Journal of Nonlinear Evolution Equations and Applications ISSN 2161-3680
Volume 2024, Number 6, pp. 81–98 (August 2024) http://www.jneea.com

ON THE EXISTENCE OF MILD SOLUTIONS FOR SOME NONLOCAL PARTIAL NEUTRAL INTEGRODIFFERENTIAL EQUATIONS IN α-NORM

JAOUAD EL MATLOUB* **KHALIL EZZINBI**
Department of Mathematics, Faculty of Sciences Semlalia, Cadi Ayyad University,
B.P: 2390 Marrakech, Morocco

Received April 25, 2024

Accepted May 3, 2024

Communicated by Gaston M. N'Guérékata

Abstract. In this work, we study the existence of mild solutions in α-norm for a class of nonlocal neutral integrodifferential equations. Our approach relies on the utilization of a fixed-point theorem of Sadovskii-Krasnosel'skii type and the theory of resolvent operators.

Keywords: Mild solution, nonlocal condition, integrodifferential equation, α-norm, resolvent operator.

2010 Mathematics Subject Classification: 45K05, 47H10, 47D06.

1 Introduction

The primary objective of this study is to examine the existence of mild solutions in α-norm for the following nonlocal integrodifferential equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[u(t)+\displaystyle\int_0^t N(t-s)u(s)\,\mathrm{d}s\right] = -Au(t)+\displaystyle\int_0^t B(t-s)u(s)\,\mathrm{d}s + f(t,u(t)), \quad t\in[0,a], \\ u(0)=u_0+g(u), \end{cases} \tag{1.1}$$

where $u(\cdot)$ is the state variable taking values in a Banach space $(X, \|\cdot\|)$. The operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on $(X, \|\cdot\|)$, $B(t)$ is a closed linear operator with domain $D(A)$, $N(t)(t \geq 0)$ is bounded linear operators on X, $f : [0, a] \times X_\alpha \longrightarrow X$,

*e-mail address: jaouad.elmatloub07@gmail.com(Corresponding author)

© 2024 Journal of Nonlinear Evolution Equations and Applications, JNEEA.com

$g : C([0,a]; X_\alpha) \longrightarrow X_\alpha$ are given functions to be specified later, and X_α stands for the domain of the fractional power operator A^α equipped with an adequate norm which will be described in the sequel.

Neutral integro-differential equations provide an abstract framework for modeling various types of partial neutral integro-differential equations. These equations are encountered in problems related to heat flow in materials with memory, viscoelasticity, heat conduction, wave propagation, and numerous other physical phenomena.

In recent years, the theory of partial neutral integrodifferential equations with nonlocal conditions received a keen interest due to their applications in many practical dynamical phenomena arising in physics, technical sciences, engineering and economy. The importance of nonlocal conditions is manifested in their better effects in applications than the classical initial conditions $u(0) = u_0$, for more details, the reader may see [6, 7] and the references given there. For that reason, Byszewski *et al.* [4, 5] initiated the study of semilinear evolution nonlocal Cauchy problem. Furthermore, many authors extended the study to integrodifferential equations. For instance, Fu *et al.* [14, 15] examined the existence results for various class of nonlocal neutral integrodifferential equations. Dos Santos *et al.* [9] proved the existence results for several class of nonlocal fractional neutral integrodifferential equations with unbounded delay. Zhu *et al.* [22] investigated the existence and regularity of solutions for the following neutral integrodifferential problem with nonlocal condition:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[x(t) + \displaystyle\int_0^t N(t-s)x(s)\,\mathrm{d}s\right] = A[x(t) + \displaystyle\int_0^t F(t-s)x(s)\,\mathrm{d}s] + f(t, x(r(t))),\ t \in [0,T], \\ x(0) + g(x) = x_0, \end{cases}$$

in a Banach space X with $A(\cdot)$ the generator of an analytic semigroup and $F(t) : D(A) \longrightarrow D(A)$ and $N(t) : X \longrightarrow X$ are two families of bounded linear operators on X. Moreover, the function $r : [0,T] \longrightarrow [0,T]$ is continuous and satisfies $0 \le r(t) \le t$ and f, g are given functions.

Differential equations with input functions depending on the spatial derivatives of the state variable are frequently used for describing distributed or spatially distributed behavior. This kind of dependency, which is frequently seen in disciplines like physics, engineering, and biology, can result in complicated and sometimes nonlinear behavior. The dynamics of fluid flow in a pipe or the spread of a chemical reaction through a medium can be modeled using these differential equations in particular, where the input function could stand in for the flow rate or concentration of the fluid or chemical, respectively, and the spatial derivative of the state variable could stand in for the spatial variation of the fluid velocity or chemical concentration. As a model for this class one may take the following nonlocal problem:

$$\begin{cases} \dfrac{\partial}{\partial t}\Big[v(t,x) + \displaystyle\int_0^t (t-s)^\delta e^{-w(t-s)} v(s,x)\,\mathrm{d}s\Big] \\ = \dfrac{\partial^2}{\partial x^2} v(t,x) + \displaystyle\int_0^t e^{-\theta(t-s)} \dfrac{\partial^2}{\partial x^2} v(s,x)\,\mathrm{d}s + h(t, \dfrac{\partial}{\partial x} v(t,x)), & 0 \le x \le \pi,\ 0 \le t \le b, \\ v(t,0) = v(t,\pi) = 0, & t \in [0,b], \\ v(0,x) = v_0(x) + \displaystyle\int_0^\pi \gamma(x,y) \dfrac{\partial v(t_0,y)}{\partial y}\,\mathrm{d}y, & 0 \le x \le \pi. \end{cases}$$

Here the situation is completely different. Particularly, if we take $X = L^2([0,\pi])$, then the second variable of h is defined on $X_{\frac{1}{2}}$ and so the solutions can not be discussed on X as it cited in many papers, which implies that the results obtained for example in [11] become invalid.

When $N(\cdot) = 0$, Ezzinbi *et al.* [11] proved the existence of mild solutions to the nonlocal problem (1.1) when g and $(T(t))_{t\geq 0}$ are compact. The analysis uses the resolvent operator theory and the Leray-Schauder alternative. Further progress were made in [13], where the compactness of g has been avoided and that of $(T(t))_{t\geq 0}$ was replaced with the norm continuity of $(T(t))_{t\geq 0}$. The authors used, in their study, the fixed point theorem of Sadovskii-Krasnosel'skii type. Motivated by the works of Ezzinbi and Ghnimi [11, 13], we investigate the existence of mild solutions of problem (1.1) by using the resolvent operator in the sense given by Dos santos *et al.* [10], fractional power operators theory and α-norm. In particular, our focus lies on the existence of mild solutions for system (1.1) within the space X_α, instead of the entire state space X, without imposing the assumption of compactness on the semigroup $(T(t))_{t\geq 0}$. This approach allows us to extend the applicability of the results, even when the nonlinear function f in system (1.1) includes spatial derivatives. The key distinction is that the function f is defined on $[0, a] \times X_\alpha$, rather than $[0, a] \times X$. As we will see later, this alteration significantly complicates the discussion at hand.

The rest of this paper is arranged as follows. In section 2, we recall some fundamental properties of the analytic resolvent operator and fractional powers of closed operators. The existence of mild solutions will be given in section 3. Finally, in section 4, we give an example to apply the abstract results.

2 Preliminaries

2.1 Fractional power of closed operators

In the following, the domain $Y = D(A)$ of A is equipped with the graph norm denoted by $\|\cdot\|_1$. We assume that the resolvent set $\rho(A)$ contains the spectral eigenvalue 0. As a result, we may define the fractional power A^α for $0 < \alpha < 1$ as a closed linear invertible operator on its domain $X_\alpha = D(A^\alpha)$. The operator A^α is given by $A^\alpha = (A^{-\alpha})^{-1}$ where

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} T(t)\,\mathrm{d}t \ \text{ with } \ \Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t}\,\mathrm{d}t.$$

Moreover, for $(A^\alpha, D(A^\alpha))$, $0 < \alpha < 1$, and its inverse $A^{-\alpha}$, one has the following famous properties.

Theorem 2.1 *[19] Let $0 < \alpha < 1$. The following properties are true.*

1. *$X_\alpha = Im(A^{-\alpha})$ is a Banach space with the norm $\|x\|_\alpha = \|A^\alpha x\|$ for $x \in X_\alpha$.*
2. *A^α is a closed linear operator with domain X_α and we have $A^\alpha = (A^{-\alpha})^{-1}$.*
3. *$A^{-\alpha}$ is a bounded linear operator on X.*
4. *If $0 < \alpha \leq \beta$ then $D(A^\beta) \hookrightarrow D(A^\alpha)$. Moreover the injection is compact if $T(t)$ is compact for $t > 0$.*
5. *$T(t) : X \to X_\alpha$ for $t > 0$.*

6. *$A^\alpha T(t)x = T(t)A^\alpha x$ for $x \in X_\alpha$ and $t \geq 0$.*
7. *For every $t > 0$, $A^\alpha T(t)$ is bounded on X and there exist $M_\alpha > 0$ and $\beta \in (0, \delta)$ such that*

$$\|A^\alpha T(t)\| \leq M_\alpha \frac{e^{-\beta t}}{t^\alpha} \quad \textit{for} \;\; t > 0.$$

2.2 Analytic resolvent operators

Next, we present the basic theory of resolvent operators for the following linear integrodifferential equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\Big[u(t) + \displaystyle\int_0^t N(t-s)u(s)\,\mathrm{d}s\Big] = Au(t) + \displaystyle\int_0^t B(t-s)u(s)\,\mathrm{d}s, \quad t \geq 0, \\ \\ u(0) = u_0 \in X. \end{cases} \tag{2.1}$$

Definition 2.1 *[10] A one-parameter family of bounded linear operators $(R(t))_{t\geq 0}$ on X is called a resolvent operator of* (2.1) *if the following conditions are verified.*

(i) The function $R(\cdot) : [0,\infty) \to \mathcal{L}(X)$ is strongly continuous, exponentially bounded and $R(0)x = x$ for all $x \in X$.

(ii) For $x \in D(A), R(\cdot)x \in C([0,\infty), Y) \cap C^1([0,\infty), X)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[R(t)x + \int_0^t N(t-s)R(s)x\,\mathrm{d}s\right] = ARx(t) + \int_0^t B(t-s)R(s)x\,\mathrm{d}s,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[R(t)x + \int_0^t R(t-s)N(s)x\,\mathrm{d}s\right] = R(t)Ax + \int_0^t R(t-s)B(s)x\,\mathrm{d}s,$$

for every $t \geq 0$.

In [10], the authors obtained an analytic resolvent operator of Eq.(2.1) under the following assumptions. The notation $\widehat{f}$ represents the Laplace transform of f.

($\mathbf{V}_1$) The operator $-A : D(A) \subseteq X \to X$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X. In this paper, $M_0 > 0$ and $\vartheta \in (\pi/2, \pi)$ are constants such that $\rho(A) \supseteq \Lambda_\vartheta = \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg(\lambda)| < \vartheta\}$ and $\|R(\lambda, A)\| \leq M_0|\lambda|^{-1}$ for all $\lambda \in \Lambda_\vartheta$.

($\mathbf{V}_2$) The function $N : [0,\infty) \to \mathcal{L}(X)$ is strongly continuous and $\widehat{N}(\lambda)x$ is absolutely convergent for $x \in X$ and $\mathrm{Re}(\lambda) > 0$. There exist $\alpha > 0$ and an analytical extension of $\widehat{N}(\lambda)$ (still denoted by $\widehat{N}(\lambda)$) to Λ_ϑ such that $\|\widehat{N}(\lambda)\| \leq N_0|\lambda|^{-\alpha}$ for every $\lambda \in \Lambda_\vartheta$, and $\|\widehat{N}(\lambda)x\| \leq N_1|\lambda|^{-1}\|x\|_1$ for every $\lambda \in \Lambda_\vartheta$ and $x \in D(A)$.

($\mathbf{V}_3$) For all $t \geq 0, B(t) : D(B(t)) \subseteq X \to X$ is a closed linear operator, $D(A) \subseteq D(B(t))$ and $B(\cdot)x$ is strongly measurable on $(0,\infty)$ for each $x \in D(A)$. There exists a $b(\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ such that $\widehat{b}(\lambda)$ exists for $\mathrm{Re}(\lambda) > 0$ and $\|B(t)x\| \leq b(t)\|x\|_1$ for all $t > 0$ and $x \in D(A)$. Moreover, the operator valued function $\widehat{B} : \Lambda_{\pi/2} \to \mathcal{L}(Y, X)$ has an analytical extension (still denoted by $\widehat{B}$) to Λ_ϑ such that $\|\widehat{B}(\lambda)x\| \leq \|\widehat{B}(\lambda)\|\|x\|_1$ for all $x \in D(A)$, and $\|\widehat{B}(\lambda)\| \to 0$ as $|\lambda| \to \infty$.

($\mathbf{V}_4$) There exist a subspace $D \subseteq D(A)$ dense in Y and positive constants $C_i, i = 1, 2$, such that $A(D) \subseteq D(A), \widehat{B}(\lambda)(D) \subseteq D(A)$, $\widehat{N}(\lambda)(D) \subseteq D(A), \|A\widehat{B}(\lambda)x\| \leq C_1\|x\|$ and $\|\widehat{N}(\lambda)x\|_1 \leq C_2|\lambda|^{-\alpha}\|x\|_1$ for every $x \in D$ and all $\lambda \in \Lambda_\vartheta$.

($\mathbf{V}_5$) The operator $\widehat{N}(\lambda) : X_\alpha \to X_\alpha$ for $\lambda \in \Lambda_\vartheta, 0 < \alpha < 1$, and $\|\widehat{N}(\lambda)\|_\alpha \to 0$ as $|\lambda| \to \infty$ uniformly for $\lambda \in \Lambda_\vartheta$.

In the following, let $r > 0$ and $\theta \in \left(\frac{\pi}{2}, \vartheta\right)$. We define the sets $\Lambda_{r,\theta}, \Gamma_{r,\theta}, \Gamma^i_{r,\theta}$ (for $i = 1, 2, 3$) as follows:

$$\begin{aligned}
\Lambda_{r,\theta} &= \{\lambda \in \mathbf{C}\backslash 0 : |\lambda| > r, |\arg(\lambda)| < \theta\},\\
\Gamma^1_{r,\theta} &= \{te^{i\theta} : t \geq r\},\\
\Gamma^2_{r,\theta} &= \{re^{i\xi} : -\theta \leq \xi \leq \theta\},\\
\Gamma^3_{r,\theta} &= \{te^{-i\theta} : t \geq r\},
\end{aligned}$$

and

$$\Gamma_{r,\theta} = \bigcup_{i=1}^{3} \Gamma^i_{r,\theta}$$

oriented counterclockwise.

Additionally, we define the sets $\Omega(F)$ and $\Omega(G)$ as follows:

$$\begin{aligned}
\Omega(F) &= \{\lambda \in \mathbb{C} : F(\lambda) = (\lambda I + \lambda\hat{N}(\lambda) - A)^{-1} \in \mathcal{L}(X)\},\\
\Omega(G) &= \{\lambda \in \mathbb{C} : G(\lambda) = (\lambda I + \lambda\widehat{N}(\lambda) - A - \widehat{B}(\lambda))^{-1} \in \mathcal{L}(X)\}.
\end{aligned}$$

Theorem 2.2 *[10] Assume that conditions* ($\mathbf{V}_1$)-($\mathbf{V}_5$) *are fulfilled. Then there exists a unique resolvent operator for Eq.*(2.1) *defined by*

$$R(t) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_{r,\theta}} e^{\lambda t} G(\lambda)\,\mathrm{d}\lambda & t > 0,\\ I & t = 0.\end{cases}$$

Moreover, there exist positive constants M, N_α such that $\|R(t)\| \leq M$ and $\|A^\alpha R(t)\| \leq \dfrac{N_\alpha}{t^\alpha}$ for $\alpha \in (0, 1)$ and $t > 0$.

The following Lemmas are essential in our work.

Lemma 2.1 *Assume that* ($\mathbf{V}_1$)-($\mathbf{V}_5$) *hold. Then $t \mapsto AR(t)$ is norm continuous (or continuous in the uniform operator topology) for $t > 0$.*

Proof. By employing the estimation (2.12) in [10, Lemma 2.2], we get the result. □

Lemma 2.2 *Let $0 < \alpha < 1$. Assume that* ($\mathbf{V}_1$)-($\mathbf{V}_5$) *hold. Then, the map $t \mapsto A^\alpha R(t)$ is norm continuous (or continuous in the uniform operator topology) for $t > 0$.*

Proof. Let $x \in X$ and $t_0 > 0$. Keeping in mind the moment formula [19], we know that there exists a positive constant ϱ such that

$$\|A^\alpha R(t)x - A^\alpha R(t_0)x\| \leq \varrho \|R(t)x - R(t_0)x\|^{1-\alpha} \|AR(t)x - AR(t_0)x\|^\alpha.$$

Since $(R(t))_{t\geq 0}$ is analytic, then by Lemma 2.1, we conclude that the map $t \mapsto A^\alpha R(t)$ is continuous in the uniform operator topology on $(0, +\infty)$. □

Remark 2.1 In general, $R(t)$ and $(-A)^\alpha$ do not commute. However, under the following condition

$$(\widehat{N}(\lambda) - \widehat{B}(\lambda))(-A)^{-\alpha}x = (-A)^{-\alpha}(\widehat{N}(\lambda) - \widehat{B}(\lambda))x \text{ for } x \in D(A),$$

this commutation can be achieved. In this paper, we make the assumption that this condition is always satisfied, ensuring that $(-A)^\alpha$ commutes with $R(t)$ for any $0 < \alpha < 1$.

2.3 The Kuratowski measure of noncompactness

In this part, we describe briefly the Kuratowski measure of noncompactness. Recall that the Kuratowski measure of noncompactness is defined by:

$$\beta(B) = \inf\{d > 0; B \text{ can be covered by a finite number of sets of diameter less than } d\}$$

for each bounded subset B of X.

The following lemmas contain several properties of $\beta(\cdot)$ that are useful for our purpose.

Lemma 2.3 *[1] Let $B, C \subseteq X$ be bounded, Then*

(1) $\beta(B) = 0$ *if and only if B is relatively compact,*

(2) $\beta(B) = \beta(\bar{B}) = \beta(\overline{\text{co}}B)$, *where $\overline{\text{co}}B$ is the closed convex hull of B,*

(3) $\beta(B) \leq \beta(C)$ *when $B \subseteq C$,*

(4) $\beta(B + C) \leq \beta(B) + \beta(C)$,

(5) $\beta(B \cup C) \leq \max\{\beta(B), \beta(C)\}$,

(6) $\beta(B(0, r)) \leq 2r$, *where $B(0, r) = \{x \in X : \|x\| \leq r\}$.*

By a measure of noncompactness on a Banach space X we mean a map $\psi : \mathcal{B}(X) \to \mathbb{R}_+$which satisfies conditions $(1) - (5)$ in Lemma 2.3 where $\mathcal{B}(X)$ stands for the collection of bounded subsets of X.

For later use, we recall the following lemmas.

Lemma 2.4 *[1] Let $G : X \longrightarrow X$ be a Lipschitz continuous map with constant k. Then $\beta(G(B)) \leq k\beta(B)$ for any bounded subset B of X.*

Lemma 2.5 *[20] Let $\mathcal{H} \subseteq C([0,a],X)$ be equicontinuous and $x_0 \in C([0,a],X)$. Then $\overline{co}\,(\mathcal{H} \cup \{x_0\})$ is also equicontinuous in $C([0,a],X)$.*

Lemma 2.6 *[1] Let $\mathcal{H} \subset C([0,a];X)$ be a bounded set. Then $\beta(\mathcal{H}(t)) \leq \beta(\mathcal{H})$ for any $t \in [0,a]$, where $\mathcal{H}(t) = \{u(t) : u \in \mathcal{H}\}$. Furthermore if $\mathcal{H}$ is equicontinuous on $[0,a]$, then $t \longmapsto \beta(\mathcal{H}(t))$ is continuous on $[0,a]$ and*

$$\beta(\mathcal{H}) = \sup\{\beta(\mathcal{H}(t)) : t \in [0,a]\}.$$

Definition 2.2 A set of functions $\mathcal{H} \subseteq L^1([0,a];X)$ is said to be uniformly integrable if there exists a positive function $\kappa \in L^1\left([0,a];\mathbb{R}^+\right)$such that $\|h(t)\| \leq \kappa(t)$ a.e. for all $h \in \mathcal{H}$.

Lemma 2.7 *[17] If $\{u_n\}_{n\in\mathbb{N}} \subseteq L^1([0,a];X)$ is uniformly integrable, then for each $n \in \mathbb{N}$ the function $t \mapsto \beta\left(\{u_n(t)\}_{n\in\mathbb{N}}\right)$ is measurable and*

$$\beta\left(\left\{\int_0^t u_n(s)\,\mathrm{d}s\right\}_{n=1}^{\infty}\right) \leq 2\int_0^t \beta\left(\{u_n(s)\}_{n=1}^{\infty}\right)\,\mathrm{d}s.$$

Lemma 2.8 *[2] Let $\mathcal{H}$ be a bounded subset of X. Then for each $\varepsilon > 0$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$ such that*

$$\beta(\mathcal{H}) \leq 2\beta\left(\{u_n\}_{n=1}^{\infty}\right) + \varepsilon.$$

We also need the following elementary result.

Lemma 2.9 *[16] For all $0 \leq m \leq n$, we denote $C_n^m = \begin{pmatrix} m \\ n \end{pmatrix}$. Let $0 < \epsilon < 1, h > 0$ and*

$$S_n = \epsilon^n + C_n^1\epsilon^{n-1}h + C_n^2\epsilon^{n-2}\frac{h^2}{2!} + \cdots + \frac{h^n}{n!},\ n \in \mathbb{N}.$$

Then, $\lim_{n\to\infty} S_n = 0$.

Now, we introduce the following sets. Let M be a nonempty closed convex subset of X and $K, S : M \to X$ be two nonlinear mappings and $x_0 \in X$. For any $D \subseteq M$, we define

$$\mathcal{F}^{(1,x_0)}(K,S,D) = \{x \in M : x = Sx + Ky,\ \text{for some } y \in D\}$$

and

$$\mathcal{F}^{(n,x_0)}(K,S,D) = \mathcal{F}^{(1,x_0)}\left(K,S,\overline{co}\left(\mathcal{F}^{(n-1,x_0)}(K,S,D) \cup \{x_0\}\right)\right)$$

for $n = 2, 3, \cdots$.

Definition 2.3 *[20]* Let X be a Banach space, M be a nonempty closed convex subset of X and ψ a measure of noncompactness on X. Let $K, S : M \to X$ be two bounded mappings (*i.e.* they take bounded sets into bounded ones) and $x_0 \in M$. We say that K is a S- convex-power condensing operator about x_0 and n_0 w.r.t. ψ if for any bounded set $D \subseteq M$ with $\psi(D) > 0$ we have

$$\psi\left(\mathcal{F}^{(n_0,x_0)}(K,S,D)\right) < \psi(D).$$

The following result is the key for our discussion.

Theorem 2.3 *[12] Let X be a Banach space and ψ be a measure of noncompactness on X. Let M be a nonempty bounded closed convex subset of X. Suppose that $K, S : M \to X$ are two continuous mappings satisfying:*

(i) *S is a strict contraction,*

(ii) *$Sx + Ky \in M$ for all $x, y \in M$,*

(iii) *there are an integer n and a vector $x_0 \in X$ such that K is S-power-convex condensing w.r.t. ψ.*

Then $S + K$ has at least one fixed point in M.

3 Existence of mild solutions

In this section, we investigate the existence of mild solutions of the nonlocal integrodifferential equation (1.1). We start by giving the definition of the so-called mild solution of (1.1).

Definition 3.1 *A function $u : [0, a] \longrightarrow X_\alpha$ is called a mild solution of Equation (1.1) if*

(i) *u is continuous on $[0, a]$,*

(ii) $u(t) = R(t)(u_0 + g(u)) + \int_0^t R(t-s) f(s, u(s))\, \mathrm{d}s$ *for* $t \in [0, a]$.

The hypotheses we need are listed below:

$(\mathbf{H}_1)$ For each $t \in [0, a]$, the function $f(t, \cdot) : X_\alpha \longrightarrow X$ is continuous and for each $u \in X_\alpha$, the function $f(\cdot, u) : [0, a] \longrightarrow X$ is measurable.

$(\mathbf{H}_2)$ There exist $\rho \in L^q([0, a]; \mathbb{R}^+)$, with $q = \frac{2-\alpha}{1-\alpha}$, and a continuous nondecreasing function $\Omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that for each $t \in [0, a]$ and $u \in X_\alpha$ we have

$$\|f(t, u)\| \leq \rho(t)\Omega(\|u\|_\alpha).$$

$(\mathbf{H}_3)$ There exists $C \in L^q([0, a]; \mathbb{R}^+)$ such that for any bounded set $D \subset X_\alpha$ and $t \in [0, a]$

$$\beta(f(t, D)) \leq C(t)\beta(D).$$

$(\mathbf{H}_4)$ There exists $L_g > 0$ such that for all $u, v \in C([0, a], X_\alpha)$

$$\|g(u) - g(v)\|_\alpha \leq L_g \|u - v\|_\alpha.$$

Now, all of this allows us to state the existence result.

Theorem 3.1 *Assume that* $(\mathbf{V}_1)$-$(\mathbf{V}_5)$ *and* $(\mathbf{H}_1)$-$(\mathbf{H}_4)$ *hold. Then the nonlocal problem* (1.1) *has at least one mild solution on* $[0,a]$ *provided that*

$$ML_g + M_\alpha \liminf_{r\to\infty} \frac{\Omega(r)}{r}\Big(\int_0^a e^{-qs}\rho^q(s)\,\mathrm{d}s\Big)^{\frac{1}{q}} < 1, \tag{3.1}$$

where

$$M_\alpha = N_\alpha \frac{e^a}{p^{\frac{1}{p}-\alpha}}\Gamma(1-\alpha p)^{\frac{1}{p}} \quad \textit{with} \quad q = \frac{2-\alpha}{1-\alpha},\ \ p = 2-\alpha.$$

Proof. We define the operators $S, K : C([0,a];X_\alpha) \longrightarrow C([0,a];X_\alpha)$ as follows

$$\begin{aligned}(Su)(t) &= R(t)\,(u_0 + g(u)) \quad \text{for} \quad t\in[0,a],\\ (Ku)(t) &= \int_0^t R(t-s)f(s,u(s))\,\mathrm{d}s \quad \text{for} \quad t\in[0,a].\end{aligned}$$

Then u is a mild solution for (1.1) if and only if u is a fixed point for the sum $S+K$. We shall prove that operators $S,\ K$ satisfy all conditions of Theorem 2.3. This will be achieved in a series of lemmas.

Lemma 3.1 K, S *are continuous on* $C([0,a];X_\alpha)$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $C([0,a];X_\alpha)$ such that $\lim\limits_{n\to\infty} u_n = u$ in $C([0,a];X_\alpha)$. By assumption $(\mathbf{H}_1)$ we know that for a.e. $s\in[0,a]$, we have

$$\lim_{n\to\infty} f\left(s,u_n(s)\right) = f(s,u(s)).$$

Thus

$$\|Ku_n - Ku\|_\alpha \le M_\alpha\Big(\int_0^a \|e^{-s}f(s,u_n(s)) - e^{-s}f(s,u(s))\|^q\,\mathrm{d}s\Big)^{\frac{1}{q}},$$

where $M_\alpha = N_\alpha \frac{e^a}{p^{\frac{1}{p}-\alpha}}\Gamma(1-\alpha p)^{\frac{1}{p}}$. Using the dominated convergence theorem, we deduce the continuity of K. Assumption $(\mathbf{H}_4)$ gives the continuity of S. □

Lemma 3.2 S *is a strict contraction.*

Proof. Let $u,\ v\in C([0,a];X_\alpha)$ and $t\in[0,a]$. By assumption $(\mathbf{H}_4)$, we have

$$\|(Su)(t) - (Sv)(t)\|_\alpha \le M\|(g(u)-g(v))\|_\alpha \le \xi\|u-v\|_\alpha,$$

where $\xi = NL_g < 1$. This proves our claim. □

Lemma 3.3 *There is a positive constant* r_0*, such that for all* $u,\ v\in B_{r_0}$

$$Su + Kv \in B_{r_0},$$

where $B_{r_0} = \{u\in C([0,a];X_\alpha) : \|u\|_\alpha \le r_0\}$.

Proof. We argue by contradiction. If it is not the case, then, for each $r > 0$, there exists $u, v \in B_r$ such that $Su + Kv \notin B_r$, that is,

$$r < \|(Su) + (Kv)\|_\alpha \leq M\left(\|u_0\|_\alpha + \|g(0)\|_\alpha + rL_g\right) + M_\alpha \Omega(r)\Big(\int_0^a e^{-qs}\rho^q(s)\,\mathrm{d}s\Big)^{\frac{1}{q}},$$

where we have used hypothesis $(\mathbf{H}_2)$. This implies, when dividing by r, that

$$1 < \frac{M}{r}\left(\|u_0\|_\alpha + \|g(0)\|_\alpha\right) + ML_g + M_\alpha \frac{\Omega(r)}{r}\Big(\int_0^a e^{-qs}\rho^q(s)\,\mathrm{d}s\Big)^{\frac{1}{q}}.$$

Taking liminf as $r \to \infty$, we obtain

$$1 \leq ML_g + M_\alpha \liminf_{r\to\infty} \frac{\Omega(r)}{r}\Big(\int_0^a e^{-qs}\rho^q(s)\,\mathrm{d}s\Big)^{\frac{1}{q}},$$

which contradicts the assumption (3.1). □

Lemma 3.4 *K is S-power-convex condensing.*

Proof. The proof of this lemma will be achieved in three steps. Let D be a subset of B_{r_0}.

Step 1: We claim that $K(D) = \Big\{Ku = \int_0^{\cdot} R(\cdot - s)f(s, u(s))\,\mathrm{d}s : u \in D\Big\}$ is equicontinuous.

For $t = 0$ and $t' > 0$

$$\begin{aligned}\|Ku(t') - Ku(0)\|_\alpha &\leq \int_0^{t'} \|R(t'-s)f(s,u(s))\|_\alpha\,\mathrm{d}s \\ &\leq M_\alpha\Omega(r_0)\Big(\int_0^{t'} e^{-qs}\rho^q(s)\,\mathrm{d}s\Big)^{\frac{1}{q}}.\end{aligned}$$

Since $s \mapsto e^{-qs}\rho^q(s)$ is integrable, then

$$\sup_{u\in B}\|Ku(t') - Ku(0)\|_\alpha \longrightarrow 0, \quad \text{as } t' \to 0.$$

Hence, $K(D)$ is equicontinuous at $t = 0$.

For $0 < t < t' \leq a$

$$\begin{aligned}\|Ku(t') - Ku(t)\|_\alpha &\leq \int_t^{t'} \|R(t'-s)f(s,u(s))\|_\alpha\,\mathrm{d}s \\ &+ \int_0^t \|(R(t'-s) - R(t-s))f(s,u(s))\|_\alpha\,\mathrm{d}s \\ &\leq N_\alpha\Omega(r_0)\int_t^{t'} \frac{\rho(s)}{(t'-s)^\alpha}\,\mathrm{d}s \\ &+ \int_0^t \|A^\alpha\Big(R(t'-s) - R(t-s)\Big)\|\|f(s,u(s))\|\,\mathrm{d}s \\ &\leq M_\alpha\Omega(r_0)\Big(\int_t^{t'} e^{-qs}\rho^q(s)\,\mathrm{d}s\Big)^{\frac{1}{q}} \\ &+ \Omega(r_0)\int_0^t \|A^\alpha\Big(R(t'-s) - R(t-s)\Big)\|\rho(s)\,\mathrm{d}s \\ &\leq I_1 + I_2,\end{aligned}$$

where

$$I_1 = M_\alpha \Omega(r_0)\Big(\int_t^{t'} e^{-qs}\rho^q(s)\,\mathrm{d}s\Big)^{\frac{1}{q}},$$
$$I_2 = \Omega(r_0)\int_0^t \|A^\alpha\Big(R(t'-s) - R(t-s)\Big)\|\rho(s)\,\mathrm{d}s.$$

It is obvious that $I_1 \to 0$ as $t' \to t$ independently of $u \in D$. Moreover, by Lemma 2.2, we deduce that $I_2 \to 0$ as $t' \to t$ independently of $u \in D$.

Step 2: We shall prove that $\mathcal{F}^{(n,0)}(K,S,D)$ is equicontinuous on $[0,a]$ for any integer $n \geq 1$. To achieve this, notice first that for $u \in \mathcal{F}^{(1,0)}(K,S,D)$ there exists $v \in D$ such that $u = Su + Kv$. Then, for $t, t' \in [0,a]$, we have

$$\begin{aligned}\left\|u(t) - u\left(t'\right)\right\|_\alpha &= \left\|Su(t) + Kv(t) - Su\left(t'\right) - Kv\left(t'\right)\right\|_\alpha \\ &\leq \left\|Su(t) - Su\left(t'\right)\right\|_\alpha + \left\|Kv(t) - Kv\left(t'\right)\right\|_\alpha \\ &\leq \xi\left\|u(t) - u\left(t'\right)\right\|_\alpha + \left\|Kv(t) - Kv\left(t'\right)\right\|_\alpha.\end{aligned}$$

Hence,

$$\left\|u(t) - u\left(t'\right)\right\|_\alpha \leq \frac{1}{1-\xi}\left\|Kv(t) - Kv\left(t'\right)\right\|_\alpha.$$

Now the equicontinuity of $\mathcal{F}^{(1,0)}(K,S,D)$ follows from **Step 1**. The same reasoning as above implies that $\mathcal{F}^{(2,0)}(K,S,D) := \mathcal{F}^{(1,0)}\left(K,S,\overline{co}\left(\mathcal{F}^{(1,0)}(K,S,D)\ \cup\{0\}\right)\right.$, is equicontinuous. By mathematical induction we can show that $\mathcal{F}^{(n,0)}(K,S,D)$ is equicontinuous for all $n \geq 1$.

Step 3: We claim that there is an integer n_0 such that $\alpha\left(\mathcal{F}^{(n_0,0)}(K,S,D)\right) < \alpha(D)$. Notice that for $t \in [0,a]$, we have

$$\begin{aligned}\mathcal{F}^{(1,0)}(K,S,D)(t) &= \Big\{u(t), u \in \mathcal{F}^{(1,0)}(K,S,D)\Big\} \\ &\subseteq \Big\{u(t) - Su(t), u \in \mathcal{F}^{(1,0)}(K,S,D)\Big\} + \Big\{Su(t), u \in \mathcal{F}^{(1,0)}(K,S,D)\Big\}.\end{aligned}$$

By using the properties of the measure of noncompactness and properties of S, we get

$$\beta\left(\mathcal{F}^{(1,0)}(K,S,D)(t)\right) \leq \beta(K(D)(t)) + \xi\alpha\left(\mathcal{F}^{(1,0)}(K,S,D)(t)\right).$$

Which implies that

$$\beta\left(\mathcal{F}^{(1,0)}(K,S,D)(t)\right) \leq \frac{1}{1-\xi}\beta(K(D)(t)). \tag{3.2}$$

On the other hand since $K(D)$ is bounded, then from Lemma 2.8 for each $\varepsilon > 0$, there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subseteq K(D)$ such that

$$\beta(K(D)(t)) \leq 2\beta\left(\{v_n(t)\}_{n=1}^\infty\right) + \varepsilon \leq 2\beta\left(\left\{\int_0^t R(t-s)f\left(s,u_n(s)\right)\,\mathrm{d}s\right\}_{n=1}^\infty\right) + \varepsilon,$$

since $s \mapsto \sup_{n\geq 1}\|R(t-s)f\left(s,u_n(s)\right)\|_\alpha$ is integrable on $[0,a]$. Then Lemma 2.7 implies that

$$\beta(K(D)(t)) \leq 4N_\alpha\int_0^t \frac{1}{(t-s)^\alpha}\beta\left(\{f\left(s,u_n(s)\right)\}_{n=1}^\infty\right)\,\mathrm{d}s + \varepsilon.$$

Thus by $(\mathbf{H}_3)$, we gain

$$\begin{aligned}\beta(K(D)(t)) &\leq 4N_\alpha \int_0^t \frac{C(s)}{(t-s)^\alpha} \beta\left(\{u_n(s)\}_{n\in\mathbb{N}}\right) \,\mathrm{d}s + \varepsilon \\ &\leq 4N_\alpha \beta(D) \int_0^t \frac{C(s)}{(t-s)^\alpha} \,\mathrm{d}s + \varepsilon \\ &\leq 4M_\alpha \beta(D) \Big(\int_0^t C^q(s) \,\mathrm{d}s \Big)^{\frac{1}{q}} + \varepsilon.\end{aligned}$$

From the density of $C([0,a];\mathbb{R}^+)$ in $L^q([0,a];\mathbb{R}^+)$ and since $C \in L^q([0,a];\mathbb{R}^+)$, then for $\delta^q < \frac{(1-k)^q}{2^{3q-1}M_\alpha^q}$ there exists $\varphi \in C([0,a];\mathbb{R}^+)$ satisfying $\int_0^a |C(s)-\varphi(s)|^q \,\mathrm{d}s < \delta^q$. Consequently

$$\begin{aligned}\beta(K(D)(t)) &\leq 4M_\alpha \beta(D) \left[2^{q-1} \int_0^t |C(s)-\varphi(s)|^q \,\mathrm{d}s + 2^{q-1} \int_0^t |\varphi(s)|^q \,\mathrm{d}s \right]^{\frac{1}{q}} + \varepsilon \\ &\leq 4M_\alpha \beta(D) 2^{1-\frac{1}{q}} [\delta^q + \tau^q t]^{\frac{1}{q}} + \varepsilon,\end{aligned}$$

where $\tau = \sup\limits_{0\leq s\leq a} \varphi(s)$. Letting $\varepsilon \to 0$, we get

$$\beta(K(D)(t)) \leq 4M_\alpha 2^{1-\frac{1}{q}} [\delta^q + \tau^q t]^{\frac{1}{q}} \beta(D).$$

This means by (3.2) that

$$\beta\left(\mathcal{F}^{(1,0)}(K,S,D)(t)\right) \leq (\lambda + \mu t)^{\frac{1}{q}} \beta(D),$$

where $\lambda = \frac{2^{3q-1}M_\alpha^q \delta^q}{(1-\xi)^q}$ and $\mu = \frac{2^{3q-1}M_\alpha^q \tau^q}{(1-\xi)^q}$. On the other hand,

$$\begin{aligned}&\mathcal{F}^{(2,0)}(K,S,D)(t) \\ &\subseteq \left\{u(t) - Su(t), u \in \mathcal{F}^{(2,0)}(K,S,D)\right\} + \left\{Su(t), u \in \mathcal{F}^{(2,0)}(K,S,D)\right\} \\ &\subseteq \left\{Kv(t), v \in \overline{co}\left(\mathcal{F}^{(1,0)}(K,S,D) \cup \{0\}\right)\right\} + \left\{Su(t), u \in \mathcal{F}^{(2,0)}(K,S,D)\right\}.\end{aligned}$$

Referring to Lemmas 2.3 and 2.4, we see that

$$\beta\left(\mathcal{F}^{(2,0)}(K,S,D)(t)\right) \leq \beta\left(K\left(\overline{\mathrm{co}}\left(\mathcal{F}^{(1,0)}(K,S,D) \cup \{0\}\right)\right)(t)\right) + \xi\beta\left(\mathcal{F}^{(2,0)}(K,S,D)(t)\right).$$

Thus

$$\beta\left(\mathcal{F}^{(2,0)}(K,S,D)(t)\right) \leq \frac{1}{1-\xi} \beta\left(K\left(\overline{co}\left(\mathcal{F}^{(1,0)}(K,S,D) \cup \{0\}\right)\right)(t)\right). \tag{3.3}$$

Furthermore, from Lemma 2.8, there exists a sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq \overline{co}\left(\mathcal{F}^{(1,0)}(K,S,D) \cup \{0\}\right)$

such that

$$\begin{aligned}
&\beta\left(K\left(\overline{co}\left(\mathcal{F}^{(1,0)}(K,S,D)\cup\{0\}\right)\right)(t)\right)\\
&\le 2\beta\left(\left\{\int_0^t R(t-s)f\left(s,w_n(s)\right)\,\mathrm{d}s\right\}_{n=1}^{\infty}\right)+\varepsilon\\
&\le 4N_\alpha\int_0^t \frac{1}{(t-s)^\alpha}\beta\left(\{f\left(s,w_n(s)\right)\}_{n=1}^{\infty}\right)\,\mathrm{d}s+\varepsilon\\
&\le 4N_\alpha\int_0^t \frac{C(s)}{(t-s)^\alpha}\beta\left(\overline{co}\left(\mathcal{F}^{(1,0)}(K,S,D)\cup\{0\}\right)(s)\right)\,\mathrm{d}s+\varepsilon\\
&\le 4N_\alpha\int_0^t \frac{C(s)}{(t-s)^\alpha}\beta\left(\mathcal{F}^{(1,0)}(K,S,D)(s)\right)\,\mathrm{d}s+\varepsilon.
\end{aligned}$$

By combining (3.2) and (3.3), we obtain

$$\begin{aligned}
&\beta\left(\mathcal{F}^{(2,0)}(K,S,D)(t)\right)\\
&\le \frac{4N_\alpha}{(1-\xi)}\Big(\int_0^t 2^{q-1}[|C(s)-\varphi(s)|^q+|\varphi(s)|^q](\lambda+\mu s)\,\mathrm{d}s\Big)^{\frac{1}{q}}\beta(D)+\frac{\varepsilon}{(1-\xi)}\\
&\le\left[\lambda(\lambda+\mu t)+\mu\left(\lambda t+\mu\frac{t^2}{2}\right)\right]^{\frac{1}{q}}\beta(D)+\frac{\varepsilon}{(1-\xi)}\\
&\le\left[\lambda^2+2\lambda\mu t+\frac{(\mu t)^2}{2}\right]^{\frac{1}{q}}\beta(D)+\frac{\varepsilon}{(1-\xi)}.
\end{aligned}$$

Letting $\varepsilon\to 0$, we gain

$$\beta\left(\mathcal{F}^{(2,0)}(K,S,D)(t)\right)\le\left[\lambda^2+2\lambda\mu t+\frac{(\mu t)^2}{2}\right]^{\frac{1}{q}}\beta(D).$$

By mathematical induction, we obtain that for all $n\ge 1$

$$\beta\left(\mathcal{F}^{(n,0)}(K,S,D)(t)\right)\le\left[\lambda^n+C_n^1\lambda^{n-1}\mu t+C_n^2\lambda^{n-2}\frac{(\mu t)^2}{2!}+\cdots+\frac{(\mu t)^n}{n!}\right]^{\frac{1}{q}}\beta(D).$$

Using the equicontinuity of $\mathcal{F}^{(n,0)}(K,S,D)$ together with Lemma 2.6, we conclude that

$$\beta\left(\mathcal{F}^{(n,0)}(K,S,D)\right)\le\left[\lambda^n+C_n^1\lambda^{n-1}\mu a+C_n^2\lambda^{n-2}\frac{(\mu a)^2}{2!}+\cdots+\frac{(\mu a)^n}{n!}\right]^{\frac{1}{q}}\beta(D).$$

Since $0<\lambda<1$ and $\mu a>0$, it follows from Lemma 2.9 that there exists $n_0\in\mathbb{N}$ such that

$$\beta\left(\mathcal{F}^{(n_0,0)}(K,S,D)\right)<\beta(D).$$

Hence K is S-power-convex condensing. This completes the proof of the Lemma. □

Consequently, applying Theorem 2.3 together with Lemmas 3.1, 3.2, 3.3, 3.4, we deduce that $S+K$ has at least one fixed point in B_{r_0} which is a mild solution of (1.1). □

4 Application

We consider the following partial integrodifferential equation with nonlocal condition

$$\begin{cases} \dfrac{\partial}{\partial t}\Big[v(t,x) + \displaystyle\int_0^t (t-s)^{\delta} e^{-w(t-s)} v(s,x)\,\mathrm{d}s\Big] \\ = \dfrac{\partial^2}{\partial x^2} v(t,x) + \displaystyle\int_0^t e^{-\theta(t-s)} \dfrac{\partial^2}{\partial x^2} v(s,x)\,\mathrm{d}s + f_1(t) f_2(\dfrac{\partial}{\partial x} v(t,x)), & 0 \le x \le \pi,\ 0 \le t \le b, \\ v(t,0) = v(t,\pi) = 0, & t \in [0,b], \\ v(0,x) = v_0(x) + \displaystyle\int_0^{\pi} \gamma(x,y) \dfrac{\partial v(t_0,y)}{\partial y}\,\mathrm{d}y, & 0 \le x \le \pi. \end{cases} \tag{5.1}$$

In this system, $\delta \in (0,1)$, w, θ are positive constants, $0 < t_0 < b$ and $v_0 \in L^2([0,\pi])$. Moreover, $\gamma : [0,\pi] \times [0,\pi] \longrightarrow \mathbb{R}$ is a C^1-function, $f_1 : [0,b] \longrightarrow \mathbb{R}$ is continuous and $f_2 : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipshitz continuous map with constant k.

In order to write Eq.(5.1) in the abstract form, take $X = L^2([0,\pi])$ and let A be defined by

$$\begin{cases} D(A) = H^2(0,\pi) \cap H_0^1(0,\pi), \\ Au = -u''. \end{cases}$$

Then $-A$ generates a strongly continuous semigroup $(T(t))_{t\ge 0}$ which is analytic. Hence, by [18, Theorem 4.6], A is sectorial and $(\mathbf{V}_1)$ is satisfied. Moreover, A has a discrete spectrum, the eigenvalues are $\{-n^2, n \in \mathbb{N}^*\}$ with the corresponding normalized eigenvectors $e_n(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(nx)$ [21]. Finally, we have the following famous properties:

(i) If $u \in D(A)$, then $Au = \displaystyle\sum_{n=1}^{+\infty} n^2 < u, e_n > e_n$.

(ii) The operator $A^{\frac{1}{2}}$ is given by

$$\begin{cases} D(A^{\frac{1}{2}}) = \left\{ u \in X, \displaystyle\sum_{n=1}^{+\infty} n < u, e_n > e_n \in X \right\}, \\ A^{\frac{1}{2}} u = \displaystyle\sum_{n=1}^{+\infty} n < u, e_n > e_n. \end{cases}$$

The kernel system $B(\cdot)$ is given by

$$\begin{cases} D(B) = D(A) \\ (B(t)u)(x) = e^{-\theta t} u''(x) \quad \text{for} \quad t \in [0,b] \ \text{ and } \ x \in [0,\pi], \end{cases}$$

and the operator $N(\cdot)$ is defined by

$$(N(t)u)(x) = t^{\delta} e^{-wt} u(x) \quad \text{for} \quad t \in [0,b] \text{ and } x \in [0,\pi].$$

In addition, one can see that conditions $(\mathbf{V}_2)-(\mathbf{V}_5)$ are satisfied with $\widehat{N}(\lambda) = \dfrac{\Gamma(\delta+1)}{(\lambda+w)^{\delta+1}}$, $b(t) = e^{-\gamma t}$ and $D = C_0^{\infty}([0,\pi])$, where $C_0^{\infty}([0,\pi])$ is the space of infinitely differentiable functions that vanish at $\xi = 0$ and $\xi = \pi$. Hence, according to [10], the linear equation (2.1) has an analytic resolvent operator $(R(t))_{t\geq 0}$.

Define the source function $f : [0,b] \times X_{\frac{1}{2}} \longrightarrow X$ by

$$f(t,u)(x) = f_1(t) f_2(u'(x))$$

and $g : C([0,b], X_{\frac{1}{2}}) \longrightarrow X_{\frac{1}{2}}$ by

$$g(u)(x) = \int_0^{\pi} \gamma(x,y) \frac{\partial u(t_0)}{\partial y}(y)\, \mathrm{d}y.$$

Then our problem can be written in the abstract form (1.1).

Lemma 4.1 *[21] If $y \in X_{\frac{1}{2}}$, then y is absolutely continuous, $y' \in X$ and*

$$\|y\|_{\frac{1}{2}} = \|y'\| = \left\| A^{\frac{1}{2}} y \right\|.$$

Let $(t,u) \in [0,b] \times X_{\frac{1}{2}}$, then we have

$$\begin{aligned}
\|f(t,u)\|^2 &= \int_0^{\pi} \left| f_1(t) f_2\left(u'(x)\right)\right|^2 \mathrm{d}x \\
&= |f_1(t)|^2 \int_0^{\pi} \left| f_2\left(u'(x)\right)\right|^2 \mathrm{d}x \\
&\leq |f_1(t)|^2 \int_0^{\pi} \left(k\left|u'(x)\right| + |f_2(0)|\right)^2 \mathrm{d}x \\
&\leq |f_1(t)|^2 \left(k^2 \int_0^{\pi} \left|u'(x)\right|^2 \mathrm{d}x + 2k|f_2(0)| \int_0^{\pi} \left|u'(x)\right| \mathrm{d}x + |f_2(0)|^2 \pi\right),
\end{aligned}$$

using Cauchy-Schwarz inequality and Lemma 4.1, we obtain

$$\|f(t,u)\|^2 \leq |f_1(t)|^2 \left(k^2 \|u\|_{\frac{1}{2}}^2 + 2k|f_2(0)|\sqrt{\pi}\|u\|_{\frac{1}{2}} + |f_2(0)|^2\pi\right),$$

then we get

$$\|f(t,u)\| \leq \rho(t)\Omega(\|u\|_{\frac{1}{2}}),$$

where

$$\rho(t) = |f_1(t)|, \quad \Omega(r) = kr + |f_2(0)|\sqrt{\pi}.$$

To prove the continuity of the map f, we need the following Lemma:

Lemma 4.2 *[3] Let $1 < p < \infty$, Ω an open set in $\mathbb{R}^n$ and $\{f_n\}_{n\geq 0}$ be a sequence in $L^p(\Omega)$. Suppose that $f_n \longrightarrow f$ as $n \longrightarrow +\infty$ in $L^p(\Omega)$. Then there exist a subsequence $\{f_{n_k}\}_{k\geq 0}$ of $\{f_n\}_{n\geq 0}$ and $h \in L^p(\Omega)$ such that*

(i) $f_{n_k} \longrightarrow f$ *a.e in* Ω.

(ii) $|f_{n_k}(x)| \leq |h(x)|$, *a.e. in* Ω.

Let $\{u_n\}_{n\geq 0} \subset X_{\frac{1}{2}}$ such that $u_n \longrightarrow u$ as $n \to +\infty$ in $X_{\frac{1}{2}}$, then $u_n' \longrightarrow u'$ as $n \to +\infty$ in X. Using Lemma 4.2, we deduce that there exist a subsequence $\{u_{n_k}\}_{k\geq 0}$ and $l \in X$ such that

$$\lim_{k\to+\infty} u_{n_k}'(x) = u'(x),$$

and

$$\left|u_{n_k}'(x)\right| \leq |l(x)| \text{ a.e.}$$

Since h is continuous, then

$$\lim_{k\to+\infty} f_2\left(u_{n_k}'(x)\right) = f_2\left(u'(x)\right).$$

By the Dominated Convergence Theorem, we conclude that

$$\lim_{k\to+\infty} f\left(t, u_{n_k}\right) = f(t,u).$$

Since that limit is independent of $\{u_{n_k}\}_{k\geq 0}$, we infer that

$$\lim_{n\to+\infty} f\left(t, u_n\right) = f(t,u).$$

Which proves the continuity of f. Hence, $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ are satisfied.

Furthermore, for any bounded subset $D \subset X_{\frac{1}{2}}$ we have

$$\beta(f(t,D)) \leq |f_1(t)|\beta(f_2(D)) \leq k|f_1(t)|\beta(D),$$

where we have applied Lemma 2.4. Thus, $(\mathbf{H}_3)$ is satisfied with $C(\cdot) = k|f_1(\cdot)|$.

On the other hand, let $u, v \in C([0,b], X_{\frac{1}{2}})$. Then, we have

$$\begin{aligned}
\|g\left(u\right) - g(v)\|_{\frac{1}{2}}^2 &= \|(g(u))' - (g(v))'\|^2 \\
&= \int_0^\pi \left| \int_0^\pi \frac{\partial}{\partial x}\gamma(x,y)\left[\frac{\partial u(t_0)}{\partial y}(y) - \frac{\partial v(t_0)}{\partial y}(y)\right] \mathrm{d}y\right|^2 \mathrm{d}x \\
&\leq \pi^2 \sup_{(x,y)\in[0,\pi]\times[0,\pi]} |\frac{\partial}{\partial x}\gamma(x,y)|^2 \int_0^\pi \left|\frac{\partial u(t_0)}{\partial y}(y) - \frac{\partial v(t_0)}{\partial y}(y)\right|^2 \mathrm{d}y.
\end{aligned}$$

This implies that g is Lipschitzian. Hence, $(\mathbf{H}_4)$ is satisfied.

Proposition 4.1 *If* $M\pi\sqrt{\sup_{(x,y)\in[0,\pi]\times[0,\pi]} |\frac{\partial\gamma(x,y)}{\partial x}|} + kM_{\frac{1}{2}}\left(\int_0^b (e^{-s}|f_1(s)|)^3\,\mathrm{d}s\right) < 1$, *then Eq.*$(5.1)$ *has at least one mild solution.*

5 Acknowledgment

The authors would like to thank the referee for the valuable comments and suggestions.

References

[1] J. Banas, K. Goebel, *Mesure of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.

[2] D. Bothe, *Multivalued perturbations of m-accretive differential inclusions*, Israel Journal of Mathematics **108** (1998), pp. 109–138.

[3] H. Brezis, *Analyse fonctionnelle, Théorie et applications*, Collection mathématiques appliquées pour la maîtrise, 1992.

[4] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, Journal of Mathematical Analysis and Applications **162** (1991), pp. 494–505.

[5] L. Byszewski, *Existence and uniqueness of solutions of semilinear evolution nonlocal Cauchy problem*, Zeszyty Naukowe Politechniki Rzeszowskiej. Matematyka i Fizyka **18** (1993), pp. 109–112.

[6] J. R. Cannon, *The One-Dimensional Heat Equation*, Addison Wesley, 1984.

[7] J. Chadam, H. M. Yin, *Determination of an unknown function in a parabolic equation with an overspecified condition*, Mathematical Methods in the Applied Sciences **13** (1990), pp. 421–430.

[8] B. Diao, K. Ezzinbi, M. Sy, *Some results in the α-norm for a class of neutral partial functional integrodifferential equations with finite delay*, Afrika Matematika **26** (2015), pp. 1621–1635 .

[9] J. P. C. Dos Santos, V. Vijayakumar, R. Murugesu, *Existence of mild solutions for nonlocal Cauchy problem for fractional neutral integro-differential equation with unbounded delay*, Communications in Mathematical Analysis **14** (2013), pp. 59–71.

[10] J. Dos Santos, H. Henriquez, E. Hernàndez, *Existence results for neutral integro-differential equations with unbounded delay*, Journal of Integral Equations and Applications **23** (2011), pp. 289–330.

[11] K. Ezzinbi, S. Ghnimi, *Existence and regularity of solutions for some partial integrodifferential equations involving the nonlocal conditions*, Numerical Functional Analysis and Optimization **40** (2019), pp. 1532–1549.

[12] K. Ezzinbi, M. A. Taoudi, *Sadovskii-Krasnosel'skii type fixed point theorems in Banach spaces with application to evolution equations*, Applied Mathematics and Computation **49** (2015), pp. 243–260.

[13] K. Ezzinbi, S. Ghnimi, *Existence results for some partialegrodifferential equations with nonlocal conditions*, Glasnik Matematicki **51** (2016), pp. 413–430.

[14] X. Fu, Y. Gao, Y. Zhang, *Existence of solutions for neutral integrodifferential equations with nonlocal conditions*, Taiwanese Journal of Mathematics **16** (2012), pp. 1879–1909.

[15] X. L. Fu, K. Ezzinbi, *Existence of solutions for neutral functional differential evolution equations with nonlocal conditions*, Nonlinear Analysis: Theory, Methods & Applications **54** (2003), pp. 215–227.

[16] L. Liu, F. Guo, C. Wu, Y. Wu, *Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces*, Journal of Mathematical Analysis and Applications **309** (2005), pp. 638–649.

[17] H. Monch, *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces*, Nonlinear Analysis **4** (1980), pp. 985–999.

[18] K. J. Engel, R. Nagel, S. Brendle, *One-Parameter Semigroups for Linear Evolution Equations*, Springer, New York, 2000.

[19] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied mathematical sciences, vol.44, Springer Science & Business Media, 2012.

[20] J. Sun, X. Zhang, *A fixed point theorem for convex-power condensing operators and its applications to abstract semilinear evolution equations*, Acta Mathematica Sinica (Chin. Ser.) **48** (2005), pp. 439–446.

[21] C. C. Travis, G. F. Webb, *Existence, stability, and compactness in the α-norm for partial functional differential equations*, Transactions of the American Mathematical Society **240** (1978), pp. 129–143.

[22] J. Zhu, X. Fu, *Existence and regularity of solutions for neutral partial integro-differential equations with nonlocal conditions*, Journal of Fixed Point Theory and Applications **22** (2020), article number 34.