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A NOTE ON A NEW CLASS OF (ω, c)-PERIODIC FUNCTIONS

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Abstract. In this paper, we introduce in a natural way a new class of (ω, c)-periodic functions and investigate some of their properties. We prove that the set of such functions is a Banach space under an appropriate norm and provide a composition result. We give also several examples of such functions.

Keywords: Banach space, (ω, c)-periodic function, $C^{(n)} - (\omega, c)$-periodic functions.

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1 Introduction

In 2018, Alvarez *et al.* [2] introduced the concept of (ω, c)-periodic functions by observing the behavior of any complex values solution $x(t)$ of the so-called Matthieu's equations

$$x'' + ax = 2q\cos(2t)x$$

which fulfills the equality $x(t+\omega) = cx(t)$, where $t \in \mathbb{R}, \omega > 0$ and c is a non zero complex number.

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This theory has rapidly attracted several researchers including Abadias *et al.* [1], Baillon *et al.* [3], Mophou and N'Guérékata [9]. Other contributions can be found in the following papers: [1, 2, 5–7, 9, 15] and references therein.

In their papers [6], Larrouy and N'Guérékata investigated more interesting properties of (ω, c)-periodic and asymptotically (ω, c)-periodic functions and studied solutions of such types of the following equations:

$$\begin{cases} {}^cD_t^{\alpha}u(t) = Au(t) + {}^cD_t^{\alpha-1}f(t, u(t)), \quad 1 < \alpha < 2, \quad t \in \mathbb{R}, \\ u(0) = 0 \end{cases} \tag{1.1}$$

and

$$\begin{cases} {}^cD_t^{\alpha}u(t) = Au(t) + {}^cD_t^{\alpha-1}f(t, u(t-h)), \quad 1 < \alpha < 2, \quad t, h \in \mathbb{R}_+, \\ u(0) = 0 \end{cases} \tag{1.2}$$

where ${}^cD_t^{\alpha}(\cdot)$ $(1 < \alpha < 2)$ stands for the Caputo derivative and A is a linear densely defined operator of sectorial type on a complex Banach space $\mathbb{X}$, and the function $f(t, u)$ is (ω, c)-periodic or asymptotically (ω, c)-periodic with respect to the first variable.

In [9], Mophou and N'Guérékata investigated further properties of the new concept of (ω, c)-periodic functions; then they applied the results to study the existence of (ω, c)-periodic mild solutions of the fractional differential equations

$$D_t^{\alpha}(u(t) - F_1(t, u(t))) = A(u(t) - F_1(t, u(t))) + D_t^{\alpha-1}F_2(t, u(t)),\ t \in \mathbb{R},$$

where $1 < \alpha < 2,\ A : D(A) \subseteq X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space X, $F_1, F_2 : \mathbb{R} \times X \to X$ are two (ω, c)-periodic functions satisfying suitable conditions in the second variable. The fractional derivative is understood in the sense of Riemann-Liouville.

Recently, Khallali *et al.* studied several classes of (ω, c)-almost periodic type functions with one and two variables and applied their results to some evolution equations.

On the other hand, Baillon *et al.* discussed the concept of $C^{(n)}$-almost periodic functions in their paper [3]. This class contains functions which are almost periodic as well as their successive derivatives up to the order n.

Motivated by the papers above, among many others, we will introduce a new class of functions called $C^{(n)} - (\omega, c)$-periodic functions and study some of their properties. We establish that the collections of all such functions turns out to be a Banach space, equipped with an appropriate norm and provide some examples of such functions.

2 Preliminaries

In this paper $\mathbb{X} = (\mathbb{X}, \|\cdot\|)$ will denote a complex Banach space and $\mathcal{B}(\mathbb{X})$ the space of all bounded linear operators $\mathbb{X} \to \mathbb{X}$, endowed with the uniform operator topology.

Definition 2.1 *([2]) Let $\omega > 0$ and c be a non-zero complex number. A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be (ω, c)-periodic if $f(t + \omega) = cf(t)$, $\forall t \in \mathbb{R}$. In this case ω is called a c-period of the function f.*

We denote by $P_{(\omega,c)}(\mathbb{R},\mathbb{X})$ the set of all (ω,c)-periodic functions from $\mathbb{R}$ to $\mathbb{X}$. When $c=1$, we write $P_{\omega}(\mathbb{R},\mathbb{X})$ instead of $P_{(\omega,1)}(\mathbb{R},\mathbb{X})$ and we say that f is ω-periodic. When $c=-1$ we obtain the class of antiperiodic functions [11].

Using the principal branch of the complex Logarithm, $c^{\frac{t}{\omega}}$ is defined as $c^{\frac{t}{\omega}} := \exp(\frac{t}{\omega}Log(c)) = c^{\wedge}(t)$ and we will use the notation $c^{\wedge}(t) := c^{\frac{t}{\omega}}$.

In [2], Alvarez *et al.* gave a useful description of the space $P_{(\omega,c)}(\mathbb{R},\mathbb{X})$. That is $P_{(\omega,c)}(\mathbb{R},\mathbb{X})$ is a translation-invariant subspace over $\mathbb{C}$ of $C(\mathbb{R},\mathbb{X})$. Then for any fixed $h>0$ and $u\in P_{(\omega,c)}(\mathbb{R},\mathbb{X})$, we have $u_h(\cdot) := u(\cdot - h) \in P_{(\omega,c)}(\mathbb{R},\mathbb{X})$.

Proposition 2.1 *([2]) Let $f\in C(\mathbb{R},\mathbb{X})$. Then, $f\in P_{(\omega,c)}(\mathbb{R},\mathbb{X})$ if and only if $f(t) = c^{\frac{t}{\omega}}u(t)$, $u(t)\in P_{\omega}(\mathbb{R},\mathbb{X})$.*

Theorem 2.1 *([9]) Let $f\in P_{(\omega,c)}(\mathbb{R},\mathbb{X})$ and $A\in\mathcal{B}(\mathbb{X})$. Then $Af\in P_{(\omega,c)}(\mathbb{R},\mathbb{X})$.*

We state and prove this basic property which follows naturally from the (ω,c)-periodicity definition. It will be very useful in the sequel.

Theorem 2.2 *([2]) $P_{(\omega,c)}(\mathbb{R},\mathbb{X})$ is a Banach space with the norm*

$$\|f\|_{\omega,c} := \sup_{t\in[0,\omega]} \| |c|^{\wedge}(-t)f(t)\|.$$

We note that if $f\in P_{(\omega,c)}(\mathbb{R},\mathbb{X})$, then $\|f\|_{\omega,c}<\infty$ and we say that f is c-bounded. The use of $\|f\|_{\omega,c}$ instead of $\|f\|_{\infty}$ will allow us to handle the (ω,c)-periodicity properties of f (see [2] for more details).

3 $C^{(n)}-(\omega,c)$-periodic functions

Let $f:\mathbb{R}\longrightarrow\mathbb{X}$. Denote by $C^{(n)}(\mathbb{R},\mathbb{X})$ (briefly $C^{(n)}(\mathbb{X})$) the space of all functions $\mathbb{R}\longrightarrow\mathbb{X}$ which have a continuous n^{th} derivative on $\mathbb{R}$. Let $C_b^{(n)}(\mathbb{R},\mathbb{X})$ (briefly $C_b^{(n)}(\mathbb{X})$) be the subspace of $C^{(n)}(\mathbb{R},\mathbb{X})$ consisting of such functions satisfying

$$\sup_{t\in\mathbb{R}}\sum_{i=0}^{n}\|f^{(i)}(t)\| < +\infty,$$

where $f^{(i)}$ denotes the $i-th$ derivative of f and $f^{(0)}=f$. Clearly $C^{(n)}(\mathbb{X})$ turns out to be a Banach space with the norm

$$\|f\|_n = \sup_{t\in\mathbb{R}}\sum_{i=0}^{n}\|f^{(i)}(t)\|$$

(cf. for instance [3]).

3.1 Generalities

Definition 3.1 *A function $f \in C^{(n)}(\mathbb{X})$ is said to be a $C^n - (\omega, c)$-periodic if $f, f', f^{(2)}, \cdots, f^{(n)}$ are (ω, c)-periodic.*

The set of all $C^{(n)} - (\omega, c)$-periodic functions will be denoted by $P^{(n)}_{(\omega,c)}(\mathbb{R}, \mathbb{X})$,(briefly $P^{(n)}_{(\omega,c)}(\mathbb{X})$). $P^{(0)}_{(\omega,c)}(\mathbb{X}) = P_{(\omega,c)}(\mathbb{X})$, is the classical space of all (ω, c)-periodic functions.

Example 3.1 *Consider the function $f(t) = ke^{\alpha t}$, where $\alpha > 0$, k and α are non-zero complex numbers. Then f is $C^{(n)} - (\frac{1}{\alpha}$,e)-periodic for any $n = 0, 1, 2, \cdots$.*

Indeed,

$$f(t + \frac{1}{\alpha}) = ke^{\alpha(t+\frac{1}{\alpha})} = ke^{\alpha t} \cdot e = ef(t),$$

then f is $(\frac{1}{\alpha}, e)$-periodic.

Since $f'(t) = \alpha ke^{\alpha t} = \alpha f(t)$, we will have

$$f'(t + \frac{1}{\alpha}) = \alpha f(t + \frac{1}{\alpha}) = \alpha ef(t) = e\alpha f(t) = ef'(t).$$

Therefore f' is $(\frac{1}{\alpha}, e)$-periodic.

Assume that $f^{(n)}(t)$ is $(\frac{1}{\alpha}, e)$-periodic. We have $f^{(n+1)}(t) = \alpha^{n+1}ke^{\alpha t}$. Therefore

$$f^{(n+1)}(t + \frac{1}{\alpha}) = \alpha f^{(n)}(t + \frac{1}{\alpha}) = \alpha ef^{(n)}(t) = e\alpha f^{(n)}(t) = ef^{(n+1)}(t).$$

So $f^{(n+1)}$ is $(\frac{1}{\alpha}, e)$-periodic. We conclude that f is $C^{(n)} - (\frac{1}{\alpha}, e)$-periodic for any $n = 0, 1, 2, \cdots$.

Example 3.2 *Consider the function $g(x) = \cos(\alpha x)$, where α is a positive real number. Then g is $C^{(n)} - (\frac{2\pi}{\alpha}, 1)$-periodic for any $n = 0, 1, 2, \cdots$.*

Indeed, $g(x) = \cos(\alpha x)$, thus

$$g(x + \frac{2\pi}{\alpha}) = \cos\left(\alpha(x + \frac{2\pi}{\alpha})\right) = g(x),$$

therefore g is $(\frac{2\pi}{\alpha}, 1)$-periodic. Since $g'(x) = -\alpha \sin(\alpha x)$ we have

$$g'(x + \frac{2\pi}{\alpha}) = -\alpha \sin\left(\alpha(x + \frac{2\pi}{\alpha})\right) = g'(x),$$

therefore g' is $(\frac{2\pi}{\alpha}, 1)$-periodic. Similarly, $g^{(n)}(x) = \alpha^n \cos(\alpha x + n \cdot \frac{\pi}{2})$, thus

$$g^{(n)}(x + \frac{2\pi}{\alpha}) = \alpha^n \cos\left(\alpha(x + \frac{2\pi}{\alpha}) + n \cdot \frac{\pi}{2}\right) = g^{(n)}(x),$$

therefore $g^{(n)}$ is $(\frac{2\pi}{\alpha}, 1)$-periodic. We conclude that g is $C^{(n)} - (\frac{2\pi}{\alpha}, 1)$-periodic for any $n = 0, 1, 2, \cdots$.

Proposition 3.1 *The set $P^{(n)}_{(\omega,c)}(\mathbb{X})$ equipped with the $\|\cdot\|^n_{(\omega,c)}$ norm below*

$$\|f\|^n_{(\omega,c)} = \sum_{i=0}^{n} \|f^{(i)}\|_{(\omega,c)},$$

turns out to be a Banach space.

Proof. Consider $f, g \in P^{(n)}_{(\omega,c)}(\mathbb{X})$, and $\lambda \in \mathbb{C}^*$.

i)

$$\begin{aligned}(f+g)^{(n)}(t+\omega) &= (f^{(n)} + g^{(n)})(t+\omega) = f^{(n)}(t+\omega) + g^{(n)}(t+\omega)\\ &= cf^{(n)}(t) + cg^{(n)}(t) = c(f^{(n)} + g^{(n)})(t)\\ &= c(f+g)^{(n)}(t),\end{aligned}$$

therefore $f+g$ is $C^{(n)} - (\omega, c)$ -periodic.

ii)

$$\begin{aligned}(\lambda f)^{(n)}(t+\omega) &= (\lambda f^{(n)})(t+\omega) = \lambda f^{(n)}(t+\omega) = \lambda c f^{(n)}(t)\\ &= c(\lambda f^{(n)})(t) = c(\lambda f)^{(n)}(t),\end{aligned}$$

therefore λf is $C^{(n)} - (\omega, c)$ -periodic.

$P^{(n)}_{(\omega,c)}(\mathbb{X})$ is a vector subspace.

Let's show that $\|\cdot\|^n_{(\omega,c)}$ is a norm.

i)

$$\begin{aligned}\|f\|^n_{(\omega,c)} = 0 &\Leftrightarrow \sum_{i=0}^{n} \|f^{(i)}\|_{(\omega,c)} = 0 \Leftrightarrow \|f^{(i)}\|_{(\omega,c)} = 0 \qquad \forall i \in [0, n]\\ &\Leftrightarrow f^{(i)}(t) = 0 \quad \forall i \in [0, n] \qquad \Leftrightarrow f(t) = 0 \qquad \forall t \in \mathbb{R}.\end{aligned}$$

ii) For $f \in P^{(n)}_{(\omega,c)}(\mathbb{X})$, and $\lambda \in \mathbb{C}^*$,

$$\|\lambda f\|^n_{(\omega,c)} = \sum_{i=0}^{n} \|\lambda f^{(i)}\|_{(\omega,c)} = \sum_{i=0}^{n} |\lambda| \|f^{(i)}\|_{(\omega,c)} = |\lambda| \sum_{i=0}^{n} \|f^{(i)}\|_{(\omega,c)} = |\lambda| \|f\|^n_{(\omega,c)}.$$

iii) For $f, g \in P^{(n)}_{(\omega,c)}(\mathbb{X})$,

$$\|f+g\|^n_{(\omega,c)} = \sum_{i=0}^{n} \|f^{(i)} + g^{(i)}\|_{(\omega,c)} \leqslant \sum_{i=0}^{n} \|f^{(i)}\|_{(\omega,c)} + \sum_{i=0}^{n} \|g^{(i)}\|_{(\omega,c)} = \|f\|^n_{(\omega,c)} + \|g\|^n_{(\omega,c)}.$$

Let us now show that $P^{(n)}_{(\omega,c)}(\mathbb{X})$ is complete for the norm $\|\cdot\|^n_{(\omega,c)}$. Consider $(f_m)_{m\in\mathbb{N}}$ the Cauchy sequence for the functions $C^{(n)}-(\omega,c)$ -periodic and suppose that $f_m \longrightarrow f$ when $m \longrightarrow \infty$.

$$\forall\epsilon>0,\ \exists m_0\in\mathbb{N},\ \forall m,p\geq m_0,\quad \|f_m-f_p\|^n_{(\omega,c)}<\epsilon,$$

but

$$\|f_m-f_p\|^n_{(\omega,c)}=\sum_{i=0}^{n}\|f_m^{(i)}-f_p^{(i)}\|_{(\omega,c)}.$$

So, $\forall i$ we have, $\|f_m^{(i)}-f_p^{(i)}\|_{(\omega,c)}<\epsilon$. Then $\forall i,\ f_m^{(i)}$ is a sequence of Cauchy in $P_{(\omega,c)}(\mathbb{X})$ which is a Banach space, which implies $f_m^{(i)}\rightarrow f^{(i)}$ and $f^{(i)}\in P_{(\omega,c)}(\mathbb{X}),\ \forall i=0,1,2,...,n$. Since $f^{(i)}\in P_{(\omega,c)}(\mathbb{X}),\ \forall i\in[0,n]$, then $f\in P^{(n)}_{(\omega,c)}(\mathbb{X})$. □

Theorem 3.1 *Let $f\in P^{(n)}_{(\omega,c_1)}(\mathbb{X})$ and $g\in P^{(n)}_{(\omega,c_2)}(\mathbb{X})$ then $(fg)\in P^{(n)}_{(\omega,c_1\cdot c_2)}(\mathbb{X})$.*

Proof.

$$(fg)(t+\omega)=f(t+\omega)\cdot g(t+\omega)=c_1f(t)\cdot c_2g(t)=c_1\cdot c_2f(t)g(t)=c_1\cdot c_2(fg)(t).$$

Therefore (fg) is $(\omega,c_1\cdot c_2)$-periodic.

It is easy to show that

$$(fg)'(t+\omega)=c_1\cdot c_2(f'g+fg')(t)=c_1\cdot c_2(fg)'(t),$$

then $(fg)'$ is also $(\omega,c_1\cdot c_2)$-periodic. Suppose that $(fg)^{(n)}$ is $(\omega,c_1\cdot c_2)$-periodic and let us prove that $(fg)^{(n+1)}$ is also $(\omega,c_1\cdot c_2)$-periodic.

Set $h=(fg)^{(n)}$ then $h'=(fg)^{(n+1)}$, if $h=(fg)^{(n)}$ is $(\omega,c_1\cdot c_2)$ then $h'=(fg)^{(n+1)}$ is also $(\omega,c_1\cdot c_2)$-periodic.

$$\begin{aligned}
h'(t)&=\lim_{\psi\to0}\frac{h(t+\psi)-h(t)}{\psi},\\
h'(t+\omega)&=\lim_{\psi\to0}\frac{h(t+\omega+\psi)-h(t+\omega)}{\psi}\\
&=\lim_{\psi\to0}\frac{c_1\cdot c_2h(t+\psi)-c_1\cdot c_2h(t)}{\psi}\\
&=c_1.c_2\lim_{\psi\to0}\frac{h(t+\psi)-h(t)}{\psi},\\
h'(t+\omega)&=c_1\cdot c_2h'(t),
\end{aligned}$$

$h'=(fg)^{(n+1)}$ is $(\omega,c_1\cdot c_2)$-periodic. Then $(fg)\in P^{(n)}_{(\omega,c_1\cdot c_2)}(\mathbb{X})$. □

In particular cases:

(a) For $c_1=c_2=c$, we obtain $(fg)\in P^{(n)}_{(\omega,c^2)}(\mathbb{X})$.

(b) for $g=f$ we have the same result.

3.2 Composition result

Let f be a continuous and bounded function. We recall that, the Nemytskii operator of superposition is defined by:

$$N_F(\varphi)(t) = F(t, \varphi(t)),$$

we have the following result:

Theorem 3.2 *Let $f \in C^{(n)}(\mathbb{R} \times \mathbb{X})$, then the following assertions are equivalent:*

(i) $\forall \varphi \in P^{(n)}_{(\omega,c)}(\mathbb{X}), \qquad N_f(\varphi) \in P^{(n)}_{(\omega,c)}(\mathbb{X});$

(ii) $\forall (t, x) \in \mathbb{R} \times \mathbb{X}, \qquad f^{(n)}(t + w, cx) = cf^{(n)}(t, x).$

Before establishing the proof of this theorem, we will prove the following very useful lemma.

Lemma 3.1 $\forall (t, x) \in \mathbb{R} \times \mathbb{X}, \exists \varphi \in P^{(n)}_{(\omega,c)}(\mathbb{X}) : \varphi(t) = x.$

Proof. Let $\psi : \mathbb{R} \longrightarrow \mathbb{X}$ be the function defined by: $\psi(t) = c^{\frac{t}{w}}$, we have $\psi \in P^{(n)}_{(\omega,c)}(\mathbb{X})$ and $\psi(0) = 1$. Set $\varphi(s) = \psi(s - t)x = c^{\frac{s-t}{w}} \cdot x = xe^{\frac{s-t}{w} Ln(c)}$, where $Ln(\cdot)$ stands for the principal value of the complex Logarithm. Then $\varphi \in P^{(n)}_{(\omega,c)}(\mathbb{X})$ and $\varphi(t) = x$. Indeed $\varphi'(s) = \frac{Ln(c)}{w}\psi(s-t)x$, $\varphi''(s) = \left(\frac{Ln(c)}{w}\right)^2 \psi(s - t)x$, step by step we have, $\varphi^{(n)}(s) = \left(\frac{Ln(c)}{w}\right)^n \psi(s - t)x$, in addition,

$$\varphi^{(n)}(s + w) = \left(\frac{Ln(c)}{w}\right)^n \psi(s + w - t)x = \left(\frac{Ln(c)}{w}\right)^n c\psi(s - t)x = c\varphi^{(n)}(s),$$

we have just shown that $\varphi \in P^{(n)}_{(\omega,c)}(\mathbb{X})$. □

Let us now prove Theorem 3.2.

Proof. $(i) \Rightarrow (ii)$ Suppose that N_f is $C^{(n)} - (\omega, c)$ -periodic ($\forall n = 1, 2, \cdots$). So, $N_f^{(n)}$ is (ω, c) -periodic. We have

$$N_f^{(n)}(\varphi)(t + w) = cN_f^{(n)}(\varphi)(t),$$
$$f^{(n)}(t + w, \varphi(t + w)) = cf^{(n)}(t, \varphi(t)),$$
$$f^{(n)}(t + w, c\varphi(t)) = cf^{(n)}(t, \varphi(t)),$$

if we set $x = \varphi(t)$ then we get $f^{(n)}(t + w, cx) = cf^{(n)}(t, x)$.

Let us now prove $(ii) \Rightarrow (i)$. Suppose that $\forall (t, x) \in \mathbb{R} \times \mathbb{X}$, $f^{(n)}(t + w, cx) = cf^{(n)}(t, x)$, by the lemma above $\forall (t, x) \in \mathbb{R} \times \mathbb{X}, \exists \varphi \in P^{(n)}_{(\omega,c)}(\mathbb{X})$ such that $x = \varphi(t)$ we have

$$f^{(n)}(t + w, c\varphi(t)) = cf^{(n)}(t, \varphi(t)),$$
$$f^{(n)}(t + w, \varphi(t + w)) = cf^{(n)}(t, \varphi(t)),$$
$$N_f^{(n)}(\varphi)(t + w) = cN_f^{(n)}(\varphi)(t).$$

Then $N_f^{(n)}(\varphi) \in P_{(\omega,c)}(\mathbb{X})$ and $N_f(\varphi)$ is $C^{(n)} - (\omega, c)$ -periodic. The proof is completed. □

Example 3.3 *Consider the heat equation*

$$u_t(x,t) = u_{xx}(x,t),$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. *Consider continuous and bounded regular solution* $u(x,t)$ *with* $u(x,0) = f(x)$. *Then it is known that*

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4t}} f(s)ds.$$

Fix $t_0 \in \mathbb{R}_+$ *and suppose that* $f(x)$ *is* $C^{(n)} - (\omega,c)$*-periodic. Then* $u(x,t_0)$ *is also* $C^{(n)} - (\omega,c)$*-periodic. Indeed, we can set,* $u(x,t_0) = f * h$ *where* $h(x) = \mathbb{I}_{[0,+\infty)} a_{t_0}(x)$ *and* $a_{t_0}(x) = \frac{1}{2\sqrt{\pi t_0}} e^{-\frac{x^2}{4t_0}}$ *then* $u(x,t_0) \in P^{(n)}_{(\omega,c)}(\mathbb{X})$ *according to Theorem 3.2 above.*

Example 3.4 *Let* f *be* $C^{(n)} - (\omega,c)$*-periodic. Then the function* $F_a(t) := \int_t^{t+a} f(s)ds$ *is* $C^{(n)} - (\omega,c)$*-periodic.*

The proof is easy as a combination of [9, Theorem 2.7], [7, Theorem 2.14] and an appropriate change of variable f.

Indeed, $F_a(t+w) := \int_{t+w}^{t+w+a} f(s)ds$. *Using the change of variable* $s = \sigma + w, ds = d\sigma$ *we obtain*

$$F_a(t+w) = \int_t^{t+a} f(\sigma + w)d\sigma = \int_t^{t+a} cf(\sigma)d\sigma = cF_a(t).$$

Then $F_a(t)$ *is* (ω,c)*-periodic. It is easy to show that*

$$\begin{aligned}
F_a'(t) &= f(t+a) - f(t),\\
F_a'(t+w) &= f(t+w+a) - f(t+w) = cf(t+a) - cf(t) = cF_a'(t),\\
F_a''(t) &= f'(t+a) - f'(t),\\
F_a''(t+w) &= f'(t+w+a) - f'(t+w) = cf'(t+a) - cf'(t) = cF_a''(t),
\end{aligned}$$

step by step we have

$$\begin{aligned}
F_a^{(n)}(t) &= f^{(n-1)}(t+a) - f^{(n-1)}(t),\\
F_a^{(n)}(t+w) &= f^{(n-1)}(t+w+a) - f^{(n-1)}(t+w) = cf^{(n-1)}(t+a) - cf^{(n-1)}(t) = cF_a^{(n)}(t).
\end{aligned}$$

Then f *is* $C^{(n)} - (\omega,c)$*-periodic* $\Rightarrow F_a(t)$ *is* $C^{(n)} - (\omega,c)$*-periodic.*

Example 3.5 *Let* $u(x,t)$ *be a regular solution in* $\mathbb{R}^2$ *of the wave equation*

$$\begin{aligned}
u_{tt}(x,t) &= u_{xx}(x,t),\\
u(x,0) &= f(x), \quad u_t(x,0) = g(x).
\end{aligned}$$

Then we have

$$u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds.$$

If we assume that f, g *are* $C^{(n)} - (\omega,c)$*-periodic, then for* $t_0 \in \mathbb{R}_+$, $u(x,t_0)$ *will be also* $C^{(n)} - (\omega,c)$*-periodic.*

Indeed,

$$u(x+w,t_0) = \frac{1}{2}[f(x+w+t_0) + f(x+w-t_0)] + \frac{1}{2}\int_{x+w-t_0}^{x+w+t_0} g(s)ds.$$

Using the change of variable $s = \sigma + w, ds = d\sigma$ we obtain

$$\begin{aligned} u(x+w,t_0) &= \frac{1}{2}[cf(x+t_0) + cf(x-t_0)] + \frac{1}{2}\int_{x-t_0}^{x+t_0} g(\sigma+w)d\sigma \\ &= \frac{1}{2}c[f(x+t_0) + f(x-t_0)] + \frac{1}{2}c\int_{x-t_0}^{x+t_0} g(\sigma)d\sigma \\ &= cu(x,t_0). \end{aligned}$$

Then $u(x,t_0)$ is (ω, c)-periodic. Similarly,

$$\begin{aligned} u'(x,t_0) &= \frac{1}{2}\left[f'(x+t_0) + f'(x-t_0)\right] + \frac{1}{2}\left[g(x+t_0) - g(x-t_0)\right], \\ u'(x+w,t_0) &= \frac{1}{2}\left[f'(x+w+t_0) + f'(x+w-t_0)\right] + \frac{1}{2}\left[g(x+w+t_0) - g(x+w-t_0)\right] \\ &= cu'(x,t_0), \\ &\vdots \\ u^{(n)}(x,t_0) &= \frac{1}{2}\left[f^{(n)}(x+t_0) + f^{(n)}(x-t_0)\right] + \frac{1}{2}\left[g^{(n-1)}(x+t_0) - g^{(n-1)}(x-t_0)\right], \\ u^{(n)}(x+w,t_0) &= cu^{(n)}(x,t_0). \end{aligned}$$

Then $u(x,t_0)$ is $C^{(n)} - (\omega, c)$-periodic.

4 Conclusion

In this note, we introduce in a natural way a new class of (ω, c)-periodic functions which includes the one presented in the pioneer work [1] and study some of their elementary properties. Our next step is to apply these results in studying the long term behavior of mild solutions to some abstract and evolution equations.

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