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EXISTENCE AND SUCCESSIVE APPROXIMATIONS FOR IMPLICIT DEFORMABLE FRACTIONAL DIFFERENTIAL BOUNDARY VALUE PROBLEMS

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Abstract. The focus of this paper is the investigation of a particular type of nonlinear deformable fractional differential equations, and analyzing their existence results. Our approach involves utilizing relevant fixed point theorems, we also explore the global convergence of successive approximations to provide additional insights into the topic. To further illustrate our findings, we provide several examples that supplement our main results.

Keywords: Implicit differential equations, fractional differential equations, deformable fractional derivative, existence, fixed point, global convergence, successive approximations.

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1 Introduction

Fractional calculus has been a captivating field of study within functional space theory for a significant period, attracting scholars owing to its diverse range of applications across various disciplines. This domain of research focuses on employing non-integer derivatives of fractional order to model and comprehend complex natural phenomena. Some of the noteworthy areas where fractional calculus has found use include electrochemistry and viscoelasticity. The application of fractional derivatives has demonstrated efficacy in extending the fundamental laws of nature, facilitating a more comprehensive and nuanced comprehension of these processes. Moreover, fractional calculus has been vital in capturing the memory and hereditary effects that emerge in several systems, which traditional integer-order derivatives fail to explain. To delve further into this subject, we recommend perusing the books [2, 16, 17, 27, 30] and the papers [5–7, 11–13, 19, 22–25] and other relevant literature.

Zulfeqarr *et al.* introduced a new concept called the "deformable fractional derivative" in their paper [31]. This derivative uses a limit technique, similar to the traditional derivative, to continuously transform a function into its derivative. The term "deformable" refers to its unique feature of allowing for continuous deformation [4]. Additional research on fractional derivatives of the conformable type can be found in papers such as [18, 20, 21, 26].

In [20], the properties of the deformable derivative were further explored by the authors, who then utilized the findings to examine the Cauchy problem with a non-local condition below:

$$\begin{aligned} &\mathfrak{D}_0^{\gamma}\xi(\zeta) = \aleph(\zeta, \xi(\zeta)), \quad \zeta \in (0, \kappa], \\ &\xi(0) + \widehat{\aleph}(\xi) = \xi_0, \end{aligned}$$

where $\mathfrak{D}_0^{\gamma}$ is the deformable derivative of order $\gamma \in (0, 1)$, and $\widehat{\aleph} : \mathcal{C} \to \mathbb{R}$ is a continuous function.

In [21], the authors considered the problem:

$$\begin{aligned} \mathfrak{D}_0^{\gamma}\xi(\zeta) &= \mathcal{Z}\xi(\zeta) + \aleph(\zeta, \xi(\zeta)), \quad \zeta \in \nabla, \\ \xi(0) &= \xi_0, \end{aligned}$$

where $\mathcal{Z} : D(\mathcal{Z}) \subset \Xi \to \Xi$ is an infinitesimal generator of a C_0-semigroup $T(\zeta)(\zeta \geqslant 0)$ on a suitable space $\Xi, \xi_0 \in \Xi$, and $\nabla = [0, \kappa], \kappa > 0$ is a constant.

In [26], the authors studied the existence and uniqueness of solutions for the implicit problem with nonlinear fractional differential equation involving the deformable fractional derivative in b-metric spaces:

$$\begin{cases} \left(\mathfrak{D}_0^{\gamma}\xi\right)(\zeta) = \aleph\left(\zeta, \xi(\zeta), \mathfrak{D}_0^{\gamma}\xi(\zeta)\right); \ \zeta \in \nabla := [0, \kappa], \\ \xi(0) = \xi_0, \end{cases}$$

where $0 < \gamma < 1$, $\aleph : \nabla \times \mathbb{R}^2 \to \mathbb{R}$ is a given function and $\xi_0 \in \mathbb{R}$.

Several research studies have investigated the convergence of successive approximations for nonlinear functional equations and the global convergence of successive approximations for functional differential equations [1–3, 28, 29]. Browder [8] established a generalization of the classical Picard-Banach contraction principle, utilizing the convergence of successive approximations in 1968. Chen [9] employed the method of successive approximations to analyze the existence of solutions for functional integral equations in 1981. Czlapiiǹski [10] investigated the global convergence of successive approximations of the Darboux problem for partial functional differential equations with infinite delay, while Faina [14] studied the generic property of global convergence of successive approximations for functional differential equations with infinite delay.

In this paper, we investigate the following class of deformable fractional differential equation with boundary conditions:

$$\begin{cases} (\mathfrak{D}_0^{\gamma}\xi)(\zeta) = \aleph(\zeta, \xi(\zeta), \mathfrak{D}_0^{\gamma}\xi(\zeta)), & \zeta \in \nabla := [0, \varpi], \\ \imath\xi(0) + \jmath\xi(\varpi) = \varrho, \end{cases} \tag{1.1}$$

where $\mathfrak{D}_0^{\gamma}\xi(\zeta)$ is the deformable fractional derivative starting from the initial time 0 of the function ξ of order $\gamma \in (0,1)$, $\aleph : \nabla \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $0 < \varpi < +\infty$ and $\imath, \jmath, \varrho$ are real constants where $\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi} \neq 0$.

This paper is structured as follows: In Section 2, we introduce the notations and offer an overview of the (k, ψ)-Hilfer fractional derivative that we will utilize throughout the manuscript. In Section 3, we present an existence result of the problem (1.1) based on Schauder's fixed point theorem. Additionally, we examine a result on the global convergence of successive approximations. Finally, in the last section, we provide various examples to reinforce the obtained results.

2 Preliminaries

Let $\nabla := [0, \varpi]$, where $0 < \varpi < +\infty$. We denote by $C(\nabla, \mathbb{R})$ the Banach space of all continuous functions from ∇ into $\mathbb{R}$, with the following norm

$$\|\xi\|_{\infty} = \sup_{\zeta \in \nabla}\{|\xi(\zeta)|\}.$$

Let $\mathcal{F} := \mathcal{F}(\nabla, \mathbb{R})$ be the Banach space defined by:

$$\mathcal{F} : \{\xi \in C(\nabla, \mathbb{R}) : \mathfrak{D}_0^{\gamma}\xi \text{ exists and continuous on } \nabla\},$$

with the norm

$$\|\xi\|_{\mathcal{F}} = \max\left\{\|\xi\|_{\infty}; \|\mathfrak{D}_0^{\gamma}\xi\|_{\infty}\right\}.$$

Consider the space $X_b^p(0, \varpi)$, $(b \in \mathbb{R},\ 1 \leq p \leq \infty)$ of those complex-valued Lebesgue measurable functions $\aleph$ on $[0, \kappa]$ for which $\|\aleph\|_{X_b^p} < \infty$, where the norm is given by:

$$\|\aleph\|_{X_b^p} = \left(\int_0^{\varpi} |\zeta^b \aleph(\zeta)|^p \frac{\mathrm{d}\,\zeta}{\zeta}\right)^{\frac{1}{p}}, \ \ (1 \leq p < \infty, b \in \mathbb{R}).$$

Definition 2.1 (The deformable fractional derivative [20,31]) *Let $\aleph : [0, +\infty) \longrightarrow \mathbb{R}$ be a given function, the deformable fractional derivative of $\aleph$ of order γ is defined by*

$$(\mathfrak{D}_0^{\gamma}\aleph)(\zeta) = \lim_{\varepsilon \to 0} \frac{(1 + \varepsilon\chi)\aleph(\zeta + \varepsilon\gamma) - \aleph(\zeta)}{\varepsilon},$$

where $\gamma + \chi = 1$ and $\gamma \in (0, 1]$. If the deformable fractional derivative of $\aleph$ of order γ exists, then we simply say that $\aleph$ is γ-differentiable.

Definition 2.2 (The γ-fractional integral [20]) *For $\gamma \in (0,1]$ and a continuous function $\aleph$, let*

$$\left(\mathcal{J}_{0^+}^{\gamma}\aleph\right)(\zeta) = \frac{1}{\gamma}e^{\frac{-\chi}{\gamma}\zeta}\int_0^{\zeta} e^{\frac{\chi}{\gamma}s}\aleph(s)\,\mathrm{d}s.$$

Lemma 2.1 ([20]) *If $\gamma, \gamma_1 \in (0,1]$ such that $\gamma+\chi=1$, $\aleph$ and $\widehat{\aleph}$ are two γ-differentiable functions at a point ζ and m, n are two given numbers, then the deformable fractional derivative satisfies the following properties:*

- $\mathfrak{D}_0^{\gamma}(\lambda) = \chi\lambda$, *for any constant λ;*
- $\mathfrak{D}_0^{\gamma}(m\aleph + n\widehat{\aleph}) = m\mathfrak{D}_0^{\gamma}(\aleph) + n\mathfrak{D}_0^{\gamma}(\widehat{\aleph})$*;*
- $\mathfrak{D}_0^{\gamma}(\aleph\widehat{\aleph}) = \widehat{\aleph}\mathfrak{D}_0^{\gamma}(\aleph) + \gamma\aleph\widehat{\aleph}'$*;*
- $\mathcal{J}_{0^+}^{\gamma}\,\mathcal{J}_{0^+}^{\gamma_1}\aleph = \mathcal{J}_{0^+}^{\gamma+\gamma_1}\aleph$.

Lemma 2.2 ([20]) *If $\gamma \in (0,1]$, $\aleph$ is continuous function, then we have:*

- $\left(\mathcal{J}_{0^+}^{\gamma}\,\mathfrak{D}_0^{\gamma}(\aleph)\right)(\zeta) = \aleph(\zeta) - e^{\frac{-\chi}{\gamma}\zeta}\aleph(0)$;
- $\mathfrak{D}_0^{\gamma}\left(\mathcal{J}_{0^+}^{\gamma}\aleph\right)(\zeta) = \aleph(\zeta)$.

Lemma 2.3 *Let $\widehat{\aleph} \in L^1(\nabla)$, $0 < \gamma \leq 1$ and $\imath, \jmath, \varrho$ are real constants where $\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi} \neq 0$. Then the problem*

$$\begin{cases} \left(\mathfrak{D}_0^{\gamma}\xi\right)(\zeta) = \widehat{\aleph}(\zeta);\ \zeta \in \nabla := [0,\varpi], \\ \imath\xi(0) + \jmath\xi(\varpi) = \varrho, \end{cases} \tag{2.1}$$

has a unique solution defined by

$$\xi(\zeta) = \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath e^{\frac{-\chi}{\gamma}(\zeta+\varpi)}}{\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}}\int_0^{\varpi} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)\,\mathrm{d}s + \frac{1}{\gamma}e^{\frac{-\chi}{\gamma}\zeta}\int_0^{\zeta} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)\,\mathrm{d}s. \tag{2.2}$$

Proof. Applying the γ-fractional integral of order γ to both sides of the equation $\left(\mathfrak{D}_0^{\gamma}\xi\right)(\zeta) = \widehat{\aleph}(\zeta)$, and by using Lemma 2.2 and if $\zeta \in \nabla$, we get

$$\xi(\zeta) - \xi(0)e^{\frac{-\chi}{\gamma}\zeta} = \frac{1}{\gamma}e^{\frac{-\chi}{\gamma}\zeta}\int_0^{\zeta} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)\,\mathrm{d}s. \tag{2.3}$$

Hence, we get

$$\xi(\zeta) = \xi(0)e^{\frac{-\chi}{\gamma}\zeta} + \frac{1}{\gamma}e^{\frac{-\chi}{\gamma}\zeta}\int_0^{\zeta} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)\,\mathrm{d}s. \tag{2.4}$$

Thus,

$$\xi(\varpi) = \xi(0)e^{\frac{-\chi}{\gamma}\varpi} + \frac{1}{\gamma}e^{\frac{-\chi}{\gamma}\varpi}\int_0^{\varpi} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)\,\mathrm{d}s.$$

From the mixed boundary conditions $\imath\xi(0) + \jmath\xi(\varpi) = \varrho$, we get

$$\imath\xi(0) + \jmath\left(\xi(0)e^{\frac{-\chi}{\gamma}\varpi} + \frac{1}{\gamma}e^{\frac{-\chi}{\gamma}\varpi}\int_0^{\varpi} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)\,\mathrm{d}s\right) = \varrho.$$

Thus,

$$\xi(0) = \frac{\varrho - \frac{\jmath}{\gamma} e^{\frac{-\chi}{\gamma}\varpi} \int_0^{\varpi} e^{\frac{\chi}{\gamma}s} \widehat{\aleph}(s)\, \mathrm{d}s}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}}.$$

Hence, we obtain

$$\xi(\zeta) = \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath e^{\frac{-\chi}{\gamma}(\zeta+\varpi)}}{\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}} \int_0^{\varpi} e^{\frac{\chi}{\gamma}s} \widehat{\aleph}(s)\, \mathrm{d}s + \frac{1}{\gamma} e^{\frac{-\chi}{\gamma}\zeta} \int_0^{\zeta} e^{\frac{\chi}{\gamma}s} \widehat{\aleph}(s)\, \mathrm{d}s.$$

Conversely, we can easily show by Lemma 2.2 that if ξ verifies equation (2.2) then it satisfies the problem (2.1). □

3 Existence and Uniqueness of Solutions

In this section, we are concerned with the existence results of the problem (1.1).

Definition 3.1 *A solution of problem* (1.1) *is a function* $\xi \in C(\nabla)$ *where*

$$\xi(\zeta) = \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath e^{\frac{-\chi}{\gamma}(\zeta+\varpi)}}{\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}} \int_0^{\varpi} e^{\frac{\chi}{\gamma}s} \widehat{\aleph}(s)\, \mathrm{d}s + \frac{1}{\gamma} e^{\frac{-\chi}{\gamma}\zeta} \int_0^{\zeta} e^{\frac{\chi}{\gamma}s} \widehat{\aleph}(s)\, \mathrm{d}s,$$

such that $\widehat{\aleph} \in C(\nabla, \mathbb{R})$*, with* $\widehat{\aleph}(\zeta) = \aleph(\zeta, \xi(\zeta), \widehat{\aleph}(\zeta))$ *and* $\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi} \neq 0$.

The hypothesis:

(H_1) There exist constants $\omega_1 > 0,\ 0 < \omega_2 < 1$ such that

$$|\aleph(\zeta, \xi_1, \Im_1) - \aleph(\zeta, \xi_2, \Im_2)| \leq \omega_1|\xi_1 - \xi_2| + \omega_2|\Im_1 - \Im_2|,$$

for any $\xi_1, \xi_2, \Im_1, \Im_2 \in \mathbb{R}$, and each $\zeta \in \nabla$.

Remark 3.1 *We note that for any* $\xi, \Im \in \mathbb{R}$*, and each* $\zeta \in \nabla$*, hypothesis* (H_1) *implies that*

$$|\aleph(\zeta, \xi, \Im)| \leq \omega_1|\xi| + \omega_2|\Im| + \aleph^*,$$

where $\aleph^* = \sup_{\zeta\in[0,\varpi]} \aleph(\zeta, 0, 0)$.

Now, we will give our existence result that is based on Schauder's fixed point theorem [15].

Theorem 3.1 *If* (H_1) *holds, and*

$$\frac{\left(e^{\frac{\chi}{\gamma}\varpi} - 1\right)|\imath + \jmath(e^{\frac{-\chi}{\gamma}\varpi} + 1)|\omega_1}{\chi|\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}|(1 - \omega_2)} < 1, \tag{3.1}$$

then problem (1.1) *has at least one solution on* $[0, \varpi]$.

Proof. Consider the operator $\mathcal{H} : C(\nabla, \mathbb{R}) \to C(\nabla, \mathbb{R})$, such that

$$(\mathcal{H}\xi)(\zeta) = \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath e^{\frac{-\chi}{\gamma}(\zeta+\varpi)}}{\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}} \int_0^{\varpi} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)\,\mathrm{d}s + \frac{1}{\gamma} e^{\frac{-\chi}{\gamma}\zeta} \int_0^{\zeta} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)\,\mathrm{d}s, \tag{3.2}$$

where $\widehat{\aleph} \in C(\nabla, \mathbb{R})$, with $\widehat{\aleph}(\zeta) = \aleph(\zeta, \xi(\zeta), \widehat{\aleph}(\zeta))$.

Let $\delta > 0$ such that

$$\delta \geq \frac{\frac{|\varrho|}{|\imath+\jmath e^{\frac{-\chi}{\gamma}\varpi}|} + \frac{\left(e^{\frac{\chi}{\gamma}\varpi}-1\right)|\imath+\jmath(e^{\frac{-\chi}{\gamma}\varpi}+1)|\aleph^*}{\chi|\imath+\jmath e^{\frac{-\chi}{\gamma}\varpi}|(1-\omega_2)}}{1 - \frac{\left(e^{\frac{\chi}{\gamma}\varpi}-1\right)|\imath+\jmath(e^{\frac{-\chi}{\gamma}\varpi}+1)|\omega_1}{\chi|\imath+\jmath e^{\frac{-\chi}{\gamma}\varpi}|(1-\omega_2)}}. \tag{3.3}$$

Consider the ball

$$\Xi_\delta = \{\xi \in C([0,\varpi], \mathbb{R}),\ \|\xi\|_\infty \leq \delta\}.$$

Claim 1. $\mathcal{H}$ is continuous.
Let $\{\xi_n\}_n$ be a sequence such that $\xi_n \to \xi$ on Ξ_δ. For each $\zeta \in \nabla$, we have

$$\begin{aligned}|(\mathcal{H}\xi_n)(\zeta) - (\mathcal{H}\xi)(\zeta)| \leq{}& \frac{|\jmath| e^{\frac{-\chi}{\gamma}(\zeta+\varpi)}}{|\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}|} \int_0^{\varpi} e^{\frac{\chi}{\gamma}s} |\widehat{\aleph}_n(s) - \widehat{\aleph}(s)|\,\mathrm{d}s \\ &+ \frac{1}{\gamma} e^{\frac{-\chi}{\gamma}\zeta} \int_0^{\zeta} e^{\frac{\chi}{\gamma}s} |\widehat{\aleph}_n(s) - \widehat{\aleph}(s)|\,\mathrm{d}s,\end{aligned}$$

where $\widehat{\aleph}_n,\ \widehat{\aleph} \in C(\nabla, \mathbb{R})$ such that

$$\widehat{\aleph}_n(\zeta) = \aleph(\zeta, \xi_n(\zeta), \widehat{\aleph}_n(\zeta)) \quad and \quad \widehat{\aleph}(\zeta) = \aleph(\zeta, \xi(\zeta), \widehat{\aleph}(\zeta)).$$

Since

$$\|\xi_n - \xi\|_\infty \to 0 \text{ as } n \to \infty$$

and $\aleph, \widehat{\aleph}$ and $\widehat{\aleph}_n$ are continuous, we deduce that

$$\|\mathcal{H}(\xi_n) - \mathcal{H}(\xi)\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Hence, $\mathcal{H}$ is continuous.

Claim 2. $\mathcal{H}(\Xi_\delta) \subset \Xi_\delta$.
Let $\xi \in \Xi_\delta$. From Remark 3.1, for each $\zeta \in \nabla$, we have

$$\begin{aligned}|\widehat{\aleph}(\zeta)| &\leq |\aleph(\zeta, \xi(\zeta), \widehat{\aleph}(\zeta))| \\ &\leq \omega_1\|\xi\|_\infty + \omega_2\|\widehat{\aleph}\|_\infty + \aleph^* \\ &\leq \omega_1\delta + \omega_2\|\widehat{\aleph}\|_\infty + \aleph^*.\end{aligned}$$

Then

$$\|\widehat{\aleph}\|_\infty \leq \frac{\delta\omega_1 + \aleph^*}{1 - \omega_2}.$$

Thus,

$$\begin{aligned}
|(\mathcal{H}\xi)(\zeta)| &\leq \left| \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath e^{\frac{-\chi}{\gamma}(\zeta+\varpi)}}{\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}} \int_0^{\varpi} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)\,\mathrm{d}s + \frac{1}{\gamma} e^{\frac{-\chi}{\gamma}\zeta} \int_0^{\zeta} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)\,\mathrm{d}s \right| \\
&\leq \frac{|\varrho|}{|\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}|} + \frac{|\jmath|}{|\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}|} \int_0^{\varpi} e^{\frac{\chi}{\gamma}s}|\widehat{\aleph}(s)|\,\mathrm{d}s + \frac{1}{\gamma} e^{\frac{-\chi}{\gamma}\zeta} \int_0^{\zeta} e^{\frac{\chi}{\gamma}s}|\widehat{\aleph}(s)|\,\mathrm{d}s \\
&\leq \frac{|\varrho|}{|\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}|} + \frac{\left(e^{\frac{\chi}{\gamma}\varpi} - 1\right)|\imath + \jmath(e^{\frac{-\chi}{\gamma}\varpi} + 1)|(\delta\omega_1 + \aleph^*)}{\chi|\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}|(1-\omega_2)} \\
&\leq \delta.
\end{aligned}$$

Hence,

$$\|\mathcal{H}(\xi)\|_\infty \leq \delta.$$

Consequently, $\mathcal{H}(\Xi_\delta) \subset \Xi_\delta$.

Claim 3. $\mathcal{H}(\Xi_\delta)$ is equicontinuous.
For $0 \leq \zeta_1 \leq \zeta_2 \leq \varpi$, and $\xi \in \Xi_\delta$, we get

$$\begin{aligned}
&|\mathcal{H}(\xi)(\zeta_2) - \mathcal{H}(\xi)(\zeta_1)| \\
&\leq \left| \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta_2}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath e^{\frac{-\chi}{\gamma}(\zeta_2+\varpi)}}{\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}} \int_0^{\varpi} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)\,\mathrm{d}s + \frac{1}{\gamma} e^{\frac{-\chi}{\gamma}\zeta_2} \int_0^{\zeta_2} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)\,\mathrm{d}s \right. \\
&\quad \left. - \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta_1}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} + \frac{\jmath e^{\frac{-\chi}{\gamma}(\zeta_1+\varpi)}}{\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}} \int_0^{\varpi} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)\,\mathrm{d}s - \frac{1}{\gamma} e^{\frac{-\chi}{\gamma}\zeta_1} \int_0^{\zeta_1} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)\,\mathrm{d}s \right| \\
&\leq \left| \left(e^{\frac{-\chi}{\gamma}\zeta_2} - e^{\frac{-\chi}{\gamma}\zeta_1}\right) \left[\frac{\varrho}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath e^{\frac{-\chi}{\gamma}\varpi}}{\gamma\imath + \gamma\jmath e^{\frac{-\chi}{\gamma}\varpi}} \int_0^{\varpi} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)\,\mathrm{d}s \right] \right| \\
&\quad + \frac{1}{\gamma} \left| e^{\frac{-\chi}{\gamma}\zeta_2} \int_{\zeta_1}^{\zeta_2} e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)\,\mathrm{d}s - \int_0^{\zeta_1} \left[e^{\frac{-\chi}{\gamma}\zeta_1} - e^{\frac{-\chi}{\gamma}\zeta_2} \right] e^{\frac{\chi}{\gamma}s}\widehat{\aleph}(s)\,\mathrm{d}s \right| \\
&\leq \left| \left(e^{\frac{-\chi}{\gamma}\zeta_2} - e^{\frac{-\chi}{\gamma}\zeta_1}\right) \left[\frac{\varrho}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath\left(1 - e^{\frac{-\chi}{\gamma}\varpi}\right)(\delta\omega_1 + \aleph^*)}{\chi\left(\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}\right)(1-\omega_2)} \right] \right| \\
&\quad + \frac{\delta\omega_1 + \aleph^*}{\chi(1-\omega_2)} \left| e^{\frac{-\chi}{\gamma}\zeta_2}\left(e^{\frac{\chi}{\gamma}\zeta_2} - e^{\frac{\chi}{\gamma}\zeta_1}\right) - \left(e^{\frac{-\chi}{\gamma}\zeta_1} - e^{\frac{-\chi}{\gamma}\zeta_2}\right)\left(e^{\frac{\chi}{\gamma}\zeta_1} - 1\right) \right| \\
&\leq \left| \left(e^{\frac{-\chi}{\gamma}\zeta_2} - e^{\frac{-\chi}{\gamma}\zeta_1}\right) \left[\frac{\varrho}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \frac{\jmath\left(1 - e^{\frac{-\chi}{\gamma}\varpi}\right)(\delta\omega_1 + \aleph^*)}{\chi\left(\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}\right)(1-\omega_2)} \right] \right| \\
&\quad + \frac{\delta\omega_1 + \aleph^*}{\chi(1-\omega_2)} \left| e^{\frac{-\chi}{\gamma}\zeta_2} - e^{\frac{-\chi}{\gamma}\zeta_1} \right|.
\end{aligned}$$

As $\zeta_2 \to \zeta_1$ then $|\mathcal{H}(\xi)(\zeta_1) - \mathcal{H}(\xi)(\zeta_2)| \to 0$. We deduce that $\mathcal{H}(\Xi_\delta)$ is equicontinuous. Consequently, Arzelá-Ascoli theorem implies that $\mathcal{H}$ is continuous and compact. Thus, by Schauder's fixed point theorem [15], we deduce that $\mathcal{H}$ has at least a fixed point which is a solution of (1.1). □

4 Successive Approximations and Uniqueness Results

This section is devoted to giving the main result of the global convergence of successive approximations of our problem (1.1). We will study the solution in $\digamma$ of our problem.

Set $\nabla_\lambda := [0, \lambda\varpi]$ for any $\lambda \in [0,1]$. In what follows, we need the following hypotheses:

(H_2) There exist a constant $\varkappa > 0$ and a continuous function $h : \nabla \times [0,\varkappa] \times [0,\varkappa] \longrightarrow \mathbb{R}_+$, such that $h(\zeta,\cdot,\cdot)$ is nondecreasing for all $\zeta \in \nabla$ and the inequality

$$|\aleph(\zeta,\xi_1,\bar{\xi_1}) - \aleph(\zeta,\xi_2,\bar{\xi_2})| \leq h\left(\zeta, |\xi_1-\xi_2|, |\bar{\xi_1}-\bar{\xi_2}|\right) \tag{4.1}$$

holds for $\zeta \in \nabla$ and $\xi_1,\xi_2,\bar{\xi_1},\bar{\xi_2} \in \mathbb{R}$, with $|\xi_1-\xi_2| \leq \varkappa$ and $|\bar{\xi_1}-\bar{\xi_2}| \leq \varkappa$.

(H_3) $R \equiv 0$ is the only function in $\digamma(\nabla_\theta, [0,\varkappa])$ which satisfies the integral inequality

$$R(\zeta) \leq \int_0^\varpi |G(\zeta,s)| h\left(s, R(s), \left(\mathfrak{D}_0^\gamma R\right)(s)\right) \mathrm{d}s,$$

with $\lambda \leq \theta \leq 1$,

$$G(\zeta,s) = \frac{1}{\gamma}\begin{cases} \dfrac{\jmath e^{\frac{\chi}{\gamma}(s-\zeta-\varpi)}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - e^{\frac{\chi}{\gamma}(s-\zeta)}, & \text{if} \quad 0 \leq s \leq \zeta \leq \varpi, \\[2ex] \dfrac{\jmath e^{\frac{\chi}{\gamma}(s-\zeta-\varpi)}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}}, & \text{if} \quad 0 \leq \zeta \leq s \leq \varpi. \end{cases}$$

Here $G(\zeta,s)$ is called the Green function of the boundary value problem (1.1).

(H_4) For each $\zeta \in \nabla$, the set

$$\{\zeta \longmapsto \aleph(\zeta,\xi_1,\bar{\xi_1}) : \xi_1,\bar{\xi_1} \in \mathbb{R}\} \text{ is equicontinuous.}$$

For $\zeta \in \nabla$, we define the successive approximations of the problem (1.1) as follows:

$$\xi_0(\zeta) = \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}},$$

$$\xi_{n+1}(\zeta) = \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \int_0^\varpi G(t,s)\aleph\left(s, \xi_n(s), \left(\mathfrak{D}_0^\gamma \xi_n\right)(s)\right) \mathrm{d}s.$$

Theorem 4.1 *Assume that the hypotheses* (H_2)-(H_4) *hold. Then, the successive approximations* ξ_n*;* $n \in \mathbb{N}$ *are well defined and converge to the unique solution of the problem* (1.1) *uniformly in* $\digamma$.

Proof. Since the function $\aleph$ is continuous, then the successive approximations are well defined. Differentiating the two sides of the successive approximations ξ_n; $n \in \mathbb{N}$ by using the improved deformable fractional derivative of order γ, by Lemma 2.2 and Lemma 2.1, we have

$$\left(\mathfrak{D}_0^\gamma \xi_0\right)(\zeta) = 0, \quad \zeta \in \nabla,$$
$$\left(\mathfrak{D}_0^\gamma \xi_{n+1}\right)(\zeta) = \aleph\left(\zeta, \xi_n(\zeta), \mathfrak{D}_0^\gamma \xi_n(\zeta)\right), \quad \zeta \in \nabla.$$

And since $\xi_n \in \digamma$, then there exist two constants $\delta_1, \delta_2 > 0$ such that

$$\|\xi_n\|_\infty \leq \delta_1 \text{ and } \|\mathfrak{D}_0^\gamma \xi_n\|_\infty \leq \delta_2.$$

Let $\zeta_1, \zeta_2 \in \nabla$, $\zeta_1 < \zeta_2$. Then,

$$\begin{aligned}
|\xi_n(\zeta_2) - \xi_n(\zeta_1)| &\leq \Bigg| \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta_2}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \int_0^\varpi G(\zeta_2, s)\aleph\left(s, \xi_{n-1}(s), \mathfrak{D}_0^\gamma \xi_{n-1}(s)\right) \mathrm{d}s \\
&\quad - \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta_1}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} + \int_0^\varpi G(\zeta_1, s)\aleph\left(s, \xi_{n-1}(s), \mathfrak{D}_0^\gamma \xi_{n-1}(s)\right) \mathrm{d}s \Bigg| \\
&\leq \left| \left(e^{\frac{-\chi}{\gamma}\zeta_2} - e^{\frac{-\chi}{\gamma}\zeta_1}\right) \left[\frac{\varrho}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}}\right] \right| \\
&\quad + \Bigg| \int_0^\varpi G(\zeta_2, s)\aleph\left(s, \xi_{n-1}(s), \mathfrak{D}_0^\gamma \xi_{n-1}(s)\right) \mathrm{d}s \\
&\quad - \int_0^\varpi G(\zeta_1, s)\aleph\left(s, \xi_{n-1}(s), \mathfrak{D}_0^\gamma \xi_{n-1}(s)\right) \mathrm{d}s \Bigg| \\
&\leq \left| \left(e^{\frac{-\chi}{\gamma}\zeta_2} - e^{\frac{-\chi}{\gamma}\zeta_1}\right) \left[\frac{\varrho}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}}\right] \right| \\
&\quad + \sup_{(\zeta,\xi,\Im)\in\nabla\times[0,\delta_1]\times[0,\delta_2]} |\aleph(\zeta,\xi,\Im)| \int_0^\varpi |G(\zeta_2, s) - G(\zeta_1, s)| \,\mathrm{d}s.
\end{aligned}$$

As $\zeta_1 \longrightarrow \zeta_2$ the right hand side of the above inequality tends to zero. On the other hand, we have

$$\begin{aligned}
&\left| \left(\mathfrak{D}_0^\gamma \xi_n\right)(\zeta_2) - \left(\mathfrak{D}_0^\gamma \xi_n\right)(\zeta_1)\right| \\
&\quad \leq \left|\aleph\left(\zeta_2, \xi_{n-1}(\zeta_2), \mathfrak{D}_0^\gamma \xi_{n-1}(\zeta_2)\right) - \aleph\left(\zeta_1, \xi_{n-1}(\zeta_1), \mathfrak{D}_0^\gamma \xi_{n-1}(\zeta_1)\right)\right| \\
&\quad \longrightarrow 0, \text{ as } \zeta_1 \longrightarrow \zeta_2.
\end{aligned}$$

Thus,

$$\left| \left(\mathfrak{D}_0^\gamma \xi_n\right)(\zeta_2) - \left(\mathfrak{D}_0^\gamma \xi_n\right)(\zeta_1)\right| \longrightarrow 0, \text{ as } \zeta_1 \longrightarrow \zeta_2.$$

As a result, the sequences $\{\xi_n(\zeta);\ n \in \mathbb{N}\}$ and $\{\left(\mathfrak{D}_0^\gamma \xi_n\right)(\zeta);\ n \in \mathbb{N}\}$ are equicontinuous on ∇.

Let

$$\vartheta := \{\lambda \in [0,1] : \{\xi_n(\zeta);\ n \in \mathbb{N}\} \text{ converges uniformly on } \nabla_\lambda\}.$$

If $\vartheta = 1$, then we have the global convergence of successive approximations. Suppose that $s < 1$, then the sequence $\{\xi_n(\zeta);\ n \in \mathbb{N}\}$ converges uniformly on ∇_ϑ. As this sequence is equicontinuous, it converges uniformly to a continuous function $\tilde{\xi}(\zeta)$. In the case that we prove that there exists $\theta \in (\vartheta, 1]$ that $\{\xi_n(\zeta);\ n \in \mathbb{N}\}$ converges uniformly on ∇_θ, this will yield a contradiction.

Put $\xi(\zeta) = \tilde{\xi}(\zeta)$ for $\zeta \in \nabla_\vartheta$. From (H_2), there exist a constant $\varkappa > 0$ and a continuous function $h : \nabla \times [0, \varkappa] \times [0, \varkappa] \longrightarrow \mathbb{R}_+$ ensuring inequality (4.1). Also, there exist $\theta \in [\vartheta, 1]$ and $n_0 \in \mathbb{N}$, such that for all $\zeta \in \nabla_\theta$ and $n, m > n_0$, we have

$$|\xi_n(\zeta) - \xi_m(\zeta)| \leq \varkappa,$$

and

$$\left|\left(\mathfrak{D}_0^\gamma \xi_n\right)(\zeta) - \left(\mathfrak{D}_0^\gamma \xi_m\right)(\zeta)\right| \leq \varkappa.$$

For all $\zeta \in \nabla_\theta$, put

$$R^{(n,m)}(\zeta) = |\xi_n(\zeta) - \xi_m(\zeta)|,$$

$$R_\jmath(\zeta) = \sup_{n,m\geq \jmath} R^{(n,m)}(\zeta),$$

$$\left(\mathfrak{D}_0^\gamma R^{(n,m)}\right)(\zeta) = \left|\left(\mathfrak{D}_0^\gamma \xi_n\right)(\zeta) - \left(\mathfrak{D}_0^\gamma \xi_m\right)(\zeta)\right|,$$

and

$$\left(\mathfrak{D}_0^\gamma R_\jmath\right)(\zeta) = \sup_{n,m\geq \jmath} \left(\mathfrak{D}_0^\gamma R^{(n,m)}\right)(\zeta).$$

Since the sequence $R_\jmath(\zeta)$ is non-increasing, it is convergent to a function $R(\zeta)$ for each $\zeta \in \nabla_\theta$. From the equi-continuity of $\{R_\jmath(\zeta)\}$, it follows that $\lim_{\jmath\to\infty} R_\jmath(\zeta) = R(\zeta)$ uniformly on ∇_θ. Furthermore, for $\zeta \in \nabla_\theta$ and $n, m \geq \jmath$, we have

$$\begin{aligned} R^{(n,m)}(\zeta) &= |\xi_n(\zeta) - \xi_m(\zeta)| \\ &\leq \sup_{s\in[0,\zeta]} |\xi_n(\zeta) - \xi_m(\zeta)| \\ &\leq \int_0^\varpi |G(\zeta,s)| \left|\aleph(s, \xi_{n-1}(s), \left(\mathfrak{D}_0^\gamma \xi_{n-1}\right)(s)) - \aleph(s, \xi_{m-1}(s), \left(\mathfrak{D}_0^\gamma \xi_{m-1}\right)(s))\right| \mathrm{d}s. \end{aligned}$$

Then, by inequality (4.1), we have

$$\begin{aligned} R^{(n,m)}(\zeta) &\leq \int_0^\varpi |G(\zeta,s)| h\left(s, |\xi_{n-1}(s) - \xi_{m-1}(s)|, \left|\left(\mathfrak{D}_0^\gamma \xi_{n-1}\right)(s) - \left(\mathfrak{D}_0^\gamma \xi_{m-1}\right)(s)\right|\right) \mathrm{d}s \\ &\leq \int_0^\varpi |G(\zeta,s)| h\left(s, R^{(n-1,m-1)}(s), \left(\mathfrak{D}_0^\gamma R^{(n-1,m-1)}\right)(s)\right) \mathrm{d}s. \end{aligned}$$

Thus,

$$R_\jmath(\zeta) \leq \int_0^\varpi |G(\zeta,s)| h\left(s, R_{\jmath-1}(s), \left(\mathfrak{D}_0^\gamma R_{\jmath-1}\right)(s)\right) \mathrm{d}s.$$

By the Lebesgue dominated convergence theorem we have

$$R(\zeta) \leq \int_0^\varpi |G(\zeta,s)| h\left(s, R(s), \left(\mathfrak{D}_0^\gamma R\right)(s)\right) \mathrm{d}s.$$

Then, by (H_3) we get $R \equiv 0$ on ∇_θ, which yields that $\lim_{\jmath\to\infty} R_\jmath(\zeta) = 0$ uniformly on ∇_θ. Thus, $\{\xi_\jmath(\zeta)\}_{\jmath=1}^\infty$ is a Cauchy sequence on ∇_θ. Therefore, $\{\xi_\jmath(\zeta)\}_{\jmath=1}^\infty$ is uniformly convergent on ∇_θ, which yields the contradiction.

Also, $\{\xi_\jmath(\zeta)\}_{\jmath=1}^\infty$ converges uniformly on ∇ to a continuous function $\xi_*(\zeta)$. We get

$$\begin{aligned} &\lim_{\jmath\to\infty} \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \int_0^\varpi G(\zeta,s) h\left(s, \xi_\jmath(s), \left(\mathfrak{D}_0^\gamma \xi_\jmath\right)(s)\right) \mathrm{d}s \\ &\quad = \frac{\varrho e^{\frac{-\chi}{\gamma}\zeta}}{\imath + \jmath e^{\frac{-\chi}{\gamma}\varpi}} - \int_0^\varpi G(\zeta,s) h\left(s, \xi_*(s), \left(\mathfrak{D}_0^\gamma \xi_*\right)(s)\right) \mathrm{d}s, \end{aligned}$$

for all $\zeta \in \nabla$. This means that ξ_* is a solution of the problem (1.1).

Let us now prove the uniqueness result of the problem (1.1). Let ξ_1 and ξ_2 be two solutions of (1.1). As above, put

$$\widehat{\vartheta} := \{\lambda \in [0,1] : \xi_1(\zeta) = \xi_2(\zeta) \text{ for } \zeta \in \nabla_\lambda\},$$

and suppose that $\widehat{\vartheta} < 1$. There exist a constant $\varkappa > 0$ and a comparison function $h : \nabla_{\widehat{\vartheta}} \times [0,\varkappa] \times [0,\varkappa] \longrightarrow \mathbb{R}_+$ verifying inequality (4.1). We take $\theta \in (\lambda, 1)$ such that

$$|\xi_1(\zeta) - \xi_2(\zeta)| \leq \varkappa,$$

and

$$|(\mathfrak{D}_0^\gamma \xi_1)(\zeta) - (\mathfrak{D}_0^\gamma \xi_2)(\zeta)| \leq \varkappa,$$

for $\zeta \in \nabla_\theta$. Then, for all $\zeta \in \nabla_\theta$, we have

$$\begin{aligned} |\xi_1(\zeta) - \xi_2(\zeta)| &\leq \int_0^\varpi |G(\zeta,s)|\, |\aleph(s,\xi_1(s),(\mathfrak{D}_0^\gamma \xi_1)(s)) - \aleph(s,\xi_2(s),(\mathfrak{D}_0^\gamma \xi_2)(s))|\, \mathrm{d}s \\ &\leq \int_0^\varpi |G(\zeta,s)| h\left(s, |\xi_1(s) - \xi_2(s)|, |(\mathfrak{D}_0^\gamma \xi_1)(s) - (\mathfrak{D}_0^\gamma \xi_2)(s)|\right) \mathrm{d}s. \end{aligned}$$

Again, by (H_3) we get $\xi_1 - \xi_2 \equiv 0$ on ∇_θ. This gives us $\xi_1 = \xi_2$ on ∇_θ, which gives a contradiction. Consequently, $\widehat{\vartheta} = 1$ and the solution of the problem (1.1) is unique on ∇. □

5 Some Examples

We give now some examples that illustrate our obtained results throughout the paper.

Example 5.1 *Consider the following problem:*

$$\begin{cases} (\mathfrak{D}_0^{\frac{1}{2}} \xi)(\zeta) = \dfrac{1}{90\,(1 + |\xi(\zeta)|)} + \dfrac{1}{30\left(1 + |(\mathfrak{D}_0^{\frac{1}{2}} \xi)(\zeta)|\right)};\ \zeta \in [0,1], \\ \xi(0) + \xi(1) = 0. \end{cases} \tag{5.1}$$

Set

$$\aleph(\zeta,\xi,\Im) = \frac{1}{90\,(1+|\xi|)} + \frac{1}{30\,(1+|\Im|)};\ \zeta \in [0,1],\ \xi,\Im \in \mathbb{R}.$$

For any $\xi, \widetilde{\xi}, \Im, \widetilde{\Im} \in \mathbb{R}$, and $\zeta \in [0,1]$, we have

$$|\aleph(\zeta,\xi,\Im) - \aleph(\zeta,\widetilde{\xi},\widetilde{\Im})| \leq \frac{1}{90}|\xi - \widetilde{\xi}| + \frac{1}{30}|\Im - \widetilde{\Im}|.$$

Hence hypothesis (H_1) is satisfied with

$$\omega_1 = \frac{1}{90} \quad and \quad \omega_2 = \frac{1}{30}.$$

Next, the condition (3.1) is verified with $\chi = 1$ and $\gamma = \frac{1}{2}$. Indeed,

$$\frac{(\imath e^{\frac{\chi}{\gamma}\varpi} + 2\jmath)\omega_1}{\chi(\imath e^{\frac{\chi}{\gamma}\varpi} + \jmath)(1-\omega_2)} = \frac{(e^2+2)\frac{1}{90}}{(e^2+1)(1-\frac{1}{30})} < 1.$$

Some calculations indicate that all of the requirements of Theorem 3.1 are verified. Thus, (5.1) has at least a solution.

Example 5.2 *We consider the following problem involving the improved Caputo-type conformable fractional derivative:*

$$\begin{cases} (\mathfrak{D}_0^{\frac{1}{3}}\xi)(\zeta) = \dfrac{8e^{\zeta}+3\zeta^3+1}{83e^{\zeta+1}(1+|\xi(\zeta)|+|(\mathfrak{D}_0^{\frac{1}{3}}\xi)(\zeta)|}, \;\; \zeta\in[0,\pi], \\ \\ \xi(0)+\xi(\pi)=0. \end{cases} \tag{5.2}$$

Set

$$\aleph(\zeta,\xi(\zeta),(\mathfrak{D}_0^{\frac{1}{3}}\xi)(\zeta)) = \frac{8e^{\zeta}+3\zeta^3+1}{83e^{\zeta+1}(1+|\xi(\zeta)|+|(\mathfrak{D}_0^{\frac{1}{3}}\xi)(\zeta)|},$$

where $\gamma=\frac{1}{3}$.

For each $\xi_1,\bar{\xi_1},\xi_2,\bar{\xi_2}\in\mathbb{R}$ *and* $\zeta\in[0,\pi]$, *we have*

$$|\aleph(\zeta,\xi_1,\xi_2)-\aleph(\zeta,\bar{\xi_1},\bar{\xi_2})| \leq \frac{8e^{\zeta}+3\zeta^3+1}{83e^{\zeta+1}}\left[|\xi_1-\bar{\xi_1}|+|\xi_2-\bar{\xi_2}|\right].$$

Therefore, (H_2) *is verified for all* $\zeta\in[0,\pi]$, $\varkappa>0$ *and the comparison function* $h:\nabla\times[0,\varkappa]\times[0,\varkappa]\longrightarrow\mathbb{R}_+$ *is defined by:*

$$h(\zeta,\xi_1,\xi_2) = \frac{8e^{\zeta}+3\zeta^3+1}{83e^{\zeta+1}}(\xi_1+\xi_2).$$

Moreover, we have

$$\lim_{\zeta_1\longrightarrow\zeta_2}\left(\aleph\left(\zeta_2,\xi_1,\xi_2\right)-\aleph\left(\zeta_1,\xi_1,\xi_2\right)\right)=0.$$

Thus, the hypothesis (H_4) *is verified. Consequently, Theorem 4.1 means that the successive approximations* ξ_n; $n\in\mathbb{N}$, *defined by*

$$\begin{aligned} &\xi_0(\zeta)=0, \quad \zeta\in[0,\pi], \\ &\xi_{n+1}(\zeta) = -\int_0^{\pi}\frac{G(\zeta,s)(8e^s+3s^3+1)}{83e^{s+1}(1+|\xi_n(s)|+|(\mathfrak{D}_0^{\frac{1}{3}}\xi_n)(s)|}\,\mathrm{d}s, \end{aligned}$$

converges uniformly on $[0,\pi]$ *to the unique solution of the problem (5.2).*

Declarations

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References

[1] S. Abbas, A. Arara, M. Benchohra, *Global convergence of successive approximations for abstract semilinear differential equations*, Panamerican Mathematical Journal **29** (2019), pp. 17–31.

[2] S. Abbas, M. Benchohra, J. R. Graef, J. Henderson, *Implicit Fractional Differential and Integral Equations: Existence and Stability*, De Gruyter, Berlin, 2018.

[3] S. Abbas, M. Benchohra, N. Hamidi, *Successive approximations for the Darboux problem for implicit partial differential equations*, Panamerican Mathematical Journal **28** (2018), pp. 1–10.

[4] P. Ahuja, A. Ujlayan, D. Sharma, H. Pratap, *Deformable Laplace transform and its applications*, Nonlinear Engineering **12** (2023): 20220278. https://doi.org/10.1515/nleng-2022-0278

[5] H. Baghani, J. J. Nieto, *On fractional Langevin equation involving two fractional orders in different intervals*, Nonlinear Analysis: Modelling and Control **24** (2019), pp. 884–897.

[6] H. Baghani, J. Alzabut, J. J. Nieto, *A coupled system of Langevin differential equations of fractional order and associated to antiperiodic boundary conditions*, Mathematical Methods in the Applied Sciences **47** (2024), pp. 10900–10910.

[7] S. Bouriah, A. Salim, M. Benchohra, *On nonlinear implicit neutral generalized Hilfer fractional differential equations with terminal conditions and delay*, Topological Algebra and its Applications **10** (2022), pp. 77–93. https://doi.org/10.1515/taa-2022-0115

[8] F. Browder, *On the convergence of successive approximations for nonlinear functional equations*, Indagationes Mathematicae **30** (1968), pp. 27–35.

[9] H. Y. Chen, *Successive approximations for solutions of functional integral equations*, Journal of Mathematical Analysis and Applications **80** (1981), pp. 19–30.

[10] T. Czlapiiǹski, *Global convergence of successive approximations of the Darboux problem for partial functional differential equations with infinite delay*, Opuscula Mathematica **34** (2014), pp. 327–338.

[11] C. Derbazi, H. Hammouche, A. Salim, M. Benchohra, *Measure of noncompactness and fractional Hybrid differential equations with Hybrid conditions*, Differential Equations & Applications **14** (2022), pp. 145–161. http://dx.doi.org/10.7153/dea-2022-14-09

[12] C. Derbazi, H. Hammouche, A. Salim, M. Benchohra, *Weak solutions for fractional Langevin equations involving two fractional orders in Banach spaces*, Afrika Matematika **34** (2023): 1. https://doi.org/10.1007/s13370-022-01035-3

[13] M. Etefa, G. M. N'Guérékata, *Existence and stability analysis of solutions for fractional differential equations with delay*, Malaya Journal of Matematik **11(S)** (2023), pp. 43–52. http://doi.org/10.26637/mjm11S/003

[14] L. Faina, *The generic property of global convergence of successive approximations for functional differential equations with infinite delay*, Communications in Applied Analysis **3** (1999), pp. 219–234.

[15] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[16] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[17] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, 2006.

[18] S. Krim, A. Salim, S. Abbas, M. Benchohra, *On implicit impulsive conformable fractional differential equations with infinite delay in b-metric spaces*, Rendiconti del Circolo Matematico di Palermo Series 2 **72** (2022), pp. 2579–2592. https://doi.org/10.1007/s12215-022-00818-8

[19] K. D. Kucche, J. J. Nieto, V. Venktesh, *Theory of nonlinear implicit fractional differential equations*, Differential Equations and Dynamical Systems **28** (2020), pp. 1–17.

[20] M. Mebrat, G. M. N'Guérékata, *A Cauchy problem for some fractional differential equation via deformable derivatives*, Journal of Nonlinear Evolution Equations and Applications **2020** (2020), pp. 55–63.

[21] A. Meraj, D. N. Pandey, *Existence and uniqueness of mild solution and approximate controllability of fractional evolution equations with deformable fractional derivative*, Journal of Nonlinear Evolution Equations and Applications **2018** (2019), pp. 85–100.

[22] W. Rahou, A. Salim, J. E. Lazreg, M. Benchohra, *Existence and stability results for impulsive implicit fractional differential equations with delay and Riesz–Caputo derivative*, Mediterranean Journal of Mathematics **20** (2023): 143. https://doi.org/10.1007/s00009-023-02356-8

[23] A. Salim, S. Abbas, M. Benchohra, E. Karapınar, *A Filippov's theorem and topological structure of solution sets for fractional q-difference inclusions*, Dynamic Systems and Applications **31** (2022), pp. 17–34. https://doi.org/10.46719/dsa202231.01.02

[24] A. Salim, S. Abbas, M. Benchohra, E. Karapınar, *Global stability results for Volterra-Hadamard random partial fractional integral equations*, Rendiconti del Circolo Matematico di Palermo Series 2 **72** (2023), pp. 1783-1795. https://doi.org/10.1007/s12215-022-00770-7

[25] A. Salim, B. Ahmad, M. Benchohra, J. E. Lazreg, *Boundary value problem for hybrid generalized Hilfer fractional differential equations*, Differential Equations & Applications **14** (2022), pp. 379–391.

[26] A. Salim, S. Krim, S. Abbas, M. Benchohra, *On deformable implicit fractional differential equations in b-metric spaces*, Journal of Mathematical Extension **17** (2023), pp. 1–17. https://doi.org/10.30495/JME.2023.2468

[27] A. A. Kilbas, O. I. Marichev, S. G. Samko, *Fractional Integrals and Derivatives (Theory and Applications)*, Gordon and Breach, Switzerland, 1993.

[28] F. S. Bachir, S. Abbas, M. Benbachir, M. Benchohra, *Successive approximations for random coupled Hilfer fractional differential systems*, Arabian Journal of Mathematics **2** (2021), pp. 301–310.

[29] F. S. Bachir, S. Abbas, M. Benbachir, M. Benchohra, C. Tunç, *Uniform convergence of successive approximations for hybrid Caputo fractional differential equations*, Palestine Journal of Mathematics **3** (2022), pp. 650–658.

[30] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg; Higher Education Press, Beijing, 2011.

[31] F. Zulfeqarr, A. Ujlayan, P. Ahuja, *A new fractional derivative and its fractional integral with some applications*, arXiv preprint (2017), arXiv: 1705.00962v1, 11 pages.