$ and $\{\beta(k)\}$ are mean almost periodic random sequences, then eq. (4.2) can be written as follows:

$$\begin{cases} A_1\, W(k + 1) + B_1\, W(k) = \alpha(k)\, \xi(k + 1) \\ A_2\, V(k + 1) + B_2\, V(k) = \beta(k)\, \xi(k + 1) . \end{cases} \tag{4.3}$$

Using the fact that both A_1 and B_2 are invertible, one can show that eq. (4.3) is equivalent to

$$\begin{cases} W(k + 1) + A_1^{-1} B_1\, W(k) = A_1^{-1}\, \alpha(k)\, \xi(k + 1) \\ B_2^{-1} A_2\, V(k + 1) + V(k) = B_2^{-1} \beta(k)\, \xi(k + 1) . \end{cases} \tag{4.4}$$

Now let us focus on the first equations appearing in eq. (4.4).

$$W(k + 1) - (-A_1^{-1}\, B_1)\, W(k) = A_1^{-1}\, \alpha(k)\, \xi(k + 1), k \in \mathbb{Z} . \tag{4.5}$$

Note that Theorem 2.7 and Theorem 2.10 imply that $\{A_1^{-1}\,\alpha(k)\,\xi(k+1)\}_{k\in\mathbb{Z}}$ is almost periodic in mean. We shall now prove that $-A_1^{-1}\,B_1$ has no eigenvalue that belongs to $\mathbb{S}^1$. From that we will deduce that eq. (4.5) has a unique mean almost periodic solution. For that, let us consider a nonzero eigenvalue λ of $-A_1^{-1}B_1$. Let $U \neq \mathbf{0}$. We have $-A_1^{-1}\,B_1\,U = \lambda U$. Consequently

$$(\lambda A + B)\,U = 0,$$

from which we deduce

$$(\lambda\widehat{A} + \widehat{B})T^{-1}\begin{bmatrix} U \\ 0 \end{bmatrix} = 0.$$

Since

$$T^{-1}\begin{bmatrix} U \\ 0 \end{bmatrix} \neq 0,$$

it follows that $\lambda\widehat{A} + \widehat{B}$ is not invertible and so is the case for $\lambda A + B$. With the assumptions made, this proves that $|\lambda| \neq 1$. Consequently, we deduce that there exists a unique mean almost periodic solution $\{W(k)\}_{k\in\mathbb{Z}}$ to the first equation of eq. (4.4).

For the second equation appearing in eq. (4.4), setting $Y(k) = V(-k)$, it becomes by changing k in $-k$,

$$Y(k) + B_2^{-1}A_2\,Y(k-1) = B_2^{-1}\beta(-k)\,\xi(-(k-1)). \tag{4.6}$$

Using similar arguments as before, we can prove that eq. (4.6) has a unique mean almost periodic solution $\{Y(k)\}_{k\in\mathbb{Z}}$, so the second equation appearing in eq. (4.4) has also a unique mean almost periodic solution $\{V(k)\}_{k\in\mathbb{Z}}$. Since eq. (4.4) and eq. (4.1) are equivalent, we obtain existence and uniqueness of a mean almost periodic solution of eq. (4.1). □

4.2 Nonlinear case

$$A\,X(k+1) + B\,X(k) = f(k, X(k))\,\xi(k+1),\; k \in \mathbb{Z} \tag{4.7}$$

where A, B are $N \times N$ square random matrices satisfying $\det A = \det B = 0$ and $f : \mathbb{Z} \times L^1(\Omega, \mathbb{R}^N) \to L^1(\Omega, \mathbb{R}^N)$ is a mean almost periodic sequence and ξ is as before.

Our setting requires the following assumption.

(H) The function $(k, W) \to f(k, W)$ is almost periodic in mean in $k \in \mathbb{Z}$ uniformly in W in $L^1(\Omega; \mathcal{O})$ where $\mathcal{O} \subset \mathbb{R}^N$ is an arbitrary bounded subset. In addition, we assume that there exists a constant $L > 0$ such that

$$\mathbf{E}\,\|f(k,U) - f(k,V)\| \leq L \cdot \mathbf{E}\,\|U - V\|, \quad \forall\, U,\, V \in L^1(\Omega, \mathcal{O}),\; k \in \mathbb{Z}\,.$$

Theorem 4.2 *Suppose that $\mathbb{S}^1 \subset \rho(A, B)$ and that (H) holds. Then for sufficiently small L, eq.* (4.7) *has a unique mean almost periodic solution.*

Proof. Letting $C(k) = f(k, X(k))$ and using Theorem 2.6 together with similar arguments presented above, one can easily obtain the existence and uniqueness of a mean almost periodic solution to eq. (4.7). □

Example 4.3 *In this example, the Euclidean space* $\mathbb{R}^2$ *is equipped with the norm* $\|\cdot\|$*, which is defined for all* $u = (u_1, u_2)$ *by* $\|u\| = |u_1| + |u_2|$.

Let α, β *be random variables taking their values in* $(0,1)$*, and let* $\eta_k = (\eta_k^1, \eta_k^2)$, $k \in \mathbb{Z}$*, be a random almost periodic (in mean) sequence in* $\mathbb{R}^2$ *independent of* X *and* $\xi(k), k \in \mathbb{Z}$ *defined in eq.* (4.7)*. Define*

$$A = \begin{bmatrix} \alpha & 1 \\ \alpha & 1 \end{bmatrix},\ B = \begin{bmatrix} \beta & \beta \\ 0 & 0 \end{bmatrix},$$

and $F : \mathbb{Z} \times L^1(\Omega, \mathcal{O}) \to \mathbb{R}^2$*:*

$$F(k, X) = \begin{bmatrix} \eta_k^1 \tan^{-1} X_1 \\ \eta_k^2 \tan^{-1} X_2 \end{bmatrix},$$

where $\mathcal{O} = \left\{ y \in \mathbb{R}^2 : \|y\| \le \delta \right\}$ *for any* $\delta > 0$ *and* $X = (X_1, X_2)^T$*. Then, by Theorem 4.2, eq.* (4.7) *has a unique almost periodic (in mean) solution.*

Indeed, we note that $\det A = \det B = 0$ *and let us verify that* F *satisfies Assumption* (H). Using the independence and the fact that $\tan^{-1}(\cdot)$ is a Lipschitz function, we can write

$$\begin{aligned} \mathbf{E}\,\|F(k,X) - F(k,Y)\| &= \mathbf{E}\,|\eta_k^1|\ \mathbf{E}\,|\tan^{-1} X_1 - \tan^{-1} Y_1| \\ &\quad + \mathbf{E}\,|\eta_k^2|\ \mathbf{E}\,|\tan^{-1} X_2 - \tan^{-1} Y_2| \\ &\le \mathbf{E}\,\|\eta_k\|\ [\mathbf{E}\,|X_1 - Y_1| + \mathbf{E}\,|X_2 - Y_2|] \\ &\le \mathbf{E}\,\|\eta_k\|\ \mathbf{E}\,\|X - Y\|\,. \end{aligned}$$

5 Almost Periodic Solutions of Stochastic Second-Order Singular Difference Equations

In this section, we study the existence of mean almost periodic solution to singular stochastic second-order difference equation of type:

$$A\,X(k+2) + B\,X(k+1) + C\,X(k) = f(k, X(k))\,\xi(k+1),\ k \in \mathbb{Z} \tag{5.1}$$

where A, B, C are $N \times N$ square random matrices satisfying $\det A = \det B = \det C = 0$ and $f : \mathbb{Z} \times L^1(\Omega, \mathbb{R}^N) \to L^1(\Omega, \mathbb{R}^N)$ is almost periodic in mean in the first variable uniformly in the second variable and ξ is as before. In order to study the existence of mean almost periodic solution to eq. (5.1), we make extensive use of the results obtained in the previous section.

For that, we rewrite eq. (5.1) as follows

$$L\,W(k+1) + M\,W(k) = F(k, W(k))\,\xi(k+1),\ k \in \mathbb{Z} \tag{5.2}$$

where

$$L = \begin{bmatrix} B & A \\ I & 0 \end{bmatrix},\ M = \begin{bmatrix} C & 0 \\ 0 & -I \end{bmatrix},\ F = \begin{bmatrix} f \\ 0 \end{bmatrix},\ \text{and W(k)} = \begin{bmatrix} X(k) \\ X(k+1) \end{bmatrix}$$

with 0 and I being the $N \times N$ zero and identity matrices.

Lemma 5.1 $\lambda L + M$ *is invertible if and only if* $\lambda^2 A + \lambda B + C$ *is invertible.*

Proof. The $N \times N$ square matrix $\lambda L + M$ is given by

$$\lambda L + M = \begin{bmatrix} \lambda B + C & \lambda A \\ \lambda I & -I \end{bmatrix}.$$

Consequently, solving

$$(\lambda L + M) \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}$$

yields $(\lambda B + C)U + \lambda AV = X$ and $\lambda U - V = Y$. If $\lambda^2 A + \lambda B + C$ is invertible, then from $V = \lambda U - Y$ it follows that $(\lambda^2 A + \lambda B + C)U = \lambda AY + X$ which yields

$$U = [\lambda^2 A + \lambda B + C]^{-1}(\lambda AY + X),\ V = [\lambda^2 A + \lambda B + C]^{-1}(X + \lambda AY) - Y.$$

The latter implies that $\lambda L + M$ is invertible.

The converse can be proven using similar arguments as before and hence is omitted. □

Set

$$\rho(A, B, C) = \left\{\lambda \in \mathbb{C} : \lambda^2 A + \lambda B + C \text{ is invertible}\right\} .$$

Using Lemma 5.1 and Theorem 4.2, we obtain the following result:

Theorem 5.2 *Suppose* $\mathbb{S}^1 \subset \rho(A, B, C)$ *and that* (H) *holds. Then for sufficiently small L, eq. (*5.1) has a unique mean almost periodic solution.

More generally, let $q \geq 2$ be an integer. The previous techniques can be easily used to study the existence of almost periodic solutions to higher order singular systems of stochastic difference equation of type

$$A_q X(k+q) + A_{q-1} X(k+q-2) + \cdots + A_1 X(k+1) + A_0 X(k) = f(k, X(k))\, \xi(k+1), \quad (5.3)$$

where A_l for $l = 0, 1, 2, \cdots, q$ are $N \times N$ square random matrices satisfying $\det A_l = 0$ for $l = 0, 1, 2, \cdots, q$ and $f : \mathbb{Z} \times L^1(\Omega; \mathbb{R}^N) \to L^1(\Omega; \mathbb{R}^N)$ is almost periodic in mean in the first variable uniformly in the second one and ξ is as before.
Setting

$$\rho(A_q, A_{q-1}, \cdots, A_0) := \big\{\lambda \in \mathbb{C} : \\ \lambda^q A_q + \lambda^{q-1} A_{q-1} + \lambda^{q-2} A_{q-2} + \cdots + \lambda_1 A_1 + A_0 \text{ is invertible}\big\},$$

the existence result can be formulated as follows.

Theorem 5.3 *Suppose* $\mathbb{S}^1 \subset \rho(A_q, A_{q-1}, \cdots, A_0)$ *and* (H) *holds. Then for sufficiently small L, eq.* (5.3) has a unique mean almost periodic solution.

6 Conclusion and Future Directions

The aim of the present paper is to extend the results of Diagana-Pennequin [6] to the stochastic case. Our study showed how to find solutions for stochastic difference equations with singular constant coefficients in the case that the random coefficients of matrices A and B are square, under the conditions that $\det A = \det B = 0$. The applications of such equations can be found in many practical areas, such as the Leontiev dynamic model of multisector economy, singular discrete optimal control problems and so forth.

The nonautonomous stochastic singular form of eq. (1.1) is delicate and remains to be investigated. It deserves our great attention.

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