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ASYMPTOTICALLY ALMOST PERIODIC SOLUTIONS OF ABSTRACT NONLINEAR VOLTERRA EQUATIONS

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Abstract. In this note, we analyze the existence and uniqueness of asymptotically almost periodic solutions for a class of the abstract nonlinear Volterra integro-differential equations with Lipschitz continuous operators. We also provide the basic results about the class of mild (a, k, C, B)-regularized resolvent families in the nonlinear setting and the well-posedness of abstract nonlinear Volterra inclusions.

Keywords: Asymptotically almost periodic functions, abstract nonlinear Volterra equations, (a, k, C, B)-regularized resolvent families.

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1 Introduction

In the existing literature concerning the abstract nonlinear Volterra integro-differential equations (cf. [2, 7, 10, 11, 17, 19] and the references quoted therein), we have not been able to locate any relevant result with regards to the question of the existence and uniqueness of asymptotically almost periodic solutions.

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In this note, we will present a new interesting result concerning the existence and uniqueness of asymptotically almost periodic solutions to the abstract nonlinear Volterra integral equations with Lipschitz continuous operators (Theorem 3.1); the proof of this result is relatively simple and it is based on the use of the Banach contraction principle. The operators under our consideration fail to be m-accretive in any sense and our results cannot be formulated for the class of asymptotically periodic functions or the class of pseudo-almost periodic functions. As mentioned in the abstract, we also introduce and analyze here the class of mild (a, k, C, B)-regularized resolvent families in the nonlinear setting and provide the basic details about the well-posedness of abstract nonlinear Volterra inclusions.

2 Asymptotically almost periodic functions

The class of almost periodic functions was introduced by H. Bohr around 1925 and later generalized by many others ([5]). Let $(X, \|\cdot\|)$ be a complex Banach space and let $f : I \to X$ be a continuous function, where $I = [0, \infty)$ or $I = \mathbb{R}$. If a number $\epsilon > 0$ is given, then we call a number $\tau > 0$ an ϵ-period for $f(\cdot)$ if $\|f(t+\tau) - f(t)\| \leq \epsilon$ for all $t \in I$; by $\vartheta(f, \epsilon)$ we denote the set consisting of all ϵ-periods for $f(\cdot)$. It is said that $f(\cdot)$ is almost periodic if for each $\epsilon > 0$ the set $\vartheta(f, \epsilon)$ is relatively dense in $[0, \infty)$, which means that for each $\epsilon > 0$ there exists a finite real number $l > 0$ such that any subinterval I' of $[0, \infty)$ of length l meets $\vartheta(f, \epsilon)$.

Following M. Fréchet (1941), we say that a continuous function $f : [0, \infty) \to X$ is asymptotically almost periodic if there exist an almost periodic function $g : \mathbb{R} \to X$ and a continuous function $q : [0, \infty) \to X$ vanishing at plus infinity such that $f(t) = g(t) + q(t)$ for all $t \geq 0$. By $AAP([0, \infty) : X)$ we denote the vector space of all asymptotically almost periodic functions $f : [0, \infty) \to X$; equipped with the sup-norm, $AAP([0, \infty) : X)$ is a Banach space. The Bochner criterion says that a continuous function $f : [0, \infty) \to X$ is asymptotically almost periodic if for each sequence (b_k) in $[0, \infty)$ there exist a subsequence (b_{k_l}) of (b_k) and a function $f^* : [0, \infty) \to X$ such that $\lim_{l \to +\infty} f(t + b_{k_l}) = f^*(t)$, uniformly for $t \geq 0$.

For further information on almost periodic type functions and their applications, we refer the reader to the research monographs [4, 6, 8, 9, 12, 13, 16] and references quoted therein.

3 The existence and uniqueness of asymptotically almost periodic solutions to (3.1)

Suppose that $B : X \to X$ and $\|Bx - By\| \leq L\|x - y\|$, $x,\ y \in X$ for some finite real constant $L > 0$. Of concern is the following abstract nonlinear Volterra integral equation

$$u(t) = b(t) + \int_0^t a(t-s)Bu(s)\ \mathrm{d}s, \quad t \geq 0. \tag{3.1}$$

It is worth noting that the Banach contraction principle can be successfully applied in the analysis of the existence of a unique asymptotically almost periodic solution of problem (3.1), provided that $a \in L^1([0, \infty))$ and $b(\cdot)$ is asymptotically almost periodic.

Towards this end, observe first that the Bochner criterion and the Lipschitz continuity of the operator B together imply that the mapping $Bf(\cdot)$ is asymptotically almost periodic, provided that

$f(\cdot)$ is asymptotically almost periodic. Further on, we have

$$\begin{aligned}\int_0^t a(t-s)f(s)\ \mathrm{d}s &= \int_0^t a(t-s)[g(s)+q(s)]\ \mathrm{d}s = \int_{-\infty}^t a(t-s)g(s)\ \mathrm{d}s \\ &+ \left[\int_0^{t/2} a(t-s)q(s)\ \mathrm{d}s + \int_{t/2}^t a(t-s)q(s)\ \mathrm{d}s - \int_{-\infty}^0 a(t-s)g(s)\ \mathrm{d}s\right] \\ &:= G(t)+Q(t), \quad t \geq 0.\end{aligned}$$

Using the integrability of $a(\cdot)$ and the standard argumentation, it follows that the mapping $t \mapsto G(t)$, $t \in \mathbb{R}$ is almost periodic, the function $Q(\cdot)$ is continuous and $\lim_{t\to+\infty} Q(t) = 0$. Therefore, the function

$$t \mapsto b(t) + \int_0^t a(t-s)f(s)\ \mathrm{d}s, \quad t \geq 0$$

is asymptotically almost periodic provided that $f(\cdot)$ is asymptotically almost periodic.

By the foregoing, the operator $\Psi : AAP([0,\infty) : X) \to AAP([0,\infty) : X)$, given by

$$[\Psi(f)](t) := b(t) + \int_0^t a(t-s)Bf(s)\ \mathrm{d}s, \\ t \geq 0,\ f \in AAP([0,\infty) : X),$$

is well-defined. Furthermore, the assumption $L\|a\|_{L^1([0,\infty))} < 1$ implies that $\Psi(\cdot)$ is a contraction and an application of the Banach contraction principle yields that the following result holds true:

Theorem 3.1 *Suppose that $B : X \to X$, $\|Bx - By\| \leq L\|x-y\|$, $x,\ y \in X$ for some finite real constant $L > 0$, $a \in L^1([0,\infty))$, $L\|a\|_{L^1([0,\infty))} < 1$ and $b(\cdot)$ is asymptotically almost periodic. Then there exists a unique asymptotically almost periodic solution $u(t)$ of* (3.1).

We can similarly analyze the existence and uniqueness of asymptotically almost automorphic solutions of (3.1), provided that the function $b(\cdot)$ is asymptotically almost automorphic.

In connection with the above analysis, we would like to introduce the following notion (by I we denote the identity operator on X):

Definition 3.2 *Suppose $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $C : X \to X$ and $B : D(B) \subseteq X \to P(X)$ is a given function. Then we say that B is a subgenerator of [the generator, if $C = \mathrm{I}$] a (local, if $\tau < \infty$) mild (a,k,C,B)-regularized resolvent family $(R(t))_{t\in[0,\tau)}$ if $R(t) : X \to X$ is continuous for every $t \in [0,\tau)$, $(R(t))_{t\in[0,\tau)}$ is strongly continuous and, for every $x \in D(B)$, there exists a locally integrable mapping $t \mapsto r_B(t;x)$, $t \in [0,\tau)$ such that $r_B(t;x) \in BR(t)x$, $t \in [0,\tau)$ and*

$$\int_0^t a(t-s)r_B(s)\ \mathrm{d}s = R(t)x - k(t)Cx, \quad t \in [0,\tau). \tag{3.2}$$

If $C = \mathrm{I}$, then we omit the term "C" from the notation.

In the nonlinear setting, we would like to emphasize the following:

(i) It is redundant to assume that $R(t)B \subseteq BR(t)$, $t \in [0,\tau)$ or $R(t)x - k(t)Cx \in B\int_0^t a(t-s)R(s)x\,ds$, $t \in [0,\tau)$, $x \in X$.

(ii) The use of function $k(\cdot)$ is discutable because we cannot prove that B generates a mild $(a*b,k,B)$-regularized resolvent family $((b*R)(t))_{t\in[0,\tau)}$ provided that B generates a mild (a,k,B)-regularized resolvent family $(R(t))_{t\in[0,\tau)}$; here, $b \in L^1_{loc}([0,\tau))$.

If $\tau = \infty$, then we say that $(R(t))_{t\geq 0}$ is exponentially non-expansive (non-expansive) if there exists $\omega \in \mathbb{R}$ $(\omega = 0)$ such that

$$\|R(t)x - R(t)y\| \leq e^{\omega t}\|x-y\|;$$

the infimum of such numbers is said to be the exponential type of $(R(t))_{t\geq 0}$. In the local setting, the notion from Definition 3.2 can be also introduced for the strongly continuous operator families defined on the closed interval $[0,\tau]$, where $0<\tau<\infty$.

Let us consider now the Banach space $C([0,\tau] : X)$, if $0<\tau<\infty$, and the Banach space $AAP([0,\infty) : X)$, if $\tau = \infty$. Keeping in mind the argumentation contained in the proof of Theorem 3.1, the Banach contraction principle and the Grönwall inequality, we can deduce the following result (we define $R(t)x$ as a solution of (3.1) with $b(t) = k(t)Cx$):

Theorem 3.3 *(i) Suppose that $k \in C[0,\tau]$, $k \geq 0$, $a \in L^1[0,\tau]$, $B : X \to X$ is a Lipschitz continuous operator, $\|Bx - By\| \leq L\|x-y\|$, $x,\ y \in X$ for some finite real constant $L>0$, and $L\int_0^\tau |a(s)|\,ds < 1$. Then there exists a unique mild (a,k,B,C)-regularized resolvent family $(R(t))_{t\in[0,\tau]}$ subgenerated by B and the following holds:*

$$\begin{aligned}&\|R(t)x - R(t)y\|\\ &\leq \left[k(t) + \int_0^t k(s)|a(t-s)|\exp\left(\int_s^t |a(t-r)|\ \mathrm{d}r\right)\mathrm{d}s\right]\cdot\|Cx - Cy\|,\end{aligned} \tag{3.3}$$

for any $t \in [0,\tau]$ and $x,\ y \in X$. Moreover, if $k(\cdot)$ is non-decreasing, then

$$\|R(t)x - R(t)y\| \leq k(t)\exp\left(\int_0^t |a(s)|\ \mathrm{d}s\right)\cdot\|Cx - Cy\|, \tag{3.4}$$

for any $t \in [0,\tau]$ and $x,\ y \in X$.

(ii) Suppose that $k \in C([0,\infty))$, $k \geq 0$, $a \in L^1([0,\infty))$, $B : X \to X$ is a Lipschitz continuous operator, $\|Bx - By\| \leq L\|x-y\|$, $x,\ y \in X$ for some finite real constant $L>0$, and $L\int_0^\infty |a(s)|\,ds < 1$. Then there exists a unique mild (a,k,B,C)-regularized resolvent family $(R(t))_{t\geq 0}$ subgenerated by B and (3.3) *holds for any $\tau > 0$, resp.,* (3.4) *holds for any $\tau>0$, provided that $k(\cdot)$ is non-decreasing. Furthermore, if $k(\cdot)$ is asymptotically almost periodic, then the mapping $t \mapsto R(t)x$, $t \geq 0$ is asymptotically almost periodic for every element $x \in X$.*

If the operator $B : D(B) \subseteq X \to X$ is not Lipschitz continuous, then we must apply some other types of fixed point theorems in order to prove the existence of a local mild (a,k,B,C)-regularized resolvent family $(R(t))_{t\in[0,\tau]}$ subgenerated by B. It would be very tempting to apply the structural results about mild (a,k,B,C)-regularized resolvent families in the analysis of the abstract nonlinear Volterra integro-differential equations which do not involve Lipschitz continuous operators (for some recent results concerning the existence and uniqueness of almost periodic type solutions to the abstract nonlinear functional differential inclusions of first order which do not involve Lipschitz continuous operators, we refer the reader to [1, 18, 20, 21, 22] and references quoted therein).

4 The well-posedness of abstract nonlinear Volterra inclusions

Suppose that $0 < \tau \leq \infty$, $a \in L^1_{loc}([0,\tau))$ and $f : [0,\tau) \to X$. By a solution of the abstract Volterra integral inclusion

$$u(t) \in f(t) + \int_0^t a(t-s)Bu(s)\,\mathrm{d}s, \quad t \in [0,\tau), \tag{4.1}$$

we mean any continuous function $t \mapsto u(t)$, $t \in [0,\tau)$ such that there exists a locally integrable mapping $t \mapsto u_B(t)$, $t \in [0,\tau)$ such that $u_B(t) \in Bu(t)$, $t \in [0,\tau)$ and

$$u(t) = f(t) + \int_0^t a(t-s)u_B(s)\,\mathrm{d}s, \quad t \in [0,\tau).$$

Hence, if B is a subgenerator of a mild (a, k, C, B)-regularized resolvent family $(R(t))_{t\in[0,\tau)}$, then for each $x \in D(B)$ the function $t \mapsto R(t)x$, $t \in [0,\tau)$ is a solution of the problem (4.1) with $f(t) \equiv k(t)Cx$. Furthermore, if the requirements of Theorem 3.3(ii) hold and $k(\cdot)$ is asymptotically almost periodic, then the mapping $t \mapsto R(t)x$, $t \geq 0$ is asymptotically almost periodic for every element $x \in X$.

On the other hand, it is said that any $(m-1)$-times continuously differentiable function $t \mapsto u(t)$, $t \in [0,\tau)$ is a solution of the abstract fractional Cauchy inclusion

$$\mathbf{D}_t^{\alpha} u(t) \in Bu(t) + h(t), \quad t \in [0,\tau); \;\; u^{(j)}(0) = u_j, \quad 0 \leq j \leq m-1, \tag{4.2}$$

where $h : [0,\tau) \to X$ is a continuous mapping, $\alpha \in (0,\infty) \setminus \mathbb{N}$, $m = \lceil \alpha \rceil$ and $\mathbf{D}_t^{\alpha} u(t)$ is the Caputo fractional derivative of order α (cf. [14] for the notion), if the initial conditions are satisfied and there exists a continuous mapping $t \mapsto u_B(t)$, $t \in [0,\tau)$ such that $u_B(t) \in Bu(t)$, $t \in [0,\tau)$ and $\mathbf{D}_t^{\alpha} u(t) = u_B(t) + h(t)$, $t \in [0,\tau)$. The following statement can be proved in exactly the same way as in the linear case ([14]; cf. also the identity [3, (1.21)]):

Proposition 4.1 *Suppose that the mapping $t \mapsto u(t)$, $t \in [0,\tau)$ is $(m-1)$-times continuously differentiable. Then $u(\cdot)$ is a solution of the abstract fractional Cauchy inclusion* (4.2) *if and only if $u(\cdot)$ is a solution of the abstract Volterra integral inclusion* (4.1) *with $a(t) \equiv g_\alpha(t)$ and $f(t) \equiv \sum_{j=0}^{m-1} g_{j+1}(t)u_j + (g_\alpha * h)(t)$, $t \in [0,\tau)$; here, $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$, $t > 0$ ($\zeta > 0$), where $\Gamma(\cdot)$ is the Euler Gamma function.*

Furthermore, we have the following:

Proposition 4.2 *Suppose that $B : X \to X$, $\|Bx - By\| \leq L\|x - y\|$, $x,\ y \in X$ for some finite real constant $L > 0$ and for each $x \in X$ there exists a solution of the abstract fractional Cauchy inclusion* (4.2) *with $h(t) \equiv 0$, $u_0 = Cx$ and $u^{(j)}(0) = 0$ for $1 \leq j \leq m-1$. Then B subgenerates a mild $(a, 1, C, B)$-regularized resolvent family on $[0,\tau)$.*

Proof. We define $R(t)x := u(t;x)$, $x \in X$, $t \in [0,\tau)$, where $u(\cdot;x)$ is a solution of the abstract fractional Cauchy inclusion (4.2) with $h(t) \equiv 0$, $u_0 = Cx$ and $u^{(j)}(0) = 0$ for $1 \leq j \leq m-1$. Then it is clear that the family $(R(t))_{t\in[0,\tau)}$ is strongly continuous as well as that Proposition 4.1 shows that, for every $x \in D(B)$, there exists a locally integrable mapping $t \mapsto r_B(t;x)$, $t \in [0,\tau)$ such that $r_B(t;x) \in BR(t)x$, $t \in [0,\tau)$ and (3.2) holds with $k(t) \equiv 1$. It remains to be proved that the mapping $R(t) : X \to X$ is continuous for every fixed number $t \in [0,\tau)$. But, this simply follows from an application of the Gronwall inequality, since we have

$$\|R(t)x - R(t)y\| \leq \|Cx - Cy\| + L\int_0^t g_\alpha(t-s)\|R(s)x - R(s)y\|\, \mathrm{d}s, \quad x,\ y \in X.$$

□

5 Conclusions and final remarks

In this note, we have investigated the existence and uniqueness of asymptotically almost periodic solutions to the abstract nonlinear Volterra integro-differential equations with Lipschitz continuous operators. We have also introduced the class of mild (a,k,C,B)-regularized resolvent families in the nonlinear setting and provided some results concerning the well-posedness of abstract nonlinear Volterra inclusions.

The Lipschitz continuity of operator B seems to be almost inevitably assumed if we want to apply the Banach contraction principle in the analysis of the existence and uniqueness of asymptotically almost periodic solutions to (3.1). We close the paper with the observation that we can similarly analyze some classes of the abstract nonlinear functional Volterra equations with Lipschitz continuous operators, involving the bounded or unbounded delays; for example, in our recent joint research study [15] with H. C. Koyuncuoğlu and T. Katıcan, we have analyzed the asymptotic behavior of solutions to the abstract non-linear fractional neutral equation

$$D^{\alpha}_{a,b}\Big[u(t) - g\big(t,u_t\big)\Big] = \sum_{j=1}^{w} B_j(t)u(t+t_j) + f\big(t,u_t\big), \quad t \geq 0; \ \ u_0 = \xi,$$

where $B_j(t)$ are Lipschitz continuous operators, $D^{\alpha}_{a,b}\cdot$ is a generalized Hilfer (a,b,α)-fractional derivative and some extra assumptions are satisfied.

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