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ℓ^p SOLUTION TO THE INITIAL VALUE PROBLEM OF THE DISCRETE NONLINEAR SCHRÖDINGER EQUATION WITH COMPLEX POTENTIAL, II

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Abstract. In this paper, we prove the global well-posedness of the initial value problem of the time-dependent discrete nonlinear Schrödinger equation with a complex potential and sufficiently general nonlinearity on a multidimensional lattice in weighted ℓ^p spaces in the case $2 < p < \infty$.

Keywords: Discrete nonlinear Schrödinger equation, semigroup, initial value problem, Lipschitz continuous, complex potential, ℓ^p solution.

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1 Introduction

The discrete nonlinear Schrödinger equation (DNLS) is a mathematical model that describes the behavior of waves in discrete systems with nonlinear interactions. This equation is of great interest in various fields, including condensed matter physics, optics, and nonlinear dynamics. In particular, when a complex potential is introduced into the DNLS equation, it gives rise to intriguing phenomena and opens up new avenues for exploring wave propagation in structured environments.

The presence of a complex potential in the DNLS equation introduces additional complexities and nonlinearity, leading to rich dynamics and novel phenomena. The complex potential can arise from various sources, such as an external field or a spatially varying refractive index in optics. It can significantly influence the propagation characteristics of waves, including wave localization, soliton formation, and wave scattering.

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Understanding the properties and dynamics of the DNLS equation is crucial for gaining insights into the behavior of discrete wave systems and exploring nonlinear effects in different physical systems. For instance, we mention nonlinear wave transmission in discrete media, propagation of localized pulses in coupled waveguides and optical fibers, and modeling Bose-Einstein condensates (see, *e.g.*, [6, 9, 10] and references therein).

Research activity in this area mainly focuses on the so-called "breathers", which are standing waves. The profile function of such a wave solves an appropriate stationary DNLS equation. Most works in this direction deal with (discrete) translation-invariant DNLS on a one-dimensional lattice and employ perturbation techniques, two-dimensional discrete-time dynamical systems, and numerical simulation (see, *e.g.*, [4–6] and references therein).

On the other hand, a series of papers [2, 13–17, 20–24] employs the theory of critical points of smooth functionals to investigate breathers for the DNLS equation with diverse nontrivial potentials. Additionally, we highlight the noteworthy contribution made in the paper [19].

The initial value problem (IVP) associated with the DNLS equation with a complex potential deals with determining the evolution of the wave function over time when its initial configuration is known. In other words, given the initial values of the wave function and its derivative at a specific time, the IVP seeks to find a solution that satisfies the DNLS equation with the given complex potential. The DNLS equation with a complex potential on a one-dimensional lattice can be written as:

$$i(\mathrm{d}\psi_n/\mathrm{d}t) + A_n\psi_n + B_n|\psi_n|^2\psi_n + C_n\psi_{n+1} + D_n\psi_{n-1} = 0,$$

where ψ_n is the complex-valued wave function at the discrete lattice site n, and A_n, B_n, C_n, and D_n represent the coefficients associated with the linear and nonlinear interactions between adjacent lattice sites.

In [25] we investigated the weighted ℓ^2 solution of the following initial value problem for the time-dependent d-dimensional discrete nonlinear Schrödinger equation

$$i\dot{u} = -\Delta u + Wu - f(n, u) + b(t, n), \tag{1.1}$$

$$u(0, n) = u^0(n)\,, \tag{1.2}$$

where the potential $W = V + i\delta$ is a complex function of

$$n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d\,,$$

$\dot{u}$ stands for the time derivative and $-\Delta$ is the d-dimensional discrete Laplacian defined by

$$\begin{aligned}\Delta u(n) = {} & u(n_1 - 1, n_2, \ldots, n_d) + u(n_1, n_2 - 1, \ldots, n_d) + \cdots + u(n_1, n_2, \ldots, n_d - 1)\\ & - 2du(n_1, n_2, \ldots, n_d)\\ & + u(n_1 + 1, n_2, \ldots, n_d) + u(n_1, n_2 + 1, \ldots, n_d) + \cdots + u(n_1, n_2, \ldots, n_d + 1).\end{aligned}$$

Note that if $\delta(n)$ is negative for all $n \in \mathbb{Z}^d$, the part δ of the potential represents dissipation effects. Additionally, our Assumption (iii) below allows the nonlinearity to contain a dissipative term. This DNLS (1.1) is the space discretization of the nonlinear Schrödinger equation in continuous media.

A limited number of papers [7, 8, 11, 12] are dedicated to equations of the form (1.1). Specifically, the paper [12] delves into the initial value problem for the DNLS equation with a zero potential and power nonlinearity on a one-dimensional lattice with a weighted ℓ^2 initial value. The principal outcome establishes global well-posedness in weighted ℓ^2 spaces with power weights.

In [7] and [8], the authors explore the DNLS with $V = 0$ and $\delta = \text{const}$. The primary findings include global well-posedness in the conservative ($\delta = 0$) and dissipative ($\delta < 0$) scenarios. Additionally, attractors' existence is demonstrated in weighted ℓ^2 spaces in the conservative case, on one-dimensional and multidimensional lattices, respectively. The paper [11] investigates the well-posedness in weighted spaces for the DNLS on a one-dimensional lattice, focusing on the case when $W = V$ is a general real potential and $b = 0$.

In the work presented in [25], we extended our previous results to the multidimensional case, accommodating a sufficiently general potential W that may not be necessarily bounded. This extension includes scenarios with a weighted ℓ^2 initial value. Furthermore, in the subsequent paper [26], we utilized the integral equation, which defines the mild solution of the DNLS as established in [25]. This application allowed us to establish the existence of a global solution for the DNLS featuring a weighted ℓ^p initial value when $1 \le p < 2$. This was achieved by leveraging the previously obtained global solutions in the ℓ^2 setting, as documented in [25]. The focus of the current paper is to demonstrate the existence of a global solution for the DNLS with a weighted ℓ^p initial value when $2 < p < \infty$. To the best of our knowledge, there has been no exploration by other mathematicians into the initial value problem for the DNLS with a weighted ℓ^p initial value. Since ℓ^p is not a Hilbert space like ℓ^2 when $p \neq 2$, we cannot rely on the features of a Hilbert space to prove our main results on ℓ^p global solutions. Instead, we employ alternative methods.

The organization of this paper is structured as follows: Section 2 provides readers with a concise review of some preliminaries on the semigroup theory of abstract differential equations for their convenience. In Section 3, we revisit the local weighted ℓ^p well-posedness result and the global weighted $\ell^p(1 \le p \le 2)$ well-posedness result established in [26]. Section 4 is dedicated to proving the existence of weighted ℓ^p global solutions when $2 < p < \infty$.

2 Semigroup theory and abstract initial value problem

We treat (1.1) as an abstract differential equation of the form

$$\dot{u} = Au + N(t, u) \tag{2.1}$$

in a complex Banach space. We always assume that A is a closed operator in a Banach space E with the domain $D(A)$, and $N : [0, \infty) \times E \to E$ is continuous. Let us provide a reminder of some elementary facts related to such equations.

A family $U(t),\ t \in [0, \infty)$, of bounded linear operators in E is a strongly continuous *semigroup of operators* if

(1) $U(t)v$ is a continuous function on $[0, \infty)$ with values in E for every $v \in E$;

(2) $U(0) = I$ is the identity operator in E;

(3) $U(t + s) = U(t)U(s)$ for all $t, s \in [0, \infty)$.

If the family $U(t)$ is defined for all $t \in \mathbb{R}$ and satisfies (1)-(3) above on the whole real line, we say that $U(t)$ is a strongly continuous *group of operators*.

If $U(t)$ is a strongly continuous semigroup of operators, then its *generator* A is defined by

$$Av = \lim_{t \to 0^+} t^{-1}(U(t) - I)v, \tag{2.2}$$

where the domain $D(A)$ consists of those $v \in E$ for which the limit in (2.2) exists.

The following result is well known (see, *e.g.*, [3, 18]).

Proposition 2.1 *If A is a generator of a strongly continuous semigroup in a Banach space E and B is a bounded linear operator in E, then $A + B$ is a generator of a strongly continuous semigroup.*

If A is a bounded linear operator, then it generates a one-parameter group e^{tA}. In general, if A is a generator of a strongly continuous semigroup, we still use the same exponential notation e^{tA} for the semigroup generated by A.

Now we discuss the abstract initial value problem for equation (2.1), with initial data

$$u(0) = u^0 \in E. \tag{2.3}$$

If A is a bounded operator, then it is sufficient to consider classical solutions, *i.e.* continuously differentiable functions with values in E that satisfy (2.1) and (2.3). In general, when the operator A is unbounded, we consider mild solutions to (2.1) and (2.3).

A continuous function u on $[0, T]$ with values in E is a *mild solution* of the initial value problem (2.1) and (2.3) if it satisfies the following integral equation

$$u(t) = e^{tA}u^0 + \int_0^t e^{(t-s)A} N(s, u(s))\, \mathrm{d}s. \tag{2.4}$$

In the case when the operator A is bounded, these are classical solutions.

We need the following well-known result (see. *e.g.*, [1, 18]).

Proposition 2.2 *Let A be a generator of a strongly continuous semigroup in a Banach space E, and $N(t, u) : [0, \infty) \times E \to E$ be continuous in t and locally Lipschitz continuous in u with the Lipschitz constant being bounded on bounded intervals of t. That is, for any $T > 0$ and $R > 0$, there exists $C = C(T, R) > 0$ such that*

$$\max_{0 \le t \le T} \|N(t, w) - N(t, w')\| \le C\|w - w'\| \tag{2.5}$$

whenever $\|w\| \le R$ and $\|w'\| \le R$.

- (a) *For every $u^0 \in E$, there exists a unique local mild solution of the initial value problem (2.1) and (2.3) defined on the maximal interval $[0, \tau_{\max})$.*
- (b) *If $\tau_{\max} < \infty$, then $\lim_{t \nearrow \tau_{\max}} \|u(t)\| = \infty$.*
- (c) *The solution $u(t)$ depends continuously on u^0 in the topology of uniform convergence on bounded closed subintervals of $[0, \tau_{\max})$.*

(d) *Assume, in addition, that the map* $N : [0,\infty) \times E \to E$ *is locally Lipschitz continuous, i.e., for any* $T > 0$ *and* $R > 0$*, there exists* $C = C(T, R) > 0$ *such that*

$$\|N(t,w) - N(t',w')\| \le C(|t - t'| + \|w - w'\|) \tag{2.6}$$

whenever $t \in [0,T]$*,* $t' \in [0,T]$*,* $\|w\| \le R$ *and* $\|w'\| \le R$*. If* $u^0 \in D(A)$*, then the mild solution of the initial value problem (2.1) and (2.3) is a classical solution.*

Remark 2.1 *Assumption (2.5) implies automatically that* N *is bounded on bounded sets.*

Remark 2.2 *If* $N(t,u)$ *is globally Lipschitz continuous in* u*, i.e. there exists a constant* $C = C(T) > 0$ *such that*

$$\max_{0\le t\le T} \|N(t,w) - N(t,w')\| \le C\|w - w'\|, \quad \textit{for all } w, w' \in E, \tag{2.7}$$

then the initial value problem (2.1) and (2.3) possesses a unique global mild solution defined on $[0,\infty)$*. Moreover, the solution* $u(t)$ *depends continuously on* u^0 *in the topology of uniform convergence on bounded closed subintervals of* $[0,\infty)$*.*

Remark 2.3 *Let* $N(t,u)$ *be of the form*

$$N(t,u) = M(u) + f(t)\,.$$

Then assumption (2.6) holds if and only if M *and* f *are locally Lipschitz continuous on* E *and* $[0,\infty)$*, respectively.*

3 Reviews on local solution and global solution

3.1 Local solution

We review local solution to the equation (1.1) under the following assumptions:

(i) *The complex potential* $W = V + i\delta$ *is such that both* V *and* δ *are real-valued functions on* $\mathbb{Z}^d$*, and*

$$\overline{\delta} = \sup\{\delta(n) | n \in \mathbb{Z}^d\} < \infty\,.$$

(ii) *The nonlinearity* $f : \mathbb{Z}^d \times \mathbb{C} \to \mathbb{C}$ *satisfies the following conditions:*

1. $f(n,0) = 0$,
2. $f(n,z) = o(z)$ *as* $z \to 0$ *uniformly with respect to* $n \in \mathbb{Z}^d$,
3. f *is uniformly locally Lipschitz continuous, that is, for every* $R > 0$*, there exists a constant* $C = C(R)$ *independent of* $n \in \mathbb{Z}^d$ *such that*

$$|f(n,z) - f(n,z')| \le C|z - z'|$$

for all $n \in \mathbb{Z}^d$ *whenever* $|z| \le R$ *and* $|z'| \le R$*.*

(iii) *The nonlinearity $f(n,z)$ is of the form $f(n,z) = g(n,|z|)z$, where $g(n,r)$ is a function and its imaginary part is nonnegative.*

Let $\Theta = (\theta_n)_{n\in\mathbb{Z}^d}$ be a sequence of positive numbers (weights). The space $\ell^p_\Theta(\mathbb{Z}^d)$ consists of all two-sided sequences of complex numbers such that the norm

$$\|u\|_{\ell^p_\Theta} = (\sum_{n\in\mathbb{Z}^d} |u(n)\theta_n|^p)^{1/p}$$

is finite. We notice that $u \in \ell^p_\Theta(\mathbb{Z}^d)$ if and only if $u\Theta \in \ell^p(\mathbb{Z}^d)$ and

$$\|u\|_{\ell^p_\Theta} = \|u\Theta\|_{\ell^p}.$$

Therefore for $1 \leq p < q \leq \infty$ we have

$$\|u\|_{\ell^q_\Theta} \leq \|u\|_{\ell^p_\Theta}$$

and

$$\ell^p_\Theta(\mathbb{Z}^d) \subset \ell^q_\Theta(\mathbb{Z}^d).$$

We always assume that the weight Θ is *regular* in the sense that:

(iv) *The sequence Θ is bounded below by a positive constant, and there exists a constant $c_0 \geq 1$ such that*

$$c_0^{-1} \leq \frac{\theta_{n+e_i}}{\theta_n} \leq c_0$$

for all $n \in \mathbb{Z}^d$ and $i = 1, \cdots, d$, where $e_i \in \mathbb{Z}^d$ has 1 at the i-th component and 0 elsewhere.

From Assumption (iv), we obtain

$$\|u\|_{\ell^p} \leq C\|u\|_{\ell^p_\Theta}, \tag{3.1}$$

which implies that $\ell^p_\Theta(\mathbb{Z}^d)$ is densely and continuously embedded into $\ell^p(\mathbb{Z}^d)$ and C is defined by

$$C = \max_{n\in\mathbb{Z}^d} 1/\theta_n.$$

Setting Θ_0 as the constant weight with unit components, we have that

$$\ell^p_{\Theta_0}(\mathbb{Z}^d) = \ell^p(\mathbb{Z}^d).$$

From the perspective of functional analysis, Assumption (iv) means that the space $\ell^p_\Theta(\mathbb{Z}^d)$ is translation invariant. More precisely, let S_i and T_i be the operators defined by

$$(S_i w)(n) = w(n - e_i), \qquad (T_i w)(n) = w(n + e_i), \quad i = 1, \cdots, d.$$

To understand the equation (1.1) in the framework of evolution equations, we interpret it as an evolution equation of the form (2.1), where $A = -iH$ and H is the Schrödinger operator defined as

$$H = -\Delta + W \tag{3.2}$$

and the operator N is given by

$$N(t,u)(n) = i\,f(n,u(t,n)) - i\,b(t,n). \tag{3.3}$$

To establish a precise interpretation, we need to analyze certain properties of the Schrödinger operator H in the space $\ell^p_\Theta(\mathbb{Z}^d)$. First, we observe that the operator (of multiplication by) $-iW = -iV + \delta$ is a diagonal operator. Since V is real and $\delta(n) \le \overline{\delta}$ for all $n \in \mathbb{Z}^d$, the operator $-iW$ generates a strongly continuous semigroup in $\ell^p_\Theta(\mathbb{Z}^d)$ given by

$$(e^{-itW}u)(n) = e^{-iV(n)t}e^{\delta(n)t}u(n), \ \ n \in \mathbb{Z}^d.$$

The domain of this operator in $\ell^p_\Theta(\mathbb{Z}^d)$ is defined as

$$D_\Theta = \{u \in \ell^p_\Theta(\mathbb{Z}^d) : \ Wu \in \ell^p_\Theta(\mathbb{Z}^d)\}, \tag{3.4}$$

where we use the notation D to represent the domain of the operator W in $\ell^p(\mathbb{Z}^d)$. It is clear that $D_\Theta \subset D$.

Based on Proposition 2.1, we derived the following lemma in [26].

Lemma 3.1 *The operator $A = -iH$ is a generator of strongly continuous group e^{tA} in the space $\ell^p_\Theta(\mathbb{Z}^d)$. Moreover, there exist two constants $M \ge 1$ and ω such that for all $t \ge 0$*

$$\|e^{tA}\| \le Me^{\omega t}. \tag{3.5}$$

We define the operator $N(t,u)$ as follows

$$N(t,u)(n) = if(n,u(n)) - ib(t,n).$$

Then the equation (1.1) can be expressed in the form of equation (2.1). The following local well-posedness result is proved in [26].

Theorem 3.1 (1) *Under Assumptions (i), (ii) and (iv), if $b \in C([0,\infty), \ell^p_\Theta(\mathbb{Z}^d))$, then for every $u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$, problem (1.1) and (1.2) has a unique local mild solution $u \in C([0,T], \ell^p_\Theta(\mathbb{Z}^d))$ for some $T > 0$.*

(2) *The mild solution $u(t) \in C([0,T], \ell^p_\Theta(\mathbb{Z}^d))$ of problem (1.1) and (1.2) obtained in part (1) is a classical solution if one of the following conditions holds*

(a) $u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$ and W is bounded;

(b) $u^0 \in D(A) = D_\Theta$ and $b : [0,\infty) \to \ell^p_\Theta(\mathbb{Z}^d)$ is locally Lipschitz continuous.

3.2 Existence of global solutions in the case $1 \le p < 2$

In this case, since we have the inclusion $\ell^p_\Theta(\mathbb{Z}^d) \subset \ell^2_\Theta(\mathbb{Z}^d)$, we established the existence of a global ℓ^2 solution by applying Theorem 3.1 in [25]. Utilizing this result, we proved the following theorem in [26].

Theorem 3.2 (1) *Under assumptions* (i), (ii), (iii), *and* (iv), *if* $\overline{\delta} \leq 0$ *and* $b \in C([0,\infty), \ell^p_\Theta(\mathbb{Z}^d)) \bigcap L^1([0,\infty), \ell^2(\mathbb{Z}^d))$, *then for every* $u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$, *problem (1.1) and (1.2) has a unique global mild solution* $u \in C([0,\infty), \ell^p_\Theta(\mathbb{Z}^d))$ *which continuously depends on* u^0 *in the topology of uniform convergence on bounded closed subintervals of* $[0,\infty)$. *Moreover, for any* $t \geq 0$

$$\|u(t)\|_{\ell^p_\Theta} \leq (\|u^0\|_{\ell^p_\Theta} + B(\omega + CM, t))e^{(\omega+CM)t}, \tag{3.6}$$

where

$$B(\omega + CM, t) = \int_0^t e^{-(\omega+CM)s} \|b(s)\|_{\ell^p_\Theta} \, \mathrm{d}s,$$

C *is the Lipschitz constant independent of* t, ω *and* M *are the constants in Lemma 3.1.*

(2) *The global mild solution* $u(t) \in C([0,\infty), \ell^p_\Theta(\mathbb{Z}^d))$ *of problem (1.1) and (1.2) obtained in* (1) *is a classical solution if one of the following conditions holds*

(a) $u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$ *and* W *is bounded;*

(b) $u^0 \in D(A) = D_\Theta$ *and* $b : [0,\infty) \to \ell^p_\Theta(\mathbb{Z}^d)$ *is locally Lipschitz continuous.*

4 Global solution in the case $2 < p < \infty$

In [26], we proved the following theorem in the case $2 < p < \infty$.

Theorem 4.1 (1) *Under assumptions* (i), (ii), (iii), *and* (iv), *if* $\overline{\delta} \leq 0$, $b \in C([0,\infty), \ell^p_\Theta(\mathbb{Z}^d)) \bigcap L^1([0,\infty), \ell^2(\mathbb{Z}^d))$, *and in addition*

$$\Theta^{-1} \in \ell^q, \quad q = 2 + \frac{4}{p-2}, \tag{4.1}$$

then for every $u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$, *problem (1.1) and (1.2) has a unique global mild solution* $u \in C([0,\infty), \ell^p_\Theta(\mathbb{Z}^d))$ *which continuously depends on* u^0 *in the topology of uniform convergence on bounded closed subintervals of* $[0,\infty)$. *Moreover, for any* $t \geq 0$

$$\|u(t)\|_{\ell^p_\Theta} \leq (\|u^0\|_{\ell^p_\Theta} + B(\omega + CM, t))e^{(\omega+CM)t}, \tag{4.2}$$

where

$$B(\omega + CM, t) = \int_0^t e^{-(\omega+CM)s} \|b(s)\|_{\ell^p_\Theta} \, \mathrm{d}s,$$

C *is the Lipschitz constant independent of* t, ω *and* M *are the constants in Lemma 3.1.*

(2) *The global mild solution* $u(t) \in C([0,\infty), \ell^p_\Theta(\mathbb{Z}^d))$ *of problem (1.1) and (1.2) obtained in* (1) *is a classical solution if one of the following conditions holds*

(a) $u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$ *and* W *is bounded;*

(b) $u^0 \in D(A) = D_\Theta$ *and* $b : [0,\infty) \to \ell^p_\Theta(\mathbb{Z}^d)$ *is locally Lipschitz continuous.*

However, the condition (4.1) in the theorem made it impossible to obtain the existence of a global ℓ^p solution as a particular case of Theorem 4.1. In this section, we will try a different way to prove the existence of a global ℓ^p solution without the technical condition (4.1). Actually, we will prove the following theorem.

Theorem 4.2 (1) *Under assumptions (i), (ii), (iii), and (iv), if $b \in C([0,\infty),\ell^p_\Theta(\mathbb{Z}^d))$, then for every $u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$, problem (1.1) and (1.2) has a unique global mild solution $u \in C([0,\infty),\ell^p_\Theta(\mathbb{Z}^d))$ which continuously depends on u^0 in the topology of uniform convergence on bounded closed subintervals of $[0,\infty)$. Moreover, for any $t \geq 0$*

$$\|u(t)\|_{\ell^p_\Theta} \leq (\|u^0\|_{\ell^p_\Theta} + B(\bar{\delta}+2dc_0,t))e^{(\bar{\delta}+2dc_0)t}\,, \tag{4.3}$$

where

$$B(\bar{\delta}+2dc_0,t) = \int_0^t e^{-(\bar{\delta}+2dc_0)s}\,\|b(s)\|_{\ell^p_\Theta}\,\mathrm{d}s,$$

$\bar{\delta}$ and c_0 are constants in assumption (i) and (iv) respectively.

(2) *The global mild solution $u(t) \in C([0,\infty),\ell^p_\Theta(\mathbb{Z}^d))$ of problem (1.1) and (1.2) obtained in (1) is a classical solution if one of the following conditions holds*

(a) $u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$ and W is bounded;

(b) $u^0 \in D(A) = D_\Theta$ and $b : [0,\infty) \to \ell^p_\Theta(\mathbb{Z}^d)$ is locally Lipschitz continuous.

Let $u^0 \in \ell^p_\Theta,\ 2 < p < \infty,\ \ell^2_\Theta \subset \ell^p_\Theta\ \ is\ \ densely\ \ embedded.$
For each positive integer k we define

$$\chi_k(n) = \begin{cases} 1, & \text{when } |n| \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$u^0_k = u^0\chi_k \in \ell^2_\Theta \subset \ell^p_\Theta,$$

then we have

$$\lim_{k\to\infty} ||u^0_k - u^0||_{\ell^p_\Theta} = 0.$$

For each $u^0_k \in \ell^2_\Theta$, there is a global classical solution u_k (see [24]), that is,

$$u_k \in C^1\big([0,\infty),\ell^2_\Theta\big) \subset C^1\big([0,\infty),\ell^p_\Theta\big).$$

We will show that the sequence $\{u_k\}$ converges to a global mild solution in an appropriately selected function space (defined later).

Let

$$N(t,u)(n) = if(n,u(t,n)) - ib(t,n).$$

For each $k \geq 1$

$$\dot{u}_k(n) = (-iH)u_k(n) + N(t,u_k)(n), \quad \text{for all } n \in \mathbb{Z}^d. \tag{4.4}$$

$p > 2$ implies $p/2 > 1$, for each $n \in \mathbb{Z}^d$, we rewrite

$$|u_k(n)|^p = \Big[|u_k(n)|^2\Big]^{\frac{p}{2}} = \Big[u_k(n)\cdot\overline{u_k(n)}\Big]^{\frac{p}{2}}. \tag{4.5}$$

Lemma 4.1 *Assume that assumptions (i), (ii), and (iii) are satisfied and $b \in C([0,\infty), \ell^p_\Theta(\mathbb{Z}^d))$. If $u^0 \in \ell^p_\Theta(\mathbb{Z}^d)$, then there exists a constant $\bar{C} = \bar{\delta} + 2dc_0$, such that for each k, the global solution $u_k \in C^1([0,\infty), \ell^p_\Theta(\mathbb{Z}^d))$ satisfies*

$$\|u_k(t)\|_{\ell^p_\Theta} \leq e^{\bar{C}t}\Big(\|u_k^0\|_{\ell^p_\Theta} + B(\bar{C},t)\Big) \tag{4.6}$$

for each $t \geq 0$.

Proof. Taking the derivative of (4.5) and using the chain rule we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|u_k(n)|^p &= \frac{p}{2}\big[|u_k(n)|^2\big]^{\frac{p}{2}-1}\frac{\mathrm{d}}{\mathrm{d}t}\big(u_k(n)\overline{u_k(n)}\big)\\ &= \frac{p}{2}\big|u_k(n)\big|^{p-2}\big(\dot{u}_k(n)\overline{u_k(n)} + u_k(n)\overline{\dot{u}_k(n)}\big)\\ &= \frac{p}{2}\big|u_k(n)\big|^{p-2}2Re\big(\dot{u}_k(n)\overline{u_k(n)}\big)\\ &= p\big|u_k(n)\big|^{p-2}Re\big(\dot{u}_k(n)\overline{u_k(n)}\big)\\ &= Re\big[p\big|u_k(n)\big|^{p-2}\overline{u_k(n)}\dot{u}_k(n)\big].\end{aligned}$$

Multiplying (4.4) by $p\big|u_k(n)\big|^{p-2}\overline{u_k(n)}$, and taking the real part we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|u_k(n)|^p &= Re\big[p|u_k(n)|^{p-2}\overline{u_k(n)}[(-iH)u_k(n)] + p|u_k(n)|^{p-2}\overline{u_k(n)}N(t,u_k)(n)\big]\\ &= Re\big[p|u_k(n)|^{p-2}\overline{u_k(n)}[(-i(-\Delta+W))u_k(n)]\\ &+ p|u_k(n)|^{p-2}\overline{u_k(n)}\ \big(if(n,u_k(t,n)) - ib(t,n)\big)\big]\\ &= Re\big[p|u_k(n)|^{p-2}\overline{u_k(n)}[(i\Delta u_k(n) - iWu_k(n)] + p|u_k(n)|^{p-2}\overline{u_k(n)}\big(if(n,u_k(t,n)\big)\\ &- p|u_k(n)|^{p-2}\overline{u_k(n)}ib(t,n)\big].\end{aligned}$$

From assumption (i) we have

$$-iW = -iV + \delta,$$

and from assumption (iii) we have

$$f(n,u_k) = g(n,|u_k|)u_k(n) = (Re\ g + iIm\ g)u_k(n).$$

Hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|u_k(n)|^p &= Re\Big[p|u_k(n)|^{p-2}\overline{u_k(n)}[(i\Delta u_k(n) - iV(n)u_k(n) + \delta(n)u_k(n)]\\ &+ p|u_k(n)|^{p-2}\overline{u_k(n)}\big(i(Re\ g + iIm\ g)u_k(n)\big)\\ &- p|u_k(n)\||^{p-2}\overline{u_k(n)}ib(t,n)\Big]\\ &= Re\Big[p|u_k(n)|^{p-2}\overline{u_k(n)}[(i\Delta u_k(n) - iV(n)u_k(n) + \delta(n)u_k(n)]\\ &+ p|u_k(n)|^{p-2}|u_k(n)|^2\big(iRe\ g - Im\ g)\big)\\ &- p|u_k(n)|^{p-2}\overline{u_k(n)}\ ib(t,n)\Big],\end{aligned}$$

$$\Delta u_k(n) = \sum_{j=1}^{d}\big[u_k(n+e_j)+u_k(n-e_j)\big] - 2du_k(n), \tag{4.7}$$

$$i\Delta u_k(n)\cdot\overline{u_k(n)} = i\sum_{j=1}^{d}\big[u_k(n+e_j)+u_k(n-e_j)\big]\overline{u_k(n)} - i2d|u_k(n)|^2, \tag{4.8}$$

$$\delta(n)u_k(n)\overline{u_k(n)} = \delta(n)|u_k(n)|^2. \tag{4.9}$$

Use (4.8) and (4.9),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|u_k(n)|^p = {} & Re\big[p|u_k(n)|^{p-2}\big(i\sum_{j=1}^{d}\big[u_k(n+e_j)+u_k(n-e_j)\big]\overline{u_k(n)}\\ & - i2d|u_k(n)|^2 - iV|u_k(n)|^2 + \delta|u_k(n)|^2\big)\\ & + p|u_k(n)|^p\big(iReg - Img\big) - p|u_k(n)|^{p-2}\overline{u_k(n)}ib(t,n)\\ = {} & p\delta(n)|u_k(n)|^p - p|u_k(n)|^{p-2}Im\sum_{j=1}^{d}\big[u_k(n+e_j)\overline{u_k(n)}\\ & + u_k(n-e_j)\overline{u_k(n)}\big] - p|u_k(n)|^p\big(Img\big) + p|u_k(n)|^{p-2}Im(\overline{u_k}b)\big].\end{aligned}$$

Multiplying both sides by θ_n^p, and by assumption (iii), $Img \geq 0$, implies $-p|u_k(n)|^p(Img) \leq 0$. Thus by assumption (i) we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|u_k(n)|^p\theta_n^p \leq p\bar{\delta}\,|u_k(n)|^p\theta_n^p - p|u_k(n)|^{p-2}Im\sum_{j=1}^{d}\big[u_k(n+e_j)\theta_n^p\overline{u_k(n)}\\ + u_k(n-e_j)\theta_n^p\overline{u_k(n)}\big] + p|u_k(n)|^{p-2}Im(\overline{u_k}b)\theta_n^p.\end{aligned}$$

Notice that the weighted ℓ^p space was defined as follows:

$$u\in\ell^p_\Theta \iff u\Theta\in\ell^p, \quad \|u\|_{\ell^p_\Theta} = \|u\Theta\|_{\ell^p}.$$

Taking summation over $n\in\mathbb{Z}^d$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{n\in\mathbb{Z}^d}|u_k(n)\theta_n|^p\Big) \leq p\bar{\delta}\sum_{n\in\mathbb{Z}^d}|u_k(n)\theta_n|^p\\ - pIm\sum_{j=1}^{d}\sum_{n\in\mathbb{Z}^d}|u_k(n)|^{p-2}\overline{u_k(n)}\big[u_k(n+e_j)\theta_n^p\\ + u_k(n-e_j)\theta_n^p\big] + p|u_k(n)|^{p-2}Im(\overline{u_k(n)}b(t,n)\theta_n^p).\end{aligned}$$

Since

$$Im(\overline{u_k(n)}b(t,n)) \leq |\overline{u_k(n)}b(t,n)| \ = |u_k(n)||b(t,n)|,$$

we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u_k\|^p_{\ell^p_\Theta} \leq p\bar{\delta}\|u_k(n)\|^p_{\ell^p_\Theta} + p\sum_{j=1}^{d}\sum_{n\in\mathbb{Z}^d}|u_k(n)\theta_n|^{p-1}|u_k(n+e_j)\theta_{n+e_j}|\frac{\theta_n}{\theta_{n+e_j}}\\ + p\sum_{j=1}^{d}\sum_{n\in\mathbb{Z}^d}|u_k(n)\theta_n|^{p-1}|u_k(n-e_j)\theta_{n-e_j}|\frac{\theta_n}{\theta_{n-e_j}} + p|u_k(n)\theta_n|^{p-1}|b(t,n)||\theta_n|.\end{aligned}$$

Since $b(t,n) \in \ell^p_\Theta$, for all $t \geq 0$ and Θ is a regular weight by assumption (iv), then by Hölder's inequality we obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u_k(t)\|^p_{\ell^p_\Theta} &\leq p\bar{\delta}\|u_k(t)\|^p_{\ell^p_\Theta} + 2dc_0p\|u_k(t)\|^p_{\ell^p_\Theta} + p\|u_k(t)\|^{p-1}_{\ell^p_\Theta}\|b(t)\|_{\ell^p_\Theta} \\ &= (p\bar{\delta} + 2dc_0p)\|u_k(t)\|^p_{\ell^p_\Theta} + p\|u_k(t)\|^{p-1}_{\ell^p_\Theta}\|b(t)\|_{\ell^p_\Theta}.\end{aligned}$$

We set

$$w(t) = \|u_k(t)\|_{\ell^p_\Theta},$$

then by the chain rule, we obtain

$$\frac{\mathrm{d}w}{\mathrm{d}t} \leq (\bar{\delta} + 2dc_0)w(t) + \|b(t)\|_{\ell^p_\Theta}.$$

Let $\bar{C} = \bar{\delta} + 2dc_0$, and using Grönwall's inequality we obtain :

$$\|u_k(t)\|_{\ell^p_\Theta} \leq e^{\bar{C}t}\big(\|u^0_k\|_{\ell^p_\Theta} + B(\bar{C},t)\big)$$

□

For any fixed $T > 0$ we introduce the following Banach space

$$X_T \equiv C\big([0,T],\ell^p_\Theta\big) \quad \|u(t,n)\|_{X_T} = \sup_{0\leq t\leq T}\|u(t)\|_{\ell^p_\Theta} < \infty.$$

Since $b \in C\big([0,\infty),\ell^p_\Theta\big)$, we have $b \in X_T$ and $u_k \in X_T$ by Lemma 4.1 for any k and $T > 0$.

Since the sequence $\{u^0_k\}$ is convergent to u^0 in ℓ^p_Θ, then $\{u^0_k\}$ is bounded in ℓ^p_Θ.

Since $B(\bar{C},t)$ and $e^{\bar{c}t}$ are continuous in t, then from Lemma 4.1 we know

$$\max_k \|u_k\|_{X_T} < \infty,$$

which means the sequence $\{u_k\}$ is bounded in X_T, for any $T > 0$.

Since $\ell^p_\Theta \subset \ell^p \subset \ell^\infty$, and $\|u\|_{\ell^\infty} \leq \|u\|_{\ell^p} \leq C\|u\|_{\ell^p_\Theta}$ for all $1 \leq p < \infty$. Then

$$\begin{aligned}\max_k \sup_{0\leq t\leq T}\sup_{n\in\mathbb{Z}^d}\big|u_k(t,n)\big| &= \max_k \sup_{0\leq t\leq T}\|u_k(t)\|_{\ell^\infty} \leq \max_k \sup_{0\leq t\leq T}\|u_k(t)\|_{\ell^p} \\ &\leq C\max_k \sup_{0\leq t\leq T}\|u_k(t)\|_{\ell^p_\Theta} = C\max_k\|u_k\|_{X_T} \equiv R(T) < \infty.\end{aligned}$$

Lemma 4.2 *Assume that assumptions (i), (ii), (iii), and (iv) are satisfied, then for any $T > 0$, there exists $\tilde{u}_T \in X_T$ such that $u_k \to \tilde{u}_T$ in X_T. Furthermore, for any $T_1 < T_2$ we have*

$$\tilde{u}_{T_2}\Big|_{[0,T_1]} = \tilde{u}_{T_1},$$

which means we can define the limit function $\tilde{u} \in C\big([0,\infty),\ell^p_\Theta\big)$ such that

$$\tilde{u}\Big|_{[0,T]} = \tilde{u}_T$$

and $u_k \to \tilde{u}$ in X_T for any $T > 0$.

Proof. Since

$$\dot{u}_k(t,n) = (-iH)u_k(t,n) + N(t,u_k(t,n)),$$
$$\dot{u}_m(t,n) = (-iH)u_m(t,n) + N(t,u_m(t,n)),$$

then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[u_m(t,n)-u_k(t,n)] &= (-iH)[u_m(t,n)-u_k(t,n)] + [N(t,u_m(t,n)) - N(t,u_k(t,n))]\\ &= (-iH)[u_m(t,n)-u_k(t,n)] + i[f(n,u_m(t,n)) - f(n,u_k(t,n))].\end{aligned}$$

Using local Lipschitz condition:

$$\max_n \Big| f(n,z_1) - f(n,z_2)\Big| \le C_R|z_1-z_2| \qquad \text{if} \quad |z_1|,\ |z_2| < R$$

and

$$\max_k \sup_{0\le t\le T} \sup_{n\in\mathbb{Z}^d} \Big|u_k(t,n)\Big| \le R(T) < \infty,$$

we have

$$\Big|f(n,u_m(t,n)) - f(n,u_k(t,n))\Big| \le C_{R(T)}\Big|u_m(t,n)-u_k(t,n)\Big| \tag{4.10}$$

uniformly with respect to n, m, k and $t \in [0,T]$.

Let

$$v_{mk}(t,n) = u_m(t,n) - u_k(t,n), \tag{4.11}$$

multiplying by $p|v_{mk}(t,n)|^{p-2}\overline{v_{mk}(t,n)}$ and taking the real part we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|v_{mk}(t,n)|^p \le\ & Re\Big[p|v_{mk}(t,n)|^{p-2}\overline{v_{mk}(t,n)}(-iHv_{mk}(t,n))\Big]\\ &+ Re\Big[p|v_{mk}(t,n)|^{p-2}\overline{v_{mk}(t,n)}(i[f(n,u_m(t,n)) - f(n,u_k(t,n))]\Big]\\ \le\ & p|v_{mk}(t,n)|^{p-2}Re\Big[\overline{v_{mk}(t,n)}(i\Delta - iW)v_{mk}(t,n)\Big] + pC_{R(T)}|v_{mk}(t,n)|^p.\end{aligned}$$

From the proof of Lemma 4.1 we have

$$\begin{aligned}Re\Big[\overline{v_{mk}(t,n)}(i\Delta - iW)v_{mk}(t,n)\Big] =\ & p\delta(n)|v_{mk}(t,n)|^p\\ &- p|v_{mk}(t,n)|^{p-2}Im\sum_{j=1}^{d}\big[v_{mk}(t,n+e_j)\overline{v_{mk}(t,n)}\\ &+ v_{mk}(t,n-e_j)\overline{v_{mk}(t,n)}\big].\end{aligned}$$

Multiplying both sides by θ_n^p we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|v_{mk}(t,n)|^p\theta_n^p \le\ & p\delta(n)\theta_n^p|v_{mk}(t,n)|^p + pC_{R(T)}|v_{mk}(t,n)|^p\theta_n^p\\ &- p|v_{mk}(t,n)|^{p-2}Im\sum_{j=1}^{d}\Big[v_{mk}(t,n+e_j)\overline{v_{mk}(t,n)}\theta_n^p\\ &+ v_{mk}(t,n-e_j)\overline{v_{mk}(t,n)}\theta_n^p\Big].\end{aligned}$$

Taking summation over $n \in \mathbb{Z}^d$ for both sides and by assumption (i) we obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v_{mk}(t)\|^p_{\ell^p_\Theta} \le\; & p\bar{\delta}\|v_{mk}(t)\|^p_{\ell^p_\Theta} + pC_{R(T)}\|v_{mk}(t)\|^p_{\ell^p_\Theta} \\ & + p\sum_{j=1}^{d}\sum_{n\in\mathbb{Z}^d}|v_{mk}(t,n)\theta_n|^{p-1}\big|v_{mk}(t,n+e_j)\theta_{n+e_j}\big|\frac{\theta_n}{\theta_{n+e_j}} \\ & + p\sum_{j=1}^{d}\sum_{n\in\mathbb{Z}^d}|v_{mk}(t,n)\theta_n|^{p-1}\big|v_{mk}(t,n-e_j)\theta_{n-e_j}\big|\frac{\theta_n}{\theta_{n-e_j}}.\end{aligned}$$

Θ is a regular weight by assumption (iv), and by Hölder inequality we obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v_{mk}(t)\|^p_{\ell^p_\Theta} &\le p\bar{\delta}\|v_{mk}(t)\|^p_{\ell^p_\Theta} + 2pdc_0\|v_{mk}(t)\|^p_{\ell^p_\Theta} + pC_{R(T)}\|v_{mk}(t)\|^p_{\ell^p_\Theta} \\ &\le p\Big(\bar{\delta} + 2dc_0 + C_{R(T)}\Big)\|v_{mk}(t)\|^p_{\ell^p_\Theta}.\end{aligned}$$

Let $\tilde{C} = \bar{\delta} + 2dc_0 + C_{R(T)}$ and $w(t) = \|v_{mk}(t)\|_{\ell^p_\Theta}$, then by the chain rule we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t) \le \tilde{C}w(t).$$

By Grönwall's inequality we obtain for any $0 \le t \le T$

$$\begin{aligned}&\|v_{mk}(t)\|_{\ell^p_\Theta} \le e^{\tilde{C}t}\|v_{mk}(0)\|_{\ell^p_\Theta}, \\ &\|u_m(t) - u_k(t)\|_{\ell^p_\Theta} \le e^{\tilde{C}t}\|u_m(0) - u_k(0)\|_{\ell^p_\Theta}, \\ &\|u_m - u_k\|_{X_T} \le \max\{1, e^{\tilde{C}T}\}\|u^0_m - u^0_k\|_{\ell^p_\Theta}.\end{aligned}$$

Therefore $\{u_k\}$ is a Cauchy sequence in X_T which implies $u_k \to \tilde{u}_T$ in X_T for some $\tilde{u}_T \in X_T$. Since for $T_2 > T_1 > 0$ we have

$$X_{T_2}\Big|_{[0,T_1]} = X_{T_1},$$

by the uniqueness of the limit of the sequence $\{u_k\}$ in X_T, we have

$$\tilde{u}_{T_2}\Big|_{[0,T_1]} = \tilde{u}_{T_1}.$$

Therefore we can define the limit function $\tilde{u} \in C\big([0,\infty), \ell^p_\Theta\big)$ such that

$$\tilde{u}\Big|_{[0,T]} = \tilde{u}_T$$

and $u_k \to \tilde{u}$ in X_T for any $T > 0$. □

Lemma 4.3 *Assume that assumptions (i), (ii), (iii), and (iv) are satisfied and* $b \in C([0,\infty), \ell^p_\Theta(\mathbb{Z}^d))$*, then for any* $T > 0$*,* $\tilde{u}$ *defined in Lemma 4.2 is the mild solution for the initial value problem (1.1) and (1.2) on* $[0,T]$*, that means* $\tilde{u}$ *satisfies*

$$\tilde{u}(t) = e^{-itH}u^0 + \int_0^t e^{-i(t-s)H}N(s,\tilde{u}(s))\,\mathrm{d}s.$$

Proof. Since for $0 \le t \le T$, and for each k we have

$$u_k(t) = e^{-itH}u_k^0 + \int_0^t e^{-i(t-s)H}N(s, u_k(s))\,\mathrm{d}s,$$

by the definition of $\tilde{u}$ in Lemma 4.2 we know

$$u_k \to \tilde{u} \quad \text{in} \quad X_T \quad \text{for any} \quad T > 0,$$

and by Lemma 3.1

$$\|e^{-itH}u_k^0 - e^{-itH}u^0\|_{\ell^p_\Theta} \le Me^{\omega t}\|u_k^0 - u^0\|_{\ell^p_\Theta},$$

which implies

$$\lim_{k\to\infty} \|e^{-itH}u_k^0 - e^{-itH}u^0\|_{X_T} \le M\max\{1, e^{\omega T}\} \lim_{k\to\infty} \|u_k^0 - u^0\|_{\ell^p_\Theta} = 0.$$

Thus letting $k \to \infty$ we obtain

$$\tilde{u}(t) = e^{-itH}u^0 + \lim_{k\to\infty}\int_0^t e^{-i(t-s)H}N(s, u_k(s))\,\mathrm{d}s.$$

Therefore in order to prove Lemma 4.3 we only need to prove

$$\lim_{k\to\infty}\int_0^t e^{-i(t-s)H}N(s, u_k(s))\,\mathrm{d}s = \int_0^t e^{-i(t-s)H}N(s, \tilde{u}(s))\,\mathrm{d}s.$$

To this end we notice that

$$N(s, u_k(s)) - N(s, \tilde{u}(s)) = i[f(n, u_k(s)) - f(n, \tilde{u}(s))]$$

and for any $T > 0$, there exists a constant $C_{R(T)} > 0$ (see the proofs in previous lemmas) such that for any $0 \le s \le T$

$$\Big|f(n, u_k(s)) - f(n, \tilde{u}(s))\Big| \le C_{R(T)}|u_k(s) - \tilde{u}(s)|.$$

Thus for any $T > 0$,

$$\begin{aligned}
&\max_{0\le t\le T}\Big\|\int_0^t e^{-i(t-s)H}N(s, u_k(s))\,\mathrm{d}s - \int_0^t e^{-i(t-s)H}N(s, \tilde{u}(s))\,\mathrm{d}s\Big\|_{\ell^p_\Theta}\\
&= \max_{0\le t\le T}\Big\|\int_0^t e^{-i(t-s)H}i[f(n, u_k(s)) - f(n, \tilde{u}(s))]\,\mathrm{d}s\Big\|_{\ell^p_\Theta}\\
&\le \max_{0\le t\le T}\int_0^t Me^{\omega(t-s)}\|f(n, u_k(s)) - f(n, \tilde{u}(s))\|_{\ell^p_\Theta}\,\mathrm{d}s\\
&\le M\max\{1, e^{\omega T}\}C_{R(T)}\int_0^T \|u_k(s) - \tilde{u}(s)\|_{\ell^p_\Theta}\,\mathrm{d}s\\
&\le M\max\{1, e^{\omega T}\}C_{R(T)}T\sup_{0\le s\le T}\|u_k(s) - \tilde{u}(s)\|_{\ell^p_\Theta}\\
&\to 0 \ \text{ as } \ k \to \infty.
\end{aligned}$$

Therefore $\tilde{u}$ satisfies the following equation

$$\tilde{u} = e^{-itH}u_0 + \int_0^t e^{-i(t-s)H}N(s, \tilde{u}(s))\,\mathrm{d}s,$$

which implies $\tilde{u}(0) = u^0$. Therefore, $\tilde{u}$ is a mild solution to the initial value problem (1.1) and (1.2) for any $T > 0$. □

Proof of Theorem 4.2

(1) From Lemma 4.3, we know $\tilde{u}$ is a mild solution to initial value problem(1.1) and (1.2) on $[0,T]$ for any $T>0$. By Theorem 3.1 there is a unique local mild solution on $[0,\tau_{\max})$. By the uniqueness of the local solution, we have $\tilde{u}(t)=u(t)$ on $[0,T]$ which implies $T<\tau_{\max}$. Therefore, $\tau_{\max}=\infty$ since $T>0$ is arbitrary. By Lemma 4.1, we have for each k

$$\|u_k(t)\|_{\ell^p_\Theta} \le e^{\bar{C}t}\Big(\|u_k^0\|_{\ell^p_\Theta} + B(\bar{C},t)\Big),$$

where $\bar{C}=\bar{\delta}+2dc_0$. Let $k\to\infty$ we obtain (4.3).

(2) The proof follows from a similar reasoning in [25].

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