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FRACTIONAL PANTOGRAPH DIFFERENTIAL EQUATIONS WITH PERIODIC CONDITIONS VIA Ψ-CAPUTO DERIVATIVE

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Abstract. In this paper, we present the existence and uniqueness criteria of the solutions for a wide class of nonlinear fractional pantograph differential equations involving Ψ-Caputo fractional derivative supplemented with periodic conditions. The main tools of our study include Mawhin's coincidence theory. An example is constructed to illustrate the application of the obtained findings.

Keywords: Coincidence degree theory, existence, uniqueness, Ψ-Caputo fractional derivative.

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1 Introduction

Fractional differential equations play a significant role in various fields of science such as physics, mechanics and engineering (see [19, 20, 28]). Recently, many scholars have explored initial and/or boundary value problems for different types of fractional differential equations. Some results on this subject can be found in [1, 2, 10, 11, 12, 13, 14, 16, 17] and the references therein.

Various definitions of fractional derivatives and integrals operators have been proposed by several mathematicians and engineers [4, 25, 30]. Recently, R. Almeida [5] introduced a new kind of fractional derivative; namely, the so called Ψ-Caputo fractional derivative. We cite some recent works in which the authors studied some nonlinear class of fractional differential equations involving this derivative [6, 7, 15].

Fractional delay differential equations, particularly those of pantograph type, motivated several mathematicians, physicists and engineers to study them due to the variety of their applications in physics and engineering [3, 9, 24, 27]. In 2017, Jalilian and Ghasmi [21] studied an initial value problem of nonlinear fractional integro-differential equation of pantograph type of the form:

$$\begin{cases} {}^cD^{\alpha}u(t) = f\left(t, u(t), u(pt)\right) + \int_0^{qt} g_1(t,s,u(s))\,\mathrm{d}s + \int_0^{t} g_2(t,s,u(s))\,\mathrm{d}s, \quad t \in [0,T], \\ u(0) = u_0, \end{cases}$$

where $0 < p, q < 1$. The symbol ${}^cD^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha \in (0,1]$. They obtained their results by using a fixed point approach.

In [29], by applying the fixed point theory, the authors studied the following nonlinear fractional pantograph equation with nonlocal boundary conditions

$$\begin{cases} {}^cD^{w,\Psi}u(t) = f\left(t, u(t), u(\eta t)\right), \quad t \in [0,T],\ \eta \in (0,1), \\ au(0) + bu(T) = c, \end{cases}$$

where ${}^cD^{w,\Psi}$ is the Ψ-Caputo factional derivative of order $0 < w < 1$ and a, b, c are real constants with $a + b \neq 0$. However, if $a + b = 0$, they have no results by this techniques.

In [8], the authors discussed the existence and uniqueness of solutions for the following equation with nonlocal conditions

$$\begin{cases} {}_cD^{\alpha;\rho}y(t) = f\left(t, y(t), y(pt)\right) + g\left(t, y(t), y((1-p)t)\right), & t \in [0,T], \\ y(0) = I^{\beta}y(\xi), & 0 < \xi < T, \end{cases}$$

where ${}_cD^{\alpha;\rho}$ denotes the Katugampola fractional derivative in Caputo sense of order $\alpha \in (0,1]$, $p \in (0,1)$, $\rho > 0$, and I^{β} is the integral operator of order $\beta > 0$.

Motivated by the aforesaid research and some well-known results on fractional pantograph differential equations, this research work investigates the existence and uniqueness results for the nonlinear fractional pantograph differential equations involving Ψ-Caputo derivative operator supplemented with periodic conditions of the form:

$$ {}^c\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{u}(\xi) = \mathcal{F}\left(\xi, \mathfrak{u}(\xi), \mathfrak{u}(p\xi)\right) + \mathcal{G}\left(\xi, \mathfrak{u}(\xi), \mathfrak{u}((1-p)\xi)\right),\ \xi \in \mathfrak{J} := [0, \mathfrak{b}], \tag{1.1}$$

$$\mathfrak{u}(0) = \mathfrak{u}(\mathfrak{b}), \tag{1.2}$$

where ${}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}$ denotes the Ψ-Caputo fractional derivative of order $0 < \alpha < 1$, $p \in (0,1)$, and $\mathcal{F}, \mathcal{G} \colon \mathfrak{J} \times \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ are given continuous functions.

The present paper is organized as follows. In Section 2, we recall some basic notions and essential preliminary results that will be used in the proofs of our main results. In Section 3, the existence and uniqueness of periodic solutions for the problem (1.1)–(1.2) are obtained via Mawhin's coincidence theory. Finally, an appropriate example is given in Section 4 to illustrate the benefit of our main findings.

2 Basic concepts

We consider the spaces $C(\mathfrak{J}, \mathfrak{R})$ and $C^{\mathfrak{m}}(\mathfrak{J}, \mathfrak{R})$ of continuous and $\mathfrak{m}$ times continuously differentiable functions on $\mathfrak{J}$, respectively. We endow $C(\mathfrak{J}, \mathfrak{R})$ with the supremum norm $\|\cdot\|_\infty$.

Definition 2.1 ([5]) *Let $\mathfrak{J} = [0, \mathfrak{b}]$, where $0 < \mathfrak{b} < +\infty$, be a finite or infinite interval and let $\alpha > 0$. Moreover, let $\mathfrak{u}$ be an integrable function defined on $\mathfrak{J}$ and let $\Psi \in C^1(\mathfrak{J}, \mathfrak{R})$ be an increasing and positive function such that $\Psi'(\xi) \neq 0$ for all $\xi \in \mathfrak{J}$. Fractional integrals and fractional derivatives of a function $\mathfrak{u}$ with respect to another function Ψ are defined as follows*:

$$\mathfrak{I}_{0^+}^{\alpha;\Psi}\mathfrak{u}(\xi) := \frac{1}{\Gamma(\alpha)} \int_0^\xi \Psi'(s)(\Psi(\xi) - \Psi(s))^{\alpha-1}\mathfrak{u}(s)\,\mathrm{d}s$$

and

$$\begin{aligned}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{u}(\xi) &:= \left(\frac{1}{\Psi'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^{\mathfrak{n}} \mathfrak{I}_{0^+}^{\mathfrak{n}-\alpha;\Psi}\mathfrak{u}(\xi) \\ &= \frac{1}{\Gamma(\mathfrak{n}-\alpha)}\left(\frac{1}{\Psi'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^{\mathfrak{n}} \int_0^\xi \Psi'(s)(\Psi(\xi) - \Psi(s))^{\mathfrak{n}-\alpha-1}\mathfrak{u}(s)\,\mathrm{d}s,\end{aligned}$$

respectively, where $\mathfrak{n} = [\alpha] + 1$.

Lemma 2.2 ([5]) *Let $\alpha > 0$ and $\beta > 0$. Then, we have*

$$\mathfrak{I}_{0^+}^{\alpha;\Psi}\mathfrak{I}_{0^+}^{\beta;\Psi}\mathfrak{u}(\xi) = \mathfrak{I}_{0^+}^{\alpha+\beta;\Psi}\mathfrak{u}(\xi) \text{ for all } \xi \in \mathfrak{J}.$$

Lemma 2.3 ([22]) *Let $\alpha > 0$, $\rho > 0$ and $\xi \in \mathfrak{J}$. If $\mathfrak{u}(\xi) = (\Psi(\xi) - \Psi(0))^{\rho-1}$, then*

$$\mathfrak{I}_{0^+}^{\alpha;\Psi}\mathfrak{u}(\xi) = \frac{\Gamma(\rho)}{\Gamma(\alpha+\rho)}\left(\Psi(\xi) - \Psi(0)\right)^{\alpha+\rho-1}.$$

Definition 2.4 ([5]) *Let $\mathfrak{n} - 1 < \alpha < \mathfrak{n}$ with $\mathfrak{n} \in \mathbb{N}$ and let $\mathfrak{u}, \Psi \in C^n(\mathfrak{J}, \mathfrak{R})$ be two functions such that Ψ is increasing and positive with $\Psi'(\xi) \neq 0$ for any $\xi \in \mathfrak{J}$. The left Ψ-Caputo fractional derivative of $\mathfrak{u}$ of order α is given by*

$${}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{u}(\xi) := \mathfrak{I}_{0^+}^{\mathfrak{n}-\alpha;\Psi}\left(\frac{1}{\Psi'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^{\mathfrak{n}}\mathfrak{u}(\xi),\ \xi \in \mathfrak{J}.$$

In particular, when $0 < \alpha < 1$, we have

$${}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{u}(\xi) = \frac{1}{\Gamma(1-\alpha)}\int_0^\xi (\Psi(\xi) - \Psi(s))^{-\alpha}\,\mathfrak{u}'(s)\,\mathrm{d}s,\ \xi \in \mathfrak{J}.$$

Theorem 2.5 ([5]) *If* $\mathfrak{u} \in C^{\mathfrak{n}}(\mathfrak{J}, \mathfrak{R})$ *and* $\mathfrak{n} - 1 < \alpha < \mathfrak{n}$*, then*

$$\mathfrak{I}_{0^+}^{\alpha;\Psi}\, {}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{u}(\xi) = \mathfrak{u}(\xi) - \sum_{k=0}^{\mathfrak{n}-1} \frac{(\Psi(\xi) - \Psi(0))^k}{k!} \left(\frac{1}{\Psi'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^k \mathfrak{u}(0).$$

In particular, when $0 < \alpha < 1$*, we have*

$$\mathfrak{I}_{0^+}^{\alpha;\Psi}\, {}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{u}(\xi) = \mathfrak{u}(\xi) - \mathfrak{u}(0).$$

Theorem 2.6 ([5]) *Let* $\mathfrak{u} \in C^1(\mathfrak{J}, \mathfrak{R})$ *and* $\alpha > 0$*. We have*

$${}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{I}_{0^+}^{\alpha;\Psi}\mathfrak{u}(\xi) = \mathfrak{u}(\xi).$$

Theorem 2.7 ([5]) *Let* $\mathfrak{u}, \upsilon \in C^{\mathfrak{n}}(\mathfrak{J}, \mathfrak{R})$ *and* $\alpha > 0$*. Then,*

$${}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{u}(\xi) = {}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\upsilon(\xi) \text{ if and only if } \mathfrak{u}(\xi) = \upsilon(\xi) + \sum_{k=0}^{\mathfrak{n}-1} c_k \left(\Psi(\xi) - \Psi(0)\right)^k,$$

where $c_k = \dfrac{1}{k!}\left(\dfrac{1}{\Psi'(\xi)}\dfrac{\mathrm{d}}{\mathrm{d}\xi}\right)^k (\mathfrak{u} - \upsilon)(0)$.

Remark 2.8 *Let* $w \in C^{\mathfrak{n}}(\mathfrak{J}, \mathfrak{R})$ *and* $\alpha > 0$*. Then,*

$${}^{c}\mathfrak{D}_{0^+}^{\alpha,\beta;\Psi}w(\xi) = 0 \text{ if and only if } w(\xi) = \sum_{k=0}^{\mathfrak{n}-1} c_k \left(\Psi(\xi) - \Psi(0)\right)^k.$$

We will present definitions and the coincidence degree theory that are essential in the proofs of our results (see [18, 23]).

Definition 2.9 *We consider the normed spaces* $\mathcal{X}$ *and* $\mathcal{Y}$*. A Fredholm operator of index zero is a linear operator* $\mathfrak{L}\colon \operatorname{dom}(\mathfrak{L}) \subset \mathcal{X} \to \mathcal{Y}$ *such that*

(a) $\dim\ker\mathfrak{L} = \operatorname{codim}\operatorname{img}\mathfrak{L} < +\infty$.

(b) $\operatorname{img}\mathfrak{L}$ *is a closed subset of* $\mathcal{Y}$*.*

By Definition 2.9, there exist continuous projectors $\mathcal{Q}\colon \mathcal{Y} \to \mathcal{Y}$ and $\mathcal{P}\colon \mathcal{X} \to \mathcal{X}$ satisfying

$$\operatorname{img}\mathfrak{L} = \ker\mathcal{Q}, \quad \ker\mathfrak{L} = \operatorname{img}\mathcal{P}, \quad \mathcal{Y} = \operatorname{img}\mathcal{Q} \oplus \operatorname{img}\mathfrak{L}, \quad \mathcal{X} = \ker\mathcal{P} \oplus \ker\mathfrak{L}.$$

Thus, the restriction of $\mathfrak{L}$ to $\operatorname{dom}\mathfrak{L} \cap \ker\mathcal{P}$, denoted by $\mathfrak{L}_{\mathcal{P}}$, is an isomorphism onto its image.

Definition 2.10 *Let* $\Omega \subseteq \mathcal{X}$ *be a bounded set and let* $\mathfrak{L}$ *be a Fredholm operator of index zero with* $\operatorname{dom}\mathfrak{L} \cap \Omega \neq \emptyset$. *Then, the operator* $\mathcal{N}\colon \overline{\Omega} \to \mathcal{Y}$ *is said to be* $\mathfrak{L}$*-compact in* $\overline{\Omega}$ *if*

(a) *the mapping* $\mathcal{Q}\mathcal{N}\colon \overline{\Omega} \to \mathcal{Y}$ *is continuous and* $\mathcal{Q}\mathcal{N}\left(\overline{\Omega}\right) \subseteq \mathcal{Y}$ *is bounded,*

(b) *the mapping* $(\mathfrak{L}_{\mathcal{P}})^{-1}(\mathrm{Id} - \mathcal{Q})\mathcal{N}\colon \overline{\Omega} \to \mathcal{X}$ *is completely continuous.*

Lemma 2.11 ([26]) *Let* $\mathcal{X}$, $\mathcal{Y}$ *be Banach spaces,* $\Omega \subset \mathcal{X}$ *a bounded and symmetric open set with* $0 \in \Omega$. *Suppose that* $\mathfrak{L}\colon \operatorname{dom}\mathfrak{L} \subset \mathcal{X} \to \mathcal{Y}$ *is a Fredholm operator of index zero with* $\operatorname{dom}\mathfrak{L} \cap \overline{\Omega} \neq \emptyset$ *and* $\mathcal{N}\colon \mathcal{X} \to \mathcal{Y}$ *is a* $\mathfrak{L}$*-compact operator on* $\overline{\Omega}$. *Assume, moreover, that*

$$\mathfrak{L}x - \mathcal{N}x \neq -\zeta(\mathfrak{L}x + \mathcal{N}(-x))$$

for any $x \in \operatorname{dom}\mathfrak{L} \cap \partial\Omega$ *and any* $\zeta \in (0, 1]$, *where* $\partial\Omega$ *is the boundary of* Ω *with respect to* $\mathcal{X}$. *Then, there exists at least one solution of the equation* $\mathfrak{L}x = \mathcal{N}x$ *on* $\operatorname{dom}\mathfrak{L} \cap \overline{\Omega}$.

3 Main results

Let the spaces

$$\mathcal{X} = \left\{\mathfrak{u} \in C(\mathfrak{J}, \mathfrak{R}) : \mathfrak{u}(\xi) = \mathfrak{I}_{0^+}^{\alpha;\Psi}\upsilon(\xi), \text{ where } \upsilon \in C(\mathfrak{J}, \mathfrak{R})\right\}$$

and

$$\mathcal{Y} = C(\mathfrak{J}, \mathfrak{R})$$

be endowed with the norms

$$\|\mathfrak{u}\|_{\mathcal{X}} = \|\mathfrak{u}\|_{\mathcal{Y}} = \|\mathfrak{u}\|_{\infty} = \sup_{\xi \in \mathfrak{J}} |\mathfrak{u}(\xi)|.$$

We give now the definition of the operator $\mathfrak{L}\colon \operatorname{dom}\mathfrak{L} \subseteq \mathcal{X} \to \mathcal{Y}$. Set

$$\mathfrak{L}\mathfrak{u} := {}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{u}, \tag{3.1}$$

where

$$\operatorname{dom}\mathfrak{L} = \{\mathfrak{u} \in \mathcal{X} : {}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{u} \in \mathcal{Y} \text{ and } \mathfrak{u}(0) = \mathfrak{u}(\mathfrak{b})\}.$$

Lemma 3.1 *Let* $\mathfrak{L}$ *be the operator given in* (3.1). *Then,*

$$\ker\mathfrak{L} = \{\mathfrak{u} \in \mathcal{X} : \mathfrak{u}(\xi) = \mathfrak{u}(0),\ \xi \in \mathfrak{J}\}$$

and

$$\operatorname{img}\mathfrak{L} = \left\{\upsilon \in \mathcal{Y} : \int_0^{\mathfrak{b}} \Psi'(s)(\Psi(\mathfrak{b}) - \Psi(s))^{\alpha-1}\upsilon(s)\,\mathrm{d}s = 0\right\}.$$

Proof. By Remark 2.8, we know that the equation $\mathfrak{L}\mathfrak{u} = {}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{u} = 0$ in $\mathfrak{J}$ has a solution of the form

$$\mathfrak{u}(\xi) = c_0 = \mathfrak{u}(0),\ \xi \in \mathfrak{J}.$$

Hence,

$$\ker \mathfrak{L} = \{\mathfrak{u} \in \mathcal{X} : \mathfrak{u}(\xi) = \mathfrak{u}(0),\ \xi \in \mathfrak{J}\}.$$

For $\upsilon \in \operatorname{img} \mathfrak{L}$ there exists $\mathfrak{u} \in \operatorname{dom} \mathfrak{L}$ such that $\upsilon = \mathfrak{L}\mathfrak{u} \in \mathcal{Y}$. Using Theorem 2.5, for every $\xi \in \mathfrak{J}$ we obtain

$$\begin{aligned}\mathfrak{u}(\xi) &= \mathfrak{u}(0) + \mathfrak{I}_{0^+}^{\alpha;\Psi}\upsilon(\xi) \\ &= \mathfrak{u}(0) + \frac{1}{\Gamma(\alpha)}\int_0^{\xi} \Psi'(s)(\Psi(\xi) - \Psi(s))^{\alpha-1}\upsilon(s)\,\mathrm{d}s.\end{aligned}$$

Since $\mathfrak{u} \in \operatorname{dom} \mathfrak{L}$, we have $\mathfrak{u}(0) = \mathfrak{u}(\mathfrak{b})$. Thus,

$$\int_0^{\mathfrak{b}} \Psi'(s)(\Psi(\mathfrak{b}) - \Psi(s))^{\alpha-1}\upsilon(s)\,\mathrm{d}s = 0.$$

Furthermore, if $\upsilon \in \mathcal{Y}$ is such that

$$\int_0^{\mathfrak{b}} \Psi'(s)(\Psi(\mathfrak{b}) - \Psi(s))^{\alpha-1}\upsilon(s)\,\mathrm{d}s = 0,$$

then for any $\mathfrak{u}(\xi) = \mathfrak{I}_{0^+}^{\alpha;\Psi}\upsilon(\xi)$, using Theorem 2.6, we get $\upsilon(\xi) = {}^c\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{u}(\xi)$. Therefore,

$$\mathfrak{u}(\mathfrak{b}) = \mathfrak{u}(0),$$

which implies that $\mathfrak{u} \in \operatorname{dom} \mathfrak{L}$. So, $\upsilon \in \operatorname{img} \mathfrak{L}$. Consequently,

$$\operatorname{img} \mathfrak{L} = \left\{\upsilon \in \mathcal{Y} : \int_0^{\mathfrak{b}} \Psi'(s)(\Psi(\mathfrak{b}) - \Psi(s))^{\alpha-1}\upsilon(s)\,\mathrm{d}s = 0\right\}.$$

This completes the proof. □

Lemma 3.2 *Let $\mathfrak{L}$ be defined by* (3.1). *Then, $\mathfrak{L}$ is a Fredholm operator of index zero, and the linear continuous projector operators $\mathcal{Q}\colon \mathcal{Y} \to \mathcal{Y}$ and $\mathcal{P}\colon \mathcal{X} \to \mathcal{X}$ can be written as*

$$\mathcal{Q}(\upsilon) = \frac{\alpha}{(\Psi(\mathfrak{b}) - \Psi(0))^{\alpha}}\int_0^{\mathfrak{b}} \Psi'(s)(\Psi(\mathfrak{b}) - \Psi(s))^{\alpha-1}\upsilon(s)\,\mathrm{d}s$$

and

$$\mathcal{P}(\mathfrak{u}) = \mathfrak{u}(0).$$

Furthermore, the operator $\mathfrak{L}_{\mathcal{P}}^{-1}\colon\ \operatorname{img}\mathfrak{L} \to \mathcal{X} \cap \ker \mathcal{P}$ can be written by

$$\mathfrak{L}_{\mathcal{P}}^{-1}(\upsilon)(\xi) = \mathfrak{I}_{0^+}^{\alpha;\Psi}\upsilon(\xi),\ \xi \in \mathfrak{J}.$$

Proof. Obviously, for each $\upsilon \in \mathcal{Y}$ we have $\mathcal{Q}^2\upsilon = \mathcal{Q}\upsilon$ and $\upsilon = \mathcal{Q}(\upsilon) + (\upsilon - \mathcal{Q}(\upsilon))$, where $(\upsilon - \mathcal{Q}(\upsilon)) \in \ker \mathcal{Q} = \operatorname{img} \mathfrak{L}$. Using the fact that $\operatorname{img} \mathfrak{L} = \ker \mathcal{Q}$ and $\mathcal{Q}^2 = \mathcal{Q}$, we obtain $\operatorname{img} \mathcal{Q} \cap \operatorname{img} \mathfrak{L} = \{0\}$. So,

$$\mathcal{Y} = \operatorname{img} \mathfrak{L} \oplus \operatorname{img} \mathcal{Q}.$$

In the same way we get that $\operatorname{img} \mathcal{P} = \ker \mathfrak{L}$ and $\mathcal{P}^2 = \mathcal{P}$. It follows for each $\mathfrak{u} \in \mathcal{X}$ that $\mathfrak{u} = (\mathfrak{u} - \mathcal{P}(\mathfrak{u})) + \mathcal{P}(\mathfrak{u})$. Hence, $\mathcal{X} = \ker \mathcal{P} + \ker \mathfrak{L}$. Clearly, we have $\ker \mathcal{P} \cap \ker \mathfrak{L} = \{0\}$. So,

$$\mathcal{X} = \ker \mathcal{P} \oplus \ker \mathfrak{L}.$$

Therefore,

$$\dim\ker\mathfrak{L} = \dim\operatorname{img}\mathcal{Q} = \operatorname{codim}\operatorname{img}\mathfrak{L}.$$

Consequently, $\mathfrak{L}$ is a Fredholm operator of index zero.

Now, we will show that the inverse of $\mathfrak{L}|_{\operatorname{dom}\mathfrak{L}\cap\ker\mathcal{P}}$ is $\mathfrak{L}_{\mathcal{P}}^{-1}$. Effectively, for $\upsilon\in\operatorname{img}\mathfrak{L}$, by Theorem 2.6, we have

$$\mathfrak{L}\mathfrak{L}_{\mathcal{P}}^{-1}(\upsilon) = {}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\left(\mathfrak{I}_{0^+}^{\alpha;\Psi}\upsilon\right) = \upsilon. \tag{3.2}$$

Furthermore, for $\mathfrak{u}\in\operatorname{dom}\mathfrak{L}\cap\ker\mathcal{P}$ we get

$$\mathfrak{L}_{\mathcal{P}}^{-1}(\mathfrak{L}(\mathfrak{u}(\xi))) = \mathfrak{I}_{0^+}^{\alpha;\Psi}\left({}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{u}(\xi)\right) = \mathfrak{u}(\xi)-\mathfrak{u}(0),\ \xi\in\mathfrak{J}.$$

Using the fact that $\mathfrak{u}\in\operatorname{dom}\mathfrak{L}\cap\ker\mathcal{P}$, we infer that

$$\mathfrak{u}(0)=0.$$

Thus,

$$\mathfrak{L}_{\mathcal{P}}^{-1}\mathfrak{L}(\mathfrak{u}) = \mathfrak{u}. \tag{3.3}$$

Using (3.2) and (3.3) together, we get $\mathfrak{L}_{\mathcal{P}}^{-1} = \left(\mathfrak{L}|_{\operatorname{dom}\mathfrak{L}\cap\ker\mathcal{P}}\right)^{-1}$. This completes the demonstration. □

Let us introduce the following hypothesis.

(A1) There exist nonnegative functions $\gamma_1,\gamma_2,\eta_1,\eta_2\in C(\mathfrak{J},\mathfrak{R}^+)$ such that

$$|\mathcal{F}(\xi,\mathfrak{u},\upsilon)-\mathcal{F}(\xi,\bar{\mathfrak{u}},\bar{\upsilon})| \leqslant \gamma_1(\xi)|\mathfrak{u}-\bar{\mathfrak{u}}|+\eta_1(\xi)|\upsilon-\bar{\upsilon}|$$

and

$$|\mathcal{G}(\xi,\mathfrak{u},\upsilon)-\mathcal{G}(\xi,\bar{\mathfrak{u}},\bar{\upsilon})| \leqslant \gamma_2(\xi)|\mathfrak{u}-\bar{\mathfrak{u}}|+\eta_2(\xi)|\upsilon-\bar{\upsilon}|$$

for every $\xi\in\mathfrak{J}$ and $\mathfrak{u},\bar{\mathfrak{u}},\upsilon,\bar{\upsilon}\in\mathfrak{R}$.

Define $\mathcal{N}\colon\mathcal{X}\to\mathcal{Y}$ by

$$\mathcal{N}\mathfrak{u}(\xi) := \mathcal{F}\left(\xi,\mathfrak{u}(\xi),\mathfrak{u}(p\xi)\right)+\mathcal{G}\left(\xi,\mathfrak{u}(\xi),\mathfrak{u}((1-p)\xi)\right),\ \xi\in\mathfrak{J} \text{ and } p\in(0,1).$$

Then, the problem (1.1)–(1.2) is equivalent to the problem $\mathfrak{L}\mathfrak{u}=\mathcal{N}\mathfrak{u}$.

Lemma 3.3 *Suppose that* **(A1)** *is satisfied. Then, for any bounded open set $\Omega\subset\mathcal{X}$, the operator $\mathcal{N}$ is $\mathfrak{L}$-compact.*

Proof. For $\mathcal{M}>0$ we consider the bounded open set $\Omega=\{\mathfrak{u}\in\mathcal{X} : \|\mathfrak{u}\|_{\mathcal{X}}<\mathcal{M}\}$. We split the proof into three steps.

Step 1: *$\mathcal{Q}\mathcal{N}$ is continuous.* Let $(\mathfrak{u}_{\mathfrak{n}})_{\mathfrak{n}\in\mathbb{N}}$ be a sequence such that $\mathfrak{u}_{\mathfrak{n}}\longrightarrow\mathfrak{u}$ in $\mathcal{Y}$. Then, for each $\xi\in\mathfrak{J}$, we have

$$\begin{aligned}&|\mathcal{Q}\mathcal{N}(\mathfrak{u}_{\mathfrak{n}})(\xi)-\mathcal{Q}\mathcal{N}(\mathfrak{u})(\xi)|\\&\quad\leqslant\frac{\alpha}{(\Psi(\mathfrak{b})-\Psi(0))^{\alpha}}\int_0^{\mathfrak{b}}\Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}\left|\mathcal{N}(\mathfrak{u}_{\mathfrak{n}})(s)-\mathcal{N}(\mathfrak{u})(s)\right|\,\mathrm{d}s.\end{aligned}$$

By **(A1)**, we have

$$\begin{aligned}
&|\mathcal{QN}(\mathfrak{u}_{\mathfrak{n}})(\xi)-\mathcal{QN}(\mathfrak{u})(\xi)| \\
&\quad\leqslant \frac{\alpha\gamma_1^*}{(\Psi(\mathfrak{b})-\Psi(0))^{\alpha}}\int_0^{\mathfrak{b}}\Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}\left|\mathfrak{u}_{\mathfrak{n}}(s)-\mathfrak{u}(s)\right|\mathrm{d}s \\
&\qquad+\frac{\alpha\eta_1^*}{(\Psi(\mathfrak{b})-\Psi(0))^{\alpha}}\int_0^{\mathfrak{b}}\Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}\left|\mathfrak{u}_{\mathfrak{n}}(ps)-\mathfrak{u}(ps)\right|\mathrm{d}s \\
&\qquad+\frac{\alpha\gamma_2^*}{(\Psi(\mathfrak{b})-\Psi(0))^{\alpha}}\int_0^{\mathfrak{b}}\Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}\left|\mathfrak{u}_{\mathfrak{n}}(s)-\mathfrak{u}(s)\right|\mathrm{d}s \\
&\qquad+\frac{\alpha\eta_2^*}{(\Psi(\mathfrak{b})-\Psi(0))^{\alpha}}\int_0^{\mathfrak{b}}\Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}\left|\mathfrak{u}_{\mathfrak{n}}((1-p)s)-\mathfrak{u}((1-p)s)\right|\mathrm{d}s \\
&\quad\leqslant \frac{\alpha\left(\gamma_1^*+\gamma_2^*+\eta_1^*+\eta_2^*\right)\|\mathfrak{u}_{\mathfrak{n}}-\mathfrak{u}\|_{\mathcal{Y}}}{(\Psi(\mathfrak{b})-\Psi(0))^{\alpha}}\int_0^{\mathfrak{b}}\Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}\mathrm{d}s \\
&\quad\leqslant \left(\gamma_1^*+\gamma_2^*+\eta_1^*+\eta_2^*\right)\|\mathfrak{u}_{\mathfrak{n}}-\mathfrak{u}\|_{\mathcal{Y}},
\end{aligned}$$

where

$$\gamma_1^*:=\|\gamma_1\|_{\infty},\ \gamma_2^*:=\|\gamma_2\|_{\infty},\ \eta_1^*:=\|\eta_1\|_{\infty},\ \eta_2^*:=\|\eta_2\|_{\infty}.$$

Thus, for each $\xi\in\mathfrak{J}$, we get

$$|\mathcal{QN}(\mathfrak{u}_{\mathfrak{n}})(\xi)-\mathcal{QN}(\mathfrak{u})(\xi)|\longrightarrow 0 \text{ as } n\longrightarrow+\infty,$$

and hence

$$\|\mathcal{QN}(\mathfrak{u}_{\mathfrak{n}})-\mathcal{QN}(\mathfrak{u})\|_{\mathcal{Y}}\longrightarrow 0 \text{ as } n\longrightarrow+\infty.$$

This means that $\mathcal{QN}$ is continuous.

Step 2: $\mathcal{QN}(\overline{\Omega})$ *is bounded.* For $\xi\in\mathfrak{J}$ and $\mathfrak{u}\in\overline{\Omega}$, we have

$$\begin{aligned}
&|\mathcal{QN}(\mathfrak{u})(\xi)| \\
&\quad\leqslant \frac{\alpha}{(\Psi(\mathfrak{b})-\Psi(0))^{\alpha}}\int_0^{\mathfrak{b}}\Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}|\mathcal{N}(\mathfrak{u})(s)|\,\mathrm{d}s \\
&\quad\leqslant \frac{\alpha}{(\Psi(\mathfrak{b})-\Psi(0))^{\alpha}}\int_0^{\mathfrak{b}}\Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}|\mathcal{F}(s,\mathfrak{u}(s),\mathfrak{u}(ps))-\mathcal{F}(s,0,0)|\,\mathrm{d}s \\
&\qquad+\frac{\alpha}{(\Psi(\mathfrak{b})-\Psi(0))^{\alpha}}\int_0^{\mathfrak{b}}\Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}|\mathcal{F}(s,0,0)|\,\mathrm{d}s \\
&\qquad+\frac{\alpha}{(\Psi(\mathfrak{b})-\Psi(0))^{\alpha}}\int_0^{\mathfrak{b}}\Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}|\mathcal{G}(s,\mathfrak{u}(s),\mathfrak{u}((1-p)s))-\mathcal{G}(s,0,0)|\,\mathrm{d}s \\
&\qquad+\frac{\alpha}{(\Psi(\mathfrak{b})-\Psi(0))^{\alpha}}\int_0^{\mathfrak{b}}\Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}|\mathcal{G}(s,0,0)|\,\mathrm{d}s \\
&\quad\leqslant \mathcal{F}^*+\mathcal{G}^*+\frac{\alpha(\gamma_1^*+\gamma_2^*)}{(\Psi(\mathfrak{b})-\Psi(0))^{\alpha}}\int_0^{\mathfrak{b}}\Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}|\mathfrak{u}(s)|\,\mathrm{d}s \\
&\qquad+\frac{\alpha\eta_1^*}{(\Psi(\mathfrak{b})-\Psi(0))^{\alpha}}\int_0^{\mathfrak{b}}\Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}|\mathfrak{u}(ps)|\,\mathrm{d}s \\
&\qquad+\frac{\alpha\eta_2^*}{(\Psi(\mathfrak{b})-\Psi(0))^{\alpha}}\int_0^{\mathfrak{b}}\Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}|\mathfrak{u}((1-p)s)|\,\mathrm{d}s \\
&\quad\leqslant \mathcal{F}^*+\mathcal{G}^*+\left(\gamma_1^*+\gamma_2^*+\eta_1^*+\eta_2^*\right)\mathcal{M},
\end{aligned}$$

where $\mathcal{F}^* := \|\mathcal{F}(\cdot,0,0)\|_\infty$ and $\mathcal{G}^* := \|\mathcal{G}(\cdot,0,0)\|_\infty$. Thus,

$$\|\mathcal{Q}\mathcal{N}(\mathfrak{u})\|_{\mathcal{Y}} \leqslant \mathcal{F}^* + \mathcal{G}^* + (\gamma_1^* + \gamma_2^* + \eta_1^* + \eta_2^*)\,\mathcal{M}.$$

So, $\mathcal{Q}\mathcal{N}(\overline{\Omega})$ is a bounded subset of $\mathcal{Y}$.

Step 3: $\mathfrak{L}_{\mathcal{P}}^{-1}(\mathrm{Id}-\mathcal{Q})\mathcal{N}\colon \overline{\Omega} \to \mathcal{X}$ *is completely continuous.* As we will use the Arzelà–Ascoli theorem, we have to show that $\mathfrak{L}_{\mathcal{P}}^{-1}(\mathrm{Id}-\mathcal{Q})\mathcal{N}(\overline{\Omega}) \subset \mathcal{X}$ is equicontinuous and bounded. Firstly, for any $\mathfrak{u} \in \overline{\Omega}$ and $\xi \in \mathfrak{J}$, we get

$$\begin{aligned}
&\mathfrak{L}_{\mathcal{P}}^{-1}(\mathcal{N}\mathfrak{u}(\xi) - \mathcal{Q}\mathcal{N}\mathfrak{u}(\xi))\\
&= \mathfrak{I}_{0^+}^{\alpha;\Psi}\Bigg[\mathcal{F}(\xi,\mathfrak{u}(\xi),\mathfrak{u}(p\xi)) + \mathcal{G}(\xi,\mathfrak{u}(\xi),\mathfrak{u}((1-p)\xi))\\
&\quad - \frac{\alpha}{(\Psi(\mathfrak{b})-\Psi(0))^\alpha}\int_0^{\mathfrak{b}} \Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}\mathcal{F}(s,\mathfrak{u}(s),\mathfrak{u}(ps))\,\mathrm{d}s\\
&\quad - \frac{\alpha}{(\Psi(\mathfrak{b})-\Psi(0))^\alpha}\int_0^{\mathfrak{b}} \Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}\mathcal{G}(s,\mathfrak{u}(s),\mathfrak{u}((1-p)s))\,\mathrm{d}s\Bigg]\\
&= \frac{1}{\Gamma(\alpha)}\int_0^{\xi} \Psi'(s)\,(\Psi(\xi)-\Psi(s))^{\alpha-1}\,\mathcal{F}(s,\mathfrak{u}(s),\mathfrak{u}(ps))\,\mathrm{d}s\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{\xi} \Psi'(s)\,(\Psi(\xi)-\Psi(s))^{\alpha-1}\,\mathcal{G}(s,\mathfrak{u}(s),\mathfrak{u}((1-p)s))\,\mathrm{d}s\\
&\quad - \frac{(\Psi(\xi)-\Psi(0))^\alpha}{\Gamma(\alpha)(\Psi(\mathfrak{b})-\Psi(0))^\alpha}\int_0^{\mathfrak{b}} \Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}\mathcal{F}(s,\mathfrak{u}(s),\mathfrak{u}(ps))\,\mathrm{d}s\\
&\quad - \frac{(\Psi(\xi)-\Psi(0))^\alpha}{\Gamma(\alpha)(\Psi(\mathfrak{b})-\Psi(0))^\alpha}\int_0^{\mathfrak{b}} \Psi'(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}\mathcal{G}(s,\mathfrak{u}(s),\mathfrak{u}((1-p)s))\,\mathrm{d}s.
\end{aligned}$$

For all $\mathfrak{u} \in \overline{\Omega}$ and $\xi \in \mathfrak{J}$, we get

$$\begin{aligned}
&|\mathfrak{L}_{\mathcal{P}}^{-1}(\mathrm{Id}-\mathcal{Q})\mathcal{N}\mathfrak{u}(\xi)|\\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^{\xi} \Psi'(s)\,(\Psi(\xi)-\Psi(s))^{\alpha-1}\,|\mathcal{F}(s,\mathfrak{u}(s),\mathfrak{u}(ps)) - \mathcal{F}(s,0,0)|\,\mathrm{d}s\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{\xi} \Psi'(s)\,(\Psi(\xi)-\Psi(s))^{\alpha-1}\,|\mathcal{F}(s,0,0)|\,\mathrm{d}s\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{\xi} \Psi'(s)\,(\Psi(\xi)-\Psi(s))^{\alpha-1}\,|\mathcal{G}(s,\mathfrak{u}(s),\mathfrak{u}((1-p)s)) - \mathcal{G}(s,0,0)|\,\mathrm{d}s\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{\xi} \Psi'(s)\,(\Psi(\xi)-\Psi(s))^{\alpha-1}\,|\mathcal{G}(s,0,0)|\,\mathrm{d}s\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}} \Psi'(s)\,(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}\,|\mathcal{F}(s,\mathfrak{u}(s),\mathfrak{u}(ps)) - \mathcal{F}(s,0,0)|\,\mathrm{d}s\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}} \Psi'(s)\,(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}\,|\mathcal{F}(s,0,0)|\,\mathrm{d}s\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}} \Psi'(s)\,(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}\,|\mathcal{G}(s,\mathfrak{u}(s),\mathfrak{u}((1-p)s)) - \mathcal{G}(s,0,0)|\,\mathrm{d}s\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}} \Psi'(s)\,(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}\,|\mathcal{G}(s,0,0)|\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{2(\mathcal{F}^*+\mathcal{G}^*)}{\alpha\Gamma(\alpha)}\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}+\frac{\gamma_1^*}{\Gamma(\alpha)}\int_0^{\xi}\Psi^{'}(s)(\Psi(\xi)-\Psi(s))^{\alpha-1}|\mathfrak{u}(s)|\,\mathrm{d}s\\
&\quad+\frac{\eta_1^*}{\Gamma(\alpha)}\int_0^{\xi}\Psi^{'}(s)(\Psi(\xi)-\Psi(s))^{\alpha-1}|\mathfrak{u}(ps)|\,\mathrm{d}s\\
&\quad+\frac{\gamma_1^*}{\Gamma(\alpha)}\int_0^{\mathfrak{b}}\Psi^{'}(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}|\mathfrak{u}(s)|\,\mathrm{d}s\\
&\quad+\frac{\eta_1^*}{\Gamma(\alpha)}\int_0^{\mathfrak{b}}\Psi^{'}(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}|\mathfrak{u}(ps)|\,\mathrm{d}s\\
&\quad+\frac{\gamma_2^*}{\Gamma(\alpha)}\int_0^{\xi}\Psi^{'}(s)(\Psi(\xi)-\Psi(s))^{\alpha-1}|\mathfrak{u}(s)|\,\mathrm{d}s\\
&\quad+\frac{\eta_2^*}{\Gamma(\alpha)}\int_0^{\xi}\Psi^{'}(s)(\Psi(\xi)-\Psi(s))^{\alpha-1}|\mathfrak{u}((1-p)s)|\,\mathrm{d}s\\
&\quad+\frac{\gamma_2^*}{\Gamma(\alpha)}\int_0^{\mathfrak{b}}\Psi^{'}(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}|\mathfrak{u}(s)|\,\mathrm{d}s\\
&\quad+\frac{\eta_2^*}{\Gamma(\alpha)}\int_0^{\mathfrak{b}}\Psi^{'}(s)(\Psi(\mathfrak{b})-\Psi(s))^{\alpha-1}|\mathfrak{u}((1-p)s)|\,\mathrm{d}s\\
&\leqslant \frac{2\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}}{\Gamma(\alpha+1)}\Big[\mathcal{F}^*+\mathcal{G}^*+\left(\gamma_1^*+\gamma_2^*+\eta_1^*+\eta_2^*\right)\mathcal{M}\Big].
\end{aligned}
$$

Therefore,

$$
\|\mathfrak{L}_{\mathcal{P}}^{-1}(\mathrm{Id}-\mathcal{Q})\mathcal{N}\mathfrak{u}\|_{\mathcal{X}}\leqslant \frac{2\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}}{\Gamma(\alpha+1)}\Big[\mathcal{F}^*+\mathcal{G}^*+\left(\gamma_1^*+\gamma_2^*+\eta_1^*+\eta_2^*\right)\mathcal{M}\Big].
$$

This means that $\mathfrak{L}_{\mathcal{P}}^{-1}(\mathrm{Id}-\mathcal{Q})\mathcal{N}(\overline{\Omega})$ is uniformly bounded in $\mathcal{X}$.

It remains to show that $\mathfrak{L}_{\mathcal{P}}^{-1}(\mathrm{Id}-\mathcal{Q})\mathcal{N}(\overline{\Omega})$ is equicontinuous. For $0<\xi_1<\xi_2\leqslant\mathfrak{b}$ and $\mathfrak{u}\in\overline{\Omega}$, we have

$$
\begin{aligned}
&|\mathfrak{L}_{\mathcal{P}}^{-1}(\mathrm{Id}-\mathcal{Q})\mathcal{N}\mathfrak{u}(\xi_2)-\mathfrak{L}_{\mathcal{P}}^{-1}(\mathrm{Id}-\mathcal{Q})\mathcal{N}\mathfrak{u}(\xi_1)|\\
&\quad\leqslant \frac{1}{\Gamma(\alpha)}\int_0^{\xi_1}\Big[\Psi^{'}(s)\Big|(\Psi(\xi_2)-\Psi(s))^{\alpha-1}-(\Psi(\xi_1)-\Psi(s))^{\alpha-1}\Big|\left|\mathcal{F}\left(s,\mathfrak{u}(s),\mathfrak{u}(ps)\right)\right|\Big]\,\mathrm{d}s\\
&\qquad+\frac{1}{\Gamma(\alpha)}\int_{\xi_1}^{\xi_2}\Psi^{'}(s)\left(\Psi(\xi_2)-\Psi(s)\right)^{\alpha-1}\left|\mathcal{F}\left(s,\mathfrak{u}(s),\mathfrak{u}(ps)\right)\right|\,\mathrm{d}s\\
&\qquad+\frac{1}{\Gamma(\alpha)}\int_0^{\xi_1}\Big[\Psi^{'}(s)\Big|(\Psi(\xi_2)-\Psi(s))^{\alpha-1}-(\Psi(\xi_1)-\Psi(s))^{\alpha-1}\Big|\left|\mathcal{G}\left(s,\mathfrak{u}(s),\mathfrak{u}((1-p)s)\right)\right|\Big]\,\mathrm{d}s\\
&\qquad+\frac{1}{\Gamma(\alpha)}\int_{\xi_1}^{\xi_2}\Psi^{'}(s)\left(\Psi(\xi_2)-\Psi(s)\right)^{\alpha-1}\left|\mathcal{G}\left(s,\mathfrak{u}(s),\mathfrak{u}((1-p)s)\right)\right|\,\mathrm{d}s\\
&\qquad+\frac{\left[(\Psi(\xi_2)-\Psi(0))^{\alpha}-(\Psi(\xi_1)-\Psi(0))^{\alpha}\right]}{\Gamma(\alpha)\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}}\int_0^{\mathfrak{b}}\Psi^{'}(s)\left(\Psi(\mathfrak{b})-\Psi(s)\right)^{\alpha-1}\left|\mathcal{F}\left(s,\mathfrak{u}(s),\mathfrak{u}(ps)\right)\right|\,\mathrm{d}s\\
&\qquad+\frac{\left[(\Psi(\xi_2)-\Psi(0))^{\alpha}-(\Psi(\xi_1)-\Psi(0))^{\alpha}\right]}{\Gamma(\alpha)\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}}\times\\
&\qquad\qquad\times\int_0^{\mathfrak{b}}\Psi^{'}(s)\left(\Psi(\mathfrak{b})-\Psi(s)\right)^{\alpha-1}\left|\mathcal{G}\left(s,\mathfrak{u}(s),\mathfrak{u}((1-p)s)\right)\right|\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^{\xi_1} \Psi'(s)\left(\Psi(\xi_1)-\Psi(s)\right)^{\alpha-1} \left|\mathcal{F}\left(s,\mathfrak{u}(s),\mathfrak{u}(ps)\right)-\mathcal{F}\left(s,0,0\right)\right| \mathrm{d}s \\
&\quad -\frac{1}{\Gamma(\alpha)} \int_0^{\xi_1} \Psi'(s)\left(\Psi(\xi_2)-\Psi(s)\right)^{\alpha-1} \left|\mathcal{F}\left(s,\mathfrak{u}(s),\mathfrak{u}(ps)\right)-\mathcal{F}\left(s,0,0\right)\right| \mathrm{d}s \\
&\quad +\frac{1}{\Gamma(\alpha)} \int_0^{\xi_1} \Psi'(s)\left[\left(\Psi(\xi_1)-\Psi(s)\right)^{\alpha-1}-\left(\Psi(\xi_2)-\Psi(s)\right)^{\alpha-1}\right] \left|\mathcal{F}\left(s,0,0\right)\right| \mathrm{d}s \\
&\quad +\frac{1}{\Gamma(\alpha)} \int_{\xi_1}^{\xi_2} \Psi'(s)\left(\Psi(\xi_2)-\Psi(s)\right)^{\alpha-1} \left|\mathcal{F}\left(s,\mathfrak{u}(s),\mathfrak{u}(ps)\right)-\mathcal{F}\left(s,0,0\right)\right| \mathrm{d}s \\
&\quad +\frac{1}{\Gamma(\alpha)} \int_{\xi_1}^{\xi_2} \Psi'(s)\left(\Psi(\xi_2)-\Psi(s)\right)^{\alpha-1} \left|\mathcal{F}\left(s,0,0\right)\right| \mathrm{d}s \\
&\quad +\frac{1}{\Gamma(\alpha)} \int_0^{\xi_1} \Psi'(s)\left(\Psi(\xi_1)-\Psi(s)\right)^{\alpha-1} \left|\mathcal{G}\left(s,\mathfrak{u}(s),\mathfrak{u}((1-p)s)\right)-\mathcal{G}\left(s,0,0\right)\right| \mathrm{d}s \\
&\quad -\frac{1}{\Gamma(\alpha)} \int_0^{\xi_1} \Psi'(s)\left(\Psi(\xi_2)-\Psi(s)\right)^{\alpha-1} \left|\mathcal{G}\left(s,\mathfrak{u}(s),\mathfrak{u}((1-p)s)\right)-\mathcal{G}\left(s,0,0\right)\right| \mathrm{d}s \\
&\quad +\frac{1}{\Gamma(\alpha)} \int_0^{\xi_1} \Psi'(s)\left[\left(\Psi(\xi_1)-\Psi(s)\right)^{\alpha-1}-\left(\Psi(\xi_2)-\Psi(s)\right)^{\alpha-1}\right] \left|\mathcal{G}\left(s,0,0\right)\right| \mathrm{d}s \\
&\quad +\frac{1}{\Gamma(\alpha)} \int_{\xi_1}^{\xi_2} \Psi'(s)\left(\Psi(\xi_2)-\Psi(s)\right)^{\alpha-1} \left|\mathcal{G}\left(s,\mathfrak{u}(s),\mathfrak{u}((1-p)s)\right)-\mathcal{G}\left(s,0,0\right)\right| \mathrm{d}s \\
&\quad +\frac{1}{\Gamma(\alpha)} \int_{\xi_1}^{\xi_2} \Psi'(s)\left(\Psi(\xi_2)-\Psi(s)\right)^{\alpha-1} \left|\mathcal{G}\left(s,0,0\right)\right| \mathrm{d}s \\
&\quad +\frac{\left[\left(\Psi(\xi_2)-\Psi(0)\right)^{\alpha}-\left(\Psi(\xi_1)-\Psi(0)\right)^{\alpha}\right]}{\Gamma(\alpha)\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}} \times \\
&\qquad\qquad \times \int_0^{\mathfrak{b}} \Psi'(s)\left(\Psi(\mathfrak{b})-\Psi(s)\right)^{\alpha-1} \left|\mathcal{F}\left(s,\mathfrak{u}(s),\mathfrak{u}(ps)\right)-\mathcal{F}\left(s,0,0\right)\right| \mathrm{d}s \\
&\quad +\frac{\left[\left(\Psi(\xi_2)-\Psi(0)\right)^{\alpha}-\left(\Psi(\xi_1)-\Psi(0)\right)^{\alpha}\right]}{\Gamma(\alpha)\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}} \int_0^{\mathfrak{b}} \Psi'(s)\left(\Psi(\mathfrak{b})-\Psi(s)\right)^{\alpha-1} \left|\mathcal{F}\left(s,0,0\right)\right| \mathrm{d}s \\
&\quad +\frac{\left[\left(\Psi(\xi_2)-\Psi(0)\right)^{\alpha}-\left(\Psi(\xi_1)-\Psi(0)\right)^{\alpha}\right]}{\Gamma(\alpha)\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}} \times \\
&\qquad\qquad \times \int_0^{\mathfrak{b}} \Psi'(s)\left(\Psi(\mathfrak{b})-\Psi(s)\right)^{\alpha-1} \left|\mathcal{G}\left(s,\mathfrak{u}(s),\mathfrak{u}((1-p)s)\right)-\mathcal{G}\left(s,0,0\right)\right| \mathrm{d}s \\
&\quad +\frac{\left[\left(\Psi(\xi_2)-\Psi(0)\right)^{\alpha}-\left(\Psi(\xi_1)-\Psi(0)\right)^{\alpha}\right]}{\Gamma(\alpha)\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}} \int_0^{\mathfrak{b}} \Psi'(s)\left(\Psi(\mathfrak{b})-\Psi(s)\right)^{\alpha-1} \left|\mathcal{G}\left(s,0,0\right)\right| \mathrm{d}s \\
&\leqslant 2\Lambda\left(\Psi(\xi_2)-\Psi(\xi_1)\right)^{\alpha}+\Lambda\left[\left(\Psi(\xi_1)-\Psi(0)\right)^{\alpha}-\left(\Psi(\xi_2)-\Psi(0)\right)^{\alpha}\right] \\
&\quad +\Lambda\left[\left(\Psi(\xi_2)-\Psi(0)\right)^{\alpha}-\left(\Psi(\xi_1)-\Psi(0)\right)^{\alpha}\right] \\
&\leqslant 2\Lambda\left(\Psi(\xi_2)-\Psi(\xi_1)\right)^{\alpha},
\end{aligned}
$$

where

$$
\Lambda := \frac{\mathcal{F}^{*}+\mathcal{G}^{*}+\left(\gamma_1^{*}+\gamma_2^{*}+\eta_1^{*}+\eta_2^{*}\right)\mathcal{M}}{\Gamma(\alpha+1)}.
$$

The operator $\mathfrak{L}_{\mathcal{P}}^{-1}(\mathrm{Id}-\mathcal{Q})\mathcal{N}(\overline{\Omega})$ is equicontinuous in $\mathcal{X}$, because the right-hand side of the above inequality tends to zero as $\xi_1 \to \xi_2$ and the limit is independent of $\mathfrak{u}$. The Arzelà–Ascoli theorem implies that $\mathfrak{L}_{\mathcal{P}}^{-1}(\mathrm{Id}-\mathcal{Q})\mathcal{N}(\overline{\Omega})$ is relatively compact in $\mathcal{X}$. As a consequence of Steps 1–3, we infer that $\mathcal{N}$ is $\mathfrak{L}$-compact in $\overline{\Omega}$. This completes the demonstration. □

Lemma 3.4 *Assume* **(A1)** *and that the condition*

$$\frac{(\gamma_1^* + \gamma_2^* + \eta_1^* + \eta_2^*)}{\Gamma(\alpha+1)} (\Psi(\mathfrak{b}) - \Psi(0))^{\alpha} < \frac{1}{2} \tag{3.4}$$

is satisfied. There exists $\mathcal{A} > 0$ *such that if*

$$\mathfrak{L}(\mathfrak{u}) - \mathcal{N}(\mathfrak{u}) = -\zeta[\mathfrak{L}(\mathfrak{u}) + \mathcal{N}(-\mathfrak{u})]$$

for some $\mathfrak{u} \in \operatorname{dom} \mathfrak{L}$ *and some* $\zeta \in (0,1]$*, then* $\|\mathfrak{u}\|_{\mathcal{X}} \leqslant \mathcal{A}$*.*

Proof. Let $\mathfrak{u} \in \mathcal{X}$ satisfy

$$\mathfrak{L}(\mathfrak{u}) - \mathcal{N}(\mathfrak{u}) = -\zeta\mathfrak{L}(\mathfrak{u}) - \zeta\mathcal{N}(-\mathfrak{u}).$$

Then,

$$\mathfrak{L}(\mathfrak{u}) = \frac{1}{1+\zeta}\mathcal{N}(\mathfrak{u}) - \frac{\zeta}{1+\zeta}\mathcal{N}(-\mathfrak{u}).$$

So, from the definitions of $\mathfrak{L}$ and $\mathcal{N}$, for any $\xi \in \mathfrak{J}$ we get

$$\begin{aligned}\mathfrak{L}\mathfrak{u}(\xi) = {}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{u}(\xi) = \frac{1}{1+\zeta}&\Big[\mathcal{F}(\xi, \mathfrak{u}(\xi), \mathfrak{u}(p\xi)) + \mathcal{G}(\xi, \mathfrak{u}(\xi), \mathfrak{u}((1-p)\xi))\Big] \\ &- \frac{\zeta}{1+\zeta}\Big[\mathcal{F}(\xi, -\mathfrak{u}(\xi), -\mathfrak{u}(p\xi)) + \mathcal{G}(\xi, -\mathfrak{u}(\xi), -\mathfrak{u}((1-p)\xi))\Big].\end{aligned}$$

By Theorem 2.5, we get

$$\begin{aligned}\mathfrak{u}(\xi) = c_0 + \frac{1}{\zeta+1}&\Big[\mathfrak{I}_{0^+}^{\alpha;\Psi}\Big(\mathcal{F}(s, \mathfrak{u}(s), \mathfrak{u}(ps) + \mathcal{G}(s, \mathfrak{u}(s), \mathfrak{u}((1-p)s))\Big)(\xi) \\ &- \zeta\mathfrak{I}_{0^+}^{\alpha;\Psi}\Big(\mathcal{F}(s, -\mathfrak{u}(s), -\mathfrak{u}(ps)) + \mathcal{G}(s, -\mathfrak{u}(s), -\mathfrak{u}((1-p)s))\Big)(\xi)\Big],\end{aligned}$$

where $c_0 = \mathfrak{u}(0)$. Thus, for every $\xi \in \mathfrak{J}$ we obtain

$$\begin{aligned}&|\mathfrak{u}(\xi)| \\ &\quad\leqslant |c_0| + \frac{1}{(\zeta+1)\Gamma(\alpha)}\int_0^{\xi} \Psi'(s)\left(\Psi(\xi) - \Psi(s)\right)^{\alpha-1} \left|\mathcal{F}\left(s, \mathfrak{u}(s), \mathfrak{u}(ps)\right)\right| \mathrm{d}s \\ &\qquad + \frac{1}{(\zeta+1)\Gamma(\alpha)}\int_0^{\xi} \Psi'(s)\left(\Psi(\xi) - \Psi(s)\right)^{\alpha-1} \left|\mathcal{G}\left(s, \mathfrak{u}(s), \mathfrak{u}((1-p)s)\right)\right| \mathrm{d}s \\ &\qquad + \frac{\zeta}{(\zeta+1)\Gamma(\alpha)}\int_0^{\xi} \Psi'(s)\left(\Psi(\xi) - \Psi(s)\right)^{\alpha-1} \left|\mathcal{F}\left(s, -\mathfrak{u}(s), -\mathfrak{u}(ps)\right)\right| \mathrm{d}s \\ &\qquad + \frac{\zeta}{(\zeta+1)\Gamma(\alpha)}\int_0^{\xi} \Psi'(s)\left(\Psi(\xi) - \Psi(s)\right)^{\alpha-1} \left|\mathcal{G}\left(s, -\mathfrak{u}(s), -\mathfrak{u}((1-p)s)\right)\right| \mathrm{d}s \\ &\leqslant |c_0| + \frac{1}{(\zeta+1)\Gamma(\alpha)}\int_0^{\xi} \Psi'(s)\left(\Psi(\xi) - \Psi(s)\right)^{\alpha-1} \left|\mathcal{F}\left(s, \mathfrak{u}(s), \mathfrak{u}(ps)\right) - \mathcal{F}(s,0,0)\right| \mathrm{d}s \\ &\qquad + \frac{1}{(\zeta+1)\Gamma(\alpha)}\int_0^{\xi} \Psi'(s)\left(\Psi(\xi) - \Psi(s)\right)^{\alpha-1} \left|\mathcal{F}(s,0,0)\right| \mathrm{d}s \\ &\qquad + \frac{1}{(\zeta+1)\Gamma(\alpha)}\int_0^{\xi} \Psi'(s)\left(\Psi(\xi) - \Psi(s)\right)^{\alpha-1} \left|\mathcal{G}\left(s, \mathfrak{u}(s), \mathfrak{u}((1-p)s)\right) - \mathcal{G}(s,0,0)\right| \mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(\zeta+1)\Gamma(\alpha)}\int_0^{\xi}\Psi'(s)\left(\Psi(\xi)-\Psi(s)\right)^{\alpha-1}\left|\mathcal{G}(s,0,0)\right|\,\mathrm{d}s\\
&+\frac{\zeta}{(\zeta+1)\Gamma(\alpha)}\int_0^{\xi}\Psi'(s)\left(\Psi(\xi)-\Psi(s)\right)^{\alpha-1}\left|\mathcal{F}\left(s,-\mathfrak{u}(s),-\mathfrak{u}(ps)\right)-\mathcal{F}(s,0,0)\right|\,\mathrm{d}s\\
&+\frac{\zeta}{(\zeta+1)\Gamma(\alpha)}\int_0^{\xi}\Psi'(s)\left(\Psi(\xi)-\Psi(s)\right)^{\alpha-1}\left|\mathcal{F}(s,0,0)\right|\,\mathrm{d}s\\
&+\frac{\zeta}{(\zeta+1)\Gamma(\alpha)}\int_0^{\xi}\Psi'(s)\left(\Psi(\xi)-\Psi(s)\right)^{\alpha-1}\left|\mathcal{G}\left(s,-\mathfrak{u}(s),-\mathfrak{u}((1-p)s)\right)-\mathcal{G}(s,0,0)\right|\,\mathrm{d}s\\
&+\frac{\zeta}{(\zeta+1)\Gamma(\alpha)}\int_0^{\xi}\Psi'(s)\left(\Psi(\xi)-\Psi(s)\right)^{\alpha-1}\left|\mathcal{G}(s,0,0)\right|\,\mathrm{d}s\\
\leqslant{}& |c_0|+\frac{2(\mathcal{F}^*+\mathcal{G}^*)\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}}{\Gamma(\alpha+1)}+\frac{2\left(\gamma_1^*+\gamma_2^*+\eta_1^*+\eta_2^*\right)}{\Gamma(\alpha+1)}\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}\|\mathfrak{u}\|_{\mathcal{X}}.
\end{aligned}
$$

Thus,

$$
\|\mathfrak{u}\|_{\mathcal{X}}\leqslant|c_0|+\frac{2(\mathcal{F}^*+\mathcal{G}^*)\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}}{\Gamma(\alpha+1)}+\frac{2\left(\gamma_1^*+\gamma_2^*+\eta_1^*+\eta_2^*\right)}{\Gamma(\alpha+1)}\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}\|\mathfrak{u}\|_{\mathcal{X}}.
$$

Consequently, we deduce that

$$
\|\mathfrak{u}\|_{\mathcal{X}}\leqslant\frac{|c_0|+\dfrac{2(\mathcal{F}^*+\mathcal{G}^*)\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}}{\Gamma(\alpha+1)}}{\left[1-\dfrac{2\left(\gamma_1^*+\gamma_2^*+\eta_1^*+\eta_2^*\right)}{\Gamma(\alpha+1)}\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}\right]}=:\mathcal{A}.
$$

The demonstration is completed. □

Lemma 3.5 *If the conditions* **(A1)** *and* (3.4) *are satisfied, then there exists a bounded open set* $\Omega\subset\mathcal{X}$ *with*

$$
\mathfrak{L}(\mathfrak{u})-\mathcal{N}(\mathfrak{u})\neq-\zeta[\mathfrak{L}(\mathfrak{u})+\mathcal{N}(-\mathfrak{u})] \tag{3.5}
$$

for any $\mathfrak{u}\in\partial\Omega$ *and any* $\zeta\in(0,1]$.

Proof. In view of Lemma 3.4, there exists a positive constant $\mathcal{A}$ such that if

$$
\mathfrak{L}(\mathfrak{u})-\mathcal{N}(\mathfrak{u})=-\zeta[\mathfrak{L}(\mathfrak{u})+\mathcal{N}(-\mathfrak{u})]
$$

holds for some $\mathfrak{u}$ and $\zeta\in(0,1]$, then $\|\mathfrak{u}\|_{\mathcal{X}}\leqslant\mathcal{A}$. So, if

$$
\Omega:=\{\mathfrak{u}\in\mathcal{X}:\|\mathfrak{u}\|_{\mathcal{X}}<\vartheta\} \tag{3.6}
$$

with $\vartheta>\mathcal{A}$, we deduce that

$$
\mathfrak{L}(\mathfrak{u})-\mathcal{N}(\mathfrak{u})\neq-\zeta[\mathfrak{L}(\mathfrak{u})-\mathcal{N}(-\mathfrak{u})]
$$

for all $\mathfrak{u}\in\partial\Omega=\{\mathfrak{u}\in\mathcal{X}:\|\mathfrak{u}\|_{\mathcal{X}}=\vartheta\}$ and $\zeta\in(0,1]$. □

Theorem 3.6 *Assume* **(A1)** *and* (3.4). *Then, there exists at least one solution for the problem* (1.1)–(1.2) *in* $\operatorname{dom}\mathfrak{L}\cap\overline{\Omega}$.

Proof. It is clear that the set Ω defined in (3.6) is symmetric, $0 \in \Omega$ and $\mathcal{X} \cap \overline{\Omega} = \overline{\Omega} \neq \emptyset$. In addition, by Lemma 3.5,

$$\mathfrak{L}(\mathfrak{u}) - \mathcal{N}(\mathfrak{u}) \neq -\zeta[\mathfrak{L}(\mathfrak{u}) - \mathcal{N}(-\mathfrak{u})]$$

for each $\mathfrak{u} \in \mathcal{X} \cap \partial\Omega = \partial\Omega$ and each $\zeta \in (0, 1]$. By Lemma 2.11, the problem (1.1)–(1.2) has at least one solution in $\operatorname{dom}\mathfrak{L} \cap \overline{\Omega}$. This completes the demonstration. □

Theorem 3.7 *Let* **(A1)** *be satisfied. Moreover, we assume that*

(A2) *there exist constants* $\overline{\gamma} > 0$ *and* $\overline{\eta} \geqslant 0$ *such that*

$$|\mathcal{F}(\xi, \mathfrak{u}, \upsilon) - \mathcal{F}(\xi, \bar{\mathfrak{u}}, \bar{\upsilon})| \geqslant \overline{\gamma}|\mathfrak{u} - \bar{\mathfrak{u}}| - \overline{\eta}|\upsilon - \bar{\upsilon}|$$

for every $\xi \in \mathfrak{J}$ *and* $\mathfrak{u}, \bar{\mathfrak{u}}, \upsilon, \bar{\upsilon} \in \mathfrak{R}$.

If

$$\frac{\overline{\eta} + \gamma_2^* + \eta_2^*}{\overline{\gamma}} + \frac{2\left(\gamma_1^* + \gamma_2^* + \eta_1^* + \eta_2^*\right)\left(\Psi(\mathfrak{b}) - \Psi(0)\right)^{\alpha}}{\Gamma(\alpha+1)} < 1, \tag{3.7}$$

then the problem (1.1)–(1.2) *has a unique solution in* $\operatorname{dom}\mathfrak{L} \cap \overline{\Omega}$.

Proof. Note that the condition (3.7) is stronger than the condition (3.4). Hence, by Theorem 3.6 we know that the problem (1.1)–(1.2) has at least one solution in $\operatorname{dom}\mathfrak{L} \cap \overline{\Omega}$.

Now, we prove its uniqueness. Suppose that the problem (1.1)–(1.2) has two different solutions $\mathfrak{u}_1, \mathfrak{u}_2 \in \operatorname{dom}\mathfrak{L} \cap \overline{\Omega}$. Then, for each $\xi \in \mathfrak{J}$ we have

$$\begin{aligned}
{}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{u}_1(\xi) &= \mathcal{F}\left(\xi, \mathfrak{u}_1(\xi), \mathfrak{u}_1(p\xi)\right) + \mathcal{G}\left(\xi, \mathfrak{u}_1(\xi), \mathfrak{u}_1((1-p)\xi)\right), \\
{}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{u}_2(\xi) &= \mathcal{F}\left(\xi, \mathfrak{u}_2(\xi), \mathfrak{u}_2(p\xi)\right) + \mathcal{G}\left(\xi, \mathfrak{u}_2(\xi), \mathfrak{u}_2((1-p)\xi)\right)
\end{aligned}$$

and

$$\mathfrak{u}_1(0) = \mathfrak{u}_1(\mathfrak{b}),\ \mathfrak{u}_2(0) = \mathfrak{u}_2(\mathfrak{b}).$$

Let $\mathfrak{U}(\xi) := \mathfrak{u}_1(\xi) - \mathfrak{u}_2(\xi)$ for all $\xi \in \mathfrak{J}$. Then,

$$\begin{aligned}
\mathfrak{L}\mathfrak{U}(\xi) &= {}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{U}(\xi) \\
&= {}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{u}_1(\xi) - {}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{u}_2(\xi) \\
&= \mathcal{F}\left(\xi, \mathfrak{u}_1(\xi), \mathfrak{u}_1(p\xi)\right) + \mathcal{G}\left(\xi, \mathfrak{u}_1(\xi), \mathfrak{u}_1((1-p)\xi)\right) \\
&\qquad - \mathcal{F}\left(\xi, \mathfrak{u}_2(\xi), \mathfrak{u}_2(p\xi)\right) - \mathcal{G}\left(\xi, \mathfrak{u}_2(\xi), \mathfrak{u}_2((1-p)\xi)\right).
\end{aligned} \tag{3.8}$$

Using the fact that $\operatorname{img}\mathfrak{L} = \ker\mathcal{Q}$, we have

$$\begin{aligned}
\int_0^{\mathfrak{b}} \Psi'(s)(\Psi(\mathfrak{b}) - \Psi(s))^{\alpha-1}\Big[&\mathcal{F}(s, \mathfrak{u}_1(s), \mathfrak{u}_1(ps)) + \mathcal{G}\left(s, \mathfrak{u}_1(s), \mathfrak{u}_1((1-p)s)\right) \\
&- \mathcal{F}(s, \mathfrak{u}_2(s), \mathfrak{u}_2(ps)) - \mathcal{G}\left(s, \mathfrak{u}_2(s), \mathfrak{u}_2((1-p)s)\right)\Big]\,\mathrm{d}s = 0.
\end{aligned}$$

Since $\mathcal{F}$ is a continuous function, there exists $\xi_0 \in [0, \mathfrak{b}]$ such that

$$\begin{aligned}
\mathcal{F}(\xi_0, \mathfrak{u}_1(\xi_0), \mathfrak{u}_1(p\xi_0)) + \mathcal{G}\left(\xi_0, \mathfrak{u}_1(\xi_0), \mathfrak{u}_1((1-p)\xi_0)\right) - \mathcal{F}(\xi_0, \mathfrak{u}_2(\xi_0), \mathfrak{u}_2(p\xi_0)) \\
- \mathcal{G}\left(\xi_0, \mathfrak{u}_2(\xi_0), \mathfrak{u}_2((1-p)\xi_0)\right) = 0.
\end{aligned}$$

In view of **(A1)** and **(A2)**, we have

$$\begin{aligned}
&\left|\mathfrak{u}_1(\xi_0)-\mathfrak{u}_2(\xi_0)\right| \\
&\quad\leqslant \frac{1}{\overline{\gamma}}\left[\overline{\eta}\left|\mathfrak{u}_1(p\xi_0)-\mathfrak{u}_2(p\xi_0)\right|+\gamma_2^*\left|\mathfrak{u}_1(\xi_0)-\mathfrak{u}_2(\xi_0)\right|+\eta_2^*\left|\mathfrak{u}_1((1-p)\xi_0)-\mathfrak{u}_2((1-p)\xi_0)\right|\right] \\
&\quad\leqslant \frac{\overline{\eta}+\gamma_2^*+\eta_2^*}{\overline{\gamma}}\left\|\mathfrak{u}_1-\mathfrak{u}_2\right\|_{\mathcal{X}}.
\end{aligned}$$

Hence,

$$\left|\mathfrak{U}(\xi_0)\right| \leqslant \frac{\overline{\eta}+\gamma_2^*+\eta_2^*}{\overline{\gamma}}\left\|\mathfrak{U}\right\|_{\mathcal{X}}. \tag{3.9}$$

On the other hand, by Theorem 2.5, we have

$$\mathfrak{I}_{0^+}^{\alpha;\Psi}\,{}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{U}(\xi)=\mathfrak{U}(\xi)-\mathfrak{U}(0),$$

which implies that

$$\mathfrak{U}(0)=\mathfrak{U}(\xi_0)-\mathfrak{I}_{0^+}^{\alpha;\Psi}\,{}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{U}(\xi_0),$$

and therefore

$$\mathfrak{U}(\xi)=\mathfrak{I}_{0^+}^{\alpha;\Psi}\,{}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{U}(\xi)+\mathfrak{U}(\xi_0)-\mathfrak{I}_{0^+}^{\alpha;\Psi}\,{}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{U}(\xi_0).$$

Using (3.9), for every $\xi\in\mathfrak{J}$, we obtain

$$\begin{aligned}
\left|\mathfrak{U}(\xi)\right| &\leqslant \left|\mathfrak{I}_{0^+}^{\alpha;\Psi}\,{}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{U}(\xi)\right|+\left|\mathfrak{U}(\xi_0)\right|+\left|\mathfrak{I}_{0^+}^{\alpha;\Psi}\,{}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{U}(\xi_0)\right| \\
&\leqslant \frac{\overline{\eta}+\gamma_2^*+\eta_2^*}{\overline{\gamma}}\left\|\mathfrak{U}\right\|_{\mathcal{X}}+\frac{2\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}}{\Gamma(\alpha+1)}\left\|{}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{U}\right\|_{\mathcal{X}}.
\end{aligned} \tag{3.10}$$

By (3.8) and **(A1)**, we find that

$$\begin{aligned}
\left|{}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{U}(\xi)\right| &= \Big|\left(\mathcal{F}\left(\xi,\mathfrak{u}_1(\xi),\mathfrak{u}_1(p\xi)\right)-\mathcal{F}\left(\xi,\mathfrak{u}_2(\xi),\mathfrak{u}_2(p\xi)\right)\right) \\
&\qquad+\left(\mathcal{G}\left(\xi,\mathfrak{u}_1(\xi),\mathfrak{u}_1((1-p)\xi)\right)-\mathcal{G}\left(\xi,\mathfrak{u}_2(\xi),\mathfrak{u}_2((1-p)\xi)\right)\right)\Big| \\
&\leqslant \left(\gamma_1^*+\gamma_2^*+\eta_1^*+\eta_2^*\right)\left\|\mathfrak{U}\right\|_{\mathcal{X}}.
\end{aligned}$$

Then,

$$\left\|{}^{c}\mathfrak{D}_{0^+}^{\alpha;\Psi}\mathfrak{U}\right\|_{\mathcal{X}} \leqslant \left(\gamma_1^*+\gamma_2^*+\eta_1^*+\eta_2^*\right)\left\|\mathfrak{U}\right\|_{\mathcal{X}}. \tag{3.11}$$

Substituting (3.11) into the right-hand side of (3.10), for every $\xi\in\mathfrak{J}$ we get

$$\left|\mathfrak{U}(\xi)\right| \leqslant \left[\frac{\overline{\eta}+\gamma_2^*+\eta_2^*}{\overline{\gamma}}+\frac{2\left(\gamma_1^*+\gamma_2^*+\eta_1^*+\eta_2^*\right)\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}}{\Gamma(\alpha+1)}\right]\left\|\mathfrak{U}\right\|_{\mathcal{X}}.$$

Therefore,

$$\left\|\mathfrak{U}\right\|_{\mathcal{X}} \leqslant \left[\frac{\overline{\eta}+\gamma_2^*+\eta_2^*}{\overline{\gamma}}+\frac{2\left(\gamma_1^*+\gamma_2^*+\eta_1^*+\eta_2^*\right)\left(\Psi(\mathfrak{b})-\Psi(0)\right)^{\alpha}}{\Gamma(\alpha+1)}\right]\left\|\mathfrak{U}\right\|_{\mathcal{X}}.$$

Hence, by (3.7), we conclude that

$$\left\|\mathfrak{U}\right\|_{\mathcal{X}}=0.$$

As a result, for any $\xi\in\mathfrak{J}$, we get

$$\mathfrak{U}(\xi)=0,$$

which implies that

$$\mathfrak{u}_1(\xi)=\mathfrak{u}_2(\xi).$$

This completes the proof. □

4 An example

We present an example of a nonlinear fractional differential equation of pantograph type with Ψ-Caputo derivative operator to illustrate our main result. Let us consider the following equation

$$
\begin{gathered}
{}^{c}\mathfrak{D}_{0^+}^{\frac{1}{5};2^\xi}\mathfrak{u}(\xi) = \mathcal{F}\left(\xi, \mathfrak{u}(\xi), \mathfrak{u}(p\xi)\right) + \mathcal{G}\left(\xi, \mathfrak{u}(\xi), \mathfrak{u}((1-p)\xi)\right), \ \xi \in \mathfrak{J}, \\
\mathfrak{u}(0) = \mathfrak{u}(1),
\end{gathered}
$$

where for any $\xi \in \mathfrak{J}$,

$$
\mathcal{F}\left(\xi, \mathfrak{u}(\xi), \mathfrak{u}\left(p\xi\right)\right) = \frac{\ln\left(e+\sqrt{\xi}\right)}{e^2+\sqrt{\xi+1}} + \frac{e^{-\xi}}{3(1+\xi)}\mathfrak{u}(\xi) + \frac{\xi}{95\sqrt{\pi}}\cos\mathfrak{u}\left(\frac{\xi}{3}\right),
$$

and

$$
\mathcal{G}\left(\xi, \mathfrak{u}(\xi), \mathfrak{u}\left((1-p)\xi\right)\right) = \frac{e^{\xi}}{13} + \frac{\xi+7}{17e^7\sqrt{\pi}}\sin\mathfrak{u}(\xi) + \frac{e^{-11-\xi}}{113\left(1+\mathfrak{u}\left(\frac{2}{3}\xi\right)\right)}.
$$

Here, $\mathfrak{J} := [0,1]$, $\alpha = \frac{1}{5}$, $\Psi(\xi) = 2^\xi$ and $p = \frac{1}{3}$.

It is clear that the functions $\mathcal{F}, \mathcal{G} \in C(\mathfrak{J} \times \mathfrak{R} \times \mathfrak{R}, \mathfrak{R})$. Furthermore, for all $\xi \in \mathfrak{J}$ and $\mathfrak{u}, \bar{\mathfrak{u}}, \upsilon, \bar{\upsilon} \in \mathfrak{R}$, we get

$$
\begin{gathered}
\left|\mathcal{F}\left(\xi, \mathfrak{u}, \upsilon\right) - \mathcal{F}\left(\xi, \bar{\mathfrak{u}}, \bar{\upsilon}\right)\right| \leqslant \frac{e^{-\xi}}{3(1+\xi)}|\mathfrak{u}-\bar{\mathfrak{u}}| + \frac{\xi}{95\sqrt{\pi}}\left|\upsilon-\bar{\upsilon}\right|, \\
\left|\mathcal{G}\left(\xi, \mathfrak{u}, \upsilon\right) - \mathcal{G}\left(\xi, \bar{\mathfrak{u}}, \bar{\upsilon}\right)\right| \leqslant \frac{\xi+7}{17e^7\sqrt{\pi}}|\mathfrak{u}-\bar{\mathfrak{u}}| + \frac{e^{-11-\xi}}{113}\left|\upsilon-\bar{\upsilon}\right|
\end{gathered}
$$

and

$$
\left|\mathcal{F}\left(\xi, \mathfrak{u}, \upsilon\right) - \mathcal{F}\left(\xi, \bar{\mathfrak{u}}, \bar{\upsilon}\right)\right| \geqslant \overline{\gamma}|\mathfrak{u}-\bar{\mathfrak{u}}| - \overline{\eta}\left|\upsilon-\bar{\upsilon}\right|,
$$

which implies that **(A1)** and **(A2)** are satisfied with

$$
\begin{aligned}
\gamma_1(\xi) &= \frac{e^{-\xi}}{3(1+\xi)}, & \gamma_2(\xi) &= \frac{\xi+7}{17e^7\sqrt{\pi}}, & \overline{\gamma} &= \frac{e^{-1}}{6}, \\
\eta_1(\xi) &= \frac{\xi}{95\sqrt{\pi}}, & \eta_2(\xi) &= \frac{e^{-11-\xi}}{113}, & \overline{\eta} &= \frac{1}{95\sqrt{\pi}}.
\end{aligned}
$$

By simple calculations, we get $\gamma_1^* = \frac{1}{3}$, $\eta_1^* = \frac{1}{95\sqrt{\pi}}$, $\gamma_2^* = \frac{8}{17e^7\sqrt{\pi}}$, $\eta_2^* = \frac{1}{113e^{11}}$ and

$$
\frac{\overline{\eta}+\gamma_2^*+\eta_2^*}{\overline{\gamma}} + \frac{2\left(\gamma_1^*+\gamma_2^*+\eta_1^*+\eta_2^*\right)\left(\Psi(\mathfrak{b})-\Psi(0)\right)^\alpha}{\Gamma(\alpha+1)} \approx 0.840359 < 1.
$$

So, by Theorem 3.7, our problem has a unique solution.

5 Conclusions

The existence and uniqueness of periodic solutions for our proposed fractional boundary value problem has been successfully investigated for the fractional pantograph differential equations with Ψ-Caputo. Our results extend and complement some existing ones. For example, By setting $\mathcal{G} \equiv 0$, our problem (1.1) equipped with periodic condition (1.2) traits the case not studied $(a+b=0)$ in [29]. An application example of our problem has been provided to validate our obtained findings.

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