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THE ILL-POSED CAUCHY PROBLEM BY CONTROLLABILITY: THE PARABOLIC CASE

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Abstract. In this paper, we are dealing with the ill-posed Cauchy problem for a parabolic operator. To do this, we interpret the problem as an inverse problem, and therefore a controllability problem. This point of view induces a regularization method that makes it possible, on the one hand, to characterize the existence of a regular solution to the problem. On the other hand, this method makes it possible to obtain a singular optimality system for the optimal control, without using any additional assumptions, such as that of non-vacuity of the interior of the sets of admissible controls, an assumption that many analyses have had to use. From this point of view, the regularization method presented here, called controllability method, is original for the analyzed problem.

Keywords: singular distributed system, optimal control, singular optimality system, the ill-posed Cauchy problem, controllability method, inverse problem.

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1 Statement of the problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded and regular domain of class $\mathcal{C}^1$, with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are disjointed, regular and with superficial positive measures. For $T \in \mathbb{R}_+^*$, we note

$$Q = \Omega\times]0,T[\qquad \text{and} \qquad \Sigma = \Gamma\times]0,T[,$$

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so

$$\Sigma = \Sigma_0 \cup \Sigma_1, \quad \text{with} \quad \Sigma_0 = \Gamma_0 \times]0, T[\quad \text{and} \quad \Sigma_1 = \Gamma_1 \times]0, T[\,.$$

Then, let us consider in Q, the boundary value problem

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z = 0 & \text{in } Q, \\ z(x,0) = 0 & \text{in } \Omega, \\ z = v_0, \quad \frac{\partial z}{\partial \nu} = v_1 & \text{on } \Sigma_0, \end{cases} \tag{1.1}$$

where, for given $v_0, v_1 \in L^2(\Sigma_0)$, $z = z(v_0, v_1)$ verifies (1.1).

Problem (1.1) is the ill-posed Cauchy problem for the heat operator; it is well known that this problem is ill-posed in Hadamard's sense. That is to say, for a given vector $v = (v_0, v_1) \in \left(L^2(\Sigma_0)\right)^2$, the problem does not always admit a solution, and it may lead to instability of the solution when it exists.

We therefore consider a priori the pairs (v, z) such as

$$v = (v_0, v_1) \in \left(L^2(\Sigma_0)\right)^2 \qquad \text{and} \qquad z \in L^2(Q), \tag{1.2}$$

where z is a solution of (1.1) for given v. It is said that such pairs constitute the control-state pairs set.

Remark 1.1 *It is important to note that, when it exists, the solution to the ill-posed Cauchy problem is unique.*

A control-state pair (v, z) will be said admissible if

$$\begin{cases} v = (v_0, v_1) \in \mathcal{U}_{ad} = \mathcal{U}_{ad}^0 \times \mathcal{U}_{ad}^1, \\ \text{where } \mathcal{U}_{ad}^0 \text{ and } \mathcal{U}_{ad}^1 \text{ are non-empty convex closed subsets of } L^2(\Sigma_0), \end{cases} \tag{1.3}$$

with (v, z) satisfying (1.1). We use the notation $(v, z) \in \mathcal{A}$ to say that $\mathcal{A}$ is the set of admissible control-state pairs and assume

$$\mathcal{A} \neq \emptyset. \tag{1.4}$$

Given $(v, z) \in \mathcal{A}$, we introduce the functional

$$J(v, z) = \frac{1}{2}\|z - z_d\|^2_{L^2(Q)} + \frac{N_0}{2}\|v_0\|^2_{L^2(\Sigma_0)} + \frac{N_1}{2}\|v_1\|^2_{L^2(\Sigma_0)}, \tag{1.5}$$

where $z_d \in L^2(Q)$ and $N_0, N_1 \in \mathbb{R}_+^*$.

Functional J is the cost function. The optimal control problem consists then to find

$$\inf \left\{ J(v, z) \;:\; (v, z) \in \mathcal{A} \right\}. \tag{1.6}$$

The assumption (1.4) and the structure of J easily show that problem (1.6) admits a unique solution, the optimal control-state pair (u, y).

The cost function J being differentiable, the first order Euler-Lagrange conditions make it possible to establish that the optimal control-state pair of (1.1), (1.5) and (1.6) satisfies the optimality condition: $\forall\, (v, z) \in \mathcal{A}$,

$$(y - z_d, z - y)_{L^2(Q)} + N_0(u_0, v_0 - u_0)_{L^2(\Sigma_0)} + N_1(u_1, v_1 - u_1)_{L^2(\Sigma_0)} \geq 0. \tag{1.7}$$

It remains to characterize the optimal pair (u, y) through a singular optimality system.

Remark 1.2 *Many authors have studied the control of the ill-posed Cauchy problem. Examples include, among others,*

- *in the elliptic case, J.-L. Lions in [4], O. Nakoulima in [6], G. Mophou and O. Nakoulima in [7], and A. Berhail and A. Omrane in [8];*
- *in the parabolic and hyperbolic case, M. Barry, G.B. Ndiaye and O. Nakoulima in [1], J.P. Kernevez in [3], and M. Barry and G.B. Ndiaye in [2].*

Nevertheless, in general, the problem remains open. Indeed, to our knowledge, almost all of the work carried out concerns only specific cases of controls (v_0, v_1)*, such as the following:*

- $\mathcal{U}_{ad}^0 = \mathcal{U}_{ad}^1 = L^2(\Sigma_0)$*, the "unconstrained" case;*
- $\mathcal{U}_{ad}^0 = \mathcal{U}_{ad}^1 = \left(L^2(\Sigma_0)\right)^+$,
- *or with the additional Slater type assumption that*

$$\text{the interiors of } \mathcal{U}_{ad}^0 \text{ and/or } \mathcal{U}_{ad}^1 \text{ are non-empty in } L^2(\Gamma_0). \tag{1.8}$$

This paper aims to constitute an argument in favor of the conjecture of J.-L. Lions. Indeed, J.-L. Lions conjectures that one should be able to solve the problem only with the usual assumptions of non-vacuity, convexity and closure of the control sets $\mathcal{U}_{ad}^0$ and $\mathcal{U}_{ad}^1$.

We succeed here, even managing to characterize the existence of a regular solution to the ill-posed Cauchy problem.

The paper is organized as follows. Section 2 is devoted to interpreting the initial problem as an inverse problem. In Section 3, we return to the control problem, starting by regularizing it via the controllability results previously obtained. After establishing the convergence of the process in Section 3.2, then the approached optimality system in Section 3.3, we end in Section 3.4 with the singular optimality system for the initial problem.

2 Controllability for the ill-posed parabolic Cauchy problem

We introduce here a point of view which, it seems to us, is new concerning the ill-posed Cauchy problem. Which point of view consists in interpreting the problem as an inverse problem, and therefore a controllability problem.

We establish that, when it exists, the solution of the ill-posed Cauchy problem is common solution of a system of two inverse problems. Then, we succeed in establishing a necessary and sufficient condition for the existence, not only of a solution, but of a regular solution to the problem.

More precisely, we consider the systems

$$\begin{cases} \frac{\partial y_1}{\partial t} - \Delta y_1 = 0 & \text{in } Q, \\ y_1(x,0) = 0 & \text{in } \Omega, \\ y_1 = v_0 & \text{on } \Sigma_0, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \frac{\partial y_2}{\partial t} - \Delta y_2 = 0 & \text{in } Q, \\ y_2(x,0) = 0 & \text{in } \Omega, \\ \frac{\partial y_2}{\partial \nu} = v_1 & \text{on } \Sigma_0, \end{cases} \tag{2.2}$$

moreover,

$$\frac{\partial y_1}{\partial \nu} = v_1 \qquad \text{and} \qquad y_2 = v_0 \quad \text{on } \Sigma_0. \tag{2.3}$$

We can then interpret (2.1), (2.2) and (2.3) as a system of inverse problems, that is to say for which we have a datum and an observation on the border Σ_0, but no information on the border Σ_1.

Then, we consider the following inverse problem: given $(v_0, v_1) \in \big(L^2(\Sigma_0)\big)^2$,
find $(w_1, w_2) \in \big(L^2(\Sigma_1)\big)^2$ such that, if y_1 and y_2 are respective solutions of

$$\begin{cases} \frac{\partial y_1}{\partial t} - \Delta y_1 = 0 & \text{in } Q, \\ y_1(x,0) = 0 & \text{in } \Omega, \\ y_1 = v_0 & \text{on } \Sigma_0, \quad \frac{\partial y_1}{\partial \nu} = w_1 \quad \text{on } \Sigma_1, \end{cases} \tag{2.4}$$

and

$$\begin{cases} \frac{\partial y_2}{\partial t} - \Delta y_2 = 0 & \text{in } Q, \\ y_2(x,0) = 0 & \text{in } \Omega, \\ \frac{\partial y_2}{\partial \nu} = v_1 & \text{on } \Sigma_0, \quad y_2 = w_2 \quad \text{on } \Sigma_1, \end{cases} \tag{2.5}$$

then y_1 and y_2 further satisfy the conditions (2.3).

Remark 2.1 *The symmetric character of the roles played by y_1 and y_2 in the formulation of the controllability problem is obvious. Consequently, one could very well be satisfied with only one of these states in the definition of the problem, thus considering one or the other of problems* (2.4) *and* (2.5) *with the corresponding observation objective in* (2.3)*. This is evidenced by the first part of the proof of Theorem 2.11.*

As far as the present analysis is concerned, it is precisely this symmetrical nature of the roles of y_1 and y_2 that motivates their simultaneous use (which facilitates, perhaps for a short time, the continuation of the analysis), but also the wish to remain faithful to the framework of Cauchy's problem.

Remark 2.2 (Well-defined nature of the controllability problem)
For $z \in L^2(Q)$ with $\frac{\partial z}{\partial t} - \Delta z = 0$, we know that

$$z|_\Sigma \in H^{-1/2,-1/4}(\Sigma)\,, \qquad \frac{\partial z}{\partial \nu}\Big|_\Sigma \in H^{-3/2,-3/4}(\Sigma) \qquad \textit{and} \qquad z(0),\, z(T) \in H^{-1}(\Omega).$$

Thus, seeking, within the framework of controllability problems, functions of $L^2(\Sigma_1)$ making it possible to reach, or if not, approaching, the targets fixed still in $L^2(\Sigma_0)$, it is necessary that the accessible states y_1 and y_2 are in $H^{3/2,3/4}(Q)$ and $H^{1/2,1/4}(Q)$, respectively.

Hence the necessity, within the framework of the problem of optimal control of the parabolic Cauchy problem, is to consider, beyond the assumption of non-vacuity $\mathcal{A} \neq \emptyset$, that the set

$$\Big\{(v,z) \in \mathcal{A} :\ z \in H^{3/2,3/4}(Q)\Big\}$$

is nonempty.

Remark 2.3 *If the system* (2.1)-(2.3) *admits a solution, then this latter verifies*

$$y_1 = z = y_2,$$

where $(v = (v_0, v_1), z)$ *constitutes a control-state pair for the Cauchy problem.*

Remark 2.4 *Problems* (2.4) *and* (2.5)*, the mixed Dirichlet-Neumann problems for the heat operator, are then two well-posed problems in the sense of Hadamard.*

With these notations, conditions (2.3) become

$$\frac{\partial y_1}{\partial \nu}(v_0, w_1)\Big|_{\Sigma_0} = v_1 \qquad \text{and} \qquad y_2(v_1, w_2)\,|_{\Sigma_0} = v_0. \tag{2.6}$$

Finally, and to fix the vocabulary, we will say that the problem (2.4)-(2.6) constitutes a problem of exact controllability and that, system (2.4)-(2.5) is exactly controllable in (v_1, v_0) if there exist $w_1, w_2 \in L^2(\Sigma_1)$ satisfying (2.6).

Remark 2.5 *By linearity of mappings*

$$(v_0, w_1) \longmapsto y_1(v_0, w_1) = y_1(v_0, 0) + y_1(0, w_1)$$

and

$$(v_1, w_2) \longmapsto y_2(v_1, w_2) = y_2(v_1, 0) + y_2(0, w_2)\,,$$

the exact controllability problem (2.4)-(2.6) *is equivalent to the following:*

$$\begin{cases} \text{Find } w_1, w_2 \in L^2(\Sigma_1) \text{ such that the solutions} \\ y_1(0, w_1) \quad \text{and} \quad y_2(0, w_2) \text{ verify} \\ \frac{\partial y_1}{\partial \nu}(0, w_1) = 0\,, \quad y_2(0, w_2) = 0 \quad \text{on } \Sigma_0, \end{cases} \tag{2.7}$$

which translates the controllability of the system $(y_1(0, w_1), y_2(0, w_2))$ *in* $(0, 0)$.

A method, to solve (2.7) is the method of approximate controllability, which consists of an approximation, by density, of the problem. This is reflecting in the following proposition.

Proposition 2.6 *Let us denote by*

$$E_1 = \left\{ \frac{\partial y_1}{\partial \nu}(0, w_1)\Big|_{\Sigma_0} ;\ w_1 \in L^2(\Sigma_1) \right\} \quad \text{and} \quad E_2 = \left\{ y_2(0, w_2)\,|_{\Sigma_0} ;\ w_2 \in L^2(\Sigma_1) \right\} \tag{2.8}$$

the sets of zero and one orders traces, on Σ_0*, of the reachable states* y_1 *and* y_2*, respectively.*

Then, we have that

$$\text{sets } E_1 \text{ and } E_2 \text{ are dense in } L^2(\Sigma_0), \tag{2.9}$$

and then we speak of the approximate controllability of the system $(y_1(0, w_1), y_2(0, w_2))$.

Proof. It is clear that E_1 and E_2 constitute vector subspaces of $L^2(\Sigma_0)$. Hence, by the Hahn-Banach Theorem, E_1 and E_2 are dense in $L^2(\Sigma_0)$ if and only if their orthogonal $E_1^\perp$ and $E_2^\perp$ are reduced to $\{0\}$.

Let $k_1 \in E_1^\perp$, so we have

$$\forall\, w_1 \in L^2(\Sigma_1), \qquad \left(k_1, \frac{\partial y_1}{\partial \nu}(0, w_1)\right)_{L^2(\Sigma_0)} = 0.$$

But, by definition of $y_1(0, w_1)$, we have

$$\begin{cases} \frac{\partial y_1}{\partial t}(0, w_1) - \Delta y_1(0, w_1) = 0 & \text{in } Q, \\ y_1(0, w_1)\,(x, 0) = 0 & \text{in } \Omega, \\ y_1(0, w_1) = 0 & \text{on } \Sigma_0, \quad \frac{\partial y_1}{\partial \nu}(0, w_1) = w_1 \quad \text{on } \Sigma_1. \end{cases}$$

This implies, for all $\varphi \in \mathcal{C}^\infty\left(\overline{Q}\right)$,

$$\left(\frac{\partial y_1}{\partial t} - \Delta y_1, \varphi\right)_{L^2(Q)} = \left(\frac{\partial y_1}{\partial t}, \varphi\right)_{L^2(Q)} - (\Delta y_1, \varphi)_{L^2(Q)} = 0,$$

which leads to

$$-\big(y_1, \frac{\partial \varphi}{\partial t}\big)_{L^2(Q)} - (y_1, \Delta\varphi)_{L^2(Q)} - \big(\frac{\partial y_1}{\partial \nu}, \varphi\big)_{L^2(\Sigma_0)} - (w_1, \varphi)_{L^2(\Sigma_1)} + \big(y_1, \frac{\partial \varphi}{\partial \nu}\big)_{L^2(\Sigma_1)} = 0. \quad (2.10)$$

Choosing φ in the above such that

$$\begin{cases} -\frac{\partial \varphi}{\partial t} - \Delta\varphi = 0 & \text{in } Q, \\ \varphi(x, T) = 0 & \text{on } \Omega, \\ \varphi = k_1 & \text{on } \Sigma_0, \quad \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on } \Sigma_1, \end{cases} \quad (2.11)$$

it comes that (2.10) is equivalent to

$$-\left(\frac{\partial y_1}{\partial \nu}, k_1\right)_{L^2(\Sigma_0)} - (w_1, \varphi)_{L^2(\Sigma_1)} = 0, \quad (2.12)$$

where

$$k_1 \in E_1^\perp \iff \left(\frac{\partial y_1}{\partial \nu}, k_1\right)_{L^2(\Sigma_0)} = 0.$$

Hence (2.12) becomes, for all $\varphi \in \mathcal{C}^\infty\left(\overline{Q}\right)$ with (2.11) and for all $w_1 \in L^2(\Sigma_1)$,

$$(w_1, \varphi)_{L^2(\Sigma_1)} = 0. \quad (2.13)$$

But we can still choose in (2.13), $w_1 = \varphi$ on Σ_1, and hence it follows

$$\|\varphi\|^2_{L^2(\Sigma_1)} = 0 \qquad \textit{i.e.} \qquad \varphi = 0 \quad \text{on } \Sigma_1.$$

Thus, with (2.11), φ verifies the Cauchy problem

$$\begin{cases} -\frac{\partial \varphi}{\partial t} - \Delta\varphi = 0 & \text{in } Q, \\ \varphi(x, T) = 0 & \text{on } \Omega, \\ \varphi = 0, \quad \frac{\partial \varphi}{\partial \nu} = 0 & \text{on } \Sigma_1. \end{cases} \quad (2.14)$$

But then, because of the uniqueness of the solution to the Cauchy problem, when it exists, we deduce that

$$\varphi \equiv 0.$$

Thereby,

$$\varphi|_{\Sigma_0} = 0 \quad \textit{i.e.} \quad k_1 = 0.$$

This last equality is valid for all $w_1 \in L^2(\Sigma_1)$, then we have

$$\forall\, k_1 \in E_1^{\perp}, \qquad k_1 = 0.$$

Which means $E_1^{\perp} = \{0\}$, that is to say E_1 is well dense in $L^2(\Sigma_0)$.

Analogously, we get the announced results for E_2. □

The following result is then immediate.

Corollary 2.7 *For all $\varepsilon > 0$, there are $w_{1\varepsilon},\ w_{2\varepsilon} \in L^2(\Sigma_1)$, such that*

$$y_{1\varepsilon} = y_1(0, w_{1\varepsilon}) \in H^{3/2,3/4}(Q) \quad \textit{and} \quad y_{2\varepsilon} = y_2(0, w_{2\varepsilon}) \in H^{1/2,1/4}(Q)$$

are unique solutions of

$$\begin{cases} \frac{\partial y_{1\varepsilon}}{\partial t} - \Delta y_{1\varepsilon} = 0 & \textit{in } Q, \\ y_{1\varepsilon}(x,0) = 0 & \textit{in } \Omega, \\ y_{1\varepsilon} = 0 & \textit{on } \Sigma_0, \quad \frac{\partial y_{1\varepsilon}}{\partial \nu} = w_{1\varepsilon} \quad \textit{on } \Sigma_1, \end{cases} \tag{2.15}$$

$$\begin{cases} \frac{\partial y_{2\varepsilon}}{\partial t} - \Delta y_{2\varepsilon} = 0 & \textit{in } Q, \\ y_{2\varepsilon}(x,0) = 0 & \textit{in } \Omega, \\ \frac{\partial y_{2\varepsilon}}{\partial \nu} = 0 & \textit{on } \Sigma_0, \quad y_{2\varepsilon} = w_{2\varepsilon} \quad \textit{on } \Sigma_1, \end{cases} \tag{2.16}$$

$$\left\| \frac{\partial y_{1\varepsilon}}{\partial \nu} \right\|_{L^2(\Sigma_0)} < \varepsilon \qquad \textit{and} \qquad \|y_{2\varepsilon}\|_{L^2(\Sigma_0)} < \varepsilon. \tag{2.17}$$

Remark 2.8 *The approximate controllability problem* (2.9) *expresses the following idea: failing to find $w_1, w_2 \in L^2(\Sigma_1)$ allowing to reach the targets*

$$\frac{\partial y_1}{\partial \nu}\Big|_{\Sigma_0} = 0 \qquad \textit{and} \qquad y_2|_{\Sigma_0} = 0$$

fixed by the exact controllability problem (2.7)*, one can obtain sequences $(w_{1\varepsilon})_\varepsilon, (w_{2\varepsilon})_\varepsilon \subset L^2(\Sigma_1)$ through which the fixed targets can be approached to ε close, and that, for all $\varepsilon > 0$.*

Starting from Remark 2.5, we deduce from the previous results, the following.

Corollary 2.9 *For all $v_0, v_1 \in L^2(\Sigma_0)$ and $\varepsilon > 0$, there are $w_{1\varepsilon},\ w_{2\varepsilon} \in L^2(\Sigma_1)$ such that*

$$y_1(v_0, w_{1\varepsilon}) \in H^{3/2,3/4}(Q) \quad \textit{and} \quad y_2(v_1, w_{2\varepsilon}) \in H^{1/2,1/4}(Q)$$

are unique solutions of

$$\begin{cases} \frac{\partial y_1}{\partial t}(v_0, w_{1\varepsilon}) - \Delta y_1(v_0, w_{1\varepsilon}) = 0 & \text{in } Q, \\ y_1(v_0, w_{1\varepsilon})\,(x,0) = 0 & \text{in } \Omega, \\ y_1(v_0, w_{1\varepsilon}) = v_0 & \text{on } \Sigma_0, \quad \frac{\partial y_1}{\partial \nu}(v_0, w_{1\varepsilon}) = w_{1\varepsilon} \quad \text{on } \Sigma_1, \end{cases} \tag{2.18}$$

$$\begin{cases} \frac{\partial y_2}{\partial t}(v_1, w_{2\varepsilon}) - \Delta y_2(v_1, w_{2\varepsilon}) = 0 & \text{in } Q, \\ y_2(v_1, w_{2\varepsilon})\,(x,0) = 0 & \text{in } \Omega, \\ \frac{\partial y_2}{\partial \nu}(v_1, w_{2\varepsilon}) = v_1 & \text{on } \Sigma_0, \quad y_2(v_1, w_{2\varepsilon}) = w_{2\varepsilon} \quad \text{on } \Sigma_1, \end{cases} \tag{2.19}$$

$$\left\| \frac{\partial y_1}{\partial \nu}(v_0, w_{1\varepsilon}) - v_1 \right\|_{L^2(\Sigma_0)} < \varepsilon \qquad \text{and} \qquad \|y_2(v_1, w_{2\varepsilon}) - v_0\|_{L^2(\Sigma_0)} < \varepsilon. \tag{2.20}$$

Remark 2.10 *Corollary 2.9 establishes the solvability of the approximate controllability problem associated with the exact controllability problem* (2.4)-(2.6).

Then we have the following theorem.

Theorem 2.11 *Given $v = (v_0, v_1) \in \left(L^2(\Sigma_0)\right)^2$, the ill-posed Cauchy problem*

$$\begin{cases} \frac{\partial z}{\partial t} - \Delta z = 0 & \text{in } Q, \\ z(x,0) = 0 & \text{in } \Omega, \\ z = v_0, \quad \frac{\partial z}{\partial \nu} = v_1 & \text{on } \Sigma_0, \end{cases} \tag{2.21}$$

admits a regular solution $z \in H^{3/2,3/4}(Q)$ if and only if either of the sequences $(w_{1\varepsilon})_\varepsilon$ or $(w_{2\varepsilon})_\varepsilon$ is bounded in $L^2(\Sigma_1)$.

Proof. 1. Let $\varepsilon > 0$. According to Corollary 2.9, there exist $w_{1\varepsilon},\ w_{2\varepsilon} \in L^2(\Sigma_1)$, such that

$$y_1(v_0, w_{1\varepsilon}) \in H^{3/2,3/4}(Q) \quad \text{and} \quad y_2(v_1, w_{2\varepsilon}) \in H^{1/2,1/4}(Q)$$

are solutions of (2.18), (2.19) and (2.20). Then, we generate

$$(w_{1\varepsilon})_\varepsilon,\ (w_{2\varepsilon})_\varepsilon \subset L^2(\Sigma_1),\ (y_1(v_0, w_{1\varepsilon}))_\varepsilon \subset H^{3/2,3/4}(Q) \text{ and } (y_2(v_1, w_{2\varepsilon}))_\varepsilon \subset H^{1/2,1/4}(Q)\,.$$

Assuming that the sequence $(w_{1\varepsilon})_\varepsilon$ is bounded in $L^2(\Sigma_1)$, it follows, the mixed Dirichlet-Neumann problem (2.18) being well defined in the sense of Hadamard, that the sequence $(y_1(v_0, w_{1\varepsilon}))_\varepsilon$ is bounded in $H^{3/2,3/4}(Q)$, and therefore again in $L^2(Q)$, by continuity of the canonical injection of $H^{3/2,3/4}(Q)$ in $L^2(Q)$. We deduce that we can extract, from $(w_{1\varepsilon})_\varepsilon$ and $(y_1(v_0, w_{1\varepsilon}))_\varepsilon$ respectively, subsequences, again denoted in the same way, which converge in $L^2(\Sigma_1)$ and $H^{3/2,3/4}(Q)$, respectively. There therefore exist

$$w_1 \in L^2(\Sigma_1) \qquad \text{and} \qquad y_1 \in H^{3/2,3/4}(Q)$$

such that

$$\begin{aligned} w_{1\varepsilon} &\longrightarrow w_1 \quad \text{weakly in } L^2(\Sigma_1), \\ y_1(v_0, w_{1\varepsilon}) &\longrightarrow y_1 \quad \text{weakly in } H^{3/2,3/4}(Q)\,. \end{aligned}$$

But then, we have on the one hand that

$$\left\| \frac{\partial y_1}{\partial \nu}(v_0, w_{1\varepsilon}) - v_1 \right\|_{L^2(\Sigma_0)} < \varepsilon$$

and

$$y_1(v_0, w_{1\varepsilon}) \longrightarrow y_1 \text{ weakly in } H^{3/2,3/4}(Q)\,,$$

involve, by continuity of the trace operator $\gamma_1 : L^2(0,T,H^{\frac{3}{2}}(\Omega)) \longrightarrow L^2(\Sigma)$,

$$\frac{\partial y_1}{\partial \nu} = v_1 \quad \text{on } \Sigma_0, \tag{2.22}$$

and on the other hand, that for all $\varphi \in \mathcal{C}^\infty(\overline{Q})$, we have

$$\frac{\partial y_1}{\partial t}(v_0, w_{1\varepsilon}) - \Delta y_1(v_0, w_{1\varepsilon}) = 0 \quad \text{in } Q \implies \left(\frac{\partial y_1}{\partial t} - \Delta y_1, \varphi\right)_{L^2(Q)} = 0,$$

so, noting $\phi_{1\varepsilon} = y_1(v_0, w_{1\varepsilon})$, it comes

$$\left(\frac{\partial \phi_{1\varepsilon}}{\partial t} - \Delta \phi_{1\varepsilon}, \varphi\right)_{L^2(Q)} = 0 \iff \left(\frac{\partial \phi_{1\varepsilon}}{\partial t}, \varphi\right)_{L^2(Q)} - (\Delta \phi_{1\varepsilon}, \varphi)_{L^2(Q)} = 0$$

that is to say

$$-\left(\phi_{1\varepsilon}, \frac{\partial \varphi}{\partial t}\right)_{L^2(Q)} - (\phi_{1\varepsilon}, \Delta\varphi)_{L^2(Q)} - \left(\frac{\partial \phi_{1\varepsilon}}{\partial \nu}, \varphi\right)_{L^2(\Sigma_0)} - (w_{1\varepsilon}, \varphi)_{L^2(\Sigma_1)} + \left(v_0, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_0)} + \left(\phi_{1\varepsilon}, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_1)} = 0.$$

Passing to the limit, it comes

$$-\left(y_1, \frac{\partial \varphi}{\partial t}\right)_{L^2(Q)} - (y_1, \Delta\varphi)_{L^2(Q)} - \left(\frac{\partial y_1}{\partial \nu}, \varphi\right)_{L^2(\Sigma_0)} - (w_1, \varphi)_{L^2(\Sigma_1)} + \left(v_0, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_0)} + \left(y_1, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_1)} = 0,$$

which is equivalent to

$$\left(\frac{\partial y_1}{\partial t} - \Delta y_1, \varphi\right)_{L^2(Q)} + \left(v_0 - y_1, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_0)} + \left(\frac{\partial y_1}{\partial \nu} - w_1, \varphi\right)_{L^2(\Sigma_1)} = 0.$$

This last equality is valid for all $\varphi \in \mathcal{C}^\infty(\overline{Q})$, it follows that

$$\begin{cases} \frac{\partial y_1}{\partial t} - \Delta y_1 = 0 & \text{in } Q, \\ y_1(x,0) = 0 & \text{in } \Omega, \\ y_1 = v_0 & \text{on } \Sigma_0, \quad \frac{\partial y_1}{\partial \nu} = w_1 \quad \text{on } \Sigma_1. \end{cases} \tag{2.23}$$

Then, (2.22) and (2.23) give, in particular

$$\begin{cases} \frac{\partial y_1}{\partial t} - \Delta y_1 = 0 & \text{in } Q, \\ y_1(x,0) = 0 & \text{in } \Omega, \\ y_1 = v_0, \quad \frac{\partial y_1}{\partial \nu} = v_1 & \text{on } \Sigma_0, \end{cases}$$

and so that $y_1 \in H^{3/2,3/4}(Q)$ is a solution of the Cauchy problem, a regular solution, due to the well-posed nature of (2.18).

Symmetrically, assuming that $(w_{2\varepsilon})_\varepsilon$ is bounded in $L^2(\Sigma_1)$, we likewise obtain

$$w_2 \in L^2(\Sigma_1) \qquad \text{and} \qquad y_2 \in H^{1/2,1/4}(Q)\,,$$

such that

$$\begin{aligned} w_{2\varepsilon} &\longrightarrow w_2 \quad \text{weakly in } L^2(\Sigma_1), \\ y_{2\varepsilon}(v_1, w_{2\varepsilon}) &\longrightarrow y_2 \quad \text{weakly in } H^{1/2,1/4}(Q)\,, \end{aligned}$$

with $y_2 \in H^{1/2,1/4}(Q) \cap L^2(Q)$, a regular solution to the Cauchy problem (see proof of Corollary 2.13).

2. We assume that the Cauchy problem admits a solution $z \in H^{3/2,3/4}(Q)$.

   So we have

$$z|_{\Sigma_1} \in H^{1/2,1/4}(\Sigma) \subset L^2(\Sigma_1) \qquad \text{and} \qquad \frac{\partial z}{\partial \nu}\Big|_{\Sigma_1} \in L^2(\Sigma_1).$$

   So that, for all $\varepsilon > 0$, we can easily choose

$$w_{1\varepsilon} = \frac{\partial z}{\partial \nu}\Big|_{\Sigma_1} \in L^2(\Sigma_1) \qquad \text{and} \qquad w_{2\varepsilon} = z|_{\Sigma_1} \in L^2(\Sigma_1)$$

   to obtain the *existence* of sequences $(w_{1\varepsilon})_\varepsilon$, $(w_{2\varepsilon})_\varepsilon \subset L^2(\Sigma_1)$ bounded in $L^2(\Sigma_1)$ since they are constants, hence the result holds.

□

Remark 2.12 *It is important to note that Corollary 2.9 only establishes the existence of functions $w_{1\varepsilon}$, $w_{2\varepsilon} \in L^2(\Sigma_1)$, such that there exist $y_1(v_0, w_{1\varepsilon}) \in H^{3/2,3/4}(Q)$ and $y_2(v_1, w_{2\varepsilon}) \in H^{1/2,1/4}(Q)$ satisfying* (2.18), (2.19) *and* (2.20). *So the statement of the previous theorem could be specified in these terms: "the Cauchy problem* (2.21) *admits a regular solution $z \in H^{3/2,3/4}(Q)$ if and only if there exists at least one sequence $(w_{1\varepsilon})_\varepsilon$ (resp. $(w_{2\varepsilon})_\varepsilon$) $\subset L^2(\Sigma_1)$ which is bounded in $L^2(\Sigma_1)$ and those terms generate $y_1(v_0, w_{1\varepsilon}) \subset H^{3/2,3/4}(Q)$ $\Big($resp. $y_2(v_1, w_{2\varepsilon}) \subset H^{1/2,1/4}(Q)\Big)$, satisfying* (2.18) *(resp.* (2.19)*) and corresponding estimation in* (2.20)*".*

The following corollary follows from Theorem 2.11.

Corollary 2.13 *Let z be a regular solution of the Cauchy problem, then*

$$y_1 = z = y_2.$$

Proof. Let (v, z) be a control-state pair for the Cauchy problem, that is to say, according to Remark 2.2, $z \in H^{3/2,3/4}(Q)$, and let $\varepsilon > 0$. By Theorem 2.11, there exist

$$\left((w_{2\varepsilon})_\varepsilon, \, (y_2(v_1, w_{2\varepsilon}))_\varepsilon\right) \subset L^2(\Sigma_1) \times H^{1/2,1/4}(Q) \quad \text{and} \quad (w_2, y_2) \in L^2(\Sigma_1) \times H^{1/2,1/4}(Q)\,,$$

such that

$$\begin{aligned} w_{2\varepsilon} &\longrightarrow w_2 \quad \text{weakly in } L^2(\Sigma_1), \\ y_2(v_1, w_{2\varepsilon}) &\longrightarrow y_2 \quad \text{weakly in } H^{1/2,1/4}(Q)\,, \end{aligned}$$

with

$$\begin{cases} \frac{\partial y_2}{\partial t}(v_1, w_{2\varepsilon}) - \Delta y_2(v_1, w_{2\varepsilon}) = 0 & \text{in } Q, \\ y_2(v_1, w_{2\varepsilon})\,(x,0) = 0 & \text{in } \Omega, \\ \frac{\partial y_2}{\partial \nu}(v_1, w_{2\varepsilon}) = v_1 & \text{on } \Sigma_0, \quad y_2(v_1, w_{2\varepsilon}) = w_{2\varepsilon} \quad \text{on } \Sigma_1, \end{cases}$$

and

$$\|y_2(v_1, w_{2\varepsilon}) - v_0\|_{L^2(\Sigma_0)} < \varepsilon.$$

Then, by continuity of the trace operator $\gamma_0 : L^2(0,T,H^{\frac{1}{2}}(\Omega)) \longrightarrow L^2(\Sigma)$, we immediately have that

$$\|y_2(v_1, w_{2\varepsilon}) - v_0\|_{L^2(\Sigma_0)} < \varepsilon \quad \text{and} \quad y_2(v_1, w_{2\varepsilon}) \longrightarrow y_2 \quad \text{weakly in } H^{1/2,1/4}(Q)$$

imply

$$y_2 = v_0 \quad \text{on } \Sigma_0. \tag{2.24}$$

On the other hand, noting $\phi_{2\varepsilon} = y_2(v_1, w_{2\varepsilon})$, we have, for all $\varphi \in \mathcal{C}^\infty(\overline{Q})$,

$$\frac{\partial \phi_{2\varepsilon}}{\partial t} - \Delta \phi_{2\varepsilon} = 0 \quad \text{in } Q \implies \left(\frac{\partial \phi_{2\varepsilon}}{\partial t} - \Delta \phi_{2\varepsilon}, \varphi\right)_{L^2(Q)} = 0,$$

that is to say

$$-\left(\phi_{2\varepsilon}, \frac{\partial \varphi}{\partial t}\right)_{L^2(Q)} - (\phi_{2\varepsilon}, \Delta\varphi)_{L^2(Q)} - (v_1, \varphi)_{L^2(\Sigma_0)} - \left(\frac{\partial \phi_{2\varepsilon}}{\partial \nu}, \varphi\right)_{L^2(\Sigma_1)} + \left(\phi_{2\varepsilon}, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_0)} + \left(w_{2\varepsilon}, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_1)} = 0.$$

Passing to the limit, it follows

$$-\left(y_2, \frac{\partial \varphi}{\partial t}\right)_{L^2(Q)} - (y_2, \Delta\varphi)_{L^2(Q)} - (v_1, \varphi)_{L^2(\Sigma_0)} - \left(\frac{\partial y_2}{\partial \nu}, \varphi\right)_{L^2(\Sigma_1)} + \left(y_2, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_0)} + \left(w_2, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_1)} = 0,$$

which is equivalent to

$$\left(\frac{\partial y_2}{\partial t} - \Delta y_2, \varphi\right)_{L^2(Q)} + \left(\frac{\partial y_2}{\partial \nu} - v_1, \varphi\right)_{L^2(\Sigma_0)} + \left(w_2 - y_2, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_1)} = 0.$$

This last equality is valid for all $\varphi \in \mathcal{C}^\infty(\overline{Q})$, it follows, in particular with (2.24),

$$\begin{cases} \frac{\partial y_2}{\partial t} - \Delta y_2 = 0 & \text{in } Q, \\ y_2(x,0) = 0 & \text{in } \Omega, \\ y_2 = v_0, \quad \frac{\partial y_2}{\partial \nu} = v_1 & \text{on } \Sigma_0, \end{cases}$$

which means y_2 is indeed a solution of the Cauchy problem.

By the uniqueness of the solution in the question, it comes, joining the result above to that obtained in the first part of the proof of Theorem 2.11,

$$y_1 = z = y_2.$$

□

3 The optimal control problem

Let us start by recalling that we are interested in controlling the Cauchy problem for the heat operator. That is to say, more precisely, we consider the problem

$$\begin{cases} \frac{\partial z}{\partial t} - \Delta z = 0 & \text{in } Q, \\ z(x,0) = 0 & \text{in } \Omega, \\ z = v_0, \quad \frac{\partial z}{\partial \nu} = v_1 & \text{on } \Sigma_0, \end{cases} \tag{3.1}$$

and, for all control-state pairs (v, z), the cost function

$$J(v,z) = \frac{1}{2}\|z - z_d\|^2_{L^2(Q)} + \frac{N_0}{2}\|v_0\|^2_{L^2(\Sigma_0)} + \frac{N_1}{2}\|v_1\|^2_{L^2(\Sigma_0)}, \tag{3.2}$$

being interested in the optimal control problem

$$\inf \{J(v,z)\,;\; (v,z) \in \mathcal{A}\}. \tag{3.3}$$

We propose here to use the controllability method (cf. [5]) to characterize the optimal control-state pair (u, y) of problem (3.1)-(3.3), without any other assumptions than the "sufficient" one of non-vacuity of the set of admissible control-state pairs (cf. Remark 2.2). To the best of our knowledge, this method seems new.

3.1 The method of controllability

Starting therefore from the assumption $\mathcal{A} \neq \emptyset$ and within the framework of Remark 2.2, we have, for all

$$v = (v_0, v_1) \in \mathcal{U}_{ad} \qquad \text{and} \qquad \varepsilon > 0,$$

there exist $w_{1\varepsilon},\, w_{2\varepsilon} \in L^2(\Sigma_1)$, $y_1(v_0, w_{1\varepsilon}) \in H^{3/2,3/4}(Q)$ and $y_2(v_1, w_{2\varepsilon}) \in H^{1/2,1/4}(Q)$ such that

$$\begin{cases} \frac{\partial y_1}{\partial t}(v_0, w_{1\varepsilon}) - \Delta y_1(v_0, w_{1\varepsilon}) = 0 & \text{in } Q, \\ y_1(v_0, w_{1\varepsilon})\,(x,0) = 0 & \text{in } \Omega, \\ y_1(v_0, w_{1\varepsilon}) = v_0 & \text{on } \Sigma_0, \quad \frac{\partial y_1}{\partial \nu}(v_0, w_{1\varepsilon}) = w_{1\varepsilon} \quad \text{on } \Sigma_1, \end{cases} \tag{3.4}$$

$$\begin{cases} \frac{\partial y_2}{\partial t}(v_1, w_{2\varepsilon}) - \Delta y_2(v_1, w_{2\varepsilon}) = 0 & \text{in } Q, \\ y_2(v_1, w_{2\varepsilon})\,(x,0) = 0 & \text{in } \Omega, \\ \frac{\partial y_2}{\partial \nu}(v_1, w_{2\varepsilon}) = v_1 & \text{on } \Sigma_0, \quad y_2(v_1, w_{2\varepsilon}) = w_{2\varepsilon} \quad \text{on } \Sigma_1, \end{cases} \tag{3.5}$$

$$\left\| \frac{\partial y_1}{\partial \nu}(v_0, w_{1\varepsilon}) - v_1 \right\|_{L^2(\Sigma_0)} < \varepsilon \qquad \text{and} \qquad \|y_2(v_1, w_{2\varepsilon}) - v_0\|_{L^2(\Sigma_0)} < \varepsilon. \tag{3.6}$$

Then we consider, for $\theta_1, \theta_2 \in \mathbb{R}^*_+ :\; \theta_1 + \theta_2 = 1$, the functional

$$\begin{aligned} J_\varepsilon(v_0, v_1) = {} & \frac{\theta_1}{2}\|y_1(v_0, w_{1\varepsilon}) - z_d\|^2_{L^2(Q)} + \frac{\theta_2}{2}\|y_2(v_1, w_{2\varepsilon}) - z_d\|^2_{L^2(Q)} \\ & + \frac{N_0}{2}\|v_0\|^2_{L^2(\Sigma_0)} + \frac{N_1}{2}\|v_1\|^2_{L^2(\Sigma_0)}, \end{aligned} \tag{3.7}$$

being interested in the control problem

$$\inf \{J_\varepsilon(v_0, v_1) \ ; \ v = (v_0, v_1) \in \mathcal{U}_{ad}\} \, . \tag{3.8}$$

The following result is then immediate.

Proposition 3.1 *For all* $\varepsilon > 0$*, the control problem* (3.8) *admits a unique solution, the optimal control* $\bar{u}_\varepsilon = (\bar{u}_{0_\varepsilon}, \bar{u}_{1_\varepsilon})$.

Remark 3.2 *Here we opt for a control of the system* (3.4)-(3.7)*, in relation only to the control vector* $v = (v_0, v_1)$*, without worrying about the selection of the sequence* $((w_{1\varepsilon}), (w_{2\varepsilon}))_\varepsilon$*. Another approach could be to foresee a hierarchical control, which would allow, beyond the control according to the vector* $v = (v_0, v_1)$*, to be interested in the question of the choice of* $((w_{1\varepsilon}), (w_{2\varepsilon}))_\varepsilon$*, furthermore, (cf. Corollary 2.9), the terms of these sequences depend on* v_0 *and* v_1.

3.2 Convergence of the method

Let $\varepsilon > 0$. Due to the existence of the optimal control $\bar{u}_\varepsilon = (\bar{u}_{0_\varepsilon}, \bar{u}_{1_\varepsilon}) \in \mathcal{U}_{ad} \subset \left(L^2(\Sigma_0)\right)^2$, and according to the results of the previous section, there exist

$$\bar{w}_{1_\varepsilon}, \, \bar{w}_{2_\varepsilon} \in L^2(\Sigma_1), \quad \bar{y}_{1_\varepsilon} \in H^{3/2,3/4}(Q) \quad \text{and} \quad \bar{y}_{2_\varepsilon} \in H^{1/2,1/4}(Q)$$

such that

$$\begin{cases} \frac{\partial \bar{y}_{1_\varepsilon}}{\partial t} - \Delta \bar{y}_{1_\varepsilon} = 0 & \text{in } Q, \\ \bar{y}_{1_\varepsilon}(x,0) = 0 & \text{in } \Omega, \\ \bar{y}_{1_\varepsilon} = \bar{u}_{0_\varepsilon} & \text{on } \Sigma_0, \quad \frac{\partial \bar{y}_{1_\varepsilon}}{\partial \nu} = \bar{w}_{1_\varepsilon}, \quad \text{on } \Sigma_1, \end{cases}$$

$$\begin{cases} \frac{\partial \bar{y}_{2_\varepsilon}}{\partial t} - \Delta \bar{y}_{2_\varepsilon} = 0 & \text{in } Q, \\ \bar{y}_{2_\varepsilon}(x,0) = 0 & \text{in } \Omega, \\ \frac{\partial \bar{y}_{2_\varepsilon}}{\partial \nu} = \bar{u}_{1_\varepsilon} & \text{on } \Sigma_0, \quad \bar{y}_{2_\varepsilon} = \bar{w}_{2_\varepsilon} \quad \text{on } \Sigma_1, \end{cases}$$

$$\left\| \frac{\partial \bar{y}_{1_\varepsilon}}{\partial \nu} - \bar{u}_{1_\varepsilon} \right\|_{L^2(\Sigma_0)} < \varepsilon \qquad \text{and} \qquad \|\bar{y}_{2_\varepsilon} - \bar{u}_{0_\varepsilon}\|_{L^2(\Sigma_0)} < \varepsilon,$$

with, for all $v \in \mathcal{U}_{ad}$,

$$J_\varepsilon(\bar{u}_{0_\varepsilon}, \bar{u}_{1_\varepsilon}) \leq J_\varepsilon(v_0, v_1) \, .$$

In particular

$$J_\varepsilon(\bar{u}_{0_\varepsilon}, \bar{u}_{1_\varepsilon}) \leq J_\varepsilon(u_0, u_1) \, , \tag{3.9}$$

where $u = (u_0, u_1)$ is the optimal solution of (3.1)-(3.3). We have in fact that $J_\varepsilon(u_0, u_1)$ is independent of ε. Indeed, let $\left(w^*_{1\varepsilon}\right)_\varepsilon$ and $\left(w^*_{2\varepsilon}\right)_\varepsilon$ be the constant sequences defined by

$$w^*_{1\varepsilon} = \frac{\partial y}{\partial \nu}\Big|_{\Sigma_1} \in L^2(\Sigma_1) \qquad \text{and} \qquad w^*_{2\varepsilon} = y|_{\Sigma_1} \in L^2(\Sigma_1).$$

So we have

- $y_1\big(u_0, w^*_{1\varepsilon}\big) = y^*_{1\varepsilon} = y$ verifies

$$\begin{cases} \frac{\partial y^*_{1\varepsilon}}{\partial t} - \Delta y^*_{1\varepsilon} = 0 & \text{in } Q, \\ y^*_{1\varepsilon}(x,0) = 0 & \text{in } \Omega, \\ y^*_{1\varepsilon} = u_0 & \text{on } \Sigma_0, \quad \frac{\partial y^*_{1\varepsilon}}{\partial \nu} = w^*_{1\varepsilon} \quad \text{on } \Sigma_1, \end{cases}$$

- $y_2\big(u_1, w^*_{2\varepsilon}\big) = y^*_{2\varepsilon} = y$ verifies

$$\begin{cases} \frac{\partial y^*_{2\varepsilon}}{\partial t} - \Delta y^*_{2\varepsilon} = 0 & \text{in } Q, \\ y^*_{2\varepsilon}(x,0) = 0 & \text{in } \Omega, \\ \frac{\partial y^*_{2\varepsilon}}{\partial \nu} = u_1 & \text{on } \Sigma_0, \quad y^*_{2\varepsilon} = w^*_{2\varepsilon} \quad \text{on } \Sigma_1, \end{cases}$$

with

$$\frac{\partial y^*_{1\varepsilon}}{\partial \nu}\Big|_{\Sigma_0} = u_1 \qquad \text{and} \qquad y^*_{2\varepsilon}|_{\Sigma_0} = u_0.$$

Consequently,

$$\begin{aligned} J_\varepsilon(u_0,u_1) &= \frac{\theta_1}{2}\|y^*_{1\varepsilon} - z_d\|^2_{L^2(Q)} + \frac{\theta_2}{2}\|y^*_{2\varepsilon} - z_d\|^2_{L^2(Q)} + \frac{N_0}{2}\|u_0\|^2_{L^2(\Sigma_0)} + \frac{N_1}{2}\|u_1\|^2_{L^2(\Sigma_0)} \\ &= \frac{\theta_1}{2}\|y - z_d\|^2_{L^2(Q)} + \frac{\theta_2}{2}\|y - z_d\|^2_{L^2(Q)} + \frac{N_0}{2}\|u_0\|^2_{L^2(\Sigma_0)} + \frac{N_1}{2}\|u_1\|^2_{L^2(\Sigma_0)}, \end{aligned}$$

i.e. $J_\varepsilon(u_0,u_1) = J(u,y)$.

Thus (3.9) becomes

$$J_\varepsilon(\bar{u}_{0_\varepsilon}, \bar{u}_{1_\varepsilon}) \le J_\varepsilon(u_0,u_1) = J(u,y), \tag{3.10}$$

and it follows that there exist constants $C_i \in \mathbb{R}^*_+$, independent of ε, such that

$$\begin{cases} \|\bar{y}_{1_\varepsilon}\|_{L^2(Q)} \le C_1, & \|\bar{y}_{2_\varepsilon}\|_{L^2(Q)} \le C_2, \\ \|\bar{u}_{0_\varepsilon}\|_{L^2(\Sigma_0)} \le C_3, & \|\bar{u}_{1_\varepsilon}\|_{L^2(\Sigma_0)} \le C_4. \end{cases} \tag{3.11}$$

Then, we deduce that

- on the one hand, there exist $\hat{u}_0, \hat{u}_1 \in L^2(\Sigma_0)$ such that

$$\begin{cases} \bar{u}_{0_\varepsilon} \longrightarrow \hat{u}_0 & \text{weakly in } L^2(\Sigma_0), \\ \bar{u}_{1_\varepsilon} \longrightarrow \hat{u}_1 & \text{weakly in } L^2(\Sigma_0), \end{cases} \tag{3.12}$$

- and on the other hand, there are $\hat{y}_1,\ \hat{y}_2 \in L^2(Q)$ such that

$$\begin{cases} \bar{y}_{1_\varepsilon} \longrightarrow \hat{y}_1 & \text{weakly in } L^2(Q), \\ \bar{y}_{2_\varepsilon} \longrightarrow \hat{y}_2 & \text{weakly in } L^2(Q). \end{cases}$$

But then it follows that

$$\begin{cases} \dfrac{\partial \bar{y}_{1_\varepsilon}}{\partial t} - \Delta \bar{y}_{1_\varepsilon} = 0 & \text{in } Q, \\ \bar{y}_{1_\varepsilon}(x,0) = 0 & \text{in } \Omega, \\ \bar{y}_{1_\varepsilon} = \bar{u}_{0_\varepsilon}, & \text{on } \Sigma_0, \\ \|\frac{\partial \bar{y}_{1_\varepsilon}}{\partial \nu} - \bar{u}_{1_\varepsilon}\|_{L^2(\Sigma_0)} < \varepsilon \end{cases} \quad \text{and} \quad \begin{cases} \bar{u}_{0_\varepsilon} \longrightarrow \hat{u}_0 & \text{weakly in } L^2(\Sigma_0), \\ \bar{u}_{1_\varepsilon} \longrightarrow \hat{u}_1 & \text{weakly in } L^2(\Sigma_0), \\ \bar{y}_{1_\varepsilon} \longrightarrow \hat{y}_1 & \text{weakly in } L^2(Q) \end{cases}$$

imply

$$\begin{cases} \frac{\partial \hat{y}_1}{\partial t} - \Delta \hat{y}_1 = 0 & \text{in } Q, \\ \hat{y}_1(x,0) = 0 & \text{in } \Omega, \\ \hat{y}_1 = \hat{u}_0, \quad \frac{\partial \hat{y}_1}{\partial \nu} = \hat{u}_1 & \text{on } \Sigma_0. \end{cases}$$

Similarly, we get

$$\begin{cases} \frac{\partial \hat{y}_2}{\partial t} - \Delta \hat{y}_2 = 0 & \text{in } Q, \\ \hat{y}_2(x,0) = 0 & \text{in } \Omega, \\ \hat{y}_2 = \hat{u}_0, \quad \frac{\partial \hat{y}_2}{\partial \nu} = \hat{u}_1 & \text{on } \Sigma_0. \end{cases}$$

Which gives $\hat{y}_1 = \hat{y} = \hat{y}_2$ is a solution of the Cauchy problem for inputs $\hat{u} = (\hat{u}_0, \hat{u}_1)$, and therefore

$$J(u,y) \leq J(\hat{u},\hat{y})\,. \tag{3.13}$$

So that, passing to the limit in (3.10), we obtain

$$J(\hat{u},\hat{y}) \leq J(u,y). \tag{3.14}$$

Finally, the uniqueness of the optimal solution (u, y) to (3.1)-(3.3), (3.13) and (3.14) leads to

$$J(\hat{u},\hat{y}) \leq J(u,y) \leq J(\hat{u},\hat{y}) \implies (\hat{u},\hat{y}) = (u,y).$$

Thereby we have just proved the following result.

Proposition 3.3 *For all $\varepsilon > 0$, the optimal control $\bar{u}_\varepsilon = (\bar{u}_{0_\varepsilon}, \bar{u}_{1_\varepsilon})$, the solution of* (3.8)*, is such that $(\bar{u}_\varepsilon, \bar{y}_\varepsilon)$ verifies*

$$\begin{cases} \bar{u}_\varepsilon \longrightarrow u & \text{weakly in } \left(L^2(\Sigma_0)\right)^2, \\ \bar{y}_\varepsilon \longrightarrow y & \text{weakly in } L^2(Q), \end{cases} \tag{3.15}$$

where (u, y) is the optimal control-state pair of (3.1)-(3.3).

3.3 Approached optimality system

Let $\varepsilon > 0$. Let us start recalling that, for the control $\bar{u}_\varepsilon = (\bar{u}_{0_\varepsilon}, \bar{u}_{1_\varepsilon}) \in \mathcal{U}_{ad}$, the optimal solution of (3.8), there exist $\bar{w}_{1_\varepsilon}$, $\bar{w}_{2_\varepsilon} \in L^2(\Sigma_1)$, $\bar{y}_{1_\varepsilon} \in H^{3/2,3/4}(Q)$ and $\bar{y}_{2_\varepsilon} \in H^{1/2,1/4}(Q)$, such that

$$\begin{cases} \frac{\partial \bar{y}_{1_\varepsilon}}{\partial t} - \Delta \bar{y}_{1_\varepsilon} = 0 & \text{in } Q, \\ \bar{y}_{1_\varepsilon}(x,0) = 0 & \text{in } \Omega, \\ \bar{y}_{1_\varepsilon} = \bar{u}_{0_\varepsilon} & \text{on } \Sigma_0, \quad \frac{\partial \bar{y}_{1_\varepsilon}}{\partial \nu} = \bar{w}_{1_\varepsilon} \quad \text{on } \Sigma_1, \end{cases} \tag{3.16}$$

$$\begin{cases} \frac{\partial \bar{y}_{2_\varepsilon}}{\partial t} - \Delta \bar{y}_{2_\varepsilon} = 0 & \text{in } Q, \\ \bar{y}_{2_\varepsilon}(x,0) = 0 & \text{in } \Omega, \\ \frac{\partial \bar{y}_{2_\varepsilon}}{\partial \nu} = \bar{u}_{1_\varepsilon} & \text{on } \Sigma_0, \quad \bar{y}_{2_\varepsilon} = \bar{w}_{2_\varepsilon} \quad \text{on } \Sigma_1, \end{cases} \tag{3.17}$$

$$\left\| \frac{\partial \bar{y}_{1_\varepsilon}}{\partial \nu} - \bar{u}_{1_\varepsilon} \right\|_{L^2(\Sigma_0)} < \varepsilon \qquad \text{and} \qquad \|\bar{y}_{2_\varepsilon} - \bar{u}_{0_\varepsilon}\|_{L^2(\Sigma_0)} < \varepsilon. \tag{3.18}$$

So let $v=(v_0,v_1)\in\mathcal{U}_{ad}$ and $\lambda\in\mathbb{R}^*$, we easily obtain that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}J_\varepsilon(\bar u_{0_\varepsilon}+\lambda\,(v_0-\bar u_{0_\varepsilon})\,,\bar u_{1_\varepsilon})\Big|_{\lambda=0}=\theta_1(\bar y_{1_\varepsilon}-z_d,\phi_{1\varepsilon})_{L^2(Q)}+N_0(\bar u_{0_\varepsilon},v_0-\bar u_{0_\varepsilon})_{L^2(\Sigma_0)} \quad (3.19)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}J_\varepsilon(\bar u_{0_\varepsilon},\bar u_{1_\varepsilon}+\lambda\,(v_1-\bar u_{1_\varepsilon}))\Big|_{\lambda=0}=\theta_2(\bar y_{2_\varepsilon}-z_d,\phi_{2\varepsilon})_{L^2(Q)}+N_1(\bar u_{1_\varepsilon},v_1-\bar u_{1_\varepsilon})_{L^2(\Sigma_0)}, \quad (3.20)$$

noting that

$$\phi_{1\varepsilon}=y_1(v_0-\bar u_{0_\varepsilon},\bar w_{1_\varepsilon})-y_1(0,\bar w_{1_\varepsilon}) \qquad \text{and} \qquad \phi_{2\varepsilon}=y_2(v_1-\bar u_{1_\varepsilon},\bar w_{2_\varepsilon})-y_2(0,\bar w_{2_\varepsilon})$$

are respective solutions of

$$\begin{cases} \frac{\partial \phi_{1\varepsilon}}{\partial t}-\Delta\phi_{1\varepsilon}=0 & \text{in } Q,\\ \phi_{1\varepsilon}(x,0)=0 & \text{in } \Omega,\\ \phi_{1\varepsilon}=v_0-\bar u_{0_\varepsilon} & \text{on } \Sigma_0, \quad \frac{\partial \phi_{1\varepsilon}}{\partial\nu}=0 \quad \text{on } \Sigma_1, \end{cases} \quad (3.21)$$

and

$$\begin{cases} \frac{\partial \phi_{2\varepsilon}}{\partial t}-\Delta\phi_{2\varepsilon}=0 & \text{in } Q,\\ \phi_{2\varepsilon}(x,0)=0 & \text{in } \Omega,\\ \frac{\partial \phi_{2\varepsilon}}{\partial\nu}=v_1-\bar u_{1_\varepsilon} & \text{on } \Sigma_0, \quad \phi_{2\varepsilon}=0 \quad \text{on } \Sigma_1. \end{cases} \quad (3.22)$$

So, with the first-order Euler-Lagrange conditions, we obtain that the optimal control $\bar u_\varepsilon$ is the unique element of $\mathcal{U}_{ad}$ satisfying, for all $v=(v_0,v_1)\in\mathcal{U}_{ad}$,

$$\begin{cases} \theta_1(\bar y_{1_\varepsilon}-z_d,\phi_{1\varepsilon})_{L^2(Q)}+N_0(\bar u_{0_\varepsilon},v_0-\bar u_{0_\varepsilon})_{L^2(\Sigma_0)} & \geq 0,\\ \theta_2(\bar y_{2_\varepsilon}-z_d,\phi_{2\varepsilon})_{L^2(Q)}+N_1(\bar u_{1_\varepsilon},v_1-\bar u_{1_\varepsilon})_{L^2(\Sigma_0)} & \geq 0. \end{cases} \quad (3.23)$$

Then, let us introduce the adjunct states $p_{1\varepsilon}$ and $p_{2\varepsilon}$ respectively defined by

$$\begin{cases} -\frac{\partial p_{1\varepsilon}}{\partial t}-\Delta p_{1\varepsilon}=\theta_1\,(\bar y_{1_\varepsilon}-z_d) & \text{in } Q,\\ p_{1\varepsilon}(x,T)=0 & \text{in } \Omega,\\ p_{1\varepsilon}=0 & \text{on } \Sigma_0, \quad \frac{\partial p_{1\varepsilon}}{\partial\nu}=0 \quad \text{on } \Sigma_1, \end{cases} \quad (3.24)$$

and

$$\begin{cases} -\frac{\partial p_{2\varepsilon}}{\partial t}-\Delta p_{2\varepsilon}=\theta_2\,(\bar y_{2_\varepsilon}-z_d) & \text{in } Q,\\ p_{2\varepsilon}(x,T)=0 & \text{in } \Omega,\\ \frac{\partial p_{2\varepsilon}}{\partial\nu}=0 & \text{on } \Sigma_0, \quad p_{2\varepsilon}=0 \quad \text{on } \Sigma_1. \end{cases} \quad (3.25)$$

So we get, from (3.21) and (3.24), that

$$\theta_1(\bar y_{1_\varepsilon}-z_d,\phi_{1\varepsilon})_{L^2(Q)}=\left(-\frac{\partial p_{1\varepsilon}}{\partial t}-\Delta p_{1\varepsilon},\phi_{1\varepsilon}\right)_{L^2(Q)}=-\left(\frac{\partial p_{1\varepsilon}}{\partial\nu},v_0-\bar u_{0_\varepsilon}\right)_{L^2(\Sigma_0)},$$

and from (3.22) and (3.25), that

$$\theta_2(\bar y_{2_\varepsilon}-z_d,\phi_{2\varepsilon})_{L^2(Q)}=-\left(\frac{\partial p_{2\varepsilon}}{\partial t},\phi_{2\varepsilon}\right)_{L^2(Q)}-(\Delta p_{2\varepsilon},\phi_{2\varepsilon})_{L^2(Q)}=(p_{2\varepsilon},v_1-\bar u_{1_\varepsilon})_{L^2(\Sigma_0)}.$$

Which gives that the optimality condition (3.23) is rewritten: $\forall v = (v_0, v_1) \in \mathcal{U}_{ad}$,

$$\begin{cases} \left(N_0 \bar{u}_{0_\varepsilon} - \frac{\partial p_{1_\varepsilon}}{\partial \nu}, v_0 - \bar{u}_{0_\varepsilon}\right)_{L^2(\Sigma_0)} & \geq 0, \\ (p_{2_\varepsilon} + N_1 \bar{u}_{1_\varepsilon}, v_1 - \bar{u}_{1_\varepsilon})_{L^2(\Sigma_0)} & \geq 0. \end{cases} \tag{3.26}$$

Hence the following theorem characterizes the approached optimal control $\bar{u}_\varepsilon = (\bar{u}_{0_\varepsilon}, \bar{u}_{1_\varepsilon})$.

Theorem 3.4 *Let $\varepsilon > 0$. The control $\bar{u}_\varepsilon = (\bar{u}_{0_\varepsilon}, \bar{u}_{1_\varepsilon})$ is the unique solution to* (3.8) *if and only if there exist*

$$\bar{w}_{1_\varepsilon}, \bar{w}_{2_\varepsilon} \in L^2(\Sigma_1), \qquad \bar{y}_{1_\varepsilon} \in H^{3/2,3/4}(Q)\,, \quad \bar{y}_{2_\varepsilon} \in H^{1/2,1/4}(Q) \quad \textit{and} \quad p_{1\varepsilon}, p_{2\varepsilon} \in L^2(Q),$$

such that the quadruplet $\{(\bar{u}_{0_\varepsilon}, \bar{u}_{1_\varepsilon}), (\bar{w}_{1_\varepsilon}, \bar{w}_{2_\varepsilon}), (\bar{y}_{1_\varepsilon}, \bar{y}_{2_\varepsilon}), (p_{1\varepsilon}, p_{2\varepsilon})\}$ is the solution of the singular optimality system defined by systems

$$\begin{cases} \frac{\partial \bar{y}_{1_\varepsilon}}{\partial t} - \Delta \bar{y}_{1_\varepsilon} = 0 & \textit{in } Q, \\ \bar{y}_{1_\varepsilon}(x,0) = 0 & \textit{in } \Omega, \\ \bar{y}_{1_\varepsilon} = \bar{u}_{0_\varepsilon} & \textit{on } \Sigma_0, \quad \frac{\partial \bar{y}_{1_\varepsilon}}{\partial \nu} = \bar{w}_{1_\varepsilon} \quad \textit{on } \Sigma_1, \end{cases} \tag{3.27}$$

$$\begin{cases} \frac{\partial \bar{y}_{2_\varepsilon}}{\partial t} - \Delta \bar{y}_{2_\varepsilon} = 0 & \textit{in } Q, \\ \bar{y}_{2_\varepsilon}(x,0) = 0 & \textit{in } \Omega, \\ \frac{\partial \bar{y}_{2_\varepsilon}}{\partial \nu} = \bar{u}_{1_\varepsilon} & \textit{on } \Sigma_0, \quad \bar{y}_{2_\varepsilon} = \bar{w}_{2_\varepsilon} \quad \textit{on } \Sigma_1, \end{cases} \tag{3.28}$$

$$\begin{cases} -\frac{\partial p_{1_\varepsilon}}{\partial t} - \Delta p_{1\varepsilon} = \theta_1 \left(\bar{y}_{1_\varepsilon} - z_d\right) & \textit{in } Q, \\ p_{1\varepsilon}(x,T) = 0 & \textit{in } \Omega, \\ p_{1\varepsilon} = 0 & \textit{on } \Sigma_0, \quad \frac{\partial p_{1_\varepsilon}}{\partial \nu} = 0 \quad \textit{on } \Sigma_1, \end{cases} \tag{3.29}$$

$$\begin{cases} -\frac{\partial p_{2_\varepsilon}}{\partial t} - \Delta p_{2\varepsilon} = \theta_2 \left(\bar{y}_{2_\varepsilon} - z_d\right) & \textit{in } Q, \\ p_{2\varepsilon}(x,T) = 0 & \textit{in } \Omega, \\ \frac{\partial p_{2\varepsilon}}{\partial \nu} = 0 & \textit{on } \Sigma_0, \quad p_{2\varepsilon} = 0 \quad \textit{on } \Sigma_1, \end{cases} \tag{3.30}$$

with the estimates

$$\left\| \frac{\partial \bar{y}_{1_\varepsilon}}{\partial \nu} - \bar{u}_{1_\varepsilon} \right\|_{L^2(\Sigma_0)} < \varepsilon \qquad \textit{and} \qquad \|\bar{y}_{2_\varepsilon} - \bar{u}_{0_\varepsilon}\|_{L^2(\Sigma_0)} < \varepsilon, \tag{3.31}$$

and the variational inequalities system: $\forall\, v = (v_0, v_1) \in \mathcal{U}_{ad}$,

$$\begin{cases} \left(N_0 \bar{u}_{0_\varepsilon} - \frac{\partial p_{1_\varepsilon}}{\partial \nu}, v_0 - \bar{u}_{0_\varepsilon}\right)_{L^2(\Sigma_0)} & \geq 0, \\ (p_{2_\varepsilon} + N_1 \bar{u}_{1_\varepsilon}, v_1 - \bar{u}_{1_\varepsilon})_{L^2(\Sigma_0)} & \geq 0. \end{cases} \tag{3.32}$$

3.4 Singular optimality system

From the results of Section 3.2, we have

$$\begin{cases} \bar{u}_{0_\varepsilon} \longrightarrow u_0 & \text{weakly in } L^2(\Sigma_0), \\ \bar{u}_{1_\varepsilon} \longrightarrow u_1 & \text{weakly in } L^2(\Sigma_0), \end{cases} \qquad \text{and} \qquad \begin{cases} \bar{y}_{1_\varepsilon} \longrightarrow y & \text{weakly in } H^{3/2,3/4}(Q)\,, \\ \bar{y}_{2_\varepsilon} \longrightarrow y & \text{weakly in } H^{1/2,1/4}(Q)\,, \end{cases}$$

where (u, y) is the optimal control-state pair of (3.1)-(3.3).

Then it follows from the fact that the mixed Dirichlet-Neumann problems (3.29) and (3.30) are well-posed, there exist

$$p_1, p_2 \in L^2(Q), \tag{3.33}$$

such that

$$\begin{cases} p_{1\varepsilon} \longrightarrow p_1 & \text{weakly in } L^2(Q), \\ p_{2\varepsilon} \longrightarrow p_2 & \text{weakly in } L^2(Q). \end{cases} \tag{3.34}$$

Thereby, the singular optimality system for the optimal solution (u, y) of (3.1)-(3.3), is as specified by the following theorem.

Theorem 3.5 *The control-state pair (u, y) is the unique solution of* (3.1)-(3.3) *if and only if there exists*

$$p = (p_1, p_2) \in \left(L^2(Q)\right)^2, \tag{3.35}$$

such that the triple $\{u, y, p\}$ is the solution of the singular optimality system defined by systems

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = 0 & \text{in } Q, \\ y(x,0) = 0 & \text{in } \Omega, \\ y = u_0, \quad \frac{\partial y}{\partial \nu} = u_1 & \text{on } \Sigma_0, \end{cases} \tag{3.36}$$

$$\begin{cases} -\frac{\partial p_1}{\partial t} - \Delta p_1 = \theta_1 \left(y - z_d\right) & \text{in } Q, \\ p_1(x,T) = 0 & \text{in } \Omega, \\ p_1 = 0 & \text{on } \Sigma_0, \quad \frac{\partial p_1}{\partial \nu} = 0 \quad \text{on } \Sigma_1, \end{cases} \tag{3.37}$$

$$\begin{cases} -\frac{\partial p_2}{\partial t} - \Delta p_2 = \theta_2 \left(y - z_d\right) & \text{in } Q, \\ p_2(x,T) = 0 & \text{in } \Omega, \\ \frac{\partial p_2}{\partial \nu} = 0 & \text{on } \Sigma_0, \quad p_2 = 0 \quad \text{on } \Sigma_1, \end{cases} \tag{3.38}$$

and the variational inequalities system

$$\begin{cases} \forall\, v = (v_0, v_1) \in \mathcal{U}_{ad}, \\ \left(N_0 u_0 - \frac{\partial p_1}{\partial \nu}, v_0 - u_0\right)_{L^2(\Sigma_0)} & \geq 0, \\ \left(p_2 + N_1 u_1, v_1 - u_1\right)_{L^2(\Sigma_0)} & \geq 0. \end{cases} \tag{3.39}$$

Proof. Indeed, we have, from (3.29), that

$$-\frac{\partial p_{1\varepsilon}}{\partial t} - \Delta p_{1\varepsilon} = \theta_1 \left(\bar{y}_{1_\varepsilon} - z_d\right) \quad \text{in } Q,$$

implies, for all $\varphi \in \mathcal{D}(Q)$,

$$\left(-\frac{\partial p_{1\varepsilon}}{\partial t} - \Delta p_{1\varepsilon}, \varphi\right)_{L^2(Q)} = \theta_1 (\bar{y}_{1_\varepsilon} - z_d, \varphi)_{L^2(Q)},$$

which is equivalent to

$$\begin{aligned}\theta_1(\bar{y}_{1_\varepsilon} - z_d, \varphi)_{L^2(Q)} &= -\left(\frac{\partial p_{1\varepsilon}}{\partial t}, \varphi\right)_{L^2(Q)} - (\Delta p_{1\varepsilon}, \varphi)_{L^2(Q)} \\ &= \left(p_{1\varepsilon}, \frac{\partial \varphi}{\partial t}\right)_{L^2(Q)} - (p_{1\varepsilon}, \Delta\varphi)_{L^2(Q)} - \left(\frac{\partial p_{1\varepsilon}}{\partial \nu}, \varphi\right)_{L^2(\Sigma_0)} + \left(p_{1\varepsilon}, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_1)},\end{aligned}$$

so, by passing to the limit, we obtain

$$\begin{aligned}\theta_1(y - z_d, \varphi)_{L^2(Q)} &= \left(p_1, \frac{\partial \varphi}{\partial t}\right)_{L^2(Q)} - (p_1, \Delta\varphi)_{L^2(Q)} - \left(\frac{\partial p_1}{\partial \nu}, \varphi\right)_{L^2(\Sigma_0)} + \left(p_1, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_1)} \\ &= \left(-\frac{\partial p_1}{\partial t} - \Delta p_1, \varphi\right)_{L^2(Q)} - \left(p_1, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_0)} + \left(\frac{\partial p_1}{\partial \nu}, \varphi\right)_{L^2(\Sigma_1)},\end{aligned}$$

which is equivalent to

$$\left(-\frac{\partial p_1}{\partial t} - \Delta p_1 - \theta_1(y - z_d), \varphi\right)_{L^2(Q)} - \left(p_1, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_0)} + \left(\frac{\partial p_1}{\partial \nu}, \varphi\right)_{L^2(\Sigma_1)} = 0.$$

This last equality is valid for all $\varphi \in \mathcal{D}(Q)$, then it follows that

$$\begin{cases} -\frac{\partial p_1}{\partial t} - \Delta p_1 = \theta_1 (y - z_d) & \text{in } Q, \\ p_1(x, T) = 0 & \text{in } \Omega, \\ p_1 = 0 & \text{on } \Sigma_0, \quad \frac{\partial p_1}{\partial \nu} = 0 \quad \text{on } \Sigma_1. \end{cases} \tag{3.40}$$

Similarly, we get from (3.30),

$$\begin{cases} -\frac{\partial p_2}{\partial t} - \Delta p_2 = \theta_2 (y - z_d) & \text{in } Q, \\ p_2(x, T) = 0 & \text{in } \Omega, \\ \frac{\partial p_2}{\partial \nu} = 0 & \text{on } \Sigma_0, \quad p_2 = 0 \quad \text{on } \Sigma_1. \end{cases} \tag{3.41}$$

Finally, we still easily pass to the limit in (3.32) to obtain that, $\forall\, v = (v_0, v_1) \in \mathcal{U}_{ad}$,

$$\begin{cases} \left(N_0 u_0 - \frac{\partial p_1}{\partial \nu}, v_0 - u_0\right)_{L^2(\Sigma_0)} & \geq 0, \\ (p_2 + N_1 u_1, v_1 - u_1)_{L^2(\Sigma_0)} & \geq 0. \end{cases} \tag{3.42}$$

Which ends up proving the announced characterization of the optimal pair (u, y). □

Remark 3.6 *As we indicated earlier, the present analysis addresses the question of the control of the Cauchy problem without using any other assumptions than the sufficient ones of non-vacuity, convexity and closure of the sets of admissible controls. The density results obtained by the interpretation made of the initial problem is enough to achieve convergence of the process.*

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