Journal of Nonlinear Evolution Equations and Applications
Volume 2024, Number 2, pp. 19–36 (April 2024)

ISSN 2161-3680
http://www.jneea.com

EXISTENCE RESULTS FOR THE MILD SOLUTIONS OF IMPULSIVE SEMI-LINEAR DIFFERENTIAL EQUATIONS

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Received January 20, 2015, revised June 2016 – March 30, 2024

Accepted on March 12, 2015, final version on April 14, 2024

Communicated by Maximilian F. Hasler

Abstract. In the present work, we consider a new class of impulsive semi-linear differential equations in an arbitrary Banach space X with non-instantaneous impulses. We prove the existence of a mild solution to the impulsive differential equation with non-instantaneous impulses by virtue of the theory of semigroups via techniques of a new fixed point theorem for convex-power condensing operators.

Keywords: C_0-semigroup, convex-power condensing operator, impulsive conditions, measure of noncompactness.

2010 Mathematics Subject Classification: 34K37, 34K30, 34G10, 47D06.

1 Introduction

In the last few decades, impulsive differential equations have received much attention of researchers mainly due to their demonstrated applications in various fields of science and engineering. Impulsive differential equations play an important role in real world problems and are used to describe processes which are characterized by the development of a sudden change in the system's state. Such processes

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have been investigated in various fields such as biology, physics, control theory, population dynamics, medicine and many others. Impulsive differential equations are an appropriate model to hereditary phenomena for which a delay argument arises in the modelling equations. For more details concerning impulsive differential equations, we refer to the monographs [3, 10] and the papers [4, 7, 8, 9, 12, 14, 20] and the references given therein.

In this paper, our purpose is to study the existence of solutions for a class of abstract differential equations with non-instantaneous impulses of the form

$$x'(t) = -Ax(t) + f\left(t, x(t), \int_0^t h(t,\tau)x(\tau)\,\mathrm{d}\tau, \int_0^t p(t,\tau)x(\tau)\,\mathrm{d}\tau\right), \quad t \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, \delta, \delta \in \mathbb{N}, \tag{1.1}$$

$$x(t) = g_i(t, x(t)), \quad t \in (t_i, s_i],\ i = 1, \cdots, \delta, \tag{1.2}$$

$$x(0) = x_0 + K(x), \tag{1.3}$$

where $-A\colon D(A) \subset X \to X$ is a closed and bounded operator which is densely defined in the Banach space X. We assume that $-A$ generates a semigroup of strongly continuous linear operators $\{T(t)\}_{t\geq 0}$ defined in the Banach space $(X, \|\cdot\|)$, $x_0 \in X$, $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_\delta \leq s_\delta < t_{\delta+1} = b$ and $g_i \in C((t_i, s_i] \times X; X)$ for all $i = 1, \cdots, \delta$. The function $f \in C([0, b] \times X^3; X)$ is an appropriate function and K is a mapping from some space of functions to be specified later.

For convenience, we set

$$(Hx)(t) = \int_0^t h(t,\tau)x(\tau)\,\mathrm{d}\tau,$$
$$(Px)(t) = \int_0^t p(t,\tau)x(\tau)\,\mathrm{d}\tau,$$

for $h \in C(D_0, \mathbb{R}^+)$ and $p \in C(D', \mathbb{R}^+)$, where $D_0 = \{(t,s) :\ 0 \leq s \leq t \leq T\}$ and $D' = \{(t,s) : 0 \leq t, s \leq T\}$.

In [7], a new class of abstract impulsive differential equations was introduced in which $f = f(t, x(t))$ without nonlocal conditions; it was also assumed that impulses are non-instantaneous. Under the assumptions that the operator A generates a C_0-semigroup of bounded linear operators and f, g_i are appropriate functions, the existence of a mild solution to the impulsive system was established. In this impulsive system, the impulses begin all of a sudden at the points t_i and continue their proceeding on a finite interval $[t_i, s_i]$. According to the authors of [7], there are many different inspirations for consideration of such an impulsive system. The hemodynamical equilibrium of a person is an example of such systems. One can prescribe some intravenous drugs (insulin) in the case of a decompensation (for example, low or high level of glucose). Since the introduction of the drugs in the bloodstream and the consequent absorbtion of the body are successive and continuous processes, we can describe this situation as an impulsive action which start suddenly and stays active on a finite time interval.

In [17], the generalization of the condensing operators as convex-power condensing operators was introduced by Sun and Zhang and a new fixed point theorem for convex-power condensing operator was established. The new fixed point theorem for convex-power condensing operators, defined by Sun and Zhang, is the generalization of the famous Sadovskii's fixed point theorem and Schauder's fixed point theorem. For more details about the measure of noncompactness we refer to [1, 2, 5, 6, 9, 11, 13, 16, 17, 18, 19, 22] and the references given therein.

We divide this paper into three sections as follows. Section 2 gives some basic definition, lemmas and theorems. Section 3 provides the existence results. The existence of the mild solution and the positive mild solution by utilizing a fixed point theorem for convex-power condensing operators is obtained in Section 3. In Section 4, an example is presented.

2 Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. By $C([0,b];X)$ we denote the Banach space of all X-valued continuous functions defined on $[0,b]$, endowed with the norm $\|x\| = \sup_{s\in[0,b]} \|x(s)\|$, where $b \in \mathbb{R}^+$. Moreover, $L^1([0,b];X)$ denotes the Banach space of X-valued Bochner integrable functions defined on $[0,b]$ endowed with the norm $\|g\|_{L^1} = \int_0^b \|g(t)\|\,\mathrm{d}t$.

The operator $-A$ is the infinitesimal generator of a uniformly continuous semigroup $\{T(t) : t \geq 0\}$ and $D(A)$ denotes the domain of A, which is dense in X. It is clear that $D(A)$ is a Banach space endowed with the graph norm. The semigroup $T(t)$ is called equicontinuous if the set $\{T(t)x : x \in \mathcal{K}\}$ is equicontinuous at t, $0 < t < \infty$, for any bounded subset $\mathcal{K} \subset X$. Throughout the paper, we assume that

(HT) the operator $-A$ generates the equicontinuous semigroup $\{T(t) : t \geq 0\}$ and there exists a positive number M such that $\|T(t)\| \leq M$ for all $t \in [0,b]$.

To define mild solutions for the impulsive differential equation (1.1)–(1.3), we consider the following space $\mathcal{PC}([0,b];X)$ which contains all the piecewise continuous functions $x\colon [0,b] \to X$ such that $x(t_i^-)$ and $x(t_i^+)$ exist for all $i = 1, 2, \cdots, \delta$. We can verify that the space $\mathcal{PC}([0,b];X)$ is a Banach space endowed with norm $||x||_{\mathcal{PC}} = \sup_{t\in[0,b]} ||x(t)||$. For a function $x \in \mathcal{PC}([0,b];X)$, define the function $\widetilde{x}_i \in C([t_i, t_{i+1}];X)$, $i = 1, \cdots, \delta$, such that

$$\widetilde{x}_i(t) = \begin{cases} x(t) & \text{for } t \in (t_i, t_{i+1}], \\ x(t_i^+) & \text{for } t = t_i. \end{cases}$$

For a set $F \subset \mathcal{PC}([0,b];X)$ and $i \in \{0, 1, \cdots, \delta\}$, we have $\widetilde{F}_i = \{\widetilde{u}_i : u \in F\}$ and we have the following Arzelà–Ascoli-type result.

Lemma 1 ([7]) *A set $F \subset \mathcal{PC}([0,b];X)$ is relatively compact in $\mathcal{PC}([0,b];X)$ if and only if each set $\widetilde{F}_i$ is relatively compact in $C([t_i, t_{i+1}];X)$.*

Now, we present the following definition of a mild solution.

Definition 1 *A piece-wise continuous function $x\colon [0,b] \to X$ is said to be a mild solution for the system* (1.1)–(1.3) *if $x(0) = x_0 + K(x)$, $x(t) = g_i(t, x(t))$ for all $t \in (t_i, s_i]$, $i = 1, \cdots, \delta$, and*

$$x(t) = T(t)[x_0 + K(x)] + \int_0^t T(t-\zeta) f(\zeta, x(\zeta), (Hx)(\zeta), (Px)(\zeta))\,\mathrm{d}\zeta \quad \textit{for } t \in [0, t_1] \tag{2.1}$$

and

$$x(t) = T(t-s_i) g_i(s_i, x(s_i)) + \int_{s_i}^t T(t-\zeta) f(\zeta, x(\zeta), (Hx)(\zeta), (Px)(\zeta))\,\mathrm{d}\zeta \tag{2.2}$$

for all $t \in (s_i, t_{i+1}]$ and every $i = 1, \cdots, \delta$.

Now, we give the definition of the Hausdorff measure of noncompactness (MNC).

Definition 2 ([2]) *The Hausdorff measure of noncompactness β of a bounded set F in a Banach space X is the greatest lower bound of those $\epsilon > 0$ for which the set F has a finite ϵ-net in the space X*, i.e.,

$$\beta(F) = \inf\{\epsilon > 0 : F \text{ has a finite } \epsilon\text{-net in } X\}.$$

Definition 3 ([2]) *The Kuratowski measure of noncompactness α of a bounded subset F of a Banach space X is given by*

$$\alpha(F) = \inf\{\epsilon > 0 : F \text{ is covered by a finite number of sets with diameter} \leq \epsilon\}.$$

The relation between the Kuratowski measure of noncompactness α and the Hausdorff measure of noncompactness β is given as

$$\beta(F) \leq \alpha(F) \leq 2\beta(F)$$

for any bounded subset $F \subset X$.

Next, we discuss some basic properties of measure of noncompactness satisfied by both the Kuratowski α and the Hausdorff measure of noncompactness β.

Lemma 2 *Let X be a real Banach space and let E, F be bounded subset of X. Then, we have the following results*:

(1) $\beta(E) = 0$ *if and only if E is relatively compact*;

(2) $\beta(E) = \beta(\text{conv}\, E) = \beta(\overline{E})$, *where* $\text{conv}(E)$ *and* $\overline{E}$ *denotes the convex hull and the closure of E, respectively*;

(3) *if $E \subset F$, then* $\beta(E) \leq \beta(F)$;

(4) $\beta(E + F) \leq \beta(E) + \beta(F)$, *where* $E + F = \{x + y : \ x \in E, \ y \in F\}$;

(5) $\beta(E \cup F) \leq \max\{\beta(E), \beta(F)\}$;

(6) $\beta(\kappa E) \leq |\kappa|\beta(E)$ *for any* $\kappa \in \mathbb{R}$;

(7) *if the map $\mathcal{Q}\colon D(\mathcal{Q}) \subset X \to Y$ is Lipschitz continuous with a Lipschitz constant μ, then $\beta_Y(\mathcal{Q}E) \leq \mu\beta(E)$ for every bounded set $E \subset D(\mathcal{Q})$; here Y is a Banach space.*

Definition 4 *Let Y be a Banach space.* A continuous and bounded map $\mathcal{Q}\colon D \subseteq Y \to Y$ is called β-condensing if $\beta(\mathcal{Q}F) < \beta(F)$ for any nonprecompact bounded subset $F \subset D$.

It should not cause any confusion if by $\beta(\cdot)$ we denote the Hausdorff measure of noncompactness in X, $C([0, b]; X)$ and $\mathcal{PC}([0, b]; X)$.

Lemma 3 ([2], Darbo–Sadovskii) *Let $D \subset X$ be a closed, bounded and convex set. If a continuous map $\mathcal{Q}\colon D \to D$ is β-condensing, then there exists a fixed point of the map $\mathcal{Q}$.*

Lemma 4 ([2, 9]) *If F is a bounded subset of $C([0,b];X)$, then we have $\beta(F(t)) \le \beta(F)$ for every $t \in [0,b]$, where $F(t) = \{x(t) : x \in F\} \subset X$. Furthermore, if F is equicontinuous on $[0,b]$, then $\beta(F(t))$ is continuous on $[0,b]$ and $\beta(F) = \sup_{t\in[0,b]} \beta(F(t))$.*

Lemma 5 ([9]) *If F is a bounded subset of $\mathcal{PC}([0,b];X)$, then we have $\beta(F(t)) \le \beta(F)$ for every $t \in [0,b]$. Furthermore, if*

(1) *F is equicontinuous on $[0,t_1]$ and each $(t_j, s_j]$, $(s_j, t_{j+1}]$, $j = 1, \cdots, \delta$;*

(2) *F is equicontinuous at the points $t = t_j^+$, $j = 1, \cdots, \delta$,*

then $\beta(F) = \sup_{t\in[0,b]} \beta(F(t))$.

Lemma 6 ([2]) *Suppose $F \subset C([0,b];X)$ is a bounded and equicontinuous set. Then $\beta(F(t))$ is continuous and*

$$\beta\bigg(\int_0^t F(\tau)\,\mathrm{d}\tau\bigg) \le \int_0^t \beta(F(\tau))\,\mathrm{d}\tau$$

for all $t \in [0,b]$, where $\int_0^t F(\tau)\,\mathrm{d}\tau = \big\{\int_0^t x(\tau)\,\mathrm{d}\tau : x \in F\big\}$.

Lemma 7 *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of functions in $L^1([0,b];\mathbb{R}_+)$. Assume that there exists $\gamma(t) \in L^1([0,b];\mathbb{R}_+)$ satisfying $\|x_n(t)\| \le \gamma(t)$ for almost all $t \in [0,b]$ and every $n \ge 1$. Then we have*

$$\beta\bigg(\bigg\{\int_0^t x_n(\tau)\,\mathrm{d}\tau : n \ge 1\bigg\}\bigg) \le 2\int_0^t \beta\big(\{x_n(\tau)\}_{n=1}^{\infty}\big)\,\mathrm{d}\tau \quad \textit{for every } t \in [0,b].$$

Lemma 8 ([21, 23]) *Assume that $0 < \epsilon < 1$ and $h > 0$. Let*

$$\mathcal{S} = \epsilon^m + C_m^1 \epsilon^{m-1} h + C_m^2 \epsilon^{m-2} \frac{(h)^2}{2!} + \cdots + \frac{(h)^m}{m!}, \quad m \in \mathbb{N}.$$

Then $\mathcal{S} = o(\frac{1}{m^s})(m \to +\infty)$, where $s > 1$ is an arbitrary real number.

In [17], the authors introduced the generalization of the definition of a condensing operator and a new fixed point theorem for that kind of operators. Firstly, we introduce some notation. Let $D \subset X$ be a bounded, closed and convex set and let $\mathcal{Q}$ be a continuous map from D into itself and $u_0 \in D$. For every $F \subset D$, we set

$$\mathcal{Q}^{(1,u_0)}(F) = \mathcal{Q}(F), \qquad \mathcal{Q}^{(n,u_0)}(F) = \mathcal{Q}(\overline{\mathrm{conv}}\{\mathcal{Q}^{(n-1,u_0)}(F), u_0\}), \quad n = 2, 3, \cdots.$$

Definition 5 ([17]) *Let $D \subset X$ be a closed, bounded and convex set. The continuous map $\mathcal{Q}\colon D \to D$ is called convex-power condensing operator, if there exist $u_0 \in D$ and an integer $n_0 > 0$ such that*

$$\beta(\mathcal{Q}^{(n_0,u_0)}(F)) < \beta(F)$$

for any bounded nonprecompact subset $F \subset D$.

Lemma 9 ([17]) *Let $D \subset X$ be a bounded, closed and convex set. If the continuous map $\mathcal{Q}\colon D \to D$ is β-convex-power condensing, then there exists at least one fixed point of the map $\mathcal{Q}$ in D.*

Lemma 10 *If* **(HT)** *holds, then the set*

$$\left\{\int_0^t T(t-s)x(s)\,\mathrm{d}s : \|x(s)\| \le \varsigma(s) \textit{ for almost every } s \in [0,b]\right\}$$

is equicontinuous for every $t \in [0,b]$.

Proof. For $0 \le t < t+\varepsilon \le b$, we get

$$\begin{aligned}\left\|\int_0^{t+\varepsilon} T(t+\varepsilon-\tau)x(\tau)\,\mathrm{d}\tau - \int_0^t T(t-\tau)x(\tau)\,\mathrm{d}\tau\right\| \\ \le \left\|\int_0^t \big(T(t+\varepsilon-\tau) - T(t-\tau)\big)x(\tau)\,\mathrm{d}\tau\right\| + \int_t^{t+\varepsilon} \|T(t+\varepsilon-\tau)x(\tau)\|\,\mathrm{d}\tau.\end{aligned} \tag{2.3}$$

When $t = 0$, we get that the right-hand side of (2.3) can be made small when ε is small independent of x. For $t > 0$ and $\sigma > 0$, we have

$$\begin{aligned}&\left\|\int_0^t T(t+\varepsilon-\tau)x(\tau)\,\mathrm{d}\tau - \int_0^t T(t-\tau)x(\tau)\,\mathrm{d}\tau\right\| \\ &\le \left\|T(\varepsilon+\sigma)\int_0^{t-\sigma} T(t-\sigma-\tau)x(\tau)\,\mathrm{d}\tau - T(\sigma)\int_0^{t-\sigma} T(t-\sigma-\tau)x(\tau)\,\mathrm{d}\tau\right\| \quad (2.4)\\ &\quad + \left\|\int_{t-\sigma}^t T(t+\varepsilon-\tau)x(\tau)\,\mathrm{d}\tau\right\| + \left\|\int_{t-\sigma}^t T(t-\tau)x(\tau)\,\mathrm{d}\tau\right\|.\end{aligned}$$

Since $T(t)$ is strongly continuous and equicontinuous, we deduce that

$$\left\|[T(\varepsilon+\sigma) - T(\sigma)]\int_0^{t-\sigma} T(t-\sigma-\tau)x(\tau)\,\mathrm{d}\tau\right\| \to 0, \tag{2.5}$$

as $\varepsilon \to 0$, uniformly for x. Since σ is arbitrarily small, the second and the third term of (2.4) tend to zero when $\sigma \to 0$.

Then, from (2.3), (2.4), (2.5), we see that $\left\{\int_0^t T(t-\tau)x(\tau)\,\mathrm{d}\tau : \|x(\tau)\| \le \varsigma(\tau) \text{ for a.e. } \tau \in [0,b]\right\}$ is equicontinuous for all $0 \le t \le b$. □

3 Existence results

Here, we establish the existence of a solution to the system (1.1)–(1.3) under some specified conditions on g_i by using a noncompact semigroup and a fixed point theorem for convex-power condensing operators.

Now, we list the following assumptions.

(Hg) The functions g_i are compact and continuous and there are positive constants K_1 and K_2 such that $\|g_i(t,x)\| \le K_1\|x\| + K_2$ for all $x \in X$ and $t \in [0,b]$.

(Hf) $f\colon [0,T]\times X\times X\times X\to X$ is a nonlinear function satisfying the Carathéodory conditions, *i.e.*:

(i) $f(t,\cdot,\cdot,\cdot)\colon X\times X\times X\to X$ is continuous for a.e. $t\in[0,b]$;

(ii) $f(\cdot,x,Hx(t),Px(t))\colon [0,b]\to X$ is measurable for all $x\in X$;

(iii) there exist a constant function $m_k\in L^1([0,b],\mathbb{R}_+)$, $k>0$, and an increasing continuous function $\Omega\colon \mathbb{R}_+\to\mathbb{R}_+$ such that

$$\|f(t,x,Hx,Px)\|\le m_k(t)\Omega(\|x\|)$$

for a.e. $t\in[0,b]$ and all $x\in X$;

(iv) there exist nonnegative Lebesgue integrable functions $L_i\in L^1([0,b];\mathbb{R}_+)$, $i=1,2,3$, such that

$$\beta(f(t,B_1(t),B_2(t),B_3(t)))\le L_1(t)\beta(B_1(t))+L_2(t)\beta(B_2(t))+L_3\beta(B_3(t))$$

for any bounded and equicontinuous $B_i\subset X$, $i=1,2,3$, and for almost every $t\in[0,b]$.

(HK) The nonlocal function $K\colon C([0,b];X)\to X$ is continuous and compact, and there exists an increasing continuous function $\mathbb{W}\colon \mathbb{R}_+\to\mathbb{R}_+$ such that

$$\|K(x)\|\le \mathbb{W}(\|x\|).$$

(Hk) $M\big[M\mathbb{W}(k)+\Omega(k)\|m_k\|_{L^1([0,b])}+kK_1+K_2+\|x_0\|\big]\le k.$

Theorem 1 *Assume that* **(HT)**, **(Hg)**, **(Hf)**, **(HK)**, **(Hk)** *hold. Then the problem* (1.1)–(1.3) *has at least one mild solution on* $[0,b]$.

Proof. Define the operator $\mathcal{Q}\colon \mathcal{PC}([0,b];X)\to\mathcal{PC}([0,b];X)$ such that $x(0)=x_0+K(x)$, $\mathcal{Q}x(t)=g_i(t,x(t))$ for $t\in(t_i,s_i]$ $(i=1,\cdots,\delta)$ and

$$\mathcal{Q}x(t)=T(t)[x_0+K(x)]+\int_0^t T(t-\zeta)f(\zeta,x(\zeta),Hx(\zeta),Px(\zeta))\,\mathrm{d}\zeta,\qquad t\in[0,t_1],\tag{3.1}$$

and

$$\mathcal{Q}x(t)=T(t-s_i)g_i(s_i,x(s_i))+\int_{s_i}^t T(t-\zeta)f(\zeta,x(\zeta),Hx(\zeta),Px(\zeta))\,\mathrm{d}\zeta\tag{3.2}$$

for $t\in(s_i,t_{i+1}]$, where $i=1,\cdots,\delta$. To prove the result, we show that there exists a fixed point of the map $\mathcal{Q}$. At first, we prove the continuity of the map $\mathcal{Q}$ on $\mathcal{PC}([0,b];X)$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{PC}([0,b];X)$ such that $\lim\limits_{n\to\infty}x_n=x$ in $\mathcal{PC}([0,b];X)$. By the continuity of g_i, it is clear that $\mathcal{Q}$ is continuous on $(t_i,s_i]$. Thus, we have

$$\begin{aligned}&\|(\mathcal{Q}x_n)(t)-(\mathcal{Q}x)(t)\|\\&\quad\le M\|g_i(s_i,x_n(s_i))-g_i(s_i,x(s_i))\|\\&\qquad+M\int_{s_i}^t\|f(\zeta,x_n(\zeta),Hx_n(\zeta),Px_n(\zeta))-f(\zeta,x(\zeta),Hx(\zeta),Px(\zeta))\|\,\mathrm{d}\zeta.\end{aligned}\tag{3.3}$$

for $t \in (s_i, t_{i+1}]$. By the continuity of f and g_i, we have

$$\lim_{n\to\infty} g_i(t_i, x_n(t_i)) = g_i(t_i, x(t_i)),\ t \in (t_i, s_i], \tag{3.4}$$

$$\lim_{n\to\infty} f(\zeta, x_n(\zeta), Hx_n(\zeta), Px_n(\zeta)) = f(\zeta, x(\zeta), Hx(\zeta), Px(\zeta)),\ \zeta \in (s_i, t_{i+1}]. \tag{3.5}$$

Therefore, using the Lebesgue dominated convergence theorem and (3.3), (3.4), (3.5), we get

$$\|\mathcal{Q}x_n(t) - \mathcal{Q}x(t)\| \to 0, \qquad \text{as } n \to \infty,$$

which implies that $\mathcal{Q}$ is continuous on $(s_i, t_{i+1}]$. For $t \in [0, t_1]$, we get

$$\begin{aligned}
&\|(\mathcal{Q}x_n)(t) - (\mathcal{Q}x)(t)\| \\
&\quad \le M\|K(x_n) - K(x)\| \\
&\qquad + M\int_0^t \|f(\zeta, x_n(\zeta), Hx_n(\zeta), Px_n(\zeta)) - f(\zeta, x(\zeta), Hx(\zeta), Px(\zeta))\|\,\mathrm{d}\zeta.
\end{aligned}$$

By the continuity of K and (3.5), we get

$$\|\mathcal{Q}x_n(t) - \mathcal{Q}x(t)\| \to 0, \qquad \text{as } n \to \infty,$$

thus $\mathcal{Q}$ is continuous on $[0, t_1]$. Hence $\mathcal{Q}$ is continuous on $[0, b]$.

Secondly, we claim that $\mathcal{Q}(B_k) \subset B_k$, where

$$B_k(\mathcal{PC}) = B_k = \{x \in \mathcal{PC}([0,b]; X) : \|x\| \le k\} \subset \mathcal{PC}([0,b]; X)$$

is the closed and convex ball with the center at the origin and radius k. For any $x \in B_k \subset \mathcal{PC}([0,b]; X)$ and $t \in [0, t_1]$, we get

$$\begin{aligned}
\|\mathcal{Q}x(t)\| &\le \|T(t)[x_0 + K(x)]\| + \int_0^t \|T(t-\zeta)\| \cdot \|f(\zeta, x(\zeta), Hx(\zeta), Px(\zeta))\|\,\mathrm{d}\zeta \\
&\le M\big[\|x_0\| + \mathbb{W}(k)\big] + M\Omega(k)\int_0^t m_k(\zeta)\,\mathrm{d}\zeta \\
&\le M\big(\|x_0\| + \mathbb{W}(k) + \Omega(k)\|m_k\|_{L^1[0,b]}\big).
\end{aligned}$$

For $t \in (s_i, t_{i+1}]$, we have

$$\begin{aligned}
\|\mathcal{Q}x(t)\| &\le \|T(t-s_i)\| \cdot \|g_i(s_i, x(s_i))\| + M\int_{s_i}^t \|f(\zeta, x(\zeta), Hx(\zeta), Px(\zeta))\|\,\mathrm{d}\zeta \\
&\le M\big(K_1 k + K_2 + \Omega(k)\|m_k\|_{L^1[s_i,t_{i+1}]}\big) \\
&= M\big(K_1 k + K_2 + \Omega(k)\|m_k\|_{L^1[0,b]}\big),
\end{aligned}$$

which implies that $\|\mathcal{Q}x\|_{\mathcal{PC}} \le M\big(K_1 k + K_2 + \Omega(k)\|m_k\|_{L^1[0,b]}\big)$ for all $i = 1, \cdots, \delta$. On the other hand, by the properties of $g_i(\cdot)$, we get

$$\|\mathcal{Q}x(t)\| \le \|g_i(t, x(t))\| \le K_1 k + K_2$$

for $t \in (t_i, s_i]$. Since $M\big[\Omega(k)\|m_k\|_{L^1[0,b]} + K_1 k + K_2 + \|x_0\| + \mathbb{W}(k)\big] \le k$, we conclude that $\|\mathcal{Q}x\|_{\mathcal{PC}} \le k$, *i.e.*, $\mathcal{Q}$ has values in B_k.

Now, we show the equicontinuity of $\mathcal{Q}(B_k)$ on $[0, b]$. Since $g_i(\cdot)$ are compact, it is obvious that $\mathcal{Q}(B_k)$ is equicontinuous on $(t_i, s_i]$. Assume that $t \in [0, t_1]$. For $x \in B_k$, $0 < h < t$ and $0 \le t < t + h \le b$, we obtain

$$\begin{aligned}\|(\mathcal{Q}x)(t+h)-(\mathcal{Q}x)(t)\| &\le \|[T(t+h)-T(t)](x_0+K(x))\| \\ &\quad + \bigg\| \int_0^{t+h} T(t+h-\zeta) f(\zeta, x(\zeta), Hx(\zeta), Px(\zeta))\,\mathrm{d}\zeta \\ &\quad - \int_0^t T(t-\zeta) f(\zeta, x(\zeta), Hx(\zeta), Px(\zeta))\,\mathrm{d}\zeta \bigg\|.\end{aligned}$$

Using the semigroup property, we have

$$\begin{aligned}\|[T(t+h)-T(t)](x_0+K(x))\| &= \|T(t)[T(t+h-t)-T(0)](x_0+K(x))\| \\ &\le M\|[T(h)-T(0)]\| \times \|x_0+K(x)\|.\end{aligned}$$

By the uniform continuity of the operator $T(\cdot)$ and Lemma 10, we conclude that $\mathcal{Q}(B_k)$ is equicontinuous on $[0,t_1]$.

For $t \in (s_i, t_{i+1}]$, we get

$$\begin{aligned}\|\mathcal{Q}x(t+h)-\mathcal{Q}x(t)\| &\le \|[T(t+h-s_i)-T(t-s_i)]g_i(s_i, x(s_i))\| \\ &\quad + \bigg\| \int_0^{t+h} T(t+h-\zeta) f(\zeta, x(\zeta), Hx(\zeta), Px(\zeta))\,\mathrm{d}\zeta \\ &\quad - \int_0^t T(t-\zeta) f(\zeta, x(\zeta), Hx(\zeta), Px(\zeta))\,\mathrm{d}\zeta \bigg\|.\end{aligned}$$

By the semigroup property and the strong continuity of $T(t)$, we get

$$\|[T(t+h-s_i)-T(t-s_i)]g_i(s_i, u(s_i))\| \le M\|[T(h)-T(0)]g_i(s_i, u(s_i))\|.$$

Since $g_i(\cdot)$ are compact and $T(\cdot)$ is strongly continuous, we infer that $\mathcal{Q}(B_k)$ is equicontinuous on $(s_i, t_{i+1}]$. Hence $\mathcal{Q}(B_k)$ is equicontinuous on each $[0,b]$.

Set $W = \overline{\mathrm{conv}}\mathcal{Q}(B_k)$, where conv and $\overline{\mathrm{conv}}$ denotes the convex hull and the closure of the convex hull, respectively. It is easy to verify that $\mathcal{Q}$ maps W into itself and that W is equicontinuous on the intervals $[0,t_1]$, $(t_i, s_i]$, $(s_i, t_{i+1}]$, $i = 1, 2, \cdots, \delta$. Next, we want to show that $\mathcal{Q}\colon W \to W$ is a convex-power condensing operator. We take $u_0 \in W$ and show that there exists a positive integer n_0 such that for every nonprecompact bounded subset $F \subset W$

$$\beta(\mathcal{Q}^{(n_0,u_0)}(F)) < \beta(F).$$

Also, by the fact that

$$\int_0^b x(s)\,\mathrm{d}s \in b\,\mathrm{conv}\{x(s) : s \in [0,b]\}, \quad x \in \mathcal{PC}([0,b];X),$$

we get that

$$\beta\bigg(\bigg\{\int_0^t h(t,s)x(s)\,\mathrm{d}s : x \in F,\ t \in [0,b]\bigg\}\bigg) \le bH_0\beta\big(\{x(t) : x \in F, t \in [0,b]\}\big),$$

$$\beta\bigg(\bigg\{\int_0^t p(t,s)x(s)\,\mathrm{d}s : x \in F,\ t \in [0,b]\bigg\}\bigg) \le bP_0\beta\big(\{x(t) : x \in F, t \in [0,b]\}\big),$$

where $H_0 = \max_{(t,s)\in D_0} |h(t,s)|$ and $P_0 = \max_{(t,s)\in D'} |p(t,s)|$.

For $t \in (s_i, t_{i+1}]$, where $i = 1, \cdots, \delta$, from Lemmas 2 and 7 we have that for $\epsilon > 0$, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset F$ such that

$$\begin{aligned}
\beta(\mathcal{Q}^{(1,u_0)}F(t)) &= \beta((\mathcal{Q}F)(t)) \\
&\leq 2\beta\big(T(t-s_i)g_i(s_i, \{x_n\}_{n=1}^{\infty})d\tau\big) \\
&\quad + 2\beta\bigg(\int_{s_i}^{t} T(t-\zeta)f\big(\zeta, \{x_n\}_{n=1}^{\infty}, H\{x_n\}_{n=1}^{\infty}, P\{x_n\}_{n=1}^{\infty}\big)\,\mathrm{d}\zeta\bigg) + \epsilon \\
&\leq 4M\int_{s_i}^{t}\big[L_1(\zeta)\beta(F(\zeta)) + L_2(\zeta)\beta((HF)(\zeta)) + L_3\beta((PF)(\zeta))\big]\,\mathrm{d}\zeta + \epsilon \\
&\leq 4M\beta(F)\int_{s_i}^{t}\big[L_1(\zeta) + bH_0L_2(\zeta) + bP_0L_3(\zeta)\big]\,\mathrm{d}\zeta + \epsilon \\
&\leq \int_{s_i}^{t} L(\zeta)\beta(F)\,\mathrm{d}\zeta + \epsilon,
\end{aligned}$$

where $L(t) = 4M[L_1(t) + bH_0L_2(t) + bP_0L_3(t)]$. Similarly, using the compactness of K, for $t \in [0, t_1]$, we get

$$\beta(\mathcal{Q}^{(1,u_0)}F(t)) \leq \int_0^t L(\zeta)\beta(F)\,\mathrm{d}\zeta + \epsilon.$$

Therefore, we have that there exists a continuous function $\varphi\colon [0,b] \to \mathbb{R}_+$ such that

$$\int_0^b |L(s) - \varphi(s)|\,\mathrm{d}s < \gamma,$$

for any $\gamma \in (0,1)$. Then, we obtain

$$\begin{aligned}
\beta(\mathcal{Q}^{(1,u_0)}F(t)) &\leq \beta(F)\bigg[\int_{s_i}^{t}|L(\zeta) - \varphi(\zeta)|\,\mathrm{d}\zeta + \int_{s_i}^{t}|\varphi(s)|\,\mathrm{d}s\bigg] + \epsilon \\
&\leq [\gamma + \widetilde{\varphi}(t-s_i)]\beta(F) + \epsilon, \quad t \in (s_i, t_{i+1}]
\end{aligned}$$

and

$$\beta(\mathcal{Q}^{(1,u_0)}F(t)) \leq [\gamma + \widetilde{\varphi}t]\beta(F) + \epsilon, \quad t \in [0, t_1],$$

where $\widetilde{\varphi} = \max\{|\varphi(t)| : t \in [0,b]\}$.

Since $\epsilon > 0$ is arbitrary, therefore it follows that

$$\begin{aligned}
&\beta(\mathcal{Q}^{(1,u_0)}F(t)) \leq (c + d(t-s_i))\beta(F), \quad t \in (s_i, t_{i+1}], \\
&\beta(\mathcal{Q}^{(1,u_0)}F(t)) \leq (c + dt)\beta(F), \quad t \in [0, t_1],
\end{aligned}$$

where $c = \gamma$, $d = \widetilde{\varphi}$.

For $t \in (t_i, s_i]$, we get $\beta(\mathcal{Q}^{(1,u_0)}F(t)) = \beta((\mathcal{Q}F)(t)) = \beta(g_i(t, F(t))) = 0$ by the fact that g_i are compact.

Furthermore, in view of Lemmas 2 and 7, for $t \in (s_i, t_{i+1}]$ we have that there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset \overline{\mathrm{conv}}\{\mathcal{Q}^{(1,u_0)}F, u_0\}$ such that

$$\begin{aligned}
\beta((\mathcal{Q}^{(2,u_0)}F)(t)) &= \beta((\mathcal{Q}\overline{\text{conv}}\{\mathcal{Q}^{(1,u_0)}F, u_0\})(t)) \\
&\le 2\beta(T(t-s_i)g_i(s_i, \{y_n\}_{n=1}^{\infty})) \\
&\quad + 2\beta\bigg(\int_{s_i}^{t} T(t-\zeta)f(\zeta, \{y_n\}_{n=1}^{\infty}, H\{y_n\}_{n=1}^{\infty}, P\{y_n\}_{n=1}^{\infty})\,\mathrm{d}\zeta\bigg) + \epsilon \\
&\le 4M\int_{s_i}^{t} \beta(f(\zeta, \{y_n\}_{n=1}^{\infty}, H\{y_n\}_{n=1}^{\infty}, P\{y_n\}_{n=1}^{\infty})\,\mathrm{d}\zeta + \epsilon \\
&\le \int_{s_i}^{t} L(\zeta)\beta(\overline{\text{conv}}\{Q^{(1,u_0)}F, u_0\}(\zeta))\,\mathrm{d}\zeta + \epsilon \\
&\le \int_{s_i}^{t} [c + d(\zeta - s_i)]\beta(F)\mathrm{d}\zeta + \epsilon \\
&\le \int_{s_i}^{t} |L(\zeta) - \varphi(\zeta)|(c + d(\zeta - s_i))\beta(F)\mathrm{d}\zeta \\
&\quad + \int_{s_i}^{t} |L(\zeta)|(c + d(\zeta - s_i))\beta(F)\mathrm{d}\zeta + \epsilon \\
&\le \bigg(c^2 + 2cd(t-s_i) + \frac{(d(t-s_i))^2}{2!}\bigg)\beta(F) + \epsilon.
\end{aligned}$$

Thus, by the mathematical induction, we deduce that for any positive integer n,

$$\begin{aligned}
\beta((\mathcal{Q}^{(n,u_0)}F)(t)) \le \bigg(a^n + C_n^1 a^{n-1} d(t-s_i) + C_n^2 a^{n-2}\frac{(d(t-s_i))^2}{2!} + \cdots \\
\cdots + \frac{(d(t-s_i))^n}{n!}\bigg)\beta(F)
\end{aligned}$$

for $t \in (s_i, t_{i+1}]$. Similarly, for $t \in [0, t_1]$, we obtain

$$\beta((\mathcal{Q}^{(n,u_0)}F)(t)) \le \bigg(a^n + C_n^1 a^{n-1}(dt) + C_n^2 a^{n-2}\frac{(dt)^2}{2!} + \cdots + \frac{(dt)^n}{n!}\bigg)\beta(F).$$

Therefore, by Lemma 5, we conclude that

$$\beta((\mathcal{Q}^{(n,u_0)}F)) \le \bigg(a^n + C_n^1 a^{n-1}(db) + C_n^2 a^{n-2}\frac{(db)^2}{2!} + \cdots + \frac{(db)^n}{n!}\bigg)\beta(F) \quad \text{for all } t \in [0, b].$$

From Lemma 8, we get that there exists a positive integer n_0 such that

$$\bigg(a^{n_0} + C_{n_0}^1 a^{n_0-1}(db) + C_{n_0}^2 a^{n_0-2}\frac{(db)^2}{2!} + \cdots + \frac{(db)^{n_0}}{n_0!}\bigg) = q < 1.$$

Since $\beta((\mathcal{Q}^{(n_0,u_0)}F)) \le q\beta(F)$ and $q < 1$, we infer that there exists at least one fixed point of the map $\mathcal{Q}$ in W which is just a mild solution of the system (1.1)–(1.3). This completes the proof of the theorem. □

In the next result, we show the existence of a mild solution to the system (1.1)–(1.3) under Lipschitz conditions imposed on g_i and K. We prove the required result with the help of Darbo–Sadovskii's fixed point theorem.

Here, we replace the condition **(Hg)** by **(Hg′)**, and **(HK)** by **(HK′)**:

(Hg′) g_i are Lipschitz continuous functions for all $i = 1, \cdots, \delta$, *i.e.*, there exist positive constants L_{g_i} such that

$$\|g_i(t,u) - g_i(t,v)\| \leq L_{g_i}\|u - v\|$$

for all $u, v \in X$;

(HK′) the nonlocal function $K\colon C([0,b];X) \to X$ is Lipschitz continuous, *i.e.*, there exists a constant $L_K > 0$ such that

$$\|K(x) - K(y)\| \leq L_K\|x - y\| \qquad \text{for all } x, y \in X;$$

(HB) $\max_{i=1,\cdots,\delta}\big[ML_K + ML_{g_i} + 4Mb(L_1 + bH_0L_2 + bP_0L_3)\big] < 1.$

Theorem 2 *Suppose that the assumptions* **(HT)**, **(Hf)**, **(Hg′)**, **(HK′)**, **(HB)** *are fulfilled. Then, the impulsive system* (1.1)–(1.3) *has a mild solution on the interval* $[0,b]$.

Proof. Consider the map $\mathcal{Q}\colon \mathcal{PC}([0,b];X) \to \mathcal{PC}([0,b];X)$ defined as $x(0) = x_0 + K(x)$, $\mathcal{Q}x(t) = g_i(t, x(t))$ for $t \in (t_i, s_i]$, $i = 1, \cdots, \delta$, and

$$\mathcal{Q}x(t) = T(t)(x_0 + K(x)) + \int_0^t T(t-\zeta)f(\zeta, x(\zeta), Hx(\zeta), Px(\zeta))\,\mathrm{d}\zeta, \qquad t \in [0, t_1],$$

and

$$\mathcal{Q}x(t) = T(t-s_i)g_i(s_i, x(s_i)) + \int_{s_i}^t T(t-\zeta)f(\zeta, x(\zeta), Hx(\zeta), Px(\zeta))\,\mathrm{d}\zeta$$

for $t \in (s_i, t_{i+1}]$, where $i = 1, \cdots, \delta$. By the proof of Theorem 1, we infer that the map $\mathcal{Q}\colon B_k \to B_k$ is continuous. Now, we prove that $\mathcal{Q}$ is a β-condensing operator on B_k.

For $x, y \in B_k$, we have

$$\|g_i(t,x) - g_i(t,y)\| \leq L_{g_i}\|x - y\|, \quad t \in (t_i, t_{i+1}],\ i = 0, \cdots, \delta. \tag{3.6}$$

Thus, using Lemma 2 (7), we conclude that

$$\beta(g_i(t, B_k)) \leq L_{g_i}\beta(B_k).$$

Let $\int_0^t T(t-\zeta)f(\zeta, x(\zeta), Hx(\zeta), Px(\zeta))\,\mathrm{d}\zeta = Q'x(t)$. By Lemmas 4 and 6, for $t \in (s_i, t_{i+1}]$, we get

$$\begin{aligned}
\beta(Q'B_k) &= \sup_{t\in(s_i,t_{i+1}]} \beta((Q'B_k)(t)) \\
&\leq \sup_{t\in(s_i,t_{i+1}]} 2\int_{s_i}^t \beta(T(t-\zeta)f(\zeta, B_k(\zeta), (HB_k)(\zeta), (PB_k)(\zeta)))\,\mathrm{d}\zeta \\
&\leq \sup_{t\in(s_i,t_{i+1}]} 4M\int_{s_i}^t (L_1 + bH_0L_2 + bP_0L_3)\beta(B_k(\zeta))\,\mathrm{d}\zeta \\
&\leq 4Mb\big[L_1 + bH_0L_2 + bP_0L_3\big]\beta(B_k).
\end{aligned}$$

Therefore, for $t \in (s_i, t_{i+1}]$, we have

$$\begin{aligned}\beta(\mathcal{Q}(B_k(t))) &= \beta\Big(T(t-s_i)g(s_i, B_k(s_i)) \\ &\quad + \int_{s_i}^{t} T(t-\zeta)f(\zeta, B_k(\zeta), (HB_k)(\zeta), (PB_k)(\zeta))\,\mathrm{d}\zeta\Big) \\ &\le \big[ML_{g_i}\beta(B_k) + 4Mb(L_1 + bH_0L_2 + bP_0L_3)\big]\beta(B_k) \\ &\le \max_{i=1,\cdots,\delta}\big\{\big[ML_{g_i} + 4Mb(L_1 + bH_0L_2 + bP_0L_3)\big]\beta(B_k)\big\}.\end{aligned}$$

For $t \in [0, t_1]$, we get

$$\begin{aligned}\beta(\mathcal{Q}(B_k(t))) &= \beta\Big(T(t)(x_0 + K(x)) + \int_0^t T(t-\zeta)f(\zeta, B_k(\zeta), (HB_k)(\zeta), (PB_k)(\zeta))\,\mathrm{d}\zeta\Big) \\ &\le \big[ML_K + 4Mb(L_1 + bH_0L_2 + bP_0L_3)\big]\beta(B_k)\end{aligned}$$

and for $t \in (t_i, s_i]$, we have

$$\beta((\mathcal{Q}B_k)) \le \max_{i=1,\cdots,\delta}\beta(g_i(t, B_k(t))) \le \max_{i=1,\cdots,\delta} L_{g_i}\beta(B_k).$$

By the assumption **(HB)**, we have that $\max_{i=1,\cdots,\delta}\big[ML_K + ML_{g_i} + 4Mb(L_1 + bH_0L_2 + bP_0L_3)\big] < 1$. Therefore, we conclude that $\beta(\mathcal{Q}B_k) < \beta(B_k)$ for $t \in [0, b]$. This shows that the solution map $\mathcal{Q}$ is β-condensing in B_k. Thus, by Darbo–Sadovskii's fixed point theorem, $\mathcal{Q}$ has a fixed point in B_k. The fixed point of the map $\mathcal{Q}$ is a mild solution for the impulsive system (1.1)–(1.3) with non-instantaneous impulses. □

Next, we will show the existence of a positive mild solution to the system (1.1)–(1.3).

Let X be a real Banach space partially ordered by a cone $\mathcal{P}$ of X, *i.e.*, for any $x_1, x_2 \in X$, $x_1 \le x_2$ if and only if $x_2 - x_1 \in \mathcal{P}$.

We assume that the operator $T(t)(t \ge 0)$ is a C_0-semigroup on X, and $T(t)(t \ge 0)$ is called a positive C_0-semigroup on X if $T(t)x \ge 0$ for any $x \ge 0$. Now we make the following assumptions:

(Hf′1) The function $f\colon [0, b] \times \mathcal{P} \times \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ satisfies Carathéodory-type conditions:

(v) f is uniformly continuous on $[0, b] \times \mathcal{P} \times \mathcal{P} \times \mathcal{P}$ and there exist nonnegative continuous functions $N_n(t)$, $n = 1, 2$, and $j(t)\colon [0, b] \to \mathcal{P}$ such that

$$f(t, u, v, w) \le N_1(t)u + N_2(t)v + j(t)$$

for any $t \in [0, b]$ and $u, v, w \in \mathcal{P}$;

(vi) there exist constants $L_i > 0$, $i = 1, 2, 3$, such that

$$\beta(f(t, D_1(t), D_2(t), D_3(t))) \le L_1\beta(D_1(t)) + L_2\beta(D_2(t)) + L_3\beta(D_3(t))$$

for any equicontinuous and bounded sets $D_n \subset C([0, b]; \mathcal{P})$, $n = 1, 2, 3$, and $t \in [0, b]$.

(HK1) The nonlocal function $K\colon C([0, b]; \mathcal{P}) \to \mathcal{P}$ is continuous and compact, and there exists a constant $L_{\mathbb{K}} > 0$ such that $\|K(y)\| \le L_{\mathbb{K}}$ for any $y \in C([0, b]; \mathcal{P})$.

(Hg2) The functions g_i are compact and continuous and there are positive constants $\mathbb{L}_{g_i}$ such that $\|g_i(t,x)\| \le \mathbb{L}_{g_i}$ for all $x \in \mathcal{P}$ and $t \in [0,b]$.

Theorem 3 *Let X be a real Banach space and let $\mathcal{P}$ be a normal cone in X. Assume that $-A$ is the generator of an equicontinuous positive C_0-semigroup, $x_0 \ge 0$ and the conditions* **(Hf′1)**, **(HK1)**, **(Hg2)** *are satisfied. Then there exists a mild solution in $\mathcal{PC}([0,b];\mathcal{P})$ of the system* (1.1)–(1.3).

Proof. We consider the operators $\mathcal{Q}$ as in (3.1)–(3.2) and $\widetilde{Q}$, defined by

$$(\widetilde{Q}x)(t) = \int_0^t T(t-\zeta)[N_1(\zeta)x(\zeta) + N_2(\zeta)(Hx)(\zeta)]\,\mathrm{d}\zeta,$$

where $Hx(t) = \int_0^t h(t,\tau)x(\tau)\,\mathrm{d}\tau$ and $t \in [0,t_1]$. For $t \in (s_i, t_{i+1}]$, $i = 1,\cdots,\delta$, we set

$$(\widetilde{Q}x)(t) = \int_{s_i}^t T(t-\zeta)[N_1 x(\zeta) + N_2(\zeta)(Hx)(\zeta)]\,\mathrm{d}\zeta.$$

Now, we show that $r(\widetilde{Q}) = 0$, where $r(\cdot)$ denotes the spectral radius of a bounded linear operator. By the definition of $\widetilde{Q}$, for any $t \in [0,t_1]$, we have

$$\begin{aligned}\|(\widetilde{Q}x)(t)\| &= \left\|\int_0^t T(t-\zeta)[N_1(\zeta)x(\zeta) + N_2(\zeta)(Hx)(\zeta)]\,\mathrm{d}\zeta\right\|\\ &\le MN^*(1+bH_0)t\|x\|_{\mathcal{PC}},\end{aligned}$$

where $N^* = \max\{\max_{\zeta\in[0,b]} N_1(\zeta), \max_{\zeta\in[0,b]} N_2(\zeta)\}$. Similarly, for $t \in (s_i, t_{i+1}]$, $i = 1,\cdots,\delta$, we have

$$\begin{aligned}\|(\widetilde{Q}x)(t)\| &= \left\|\int_{s_i}^t T(t-\zeta)[N_1 x(\zeta) + N_2(\zeta)(Hx)(\zeta)]\,\mathrm{d}\zeta\right\|\\ &\le MN^*(1+bH_0)(t-s_i)\|x\|_{\mathcal{PC}}.\end{aligned}$$

Further,

$$\begin{aligned}\|(\widetilde{Q}^2x)(t)\| &\le MN^*\int_0^t \left[\|\widetilde{Q}x(s)\| + \int_0^s h(s,\zeta)u(\zeta)\ \mathrm{d}\zeta\right]\mathrm{d}s\\ &\le (MN^*)^2\int_0^t \left[(1+bH_0)s\|x\|_{\mathcal{PC}} + H_0\int_0^s (1+bH_0)\zeta\|x\|_{\mathcal{PC}}\ \mathrm{d}\zeta\right]\mathrm{d}s\\ &\le (MN^*)^2(1+bH_0)\|x\|_{\mathcal{PC}}\left[\frac{t^2}{2} + H_0 b\frac{t^2}{2}\right]\\ &\le (MN^*(1+bH_0))^2\frac{t^2}{2!}\|x\|_{\mathcal{PC}}.\end{aligned}$$

Similarly, for $t \in (s_i, t_{i+1}]$, we have

$$\|(\widetilde{Q}^2x)(t)\| \le (MN^*(1+bH_0))^2\frac{(t-s_i)^2}{2!}\|x\|_{\mathcal{PC}}.$$

By the mathematical induction, for any positive integer n, we get

$$\|(\widetilde{Q}^nx)(t)\| \le (MN^*(1+bH_0))^n\frac{t^n}{n!}\|x\|_{\mathcal{PC}}, \quad t \in [0,t_1],$$

and for $t \in (s_i, t_{i+1}]$, $i = 1, \cdots, \delta$, we have

$$\|(\widetilde{Q}^n x)(t)\| \leq (MN^*(1+bH_0))^n \frac{(t-s_i)^n}{n!}\|x\|_{\mathcal{PC}}.$$

Hence, we obtain

$$\|(\widetilde{Q}^n x)\|_{\mathcal{PC}} \leq (MN^*(1+bH_0))^n \frac{b^n}{n!}\|x\|_{\mathcal{PC}}.$$

Thus, $\|\widetilde{Q}^n\| \leq (MN^*(1+bH_0))^n \frac{b^n}{n!}$. Therefore, we get $r(\widetilde{Q}) = \lim_{n\to\infty} \|\widetilde{Q}^n\|^{1/n} = 0$.

Let $0 < MN^*(1+bH_0) < G^{-1}$, where G is the normal constant of $\mathcal{G}$. Hence, we get that there is an equivalent norm $\|\cdot\|^*$ in X such that $\|\widetilde{Q}\|^* \leq r(\widetilde{Q}) + MN^*(1+bH_0) = MN^*(1+bH_0)$, where $\|\widetilde{Q}\|^*$ means the operator norm of $\widetilde{Q}$ with respect to the norm $\|\cdot\|^*$.

Consider $M^* = \sup_{t\in[0,b]}\{\|T(t)\|^*\}$,

$$r^* \geq \max_{i=1,\cdots,\delta}\Big\{GM^*\big[\|x_0\|^* + \mathbb{L}_K + b\|j(t)\|^*\big]\big(1 - G(MN^*(1+bH_0))\big)^{-1},$$
$$GM^*\big[\mathbb{L}_{g_i} + b\|j(t)\|^*\big]\big(1 - G(MN^*(1+bH_0))\big)^{-1}\Big\},$$

where $\|x\|^* = \max_{t\in[0,b]}\|x(t)\|^*$, and $B_{r^*}(\mathcal{P}) = \{x \in \mathcal{PC}([0,b];\mathcal{P}) : \|x\|^*_{\mathcal{PC}} \leq r^*\}$. Since f is uniformly continuous, from the definition of the map $\mathcal{Q}$, it follows that for any $u \in B_{r^*}(\mathcal{P})$, $(\mathcal{Q}x)(t) \geq 0$ and

$$\begin{aligned}(\mathcal{Q}x)(t) &\leq T(t)(x_0 + K(x)) + \int_0^t T(t-\zeta)\big[N_1(\zeta)x(\zeta) + N_2(\zeta)(Hx)(\zeta) + j(\zeta)\big]\,\mathrm{d}\zeta\\ &\leq T(t)(x_0 + L_{\mathbb{K}}) + \int_0^t T(t-\zeta)\big[N_1(\zeta)x(\zeta) + N_2(\zeta)(Hx)(\zeta)\big]\,\mathrm{d}\zeta\\ &\quad + \int_0^t T(t-\zeta)j(\zeta)\,\mathrm{d}\zeta, \quad t \in [0,t_1],\end{aligned}$$

and

$$\mathcal{Q}x(t) \leq T(t)\mathbb{L}_{g_i} + \int_{s_i}^t T(t-\zeta)\big[N_1(\zeta)x(\zeta) + N_2(\zeta)(Hx)(\zeta)\big]\,\mathrm{d}\zeta + \int_0^t T(t-\zeta)j(\zeta)\,\mathrm{d}\zeta$$

for all $t \in (s_i, t_{i+1}]$, $i = 1, \cdots, \delta$. Since $\mathcal{P}$ is a normal cone, we get

$$\begin{aligned}\|(\mathcal{Q}x)(t)\|^* &\leq G\bigg(\|T(t)(x_0 + K(x))\|^* + \|(\widetilde{Q}x)(t)\|^* + \bigg\|\int_0^t T(t-\zeta)j(\zeta)\,\mathrm{d}\zeta\bigg\|^*\bigg)\\ &\leq G\bigg(M^*(\|x_0\|^* + L_{\mathbb{K}}) + \|\widetilde{Q}\|^*\|x\|^*_{\mathcal{PC}} + \int_0^b \|T(t-\zeta)\|^*\|j(\zeta)\|^*\,\mathrm{d}\zeta\bigg)\\ &\leq GM^*(\|x_0\|^* + L_{\mathbb{K}}) + GMN^*(1+bH_0)r^* + GbM^*\|j\|^* \leq r^*\end{aligned}$$

for $t \in [0,t_1]$, and for $t \in (s_i, t_{i+1}]$

$$\|\mathcal{Q}x(t)\|^* \leq GM^*\mathbb{L}_{g_i} + GMN^*(1+bH_0)r^* + GbM^*\|j\|^* \leq r^*.$$

Therefore, we deduce that $\mathcal{Q}\colon B_{r^*}(\mathcal{P}) \to B_{r^*}(\mathcal{P})$. Let $\widetilde{B} = \overline{\operatorname{conv}}(B_{r^*}(\mathcal{P}))$. Then $\overline{B}$ is a bounded convex closed set in $\mathcal{PC}([0,b];X)$ and $\mathcal{Q}\colon \overline{B} \to \overline{B}$. Thus, similarly to the proof of Theorem 1, we can show that $\mathcal{Q}$ is a convex-power condensing operator. Thus, we get the required result by using the convex-power condensing fixed point theorem. □

4 Application

Let us consider the following impulsive problem

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + F\left(t, u(t,x), \int_0^t h(t,\tau,x)u(\tau,x)\,\mathrm{d}\tau, \int_0^t p(t,\tau,x)x(\tau,x)\,\mathrm{d}\tau\right),$$
$$(t,x) \in \bigcup_{i=1}^{N}[s_i, t_{i+1}] \times [0,\pi], \tag{4.1}$$
$$u(t,0) = u(t,\pi) = 0, \qquad t \in [0,b], \tag{4.2}$$
$$u(0,x) = u_0 + K(u), \qquad x \in [0,\pi], \tag{4.3}$$
$$u(t,x) = G_i(t, u(t,x)), \quad x \in [0,\pi],\ t \in (t_i, s_i], \tag{4.4}$$

where $0 = t_0 = s_0 < t_1 \leq s_1 < \cdots < t_\delta \leq s_\delta < t_{\delta+1} = b$ are fixed numbers, $u_0 \in X$, $F \in C([0,b] \times X \times X \times X, X)$, $K\colon C([0,b], X) \to X$ and $G_i \in C((t_i, s_i] \times X, X)$ for all $i = 1, \cdots, \delta$.

We consider $X = L^2[0,\pi]$ and the operator A defined by $A = u''$ with the domain

$$D(A) = \{x \in X : x, x' \text{ are absolutely continuous and } x'' \in X, x(0) = x(\pi) = 0\}.$$

It is well-known from [15] that A is the infinitesimal generator of an analytic semigroup $T(t)$, $t \geq 0$, and that an analytic semigroup is equicontinuous. This means that A satisfies the assumption **(HT)**.

To convert the problem (4.1)–(4.4) into the abstract form (1.1)–(1.3), we introduce the functions $f\colon [0,b] \times X^3 \to X$ and $g_i\colon (t_i, s_i] \times X \to X$ such that $f(t,y)(x) = F(t, y(x), Hy(x), Py(x))$ and $g_i(t,y)(x) = G_i(t, y(x))$.

Thus, the results of the earlier sections which guarantee the existence of a mild solution and a positive solution may be applied with appropriate functions f and g_i and $K(u)$ satisfying suitable conditions.

5 Acknowledgement

The authors would like to thank the referee for valuable comments and suggestions.

References

[1] R. P. Agarwal, M. Benchohra, D. Seba, *On the application of measure of noncompactness to the existence of solutions for fractional differential equations*, Results in Mathematics **55** (2009), no. 3-4, 221–230.

[2] J. Banaś, K. Goebel, *Measure of noncompactness in Banach spaces*, Lecture Notes in Pure and Applied Mathematics, vol. 60, Marcel Dekker, Inc., New York, 1980.

[3] M. Benchohra, J. Henderson, S. Ntouyas, *Impulsive differential equations and inclusions*, Contemporary Mathematics and Its Applications, vol. 2, Hindawi Publishing Corporation, New York, 2006.

[4] Y. K. Chang, V. Kavitha, M. Mallika Arjunan, *Existence results for impulsive neutral differential and integrodifferential equation with nonlocal conditions via fractional operators*, Nonlinear Analysis. Hybrid System **4** (2010), no. 1, 32–43.

[5] P. Chen, Y. Li, *Nonlocal problem for fractional evolution equations of mixed type with the measure of noncompactness*, Abstract and Applied Analysis, 2013 (2013), 12 pp., Art. ID 784816.

[6] Z. Fan, G. Li, *Existence results for semilinear differential equations with nonlocal and impulsive conditions*, Journal of Functional Analysis **258** (2010), no. 5, 1709–1727.

[7] E. Herández, D. O'Regan, *On a new class of abstract impulsive differential equations*, Proceedings of the American Mathematical Society **141** (2012), no. 5, 1641–1649.

[8] E. Hernández, M. Rabello, H. R. Henríquez, *Existence of solutions for impulsive partial neutral functional differential equations*, Journal of Mathematical Analysis and Applications **331** (2007), no. 2, 1135–1158.

[9] S. Ji, G. Li, *A unified approach to nonlocal impulsive differential equations with the measure of noncompactness*, Advances in Difference Equations **2012** (2012), 14 pp.

[10] V. Lakshmikantham, D. D. Baĭnov, P. S. Simeonov, *Theory of impulsive differential equations*, Series in Modern Applied Mathematics, vol. 6, World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[11] K. Li, J. Peng, J. Gao, *Nonlocal fractional semilinear differential equations in separable Banach spaces*, Electronic Journal of Differential Equations **2013** (2013), 7 pp.

[12] J. H. Liu, *Nonlinear impulsive evolution equations*, Dynamics of Continuous, Discrete and Impulsive Systems **6** (1999), no. 1, 77–85.

[13] L. Liu, F. Guo, C. Wu, Y. Wu, *Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces*, Journal of Mathematical Analysis and Applications **309** (2005), no. 2, 638–649.

[14] J. J. Nieto, R. Rodríguez-López, *Periodic boundary value problem for non-Lipschitzian impulsive functional differential equations*, Journal of Mathematical Analysis and Applications **318** (2006), no. 2, 593–610.

[15] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.

[16] H.-B. Shi, W.-T. Li, H.-R. Sun, *Existence of mild solutions for abstract mixed type semilinear evolution equations*, Turkish Journal of Mathematics **35** (2011), no. 3, 457–472.

[17] J. X. Sun, X. Y. Zhang, *The fixed point theorems of convex-power condensing operator and applications to abstract semilinear evolution equations*, Acta Mathematica Sinica, Chinese Series **48** (2005), no. 3, 439–446.

[18] X. Xue, *Nonlinear differential equations with nonlocal conditions in Banach spaces*, Nonlinear Analysis. Theory, Methods & Applications **63** (2005), no. 4, 575–586.

[19] X. Xue, *Nonlocal nonlinear differential equations with a measure of noncompactness in Banach spaces*, Nonlinear Analysis. Theory, Methods & Applications **70** (2009), no. 7, 2593–2601.

[20] Z. Yan, *Existence of solutions for nonlocal impulsive partial functional integrodifferential equation via fractional operators*, Journal of Computational and Applied Mathematics **235** (2011), no. 8, 2252–2262.

[21] X. Zhang, L. S. Liu, C. X. Wu, *Global solutions of nonlinear second-order impulsive integro-differential equations of mixed type in Banach spaces*, Nonlinear Analysis. Theory, Methods & Applications **67** (2007), no. 8, 2335–2349.

[22] L. Zhu, G. Li, *Existence results of semilinear differential equations with nonlocal initial conditions in Banach spaces*, Nonlinear Analysis. Theory, Methods & Applications **74** (2011), no. 15, 5133–5140.

[23] T. Zhu, C. Song, G. Li, *Existence of mild solutions for abstract semilinear evolution equations in Banach spaces*, Nonlinear Analysis. Theory, Methods & Applications **75** (2012), no. 1, 177–181.