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ULAM-HYERS STABILITY OF NON-INSTANTANEOUS IMPULSIVE INTEGRO-DIFFERENTIAL EQUATION OF REAL-ORDER WITH CAPUTO DERIVATIVE WITH APPLICATION TO CIRCUITS

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Abstract. In this work, we first discuss the existence of a mild solution of the Caputo fractional non-instantaneous impulsive integro-differential equation and then discuss its stability in the sense of Ulam-Hyers. We establish our main results by using the well-known Banach fixed point theorem. Two suitable examples are presented to authenticate the results. In addition, with the help of the obtained results, the bound for a non-instantaneous impulsive fractional-order RLC circuit current is estimated, and it is found that the bound primarily depends upon the bandwidth and the fractional order of the RLC system. Further, for a given bandwidth, we show how the fractional order of the system influences the behavior and magnitude of the current.

Keywords: Ulam-Hyers stability, Non-instantaneous impulsive fractional differential equation, Banach fixed point theorem, Fractional-order RLC circuits, Ulam-Hyers constant.

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1 Introduction

Although the theory of impulsive differential equations has been considered to be of paramount importance for a very long time, its actual importance has been realized only in the $1980s$ since

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when a substantial development in this regard has taken place. More advancement has presently been taking place as evident from the works in [3, 7, 8, 14, 48]. Such development has allowed more accurate modeling of some real-world problems through the consideration of impulsive differential equations. Consequently, impulsive differential equation has carved a niche for itself in tackling physical phenomena that arise in almost all disciplines of science and engineering such as biophysics, chemical engineering, electrical engineering, population dynamics, financial mathematics [5, 9, 17, 21, 31], to name a few. Usually, there are two familiar impulses that are considered for most of the works, viz., instantaneous impulses and non-instantaneous impulses. An instantaneous impulse is an impulse in which the length of time interval of the action is vanishingly small. On the other hand, a non-instantaneous impulse is an impulse that comprises an impulsive action starting at some arbitrary fixed time and remaining active for a finite duration (for details, one can refer to [6, 39]). The impetus to study impulsive differential equations may be judged from a practical example as follows. While considering the hemodynamical equilibrium of a human, imagine a simple but practical situation: in the case of the occurrence of a decompensation, say, high level of sugar, an intravenous drug (usually insulin) may be injected. Consequently, the introduction of drug in the bloodstream and the associated absorption by the body can be considered as gradual and continuous processes. This can be seen as an impulsive action with an abrupt start but which stays active for a finite period of time. Obviously, this situation can indeed be mathematically expressed in terms of a non-instantaneous impulsive differential equation as can also be found in [1, 10, 28, 35, 43].

It may be deemed natural to pose a fundamental question arising in mathematical analysis: What may be the condition(s) under which a mathematical object that satisfies a certain property approximately is very likely to be close to another object which satisfies the property exactly? Considering a functional equation, the above may take the form: when must its solution that differs slightly from a given solution be close to it? A fundamental question like this is pivotal while studying the stability feature of functional equations for which Ulam [42] initiated a problem. He derived some conditions so as to ensure the existence of a linear function closer to an approximately linear function. In continuation and response to it, Hyers [19] considered Ulam's problem by accounting for additive functions on Banach spaces as follows:
Consider real Banach spaces $(U, ||.||_U)$ and $(V, ||.||_V)$. Further, assume that, with $\varepsilon > 0$ and for every function $h : U \to V$ satisfying

$$||h(u_1 + u_2) - h(u_1) - h(u_2)||_V \leq \varepsilon, \;\; \forall u_1, u_2 \in U,$$

there exists a unique additive function $L : U \to V$ obeying

$$||h(u_1) - L(u_1)||_V \leq \varepsilon, \quad \forall u_1 \in U.$$

After Hyers published the above result, a good number of researchers started considering extensions to Ulam's problem to different functional equations, and also generalizing Hyers' result from different perspectives (*e.g.*, [20] and [22]).

Hernández *et al.* [18] presented a new idea for a class of abstract differential equations subject to non-instantaneous impulses in which the existence of both mild and classical solutions was accomplished. Wang *et al.* [46] came up with the concept of a piecewise (PC) mild impulsive Cauchy problem by defining the notion of a mild solution and compared the weak and classical solutions. An abstract impulsive differential equation can be considered by associating abrupt and instantaneous impulses. Researchers have investigated such problems and examined the existence and the other qualitative properties of the solutions from different perspectives. For a comprehensive study on

Caputo fractional differential equations with non-instantaneous impulses, one can refer to Agarwal *et al.* [2].

Wang *et al.* [44] studied the following nonlinear instantaneous impulsive Caputo fractional differential equation to examine the Ulam-Hyers stability in a finite interval $\mathbb{J} = [0, T]$:

$$\begin{aligned} {}_0^C\mathcal{D}_\theta^\alpha x(\theta) &= f(\theta, x(\theta)) \quad \theta \in J' = \mathbb{J} \setminus \{\theta_1, \theta_2, \dots, \theta_m\}, \\ \Delta x(\theta_k) &= x(\theta_k^+) - x(\theta_k^-) = I_k(x(\theta_k^-)), \quad k = 1, 2, \dots, m, \\ x(0) &= x_0, \end{aligned}$$

where $\alpha \in (0,1)$, $f : \mathbb{J} \times \mathbb{R} \to \mathbb{R}$ is jointly continuous, $I_k : \mathbb{R} \to \mathbb{R}$ and θ_k satisfy $0 = \theta_0 < \theta_1 < \cdots < \theta_m < \theta_{m+1} = T$, $x(\theta_k^+) = \lim_{\epsilon \to 0^+} x(\theta_k + \epsilon)$ and $x(\theta_k^-) = \lim_{\epsilon \to 0^-} x(\theta_k + \epsilon)$.

Zhou *et al.* [45] studied the Caputo fractional linear differential equation by taking into account a periodic boundary condition as follows:

$$\begin{aligned} {}_0^C\mathcal{D}_\theta^\alpha x(\theta) &= f(\theta, x(\theta)), \quad \theta \in (\varrho_i, \theta_{i+1}], \; i = 0, 1, 2, \dots, N, \\ x(\theta) &= g_i(\theta, x(\theta)), \quad \theta \in (\theta_i, \varrho_i], \; i = 1, 2, \dots, N, \\ x(0) &= x(a), \end{aligned} \tag{1.1}$$

where $0 < \alpha < 1$, $0 = \theta_0 = \varrho_0 < \theta_1 \leq \varrho_1 \leq \theta_2 < \cdots < \theta_N \leq \varrho_N < \theta_{N+1} = a$ are given fixed numbers, $\mathbb{J} = [0, a]$. Here, $g_i \in C\big([\theta_i, \varrho_i] \times \mathbb{R}, \mathbb{R}\big)$, $\quad i = 1, 2, \dots, N$, and $f : \mathbb{J} \times \mathbb{R} \to \mathbb{R}$ is a given function. They introduced a general framework to find the solutions for impulsive fractional boundary value problems and established some sufficient conditions for the existence of the solutions by using fixed point theorem.

Lin *et al.* [27], by utilizing Banach fixed point theorem and assuming suitable conditions on the functions, established the existence of a solution for a new class of non-instantaneous impulsive integro-differential equations and also the generalized Ulam-Hyers-Rassias stability:

$$x'(\theta) = f\Big(\theta, x(\theta), \int_0^\theta k(\varrho, x(\varrho))\,\mathrm{d}\varrho\Big), \quad \theta \in (\varrho_i, \theta_{i+1}], \quad i = 0, 1, 2, \dots, m, \tag{1.2}$$

$$x(\theta) = g_i\Big(\theta, x(\theta), \int_0^\theta l(\varrho, x(\varrho))\,\mathrm{d}\varrho\Big), \quad \theta \in (\theta_i, \varrho_i], \quad i = 1, 2, 3, \dots, m, \tag{1.3}$$

$$x(0) = x_0, \tag{1.4}$$

where $0 = \theta_0 = \varrho_0 < \theta_1 \leq \varrho_1 \leq \theta_2 < \cdots < \theta_m \leq \varrho_m < \theta_{m+1} = T$ are given fixed numbers, $\mathbb{J} = [0, T]$. Here, $g_i \in C\big([\theta_i, \varrho_i] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big)$, $\quad i = 1, 2, \dots, m$, $f : \mathbb{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $k, l : \mathbb{J} \times \mathbb{R} \to \mathbb{R}$ are given functions.

In a similar manner, Rus [38] deduced a fundamental Ulam-Hyers stability result for ordinary differential equations. Ding [15] investigated the existence, uniqueness and Ulam-Hyers stability for a Caputo fractional impulsive delay differential equation. For a detailed study on the concepts for Ulam-Hyers stability for functional equations, readers are referred to the works by Ulam [42], Rassias [37], Hyers [19] and Jung [22]. On the other hand, with respect to the stability analysis for various classes of fractional differential equations with a non-singular kernel fractional differential operators, Khan *et al.* [24] established the existence and stability of the solution in the sense of Ulam-Hyers for a nonlinear fractional differential equation involving a non-singular kernel fractional differential operator. Fuentes *et al.* [16] analysed the dynamics and Lyapunov stability of a general class of fractional order differential system with non-singular kernel by using the Laplace transform and Lyapunov functions. By using the comparison theorem, they established some bound estimation for the solution of the system.

Motivated by the above works on Ulam-Hyers stability, our objective is to examine the existence and uniqueness, along with Ulam-Hyers stability, of the mild solution of the following Caputo fractional integro-differential equation subject to non-instantaneous impulses:

$$ {}_0^C\mathcal{D}_\theta^\alpha v(\theta) = \mathcal{G}\big(\theta, v(\theta), \int_0^\theta g(\varrho, v(\varrho))\,\mathrm{d}\varrho\big), \quad \theta \in (\varrho_i, \theta_{i+1}], \quad 0 \le i \le m, \tag{1.5} $$

$$ v(\theta) = \mathcal{H}_i\big(\theta, v(\theta), \int_0^\theta h(\varrho, v(\varrho))\,\mathrm{d}\varrho\big), \quad \theta \in (\theta_i, \varrho_i], \quad 1 \le i \le m, \tag{1.6} $$

$$ v(0) = v_0, \tag{1.7} $$

where $0 < \alpha < 1$, $0 = \theta_0 = \varrho_0 < \theta_1 \le \varrho_1 \le \theta_2 < \cdots < \theta_m \le \varrho_m < \theta_{m+1} = T$ are fixed given numbers, $\mathbb{J} = [0, T]$. Here, $\mathcal{H}_i \in C\big([\theta_i, \varrho_i] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big)$, $\quad i = 1, 2, \ldots, m$, $\mathcal{G} : \mathbb{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g, h : \mathbb{J} \times \mathbb{R} \to \mathbb{R}$ are given functions.

In the above considered system given by (1.5)-(1.7), a Caputo fractional-order differential operator ${}_0^C\mathcal{D}$ is taken into consideration since, compared to the other fractional-order differential operators, a system containing the Caputo fractional-order differential operator conveys a clear physical meaning of initial conditions such as $v(0), v'(0)$ and so on. It makes a system more realistic because of an initial condition with physical meaning. Inspired by the nature of the Caputo derivative operator, we pose an initial condition to the considered system at the initial time 0. One may pose the initial condition at any other initial time too, say θ_0, not equal to 0. To the best of the authors' knowledge, the stability in the sense of Ulam-Hyers has not been studied till date for such a class of non-instantaneous fractional-order integro-differential equations (1.5)-(1.7). Further, after establishing the theoretical results for the system given by (1.5)-(1.7), and estimating the Ulam-Hyers constant, we are motivated by a number of works such as [13,15,16,41] to estimate the bound for the solution corresponding to the initial value problem (1.5)-(1.7). This article also provides a bound estimation for the circuit current of the non-instantaneous fractional-order RLC circuit in terms of a Ulam-Hyers constant and this shows the practical applicability of the Ulam-Hyers constant in some important problems. This work has the potential to occupy an important place in fractional-order RLC circuit problems as well as in the application of Ulam-Hyers stability.

The paper is structured as follows. In Section 2, mathematical preliminaries are given. In Section 3, some useful inequalities are derived, and the main theoretical results are established under suitable assumptions. In Section 4, two suitable examples are provided to illustrate the obtained results. In Section 5, an application to fractional-order RLC circuits is presented to illustrate the importance and novel significance of the proposed results. Finally, conclusions are drawn in Section 6.

2 Preliminary results

Define

$$ \begin{aligned} & PC(\mathbb{J}, \mathbb{R}) \\ & = \big\{\Phi : \mathbb{J} \to \mathbb{R} : \Phi \in C\big((\theta_i, \theta_{i+1}], \mathbb{R}\big), i = 0, 1, \ldots, m; \Phi(\theta_i^+), \Phi(\theta_i^-) \text{ exist with } \Phi(\theta_i) = \Phi(\theta_i^-)\big\} \end{aligned} $$

which is a Banach space with the norm

$$\|\Phi\|_{PC} = \sup_{\theta\in\mathbb{J}} |\Phi(\theta)|.$$

Now, some definitions and properties of fractional operator are recalled [25, 29, 34] since they will be required during the course of establishing the results.

Definition 2.1 *[25] Let us denote by $AC^n(\mathbb{I})$ ($n = 1, 2, \ldots$, $\mathbb{I}$ an interval) the space of those functions Φ having continuous derivatives up to order $(n-1)$ on $\mathbb{I}$ with $\Phi^{(n-1)} \in AC(\mathbb{I})$, where $AC(\mathbb{I})$ is the space of absolutely continuous functions in $\mathbb{I}$.*

Definition 2.2 *[25] The Gamma function $\Gamma(z)$ is defined by*

$$\Gamma(z) = \int_0^\infty e^{-\varrho}\varrho^{z-1}\,\mathrm{d}\varrho, \tag{2.1}$$

with the integral converging in the right half of the complex plane, i.e., Re$(z) > 0$.

Definition 2.3 *[25] The Riemann-Liouville left-sided fractional integral ${}_a\mathcal{I}_\theta^\alpha$ of order $\alpha > 0$ of the function $v \in L^1([a,b],\mathbb{R})$ is defined by*

$${}_a\mathcal{I}_\theta^\alpha v(\theta) = \frac{1}{\Gamma(\alpha)}\int_a^\theta (\theta-\varrho)^{\alpha-1}v(\varrho)\,\mathrm{d}\varrho, \quad \theta > a. \tag{2.2}$$

Definition 2.4 *[25] The Riemann-Liouville fractional-order derivative ${}_a\mathcal{D}_\theta^\alpha$ of order α of a function $v \in AC^n([a,b],\mathbb{R})$ where $n = \lfloor\alpha\rfloor + 1$, is defined as*

$${}_a\mathcal{D}_\theta^\alpha v(\theta) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\dfrac{\mathrm{d}^n}{\mathrm{d}\theta^n}\displaystyle\int_a^\theta (\theta-\varrho)^{n-\alpha-1}v(\varrho)\,\mathrm{d}\varrho, & n-1 < \alpha < n, \\ \dfrac{\mathrm{d}^n v(\theta)}{\mathrm{d}\theta^n}, & \alpha = n. \end{cases} \tag{2.3}$$

Definition 2.5 *[34] The Caputo fractional derivative ${}_a^C\mathcal{D}_\theta^\alpha$ of order α of a function $v \in AC^n([a,b],\mathbb{R})$, where $n = \lfloor\alpha\rfloor + 1$, is defined by*

$${}_a^C\mathcal{D}_\theta^\alpha v(\theta) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_a^\theta (\theta-\varrho)^{n-\alpha-1}v^{(n)}(\varrho)\,\mathrm{d}\varrho, & n-1 < \alpha < n, \\ \dfrac{\mathrm{d}^n v(\theta)}{\mathrm{d}\theta^n}, & \alpha = n, \end{cases} \tag{2.4}$$

where $v^{(n)}(\theta) = \dfrac{\mathrm{d}^n v(\theta)}{\mathrm{d}\theta^n}$.

Lemma 2.1 *[11] Let $K \in C(\mathbb{J},\mathbb{R})$. Then, a function $v \in C(\mathbb{J},\mathbb{R})$ is termed a solution of the integral equation*

$$v(\theta) = v_b - \frac{1}{\Gamma(\alpha)}\int_0^b (b-\varrho)^{\alpha-1}K(\varrho)\,\mathrm{d}\varrho + \frac{1}{\Gamma(\alpha)}\int_0^\theta (\theta-\varrho)^{\alpha-1}K(\varrho)\,\mathrm{d}\varrho, \tag{2.5}$$

iff v satisfies the following fractional Cauchy problem:

$${}_0^C\mathcal{D}_\theta^\alpha v(\theta) = K(\theta), \quad \theta \in \mathbb{J} = [0,T], \tag{2.6}$$

$$v(b) = v_b, \quad 0 < b < T. \tag{2.7}$$

Theorem 2.1 *[11] (Banach fixed point theorem) Let (N, d_N) be a complete metric space and $\Lambda : N \to N$ be a contraction map with the Lipschitz constant $L < 1$. Suppose there exists a non-negative integer k satisfying $d_N(\Lambda^{k+1}y, \Lambda^k y) < +\infty$ for some $y \in N$. Then,*
(i) The sequence $\{\Lambda^n y\}$ converges to a fixed point x^* of Λ.
(ii) x^* is the unique fixed point of Λ in $N^* = \{z \in N | d_N(\Lambda^k y, z) < \infty\}$.
(iii) For $z \in N^*$, one has $d_N(z, x^*) \leq \frac{1}{1-L} d_N(\Lambda z, z)$.

3 Main Results

3.1 Background

Here we introduce some definitions, lemmas and theorems connected to problem $(1.5)-(1.7)$ which will be required for establishing the stability for the problem undertaken in the sense of Ulam-Hyers.

Definition 3.1 *A function $v \in PC(\mathbb{J}, \mathbb{R})$ is called a mild solution of problem $(1.5)-(1.7)$ if v satisfies*

$$v(\theta) = \begin{cases} v_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^\theta (\theta - \varrho)^{\alpha-1} \mathcal{G}\big(\varrho, v(\varrho), \int_0^\varrho g(\rho, v(\rho))\, \mathrm{d}\rho\big)\, \mathrm{d}\varrho, \quad \theta \in [0, \theta_1], \\ \mathcal{H}_i\big(\theta, v(\theta), \displaystyle\int_0^\theta h(\varrho, v(\varrho))\, \mathrm{d}\varrho\big), \qquad \theta \in (\theta_i, \varrho_i],\ i = 1, 2, 3, \ldots, m, \\ \mathcal{H}_i\big(\varrho_i, v(\varrho_i), \displaystyle\int_0^{\varrho_i} h(\varrho, v(\varrho))\, \mathrm{d}\varrho\big) \\ -\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^{\varrho_i} (\varrho_i - \varrho)^{\alpha-1} \mathcal{G}\big(\varrho, v(\varrho), \int_0^\varrho g(\rho, v(\rho))\, \mathrm{d}\rho\big)\, \mathrm{d}\varrho \\ +\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^\theta (\theta - \varrho)^{\alpha-1} \mathcal{G}\big(\varrho, v(\varrho), \int_0^\varrho g(\rho, v(\rho))\, \mathrm{d}\rho\big)\, \mathrm{d}\varrho, \quad \theta \in (\varrho_i, \theta_{i+1}]. \end{cases} \tag{3.1}$$

Let $\varepsilon > 0$ and consider the following inequalities:

$$\begin{cases} |{}_0^C\mathcal{D}_\theta^\alpha w(\theta) - \mathcal{G}\big(\theta, w(\theta), \int_0^\theta g(\varrho, w(\varrho))\, \mathrm{d}\varrho\big)| \leq \varepsilon, \quad \theta \in (\varrho_i, \theta_{i+1}],\ 0 \leq i \leq m, \\ |w(\theta) - \mathcal{H}_i\big(\theta, w(\theta), \int_0^\theta h(\varrho, w(\varrho))\, \mathrm{d}\varrho\big)| \leq \varepsilon, \qquad \theta \in (\theta_i, \varrho_i],\ 1 \leq i \leq m. \end{cases} \tag{3.2}$$

Theorem 3.1 *[3] A function $w \in PC(\mathbb{J}, \mathbb{R})$ is a solution of the inequalities (3.2) if and only if there exist a function $H \in PC(J, \mathbb{R})$ and a sequence $\{H_i\}$, $i = 1, 2, \ldots, m$, (which depends on v) such that*
(i) $|H(\theta)| \leq \varepsilon, \quad \theta \in \mathbb{J},$ and $|H_i| \leq \varepsilon, \quad i = 1, 2, \ldots, m,$
(ii) ${}_0^C\mathcal{D}_\theta^\alpha w(\theta) = \mathcal{G}\big(\theta, w(\theta), \int_0^\theta g(\varrho, w(\varrho))\, \mathrm{d}\varrho\big) + H(\theta), \quad \theta \in (\varrho_i, \theta_{i+1}],$
(iii) $w(\theta) = \mathcal{H}_i\big(\theta, w(\theta), \int_0^\theta h(\varrho, w(\varrho))\, \mathrm{d}\varrho\big) + H_i, \quad \theta \in (\theta_i, \varrho_i], \quad 1 \leq i \leq m.$

Lemma 3.1 *Suppose $\mathcal{G} \in C(\mathbb{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\mathcal{H}_i \in C([\theta_i, \varrho_i] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $i = 1, 2, \ldots, m$, and $g, h : \mathbb{J} \times \mathbb{R} \to \mathbb{R}$ are continuous. If $w \in PC(\mathbb{J}, \mathbb{R})$ is a solution of the inequalities (3.2), then it* *satisfies the following integral inequalities:*

$$\begin{cases} |w(\theta) - w(0) - \frac{1}{\Gamma(\alpha)}\int_0^\theta (\theta-\varrho)^{\alpha-1}\mathcal{G}\big(\varrho, w(\varrho), \int_0^\varrho g(\rho, w(\rho))\,\mathrm{d}\rho\big)\,\mathrm{d}\varrho| \le \frac{\varepsilon\theta_1^\alpha}{\Gamma(\alpha+1)}, & \theta\in(0,\theta_1],\\ |w(\theta) - \mathcal{H}_i\big(\theta, w(\theta), \int_0^\theta h(\varrho, w(\varrho))\,\mathrm{d}\varrho\big)| \le \varepsilon, & \theta\in(\theta_i,\varrho_i],\quad 1\le i\le m,\\ |w(\theta) - \mathcal{H}_i\big(\varrho_i, w(\varrho_i), \int_0^{\varrho_i} h(\varrho, w(\varrho))\,\mathrm{d}\varrho\big) & \\ +\frac{1}{\Gamma(\alpha)}\int_0^{\varrho_i} (\varrho_i-\varrho)^{\alpha-1}\mathcal{G}\big(\varrho, w(\varrho), \int_0^\varrho g(\rho, w(\rho))\,\mathrm{d}\rho\big)\,\mathrm{d}\varrho & \\ -\frac{1}{\Gamma(\alpha)}\int_0^{\theta} (\theta-\varrho)^{\alpha-1}\mathcal{G}\big(\varrho, w(\varrho), \int_0^\varrho g(\rho, w(\rho))\,\mathrm{d}\rho\big)\,\mathrm{d}\varrho| & \\ \le \Big(1+\frac{\varrho_i^\alpha+\theta_{i+1}^\alpha}{\Gamma(\alpha+1)}\Big)\varepsilon, & \theta\in(\varrho_i,\theta_{i+1}],\quad i=1,2,3,\dots,m. \end{cases} \tag{3.3}$$

Proof. From Lemma 2.1 and by Theorem 3.1, for $\theta \in [0, \theta_1]$, we can get

$$\begin{aligned} w(\theta) = w(0) &+ \frac{1}{\Gamma(\alpha)}\int_0^\theta (\theta-\varrho)^{\alpha-1}\mathcal{G}\big(\varrho, w(\varrho), \int_0^\varrho g(\rho, w(\rho))\,\mathrm{d}\rho\big)\,\mathrm{d}\varrho\\ &+ \frac{1}{\Gamma(\alpha)}\int_0^\theta (\theta-\varrho)^{\alpha-1} H(\varrho)\ \mathrm{d}\varrho. \end{aligned}$$

Then,

$$|w(\theta) - w(0) - \frac{1}{\Gamma(\alpha)}\int_0^\theta (\theta-\varrho)^{\alpha-1}\mathcal{G}\big(\varrho, w(\varrho), \int_0^\varrho g(\rho, w(\rho))\,\mathrm{d}\varrho\big)\,\mathrm{d}\varrho| \le \frac{\varepsilon\theta_1^\alpha}{\Gamma(\alpha+1)}.$$

For $\theta \in (\theta_i, \varrho_i]$, $i = 1, 2, \dots, m$, we have

$$|w(\theta) - \mathcal{H}_i\big(\theta, w(\theta), \int_0^\theta h(\varrho, w(\varrho))\,\mathrm{d}\varrho\big)| \le |H_i| \le \varepsilon, \quad \theta\in(\theta_i,\varrho_i], \quad 1\le i\le m.$$

For $\theta \in (\varrho_i, \theta_{i+1}]$, $i = 1, 2, \dots, m$, we get

$$\begin{aligned} w(\theta) = &\frac{1}{\Gamma(\alpha)}\int_0^\theta (\theta-\varrho)^{\alpha-1}\mathcal{G}\big(\varrho, w(\varrho), \int_0^\varrho g(\rho, w(\rho))\,\mathrm{d}\rho\big)\ \mathrm{d}\varrho + H_i\\ &- \frac{1}{\Gamma(\alpha)}\int_0^{\varrho_i} (\varrho_i-\varrho)^{\alpha-1}\mathcal{G}\big(\varrho, w(\varrho), \int_0^\varrho g(\rho, w(\rho))\,\mathrm{d}\rho\big)\,\mathrm{d}\varrho\cdot\frac{1}{\Gamma(\alpha)}\int_0^\theta (\theta-\varrho)^{\alpha-1}H(\varrho)\ \mathrm{d}\varrho\\ &+ \mathcal{H}_i\big(\varrho_i, w(\varrho_i), \int_0^{\varrho_i} h(\varrho, w(\varrho))\,\mathrm{d}\varrho\big) - \frac{1}{\Gamma(\alpha)}\int_0^{\varrho_i} (\varrho_i-\varrho)^{\alpha-1}H(\varrho)\,\mathrm{d}\varrho. \end{aligned}$$

Thus,

$$\begin{aligned} |w(\theta) - &\mathcal{H}_i\big(\varrho_i, w(\varrho_i), \int_0^{\varrho_i} g(\varrho, w(\varrho))\,\mathrm{d}\varrho\big)\\ &+ \frac{1}{\Gamma(\alpha)}\int_0^{\varrho_i} (\varrho_i-\varrho)^{\alpha-1}\mathcal{G}\big(\varrho, w(\varrho), \int_0^\varrho g(\rho, w(\rho))\,\mathrm{d}\rho\big)\,\mathrm{d}\varrho\\ &- \frac{1}{\Gamma(\alpha)}\int_0^\theta (\theta-\varrho)^{\alpha-1}\mathcal{G}\big(\varrho, w(\varrho), \int_0^\varrho g(\rho, w(\rho))\,\mathrm{d}\rho\big)\,\mathrm{d}\varrho| \le \Big(1+\frac{\varrho_i^\alpha+\theta_{i+1}^\alpha}{\Gamma(\alpha+1)}\Big)\varepsilon. \end{aligned}$$

Hence, the result is established. □

Definition 3.2 *The problem comprising equations* $(1.5)-(1.7)$ *is termed stable in the sense of Ulam-Hyers if there exists some constant* $c_\alpha > 0$ *such that, for each* $\varepsilon > 0$ *and for every solution* $w \in PC(\mathbb{J}, \mathbb{R})$ *of* (3.2)*, there exists a solution* $v \in PC(J, \mathbb{R})$ *(mild solution) of* $(1.5)-(1.7)$ *with*

$$|w(\theta) - v(\theta)| \le c_\alpha\varepsilon, \quad \textit{for all}\ \ \theta\in\mathbb{J}.$$

3.2 Existence and stability of the problem

To establish the desired results on the existence and stability of the solution of the problem under consideration, we take into account some hypotheses as follows:
(A1) $\mathcal{G} \in C(\mathbb{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exists $L_{\mathcal{G}} > 0$ satisfying

$$|\mathcal{G}(\theta, x_1, y_1) - \mathcal{G}(\theta, x_2, y_2)| \leq L_{\mathcal{G}}(|x_1 - x_2| + |y_1 - y_2|)$$

for each $\theta \in \mathbb{J}$, and for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.
(A2) $\mathcal{H}_i \in C\big([\theta_i, \varrho_i] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big)$ and there exists $L_{\mathcal{H}_i} > 0$, $i = 1, 2, \ldots, m$, satisfying

$$|\mathcal{H}_i(\theta, x_1, y_1) - \mathcal{H}_i(\theta, x_2, y_2)| \leq L_{\mathcal{H}_i}(|x_1 - x_2| + |y_1 - y_2|)$$

for each $\theta \in (\theta_i, \varrho_i]$, and for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.
(A3) $g, h \in C(\mathbb{J} \times \mathbb{R}, \mathbb{R})$ and there exist $G_g > 0$, $H_h > 0$ satisfying

$$|g(\theta, x_1) - g(\theta, x_2)| \leq G_g|x_1 - x_2|, \quad |h(\theta, y_1) - h(\theta, y_2)| \leq H_h|y_1 - y_2|$$

for each $\theta \in \mathbb{J}$, and for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

We are now in a position to proceed to establish our objectives with respect to problem $(1.5) - (1.7)$ through the concept of Ulam-Hyers stability.

Theorem 3.2 *Assume that assumptions (A1), (A2) and (A3) hold and*

$$\Theta = \max_{1 \leq i \leq m} \bigg\{ \frac{L_{\mathcal{G}}\theta_1^{\alpha}}{\Gamma(1+\alpha)}\Big(1 + \frac{\theta_1 G_g}{1+\alpha}\Big), L_{\mathcal{H}_i}\big(1 + \varrho_i H_h\big) + \frac{L_{\mathcal{G}}}{\Gamma(1+\alpha)}\big(\varrho_i^{\alpha} + \theta_{i+1}^{\alpha}\big) + \frac{G_g L_{\mathcal{G}}}{\Gamma(2+\alpha)}\big(\varrho_i^{\alpha+1} + \theta_{i+1}^{\alpha+1}\big) \bigg\} < 1.$$

Then, the problem $(1.5) - (1.7)$ *is Ulam-Hyers stable, i.e., there exists a unique mild solution* $v^* \in PC(J, \mathbb{R})$ *for the problem* $(1.5) - (1.7)$ *such that*

$$v^*(\theta) = \begin{cases} v_0 + \frac{1}{\Gamma(\alpha)} \int_0^{\theta} (\theta - \varrho)^{\alpha-1} \mathcal{G}\big(\varrho, v^*(\varrho), \int_0^{\varrho} g(\rho, v^*(\rho))\, \mathrm{d}\rho\big)\, \mathrm{d}\varrho, & \theta \in [0, \theta_1], \\ \mathcal{H}_i\big(\theta, v^*(\theta), \int_0^{\theta} h(\varrho, v^*(\varrho))\, \mathrm{d}\varrho\big), & \theta \in (\theta_i, \varrho_i], \quad i = 1, 2, 3, \ldots, m, \\ \mathcal{H}_i\big(\varrho_i, v^*(\varrho_i), \int_0^{\varrho_i} h(\varrho, v^*(\varrho))\, \mathrm{d}\varrho\big) & \\ +\frac{1}{\Gamma(\alpha)} \int_0^{\theta} (\theta - \varrho)^{\alpha-1} \mathcal{G}\big(\varrho, v^*(\varrho), \int_0^{\varrho} g(\rho, v^*(\rho))\, \mathrm{d}\rho\big)\, \mathrm{d}\varrho & \\ -\frac{1}{\Gamma(\alpha)} \int_0^{\varrho_i} (\varrho_i - \varrho)^{\alpha-1} \mathcal{G}\big(\varrho, v^*(\varrho), \int_0^{\varrho} g(\rho, v^*(\rho))\, \mathrm{d}\rho\big)\, \mathrm{d}\varrho, & \theta \in (\varrho_i, \theta_{i+1}], \end{cases} \tag{3.4}$$

and for each $w \in PC(J, \mathbb{R})$ *satisfying the inequality* (3.2) *with* $w(0) = v_0$*, we have*

$$|w(\theta) - v^*(\theta)| \leq \frac{r\varepsilon}{1 - \Theta}, \quad \textit{for all } \theta \in \mathbb{J},$$

where $r = \max_{1 \leq i \leq m} \Big\{ \frac{\theta_1^{\alpha}}{\Gamma(1+\alpha)}, 1 + \frac{\varrho_i^{\alpha} + \theta_{i+1}^{\alpha}}{\Gamma(1+\alpha)} \Big\}.$

Proof. We consider

$$N = PC(\mathbb{J}, \mathbb{R}), \tag{3.5}$$

with the metric

$$d_N(u_1,u_2) = ||u_1-u_2||_{PC} = \sup_{\theta\in\mathbb{J}} |u_1(\theta)-u_2(\theta)|, \quad u_1,u_2\in N. \tag{3.6}$$

Define an operator $\Lambda : N \to N$ by

$$\Lambda v(\theta) = \begin{cases} v_0 + \frac{1}{\Gamma(\alpha)}\int_0^\theta (\theta-\varrho)^{\alpha-1}\mathcal{G}\big(\varrho, v(\varrho), \int_0^\varrho f(\rho,v(\rho))\,\mathrm{d}\rho\big)\,\mathrm{d}\varrho, \quad \theta\in[0,\theta_1],\\ \mathcal{H}_i\big(\theta, v(\theta), \int_0^\theta g(\varrho,v(\varrho))\,\mathrm{d}\varrho\big), \qquad \theta\in(\theta_i,\varrho_i], \quad i=1,2,3,\dots,m,\\ \mathcal{H}_i\big(\varrho_i, v(\varrho_i), \int_0^{\varrho_i} g(\varrho,v(\varrho))\,\mathrm{d}\varrho\big)\\ -\frac{1}{\Gamma(\alpha)}\int_0^{\varrho_i}(\varrho_i-\varrho)^{\alpha-1}\mathcal{G}\big(\varrho, v(\varrho), \int_0^\varrho f(\rho,v(\rho))\,\mathrm{d}\rho\big)\,\mathrm{d}\varrho\\ +\frac{1}{\Gamma(\alpha)}\int_0^\theta(\theta-\varrho)^{\alpha-1}\mathcal{G}\big(\varrho, v(\varrho), \int_0^\varrho f(\rho,v(\rho))\,\mathrm{d}\rho\big)\,\mathrm{d}\varrho, \quad \theta\in(\varrho_i,\theta_{i+1}]. \end{cases}$$

Next, we prove that the operator Λ is a contraction map. Let $u_1,u_2\in N$. For $\theta\in[0,\theta_1]$, we have

$$\begin{aligned} &|\Lambda u_1(\theta)-\Lambda u_2(\theta)|\\ &\le \frac{1}{\Gamma(\alpha)}\int_0^\theta(\theta-\varrho)^{\alpha-1}|\mathcal{G}\big(\varrho,u_1(\varrho),\int_0^\varrho g(\rho,u_1(\rho))\,\mathrm{d}\rho\big)-\mathcal{G}\big(\varrho,u_2(\varrho),\int_0^\varrho g(\rho,u_2(\rho))\,\mathrm{d}\rho\big)|\,\mathrm{d}\varrho\\ &\le \frac{L_{\mathcal{G}}\theta_1^\alpha}{\Gamma(\alpha+1)}\Big(1+\frac{\theta_1 G_g}{\alpha+1}\Big)d_N(u_1,u_2). \end{aligned}$$

Thus,

$$||\Lambda u_1-\Lambda u_2||_{C([0,\theta_1],\mathbb{R})} \le \frac{L_{\mathcal{G}}\theta_1^\alpha}{\Gamma(\alpha+1)}\Big(1+\frac{\theta_1 G_g}{\alpha+1}\Big)||u_1-u_2||_{PC}. \tag{3.7}$$

For $\theta\in(\theta_i,\varrho_i], \ \ i=1,2,\dots,m$, we have

$$\begin{aligned} |\Lambda u_1-\Lambda u_2| &= |\mathcal{H}_i\big(\theta,u_1(\theta),\int_0^\theta h(\varrho,u_1(\varrho))\,\mathrm{d}\varrho\big)-\mathcal{H}_i\big(\theta,u_2(\theta),\int_0^\theta h(\varrho,u_2(\varrho))\,\mathrm{d}\varrho\big)|\\ &\le L_{\mathcal{H}_i}\big(1+\varrho_i H_h\big)d_N(u_1,u_2). \end{aligned}$$

Therefore,

$$||\Lambda u_1-\Lambda u_2||_{C((\theta_i,\varrho_i],\mathbb{R})} \le L_{\mathcal{H}_i}\big(1+\varrho_i H_h\big)||u_1-u_2||_{PC}. \tag{3.8}$$

For $\theta\in(\varrho_i,\theta_{i+1}], \ \ i=1,2,\dots,m$, we get

$$\begin{aligned} &|\Lambda u_1-\Lambda u_2|\\ &\le |\mathcal{H}_i\big(\varrho_i,u_1(\varrho_i),\int_0^{\varrho_i} h(\varrho,u_1(\varrho))\,\mathrm{d}\varrho\big)-\mathcal{H}_i\big(\varrho_i,u_2(\varrho_i),\int_0^{\varrho_i} h(\varrho,u_2(\varrho))\,\mathrm{d}\varrho\big)|\\ &+\frac{1}{\Gamma(\alpha)}\int_0^{\varrho_i}(\varrho_i-\varrho)^{\alpha-1}|\mathcal{G}\big(\varrho,u_1(\varrho),\int_0^\varrho g(\rho,u_1(\rho))\,\mathrm{d}\rho\big)-\mathcal{G}\big(\varrho,u_2(\varrho),\int_0^\varrho g(\rho,u_2(\rho))\,\mathrm{d}\rho\big)|\,\mathrm{d}\varrho\\ &+\frac{1}{\Gamma(\alpha)}\int_0^{\theta}(\theta-\varrho)^{\alpha-1}|\mathcal{G}\big(\varrho,u_1(\varrho),\int_0^\varrho g(\rho,u_1(\rho))\,\mathrm{d}\rho\big)-\mathcal{G}\big(\varrho,u_2(\varrho),\int_0^\varrho g(\rho,u_2(\rho))\,\mathrm{d}\rho\big)|\,\mathrm{d}\varrho. \end{aligned}$$

Consequently,

$$|\Lambda u_1-\Lambda u_2| \le \Big[L_{\mathcal{H}_i}\big(1+\varrho_i H_h\big)+\frac{L_{\mathcal{G}}}{\Gamma(1+\alpha)}\big(\varrho_i^\alpha+\theta_{i+1}^\alpha\big)+\frac{G_g L_{\mathcal{G}}}{\Gamma(2+\alpha)}\big(\varrho_i^{\alpha+1}+\theta_{i+1}^{\alpha+1}\big)\Big]d_N(u_1,u_2),$$

which ultimately gives

$$||\Lambda u_1 - \Lambda u_2||_{C([\varrho_i,\theta_{i+1}],\mathbb{R})} \tag{3.9}$$
$$\leq \left[L_{\mathcal{H}_i}\big(1+\varrho_i H_h\big) + \frac{L_{\mathcal{G}}}{\Gamma(1+\alpha)}\big(\varrho_i^{\alpha}+\theta_{i+1}^{\alpha}\big) + \frac{G_g L_{\mathcal{G}}}{\Gamma(2+\alpha)}\big(\varrho_i^{\alpha+1}+\theta_{i+1}^{\alpha+1}\big)\right] \times ||u_1-u_2||_{PC}.$$

Thus, we observe that

$$d_N(\Lambda u_1, \Lambda u_2) \leq \Theta d_N(u_1,u_2), \quad \text{for all } u_1,u_2 \in N. \tag{3.10}$$

By assumption, we know that $\Theta < 1$ and hence, Λ is a contraction map. Subsequently, Banach fixed point theorem confirms that there exists a unique mild solution $v^* \in N$ as defined in (3.4) to the problem $(1.5) - (1.7)$.

Stability:
We consider a function $w \in N$ which satisfies the inequality (3.2) with $w(0) = v_0$. Then, according to Lemma 3.1, we obtain for $\theta \in [0,\theta_1]$,

$$\begin{aligned}|\Lambda w(\theta) - w(\theta)| &\leq |w(\theta) - \frac{1}{\Gamma(\alpha)}\int_0^{\theta}(\theta-\varrho)^{\alpha-1}\mathcal{G}\big(\varrho, w(\varrho), \int_0^{\varrho} g(\rho, v(\rho))\,\mathrm{d}\rho\big)\,\mathrm{d}\varrho - v_0| \\ &\leq \frac{\varepsilon\theta_1^{\alpha}}{\Gamma(\alpha+1)}.\end{aligned}$$

Thus,

$$||\Lambda w - w||_{C([0,\theta_1],\mathbb{R})} \leq \frac{\varepsilon\theta_1^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.11}$$

Similarly, for $\theta \in (\theta_i, \varrho_i],\ i = 1,2,\ldots,m$, we have

$$|\Lambda w(\theta) - w(\theta)| \leq |w(\theta) - \mathcal{H}_i\big(\theta, w(\theta), \int_0^{\theta} h(\varrho, w(\varrho))\,\mathrm{d}\varrho\big)| \leq \varepsilon.$$

This gives us

$$||\Lambda w - w||_{C((\theta_i,\varrho_i],\mathbb{R})} \leq \varepsilon. \tag{3.12}$$

For $\theta \in (\varrho_i, \theta_{i+1}],\ i = 1,2,3,\ldots,m$, we get

$$\begin{aligned}|\Lambda w(\theta) - w(\theta)| \leq |w(\theta) &- \mathcal{H}_i\big(\varrho_i, w(\varrho_i), \int_0^{\varrho_i} h(\varrho, w(\varrho))\,\mathrm{d}\varrho\big) \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^{\varrho_i}(\varrho_i-\varrho)^{\alpha-1}\mathcal{G}\big(\varrho, w(\varrho), \int_0^{\varrho} g(\rho, w(\rho))\,\mathrm{d}\rho\big)\,\mathrm{d}\varrho \\ &- \frac{1}{\Gamma(\alpha)}\int_0^{\theta}(\theta-\varrho)^{\alpha-1}\mathcal{G}\big(\varrho, w(\varrho), \int_0^{\varrho} g(\rho, w(\rho))\,\mathrm{d}\rho\big)\,\mathrm{d}\varrho| \leq \Big(1 + \frac{\varrho_i^{\alpha}+\theta_{i+1}^{\alpha}}{\Gamma(\alpha+1)}\Big)\varepsilon.\end{aligned}$$

Therefore, we obtain

$$||\Lambda w - w||_{C((\varrho_i,\theta_{i+1}],\mathbb{R})} \leq \Big(1 + \frac{\varrho_i^{\alpha}+\theta_{i+1}^{\alpha}}{\Gamma(\alpha+1)}\Big)\varepsilon. \tag{3.13}$$

Hence, from above inequalities $(3.11) - (3.13)$, we have

$$d_N(\Lambda w, w) \leq r\varepsilon < +\infty. \tag{3.14}$$

Here, if $k = 0$, then we consider the space

$$N^* = \{z \in N | d_N(w, z) < \infty\}. \tag{3.15}$$

Taking $z(\theta) = w(\theta)$, it follows that $d_N(w, w) = 0 < \infty$ which implies $w \in N^*$. Hence, by Theorem 2.1 (part (iii)), if v^* is a unique fixed point of the operator Λ, the following is obtained:

$$|w(\theta) - v^*(\theta)| \leq \frac{d_N(\Lambda w, w)}{1 - \Theta} \leq \frac{r\varepsilon}{1 - \Theta}, \quad \textit{for all } \theta \in \mathbb{J}. \tag{3.16}$$

Therefore, it can be concluded that problem $(1.5) - (1.7)$ is Ulam-Hyers stable. □

4 Examples

Two examples are presented to authenticate the results obtained in the preceding section.

Example 4.1 *The following Caputo fractional differential equation with non-instantaneous impulse is considered:*

$$\begin{cases} {}_0^C\mathcal{D}_\theta^{\frac{1}{3}} v(\theta) = \frac{1}{5+3\theta^2}\Big(|v(\theta)| + \int_0^\theta \frac{|v(\varrho)|}{10+7\varrho^2}\,\mathrm{d}\varrho\Big), & \theta \in (0,1] \cup (2,3], \\ v(\theta) = \frac{1}{(5+3(\theta-1)^2)(1+|v(\theta)|)}\Big(|v(\theta)| + \int_0^\theta \frac{|v(\varrho)|}{15+11\varrho^2}\,\mathrm{d}\varrho\Big), & \theta \in (1,2], \end{cases} \tag{4.1}$$

and the corresponding non-instantaneous impulsive fractional differential inequalities with impulse interval (1,2], and for $\varepsilon > 0$, is given by

$$\begin{cases} |{}_0^C\mathcal{D}_\theta^{\frac{1}{3}} v(\theta) - \frac{1}{5+3\theta^2}\Big(|v(\theta)| + \int_0^\theta \frac{|v(\varrho)|}{10+7\varrho^2}\,\mathrm{d}\varrho\Big)| \leq \varepsilon, & \theta \in (0,1] \cup (2,3], \\ |v(\theta) - \frac{1}{(5+3(\theta-1)^2)(1+|v(\theta)|)}\Big(|v(\theta)| + \int_0^\theta \frac{|v(\varrho)|}{15+11\varrho^2}\,\mathrm{d}\varrho\Big)| \leq \varepsilon, & \theta \in (1,2]. \end{cases} \tag{4.2}$$

Here, $J = [0,3]$ and $0 = \theta_0 = \varrho_0 < \theta_1 = 1 < \varrho_1 = 2 < \theta_2 = 3$, $f(\theta, v(\theta)) = \frac{|v(\theta)|}{10+7\theta^2}$ with $F = \frac{1}{10}$ and

$$\mathcal{G}\Big(\theta, v(\theta), \int_0^\theta f(\varrho, v(\varrho))\,\mathrm{d}\varrho\Big) = \frac{1}{5+3\theta^2}\Big(|v(\theta)| + \int_0^\theta \frac{|v(\varrho)|}{10+7\varrho^2}\,\mathrm{d}\varrho\Big)$$

with $L_{\mathcal{G}} = \frac{1}{5}$. Also, $g(\theta, v(\theta)) = \frac{|v(\theta)|}{15+11\theta^2}$ with $G = \frac{1}{15}$, and

$$\mathcal{G}_1\Big(\theta, v(\theta), \int_0^\theta g(\varrho, v(\varrho))\,\mathrm{d}\varrho\Big) = \frac{1}{(5+3(\theta-1)^2)(1+|v(\theta)|)}\Big(|v(\theta)| + \int_0^\theta \frac{|v(\varrho)|}{15+11\varrho^2}\,\mathrm{d}\varrho\Big),$$

with $L_{\mathcal{G}_1} = \frac{1}{5}$ for $\theta \in (1,2]$. Now,

$$\Theta = \max\Big\{\frac{\theta_1^\alpha L_{\mathcal{G}}}{\Gamma(1+\alpha)}\Big(1 + \frac{\theta_1 F}{1+\alpha}\Big), L_{\mathcal{G}_1}\big(1 + \varrho_1 G\big) + \frac{L_{\mathcal{G}}}{\Gamma(1+\alpha)}\big(\varrho_1^\alpha + \theta_2^\alpha\big) + \frac{F L_{\mathcal{G}}}{\Gamma(2+\alpha)}\big(\varrho_1^{\alpha+1} + \theta_2^{\alpha+1}\big)\Big\}.$$

Then,

$$\Theta = \max\Big\{\frac{\frac{1}{5}}{\Gamma(\frac{4}{3})}\Big(1 + \frac{3}{40}\Big), \frac{1}{5}\big(1 + \frac{2}{15}\big) + \frac{\frac{1}{5}}{\Gamma(\frac{4}{3})}\big(\sqrt[3]{2} + \sqrt[3]{3}\big) + \frac{\frac{1}{50}}{\Gamma(\frac{7}{3})}\big(2\sqrt[3]{2} + 3\sqrt[3]{3}\big)\Big\},$$

which implies

$$\Theta = \max\Big\{0.240767, 0.946876905\Big\} = 0.946876905 \cong 0.947 < 1.$$

Here, $r = \max\Big\{\frac{1}{\Gamma(\frac{4}{3})}, 1 + \frac{\sqrt[3]{2}+\sqrt[3]{3}}{\Gamma(\frac{4}{3})}\Big\} = \max\Big\{0.89297951, 4.02601637\Big\}$, *i.e.,* $r = 4.02601637 \cong 4.03$. *Theorem 3.2 confirms that there exists a unique solution* $v^* : [0,3] \to \mathbb{R}$ *such that*

$$v^*(\theta) = \begin{cases} u_0 + \frac{1}{\Gamma(\frac{1}{3})} \int_0^\theta (\theta-\varrho)^{-\frac{2}{3}} \frac{1}{5+3\varrho^2}\Big(|v^*(\varrho)| + \int_0^\varrho \frac{|v^*(\rho)|}{10+7\rho^2}\,\mathrm{d}\rho\Big)\,\mathrm{d}\varrho, & \theta \in [0,1],\\ \frac{1}{(5+3(\theta-1)^2)(1+|v^*(\theta)|)}\Big(|v^*(\theta)| + \int_0^\theta \frac{|v^*(\varrho)|}{15+11\varrho^2}\,\mathrm{d}\varrho\Big), & \theta \in (1,2],\\ \frac{1}{8(1+|v^*(2)|)}\Big(|v^*(2)| + \int_0^2 \frac{|v^*(\varrho)|}{15+11\varrho^2}\,\mathrm{d}\varrho\Big) & \\ -\frac{1}{\Gamma(\frac{1}{3})} \int_0^2 (2-\varrho)^{\frac{-2}{3}} \frac{1}{5+3\varrho^2}\Big(|v^*(\varrho)| + \int_0^\varrho \frac{|v^*(\rho)|}{10+7\rho^2}\,\mathrm{d}\rho\Big)\,\mathrm{d}\varrho & \\ +\frac{1}{\Gamma(\frac{1}{3})} \int_0^\theta (\theta-\varrho)^{-\frac{2}{3}} \frac{1}{5+3\varrho^2}\Big(|v^*(\varrho)| + \int_0^\varrho \frac{|v^*(\rho)|}{10+7\rho^2}\,\mathrm{d}\rho\Big)\,\mathrm{d}\varrho, & \theta \in (2,3], \end{cases} \tag{4.3}$$

with

$$|v(\theta) - v^*(\theta)| \leq \frac{r\varepsilon}{1-\Theta} = 76.04\varepsilon, \quad \textit{for all } \theta \in J = [0,3],$$

where $v(\theta)$ *is the solution of the fractional differential inequalities* (4.2).

Example 4.2 *Consider the following Caputo fractional differential equation:*

$$\begin{cases} {}_0^C\mathcal{D}_t^{\frac{1}{4}} u(t) = \frac{u(t)}{5} + \frac{1}{8}\int_0^t \frac{u(\varrho)}{3+\varrho}\,\mathrm{d}\varrho + F(t), & t \in (0,1] \cup (2,3],\\ u(t) = \frac{\sin(u(t))}{5} + \frac{1}{6}\int_0^t \frac{u(\varrho)}{2+\varrho}\,\mathrm{d}\varrho + G_1(t), & t \in (1,2],\\ u(0) = 1, & \end{cases} \tag{4.4}$$

where $F(t)$ *and* $G_1(t)$ *are functions of t chosen such that*

$$u(t) = \begin{cases} (t+3)(\frac{t}{10} - \frac{t^{0.25}}{4} + 1), & t \in [0,1],\\ (t+2)(t^{0.25}+1), & t \in (1,2],\\ (t+3)(\frac{t}{10} - \frac{t^{0.25}}{4} + 1) + \frac{21}{2^{1.75}} - 2, & t \in (2,3], \end{cases} \tag{4.5}$$

is the solution of problem (4.4)*. Clearly, the function in equation* (4.5) *is a unique solution for problem* (4.4) *by Theorem 3.2, since* $\Theta = 0.833 < 1$, *and all the conditions are satisfied. Further, the Ulam-Hyers constant is given by*

$$c_{f,\alpha} = \begin{cases} 6.59, & t \in [0,1],\\ 5.98, & t \in (1,2],\\ 22.49, & t \in (2,3]. \end{cases} \tag{4.6}$$

Thus, by Theorem 3.2*, problem* (4.4) *is Ulam-Hyers stable which means that the inequality* $|v(t) - u(t)| \leq c_{f,\alpha}\varepsilon$ *holds for all* $t \in J = [0,3]$ *and any solution* $v(t)$ *satisfying the inequalities* (3.2)*. Subsequently, according to Theorem* 3.1*, it is the solution of*

$$\begin{cases} {}_0^C\mathcal{D}_t^{\frac{1}{4}} v(t) = \frac{v(t)}{5} + \frac{1}{8}\int_0^t \frac{v(\varrho)}{3+\varrho}\,\mathrm{d}\varrho + F(t) + G(t), & t \in (0,1] \cup (2,3],\\ v(t) = \frac{\sin(v(t))}{5} + \frac{1}{6}\int_0^t \frac{v(\varrho)}{2+\varrho}\,\mathrm{d}\varrho + G_1(t) + u_1 & t \in (1,2], \end{cases} \tag{4.7}$$

where $G \in PC(J,\mathbb{R})$ *and* $u_1 \in \mathbb{R}$ *with* $|G(t)| \leq \varepsilon$, $t \in J$, $|u_1| \leq \varepsilon$*. In particular, for* $\varepsilon = 0.5$*, take* $G(t) = \varepsilon$, $u_1 = \varepsilon$*. We plot the graphs of the solution* $u(t)$ *of problem* (4.4) *and the solution* $v(t)$ *of problem* (4.7) *in figure* 1*. We also present the difference function* $|v(t) - u(t)|$ *for* $\varepsilon = 0.5$ *and the upper bound* $c_{f,\alpha}\varepsilon$ *graphically in figure 2. It points towards a difference which is almost uniform.*

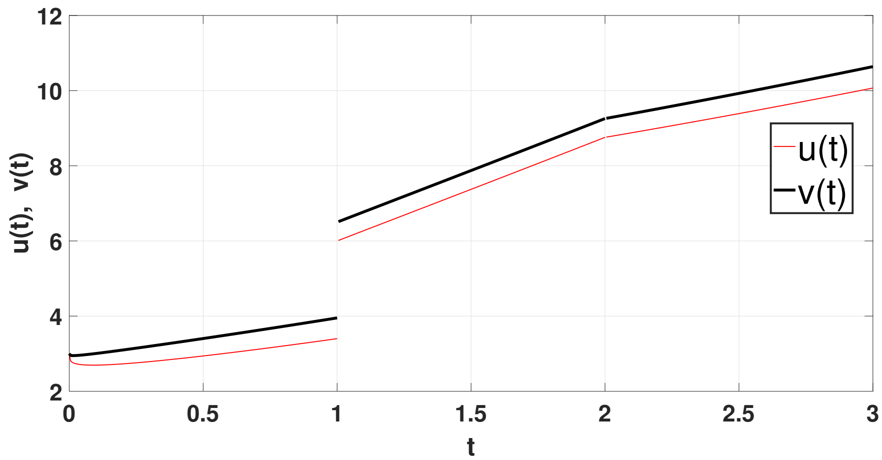

Figure 1: Solution $u(t)$ of the problem (4.4) and solution $v(t)$ of the problem (4.7) for $\varepsilon = 0.5$.

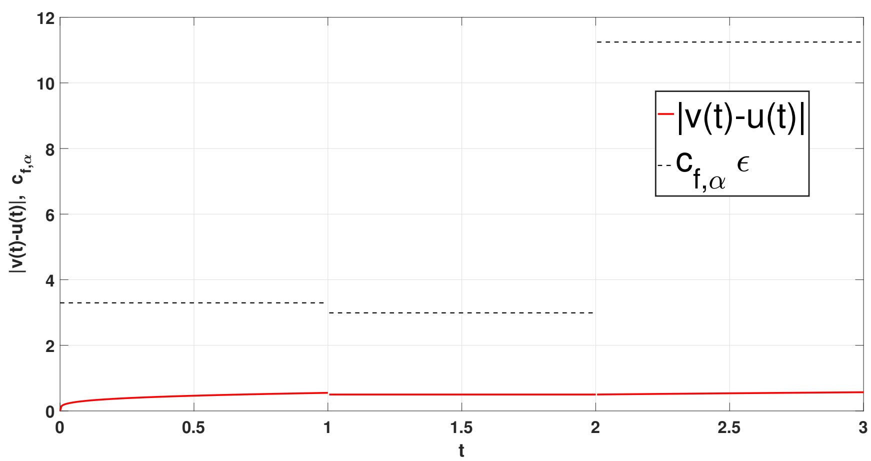

Figure 2: The difference function $|v(t) - u(t)|$ and upper bound $c_{f,\alpha}\varepsilon$ for $\varepsilon = 0.5$.

5 An Application to Fractional-order RLC

RLC circuit is one of the most basic and fundamental circuits in various electronic devices and there is a huge literature describing the RLC circuit concerning integer-order differential equations. Fractional-order RLC circuit model is the generalization of the classical integer-order RLC circuit. This fractional model has a number of advantages over its integer-order counterpart because of the fractional order. This provides substantial flexibility in the design and control of circuit which enhances the performance and exhibits the novel behavior. Radwan *et al.* [36] presented a broad view of the fractional-order RLC circuit model and illustrated that it was not possible to observe some novel phenomena in the absence of the fractional-order model. They also established that the fractional-order impedance was purely imaginary, and it enabled the modeling of a huge capacitance/inductance by considering a very small fractional order. Under the variant and non-variant voltage source, Khader *et al.* [23] carried out a numerical study of the fractional-order RLC circuit and graphically compared the solution of the fractional-order circuit with the corresponding integer-order one. For complete information on fractional-order RLC circuit, the readers are referred to the works in [4, 40, 47]. Shankar and Bora [41] estimated the bound for the difference between the fractional-order and the integer-order non-instantaneous impulsive RLC circuit currents. They established that the bound was primarily dependent on the bandwidth of the circuit.

Dhaneliya *et al.* [13] computed the analytical solution of the fractional-order RLC circuit in terms of an infinite series in which, each term was a generalized Mittag-Leffler function. Hence, it is a challenging task to estimate the bound for the solution; and computation of the solution may increase the round-off error. Therefore, to overcome such problems, we propose to find some bound estimate for the solution. The following non-instantaneous fractional impulsive RLC circuit under

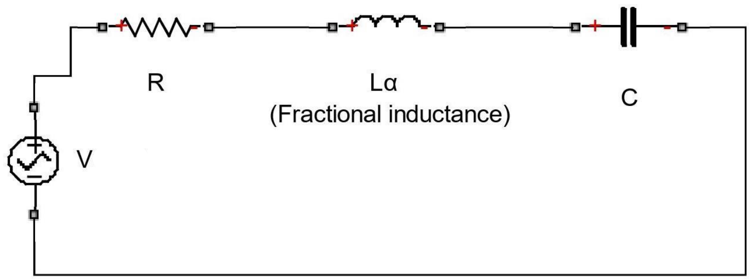

Figure 3: Fractional-order RLC circuit under input voltage V, with fractional-order influence on the inductance

a given input voltage $V(\theta)$ is considered:

$$L_\alpha \, {}_0^C D_\theta^\alpha I_\alpha(\theta) + RI_\alpha(\theta) + \frac{1}{C}\int_0^\theta I_\alpha(\varrho)\,\mathrm{d}\varrho = V(\theta), \quad \theta \in \Big(0, \frac{1}{4}\Big] \cup \Big(\frac{1}{2}, 1\Big], \tag{5.1}$$

$$RI_\alpha(\theta) + \frac{1}{C}\int_0^\theta I_\alpha(\varrho)\,\mathrm{d}\varrho = V(\theta), \quad \theta \in \Big(\frac{1}{4}, \frac{1}{2}\Big], \tag{5.2}$$

$$I_\alpha(0) = 0, \tag{5.3}$$

where L_α, R and C are inductor, resistor and capacitor, respectively. Here $0 = \varrho_0 < \theta_1 = \frac{1}{4} < \varrho_1 = \frac{1}{2} < \theta_2 = 1$, and we also assume that $R^2 = 4L_\alpha/C$. In the above circuit (Fig. 3), for the time interval $\big(\frac{1}{4}, \frac{1}{2}\big]$, the effect of the inductor is negligible but its effect is clearly visible when the time interval $\big(\frac{1}{2}, 1\big]$ is considered.

Theorem 5.1 *Consider the non-instantaneous impulsive fractional-order RLC circuit* $(5.1)-(5.3)$ *and assume that the input voltage* $V(\theta)$ *is bounded, i.e., there exists* $B > 0$ *such that* $|V(\theta)| < B$, *for all* $\theta \in [0, 1]$. *Let the bandwidth* $a = \dfrac{R}{2L_\alpha}$ *of the* RLC *circuit satisfy*

$$\frac{a}{2}\left[1 + \frac{a}{4} + \frac{2}{\Gamma(1+\alpha)}\left\{\frac{a^2}{1+\alpha}\Big(1 + \frac{1}{2^{\alpha+1}}\Big) + \frac{1}{2^\alpha} + 1\right\}\right] = \Theta < 1. \tag{5.4}$$

Then, the circuit current I_α *of the system* $(5.1) - (5.3)$ *is bounded, i.e.,*

$$|I_\alpha(\theta)| \le L_U B, \quad \textit{for all } \theta \in [0, 1], \tag{5.5}$$

where $L_U = \dfrac{1 + \dfrac{(1 + \frac{1}{2^\alpha})}{\Gamma(1+\alpha)}}{1 - \Theta}$.

Proof. *Here,*

$$\mathcal{G}\big(\theta, I_\alpha(\theta), \int_0^\theta g(\varrho, I_\alpha(\varrho))\,\mathrm{d}\varrho\big) = \frac{1}{L_\alpha}\Bigg(V(\theta) - RI_\alpha(\theta) - \frac{1}{C}\int_0^\theta I_\alpha(\varrho)\ \mathrm{d}\varrho\Bigg),$$

with $L_{\mathcal{G}} = \max\Big\{\frac{R}{L_\alpha}, \frac{1}{CL_\alpha}\Big\}$. *By using the given condition* $\Theta < 1$, *it implies that* $a < 1$. *Under the given assumption* $R^2 = 4L_\alpha/C$, *we get* $CL_\alpha = \frac{1}{a^2}$. *Thus, we have*

$$L_{\mathcal{G}} = \max\{2a, a^2\} = 2a. \tag{5.6}$$

Further,

$$\mathcal{H}_1\big(\theta, I_\alpha(\theta), \int_0^\theta h(\varrho, I_\alpha(\varrho))\,\mathrm{d}\varrho\big) = \frac{1}{R}\Bigg(V(\theta) - \frac{1}{C}\int_0^\theta I_\alpha(\varrho)\ \mathrm{d}\varrho\Bigg)$$

with $L_{\mathcal{H}_1} = \frac{1}{RC} = \frac{a}{2}$. *We also take* $G_g = \frac{1}{CL_\alpha} = a^2$ *and* $H_h = \frac{1}{RC} = \frac{a}{2}$. *Here, we observe that*

$$\begin{aligned}
\Theta &= \max\Bigg\{\frac{L_{\mathcal{G}}\theta_1^\alpha}{\Gamma(1+\alpha)}\Big(1 + \frac{\theta_1 G_g}{1+\alpha}\Big), L_{\mathcal{H}_1}\big(1 + \varrho_1 H_h\big) + \frac{L_{\mathcal{G}}}{\Gamma(1+\alpha)}\big(\varrho_1^\alpha + \theta_2^\alpha\big) \\
&\quad + \frac{G_g L_{\mathcal{G}}}{\Gamma(2+\alpha)}\big(\varrho_1^{\alpha+1} + \theta_2^{\alpha+1}\big)\Bigg\} \\
&= \max\Bigg\{\frac{2a}{4^\alpha\Gamma(1+\alpha)}\Bigg(1 + \frac{a^2}{4(1+\alpha)}\Bigg), \frac{a}{2}\Big(1 + \frac{a}{4}\Big) + \frac{2a}{\Gamma(1+\alpha)}\Big(1 + \frac{1}{2^\alpha}\Big) \\
&\quad + \frac{2a^3}{\Gamma(2+\alpha)}\Big(1 + \frac{1}{2^{\alpha+1}}\Big)\Bigg\} \\
&= \frac{a}{2}\Bigg[1 + \frac{a}{4} + \frac{2}{\Gamma(1+\alpha)}\Bigg\{\frac{a^2}{1+\alpha}\Big(1 + \frac{1}{2^{\alpha+1}}\Big) + \frac{1}{2^\alpha} + 1\Bigg\}\Bigg].
\end{aligned} \tag{5.7}$$

By given condition (5.4), *we have* $\Theta < 1$. *Thus, by Theorem 3.2, the fractional-order* RLC *system* $(5.1) - (5.3)$ *is Ulam-Hyers stable with a Ulam-Hyers constant given by*

$$L_U = \frac{1 + \dfrac{(1 + \frac{1}{2^\alpha})}{\Gamma(1+\alpha)}}{1 - \Theta}. \tag{5.8}$$

Now, take $I_\alpha(\theta) = I(\theta) = 0$, *for all* $\theta \geq 0$, *in the system* $(5.1) - (5.3)$ *to get*

$$|L_\alpha\, {}_0^C D_\theta^\alpha I_\alpha(\theta) + RI_\alpha(\theta) + \frac{1}{C}\int_0^\theta I_\alpha(\varrho)\,\mathrm{d}\varrho - V(\theta)| = |V(\theta)| \leq B, \quad \theta \in \Big(0, \frac{1}{4}\Big] \cup \Big(\frac{1}{2}, 1\Big], \tag{5.9}$$

and

$$|RI_\alpha(\theta) + \frac{1}{C}\int_0^\theta I_\alpha(\varrho)\,\mathrm{d}\varrho - V(\theta)| = |V(\theta)| \leq B, \quad \theta \in \Big(\frac{1}{4}, \frac{1}{2}\Big]. \tag{5.10}$$

Thus, we take $\varepsilon = B$, *and from equations* $(5.9), (5.10)$, *we observe that the function* $I(\theta) = 0$ *satisfies the inequality* (3.2). *Thus, by Theorem 3.2, we have the following bound for the fractional-order* RLC *circuit current* $I_\alpha(\theta)$:

$$|I_\alpha(\theta)| = |I_\alpha(\theta) - I(\theta)| \leq L_U B, \quad \textit{for all } \theta \in [0, 1], \tag{5.11}$$

where L_U *is a Ulam-Hyers constant given in equation* (5.8). □

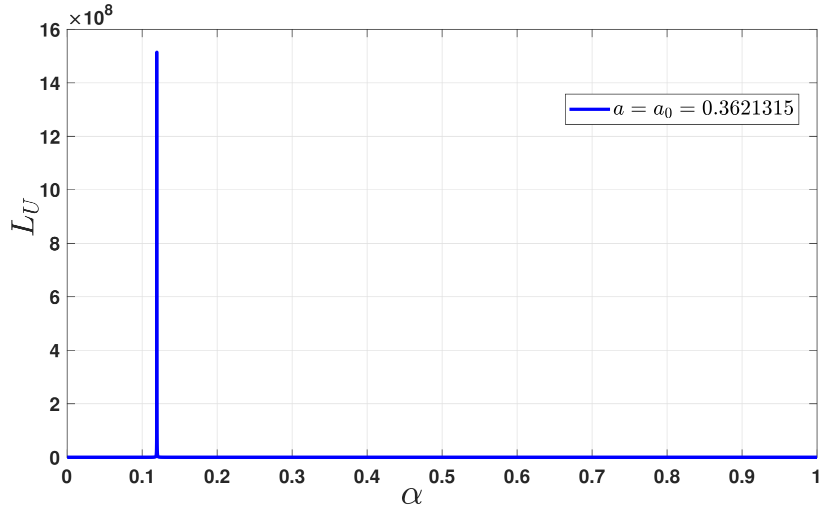

Figure 4: Graph of Ulam-Hyers constant L_U given by (5.8) for $a = a_0 = 0.3621315$ (threshold value).

Remark 5.1 *From the numerical computation, we observe that, for* $a \leq 0.3621315$, *the condition* (5.4) *of Theorem 5.1 holds for all* $\alpha \in [0, 1]$. *So, we call* $a_0 = 0.3621315$ *the threshold value of the bandwidth* a *in the sense that, for* $a \leq a_0$, *the condition* (5.4) *of Theorem 5.1 holds for all* $\alpha \in [0, 1]$.

First, we plot the graph of the Ulam-Hyers constant L_U with $\alpha \in [0, 1]$ and for $a = a_0$. From Fig. 4, we observe the graph of L_U attains its maximum point at $\alpha = 0.119612$, which means that, if the bandwidth a of the given RLC system $(5.1) - (5.3)$ is equal to the threshold value $a_0 = 0.3621315$, then from inequality (5.5) and under the given input voltage, the absolute value of the circuit current is maximum for $\alpha = 0.119612$, compared to the fractional order $\alpha \in [0, 1] \setminus \{0.119612\}$ of the system.

Next, we study the behavior of the Ulam-Hyers constant for the bandwidth a closer to the threshold value a_0. From Fig. 5, we observe that the variation of the Ulam-Hyers constant is sensitive for the bandwidth a closer to the threshold value a_0. For each value of a, L_U attains its maximum value for the fractional order satisfying $0.1 < \alpha < 0.2$.

Next, we plot the graph of the Ulam-Hyers constant L_U for different values of $a < a_0$. From Fig. 6, it is clear that the maximum value of L_U is not as high as compared to the case when the values of the bandwidth are near the threshold value a_0. In comparison to the larger values of the bandwidth, the values of L_U on $0 \leq \alpha \leq 1$ are smaller corresponding to the smaller values of the bandwidth. In other words, if we denote $L_U(a, \alpha)$ as the Ulam-Hyers constant for a given bandwidth a on $0 \leq \alpha \leq 1$, then $L_U(a_1, \alpha) < L_U(a_2, \alpha)$ for all $0 \leq \alpha \leq 1$ with $a_1 < a_2 \leq a_0$. Here, we also observe that, for smaller values of $a(\ll a_0)$, L_U remains almost constant, which means that, for smaller values of the bandwidth, the absolute values of the circuit current I_α are bounded with a bound independent of the fractional-order α.

Finally, from the above observations, we conclude that the absolute values of the circuit current I_α of the fractional-order RLC system $(5.1) - (5.3)$ are large corresponding to larger values of the bandwidth $a(< a_0)$, and for each value of $a(\leq a_0)$, the absolute value of the circuit current I_α is the maximum for the fractional order $0.1 < \alpha < 0.2$. In particular, for $a = a_0$ and $\alpha = 0.119612$,

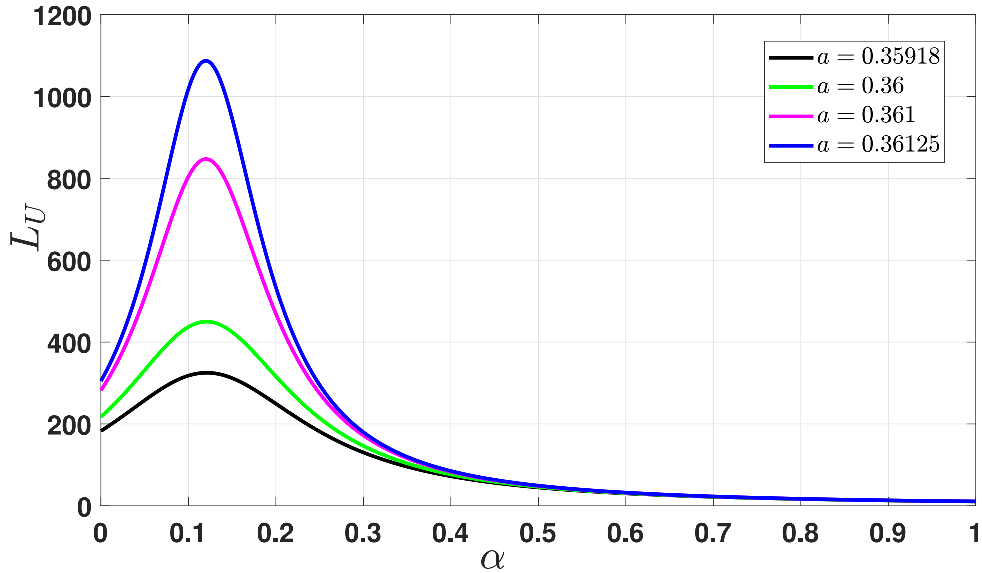

Figure 5: Graph of Ulam-Hyers constant L_U given by (5.8) for the bandwidth close to the threshold value a_0

under a given input voltage, we can generate the circuit current with the largest absolute value for the fractional order $\alpha = 0.119612$, compared to the fractional order $\alpha \in [0,1]\backslash\{0.119612\}$. This shows that the fractional-order RLC circuit model has many advantages over its integer-order counterpart because of the inclusion of the order α included in the model which affects the performance and enhances the novel behavior lending more flexibility in the design and control of the circuit. Further, this shows the practical applicability of the Ulam-Hyers constant in some important problems. We firmly believe this finding has the potential to occupy an important place in fractional-order RLC circuit problems as well as in the application of Ulam-Hyers stability.

6 Conclusion

In this work, we have displayed the existence and Ulam-Hyers stability results of the mild solution of the fractional non-instantaneous impulsive integro-differential equation with Caputo derivative. The main result was established by using Banach fixed point theorem under appropriate assumptions. Two demonstrated examples ascertain the applicability of the obtained results. The main result was used to estimate the bound for the non-instantaneous impulsive fractional-order RLC circuit current I_α, and it is found that the bound mainly depends on the bandwidth and fractional-order of the system. Further, by understanding the behaviour of the Ulam-Hyers constant L_U numerically, we determined that the absolute values of the circuit current I_α of the fractional-order RLC system are larger for larger values of the bandwidth $a(< a_0)$, and for each value of $a(\leq a_0)$, the absolute value of the circuit current I_α is the maximum for the fractional order between $0.1 < \alpha < 0.2$. In particular, for $a = a_0$ and $\alpha = 0.119612$, under a given input voltage, we can generate the circuit current with the largest absolute value for the fractional order $\alpha = 0.119612$, in comparison to the fractional-order $\alpha \in [0,1] \setminus \{0.119612\}$.

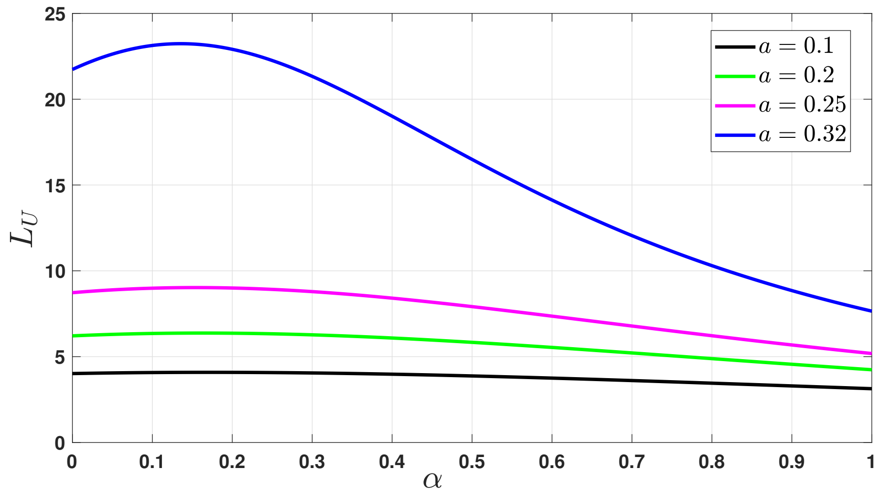

Figure 6: Graph of Ulam-Hyers constant L_U given by (5.8) for the bandwidth $a < a_0$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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