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STEPANOV MULTI-DIMENSIONAL ALMOST AUTOMORPHIC TYPE FUNCTIONS AND APPLICATIONS

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Abstract. The main purpose of this paper is to analyze various notions of Stepanov multi-dimensional almost automorphy in Lebesgue spaces with variable exponents. The introduced notion seems to be new even in the one-dimensional setting, with the exponent $p(\cdot)$ being constant. We provide some applications of our results to the abstract Volterra integro-differential equations.

Keywords: Stepanov multi-dimensional almost automorphic type functions, Stepanov multi-dimensional almost periodic type functions, Bochner transform, Lebesgue spaces with variable exponents, abstract Volterra integro-differential equations.

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1 Introduction and preliminaries

The notion of almost periodicity was introduced by the Danish mathematician Harald Bohr around 1925 and later generalized by many others. The notion of almost automorphy, which generalizes the notion of almost periodicity, was discovered by the American mathematician S. Bochner in 1955. Starting presumably with the papers of W. A. Veech [30,31], many authors have deeply investigated the concept of almost automorphy on various classes of (semi-)topological groups.

Suppose that $(X, \| \cdot \|)$ is a complex Banach space and $F : \mathbb{R}^n \to X$ is a continuous mapping. Then we say that the function $F(\cdot)$ is almost automorphic if and only if for every sequence $(\mathbf{b}_k)$ in $\mathbb{R}^n$ there exist a subsequence $(\mathbf{a}_k)$ of $(\mathbf{b}_k)$ and a mapping $G : \mathbb{R}^n \to X$ such that

$$\lim_{k\to\infty} F\big(\mathbf{t} + \mathbf{a}_k\big) = G(\mathbf{t}) \text{ and } \lim_{k\to\infty} G\big(\mathbf{t} - \mathbf{a}_k\big) = F(\mathbf{t}), \tag{1.1}$$

pointwisely for $\mathbf{t} \in \mathbb{R}^n$. If this is the case, then the range of $F(\cdot)$ is relatively compact in X and the limit function $G(\cdot)$ is bounded on $\mathbb{R}^n$ but not necessarily continuous on $\mathbb{R}^n$. Further on, if the convergence of limits appearing in (1.1) is uniform on compact subsets of $\mathbb{R}^n$, resp. the whole space $\mathbb{R}^n$, then we say that the function $F(\cdot)$ is compactly almost automorphic, resp. almost periodic. It is well known that an almost automorphic function $F(\cdot)$ is compactly almost automorphic if and only if $F(\cdot)$ is uniformly continuous. The function $F(\cdot)$ is said to be asymptotically almost automorphic (asymptotically almost periodic) if and only if there exist an almost automorphic function (almost periodic function) $G : \mathbb{R}^n \to X$ and a function $Q \in C_0(\mathbb{R}^n : X)$ such that $F(\mathbf{t}) = G(\mathbf{t}) + Q(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^n$; here, $C_0(\mathbb{R}^n : X)$ denotes the Banach space of all continuous functions from $\mathbb{R}^n$ into X vanishing at infinity. For more details about almost periodic functions, almost automorphic functions, various generalizations and applications, we refer the reader to the research monographs [6,7,15,16,19,20,26] and [32].

In our recent research study [3], we have investigated various notions of almost periodicity for a continuous function $F : I \times X \to Y$, where Y is a complex Banach space equipped with the norm $\| \cdot \|_Y$ and $\emptyset \neq I \subseteq \mathbb{R}^n$ (in this paper, $\emptyset \neq I \subseteq \mathbb{R}^n$ generally does not satisfy the semigroup property $I + I \subseteq I$ or contain the zero vector). In [5], we have investigated various notions of almost automorphy for a continuous function $F : \mathbb{R}^n \times X \to Y$. The notions of almost periodicity and almost automorphy on (semi-)topological groups were analyzed by numerous authors; see the research monographs [26] by B. M. Levitan, [28] by A. A. Pankov and the reference list given in the forthcoming monograph [20] by M. Kostić for more details on the subject.

It would be really difficult to summarize here all obtained results about the one-dimensional Stepanov almost periodic functions and their applications. Our recent paper [4] investigates various classes of Stepanov multi-dimensional almost periodic type functions $F : I \times X \to Y$, where Y is a complex Banach space and $\emptyset \neq I \subseteq \mathbb{R}^n$. The main aim of this paper is to continue the research studies [4, 5] by investigating various classes of Stepanov multi-dimensional almost automorphic functions in Lebesgue spaces with variable exponents (see also the recent research articles by T. Diagana, M. Kostić [8, 9] and M. Kostić, W.-S. Du [24, 25] for related studies carried out in the dimensional setting).

In support of our investigation of the Stepanov multi-dimensional almost automorphy, we would like to present the following illustrative example (the notion and terminology used will be explained a bit later):

Example 1.1 *The existence and uniqueness of the spatially almost automorphic solutions of the homogeneous heat equation with nonlocal diffusion*

$$\begin{aligned} u_t - \Delta u = 0 \ \text{ in } \ [0,+\infty) \times \mathbb{R}^n, \\ u(0,x) = F(x) \ \text{ in } \ \mathbb{R}^n \times \{0\}, \end{aligned} \tag{1.2}$$

has recently been initiated in [5, Subsection 3.2]. We can similarly consider the Stepanov spatially almost automorphic solutions of (1.2).

More precisely, let $X = C_b(\mathbb{R}^n : \mathbb{C})$, the Banach space of bounded continuous functions on $\mathbb{R}^n$ equipped with the sup-norm. Then we know that the Gaussian semigroup

$$(G(t)F)(x) := \bigl(4\pi t\bigr)^{-(n/2)} \int_{\mathbb{R}^n} F(x-y) e^{-\frac{|y|^2}{4t}} \, dy, \quad t > 0, \ f \in X, \ x \in \mathbb{R}^n,$$

is a bounded holomorphic semigroup which is not strongly continuous at zero, generated by the Laplacian Δ_x with maximal distributional domain (see [1, Example 3.7.6, Example 3.7.8] for more details). Under certain conditions, the unique solution of (1.2) *is given by $(t,x) \mapsto (G(t)F)(x)$, $t \geq 0$, $x \in \mathbb{R}^n$. Let a number $t_0 > 0$ be fixed, and let $p \in D_+(\Omega)$. Then Proposition 2.11 below shows that the function $\mathbb{R}^n \ni x \mapsto u(x, t_0) \equiv (G(t_0)F)(x) \in \mathbb{C}$ is bounded, Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic provided that R is any non-empty collection of sequences in $\mathbb{R}^n$ and the function $F(\cdot)$ is bounded, Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic. The conclusions obtained in [5, Example 1] and the fifth application of [4, Section 6] can be also reconsidered for the Stepanov multi-dimensional almost automorphic type inhomogeneities.*

The organization and main ideas of this paper can be briefly described as follows. The main aim of Subsection 1.1 is to remind the readers of the basic definitions and results from the theory of Lebesgue spaces with variable exponents; in Subsection 1.2, we collect the necessary definitions and results about multi-dimensional almost periodic functions and multi-dimensional almost automorphic functions, while in Subsection 1.3 we recall the basic definitions with regards to Stepanov multi-dimensional almost periodic functions in Lebesgue spaces with variable exponents.

In the one-dimensional setting, the notion of a Stepanov p-almost automorphy was introduced by V. Casarino [2] in 2000 and later reconsidered, in a slightly different form, by G. M. N'Guérékata and A. Pankov [17] in 2008 ($1 \leq p < \infty$). In [9, Definition 6, Definition 7], we have recently made the first steps in the analysis of (asymptotical) Stepanov $p(x)$-almost automorphy in the one-dimensional setting. The main aim of Section 2 is to investigate the Stepanov multi-dimensional almost automorphic type functions in Lebesgue spaces with variable exponents; here, among many other topics, we investigate the pointwise products of Stepanov multi-dimensional almost automorphic functions, the convolution invariance of Stepanov multi-dimensional almost automorphy and provide several illustrative examples. In Subsection 2.1, we provide certain applications of our results to the abstract Volterra integro-differential equations in Banach spaces, considering primarily the multi-dimensional heat equation and the multi-dimensional wave equation. Although we work with Lebesgue spaces with variable exponents, it is worth noting that the introduced classes of Stepanov multi-dimensional almost automorphic functions seem to be not analyzed elsewhere even in the case that the exponent $p(\cdot)$ has a constant value.

Before explaining the notation, we would like to emphasize that this is the first research study of spatially Stepanov almost automorphic solutions of the partial differential equations (for some results concerning the Stepanov almost automorphic solutions in time for various classes of the partial differential equations, one may refer *e.g.* to the works [7, 9, 11, 12, 16–19, 21] and references quoted therein).

Unless stated otherwise, we will always assume that Ω is a fixed compact subset of $\mathbb{R}^n$ with positive Lebesgue measure as well as that $p : \Omega \to [1,\infty]$ is an element of the space $\mathcal{P}(\Omega)$, introduced in Subsection 1.1. We will also always assume henceforth that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces, R is a non-empty collection of sequences in $\mathbb{R}^n$ and R_X is a non-empty collection of sequences in $\mathbb{R}^n \times X$; $L(X,Y)$ denotes the Banach algebra of all bounded linear operators from X into Y with $L(X,X)$ being denoted $L(X)$. By $\mathcal{B}$ we denote a certain collection of non-empty subsets of X satisfying that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$. By $P(A)$ we denote the power set of A and by $(e_1, \cdot \cdot \cdot, e_n)$ we denote the canonical basis of $\mathbb{R}^n$.

In this paper, we will always assume that Ω is a fixed compact subset of $\mathbb{R}^n$ with positive Lebesgue measure and $p \in \mathcal{P}(\Omega)$. If $F : \mathbb{R}^n \times X \to Y$, then we introduce the multi-dimensional Bochner transform $\hat{F}_\Omega : \mathbb{R}^n \times X \to Y^\Omega$ by

$$\big[\hat{F}_\Omega(\mathbf{t}; x)\big](u) := F(\mathbf{t} + \mathbf{u}; x), \quad \mathbf{t} \in \mathbb{R}^n,\ \mathbf{u} \in \Omega,\ x \in X;$$

here, Y^Ω denotes the set of all functions from Ω into Y.

1.1 Lebesgue spaces with variable exponents $L^{p(x)}$

Basic source of information about Lebesgue spaces with variable exponents $L^{p(x)}$ can be obtained by consulting the research monograph [10] by L. Diening, P. Harjulehto, P. Hästüso and M. Ruzicka.

Suppose that $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a non-empty Lebesgue measurable subset and $M(\Omega : X)$ is the collection of all measurable functions $f : \Omega \to X$; $M(\Omega) := M(\Omega : \mathbb{R})$. By $\mathcal{P}(\Omega)$ we denote the vector space of all Lebesgue measurable functions $p : \Omega \to [1,\infty]$. If $p \in \mathcal{P}(\Omega)$ and $f \in M(\Omega : X)$, then we define

$$\varphi_{p(x)}(t) := \begin{cases} t^{p(x)}, & t \geq 0,\ 1 \leq p(x) < \infty, \\ 0, & 0 \leq t \leq 1,\ p(x) = \infty, \\ \infty, & t > 1,\ p(x) = \infty \end{cases}$$

and

$$\rho(f) := \int_\Omega \varphi_{p(x)}(\|f(x)\|)\ \mathrm{d}x.$$

The Lebesgue space $L^{p(x)}(\Omega : X)$ with variable exponent is defined by

$$L^{p(x)}(\Omega : X) := \Big\{ f \in M(\Omega : X) : \lim_{\lambda \to 0+} \rho(\lambda f) = 0 \Big\}.$$

Equivalently,

$$L^{p(x)}(\Omega : X) = \Big\{ f \in M(\Omega : X) : \text{ there exists } \lambda > 0 \text{ such that } \rho(\lambda f) < \infty \Big\};$$

see, *e.g.*, [10, p. 73]. For every $u \in L^{p(x)}(\Omega : X)$, we introduce the Luxemburg norm of $u(\cdot)$ by

$$\|u\|_{p(x)} := \|u\|_{L^{p(x)}(\Omega:X)} := \inf\Big\{ \lambda > 0 : \rho(u/\lambda) \leq 1 \Big\}.$$

Equipped with the above norm, the space $L^{p(x)}(\Omega : X)$ is a Banach space, coinciding with the usual Lebesgue space $L^p(\Omega : X)$ in the case that $p(x) \equiv p \geq 1$ is a constant function. Further on, if $p \in M(\Omega) \equiv M(\Omega : \mathbb{C})$, then we define

$$p^- := \text{essinf}_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \text{esssup}_{x\in\Omega} p(x).$$

Set

$$D_+(\Omega) := \big\{p \in M(\Omega) : 1 \leq p^- \leq p(x) \leq p^+ < \infty \text{ for a.e. } x \in \Omega\big\}.$$

Let us recall that, for every $p \in D_+([0,1])$, we have

$$L^{p(x)}(\Omega : X) = \Big\{f \in M(\Omega : X)\,;\ \text{for all } \lambda > 0 \text{ we have } \rho(\lambda f) < \infty\Big\}.$$

Set

$$E^{p(x)}(\Omega : X) := \Big\{f \in L^{p(x)}(\Omega : X)\,;\ \text{for all } \lambda > 0 \text{ we have } \rho(\lambda f) < \infty\Big\};$$

$E^{p(x)}(\Omega) \equiv E^{p(x)}(\Omega : \mathbb{C})$.

We will use the following lemma:

Lemma 1.2 *(i) (The Hölder inequality) Let $p,\ q,\ r \in \mathcal{P}(\Omega)$ such that*

$$\frac{1}{q(x)} = \frac{1}{p(x)} + \frac{1}{r(x)}, \quad x \in \Omega.$$

Then, for every $u \in L^{p(x)}(\Omega : X)$ and $v \in L^{r(x)}(\Omega)$, we have $uv \in L^{q(x)}(\Omega : X)$ and

$$\|uv\|_{q(x)} \leq 2\|u\|_{p(x)}\|v\|_{r(x)}.$$

(ii) Let Ω be of a finite Lebesgue's measure and let $p,\ q \in \mathcal{P}(\Omega)$ such $q \leq p$ a.e. on Ω. Then $L^{p(x)}(\Omega : X)$ is continuously embedded in $L^{q(x)}(\Omega : X)$, and the constant of embedding is less than or equal to $2(1 + m(\Omega))$.

(iii) Let $f \in L^{p(x)}(\Omega : X)$, $g \in M(\Omega : X)$ and $0 \leq \|g\| \leq \|f\|$ a.e. on Ω. Then $g \in L^{p(x)}(\Omega : X)$ and $\|g\|_{p(x)} \leq \|f\|_{p(x)}$.

(iv) (The dominated convergence theorem) Let $p \in \mathcal{P}(\Omega)$, and let $f_k,\ f \in M(\Omega : X)$ for all $k \in \mathbb{N}$. If $\lim_{k\to\infty} f_k(x) = f(x)$ for a.e. $x \in \Omega$ and there exists a real-valued function $g \in E^{p(x)}(\Omega)$ such that $\|f_k(x)\| \leq g(x)$ for a.e. $x \in \Omega$, then $\lim_{k\to\infty} \|f_k - f\|_{L^{p(x)}(\Omega:X)} = 0$.

For further information concerning the Lebesgue spaces with variable exponents $L^{p(x)}$, see also [13] and [27].

1.2 Multi-dimensional almost periodic functions and multi-dimensional almost automorphic functions

Let us recall that, throughout this paper, we assume that $n \in \mathbb{N}$, $\mathcal{B}$ is a non-empty collection of subsets of X, R is a non-empty collection of sequences in $\mathbb{R}^n$ and R_X is a non-empty collection of sequences in $\mathbb{R}^n \times X$; usually, $\mathcal{B}$ denotes the collection of all bounded subsets of X or all compact subsets of X. Henceforth we will always assume that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.

In this subsection, we recall the basic facts about multi-dimensional almost periodic functions and multi-dimensional almost automorphic functions; see [3] and [5] for more details.

The following definition is extremely important in our study:

Definition 1.3 *Suppose that $F : \mathbb{R}^n \times X \to Y$ is a continuous function as well as that for each $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot \cdot \cdot, b_k^n)) \in \mathrm{R}$ we have that $W_{B,(\mathbf{b}_k)} : B \to P(P(\mathbb{R}^n))$ and $\mathrm{P}_{B,(\mathbf{b}_k)} \in P(P(\mathbb{R}^n \times B))$. Then we say that the function $F(\cdot;\cdot)$ is:*

- *(i)* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot \cdot \cdot, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \cdot \cdot \cdot, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that*

$$\lim_{l \to +\infty} F\bigl(\mathbf{t} + (b_{k_l}^1, \cdot \cdot \cdot, b_{k_l}^n); x\bigr) = F^*(\mathbf{t}; x)$$

  uniformly for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$;

- *(ii)* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot \cdot \cdot, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \cdot \cdot \cdot, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that*

$$\lim_{l \to +\infty} F\bigl(\mathbf{t} + (b_{k_l}^1, \cdot \cdot \cdot, b_{k_l}^n); x\bigr) = F^*(\mathbf{t}; x) \tag{1.3}$$

  and

$$\lim_{l \to +\infty} F^*\bigl(\mathbf{t} - (b_{k_l}^1, \cdot \cdot \cdot, b_{k_l}^n); x\bigr) = F(\mathbf{t}; x), \tag{1.4}$$

  pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$. If for each $x \in B$ the above limits converge uniformly on compact subsets of $\mathbb{R}^n$, then we say that $F(\cdot;\cdot)$ is compactly $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic;*

- *(iii)* $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot \cdot \cdot, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \cdot \cdot \cdot, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that* (1.3)-(1.4) *hold pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ as well as that for each $x \in B$ the convergence in $\mathbf{t}$ is uniform for any element of the collection* $W_{B,(\mathbf{b}_k)}(x)$;

- *(iv)* $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \cdot \cdot \cdot, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \cdot \cdot \cdot, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that* (1.3)-(1.4) *hold pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ as well as that the convergence in* (1.3)-(1.4) *is uniform in $(\mathbf{t}; x)$ for any set of the collection* $\mathrm{P}_{B,(\mathbf{b}_k)}$.

Further on, the notion introduced above is a special case of the notion introduced in the following definition, with

$$\mathrm{R}_X := \bigl\{ b : \mathbb{N} \to \mathbb{R}^n \times X \, ; \, (\exists a \in \mathrm{R}) \, b(l) = (a(l); 0) \text{ for all } l \in \mathbb{N} \bigr\}. \tag{1.5}$$

Definition 1.4 *Suppose that $F : \mathbb{R}^n \times X \to Y$ is a continuous function as well as that for each $B \in \mathcal{B}$ and $(\mathbf{b}; \mathbf{x}) = ((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \cdot\cdot\cdot, b_k^n); x_k)) \in \mathrm{R}_X$ we have $W_{B,(\mathbf{b};\mathbf{x})} : B \to P(P(\mathbb{R}^n))$ and $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})} \in P(P(\mathbb{R}^n \times B))$. Then we say that the function $F(\cdot;\cdot)$ is:*

(i) $(\mathrm{R}_X, \mathcal{B})$*-multi-almost periodic if and only if for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \cdot\cdot\cdot, b_k^n); x_k)) \in \mathrm{R}_X$ there exist a subsequence $((\mathbf{b}; \mathbf{x})_{k_l} = ((b_{k_l}^1, b_{k_l}^2, \cdot\cdot\cdot, b_{k_l}^n); x_{k_l}))$ of $((\mathbf{b}; \mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that*

$$\lim_{l\to+\infty} F\bigl(\mathbf{t} + (b_{k_l}^1, \cdot\cdot\cdot, b_{k_l}^n); x + x_{k_l}\bigr) = F^*(\mathbf{t}; x)$$

uniformly for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$;

(ii) $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \cdot\cdot\cdot, b_k^n); x_k)) \in \mathrm{R}_X$ there exist a subsequence $((\mathbf{b}; \mathbf{x})_{k_l} = ((b_{k_l}^1, b_{k_l}^2, \cdot\cdot\cdot, b_{k_l}^n); x_{k_l}))$ of $((\mathbf{b}; \mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that*

$$\lim_{l\to+\infty} F\bigl(\mathbf{t} + (b_{k_l}^1, \cdot\cdot\cdot, b_{k_l}^n); x + x_{k_l}\bigr) = F^*(\mathbf{t}; x) \tag{1.6}$$

and

$$\lim_{l\to+\infty} F^*\bigl(\mathbf{t} - (b_{k_l}^1, \cdot\cdot\cdot, b_{k_l}^n); x - x_{k_l}\bigr) = F(\mathbf{t}; x), \tag{1.7}$$

pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$. We say that the function $F(\cdot;\cdot)$ is compactly $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic if and only if the convergence of limits in (1.6)-(1.7) *is uniform on any compact subset K of $\mathbb{R}^n \times X$ which belongs to $\mathbb{R}^n \times B$;*

(iii) $(\mathrm{R}_X, \mathcal{B}, W_{\mathcal{B},\mathrm{R}_X})$*-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \cdot\cdot\cdot, b_k^n); x_k)) \in \mathrm{R}_X$ there exist a subsequence $((\mathbf{b}; \mathbf{x})_{k_l})$ of $((\mathbf{b}; \mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that* (1.6)-(1.7) *hold pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ as well as that for each $x \in B$ the convergence in* (1.6)-(1.7) *is uniform in $\mathbf{t}$ for any set of the collection $W_{B,(\mathbf{b};\mathbf{x})}(x)$;*

(iv) $(\mathrm{R}_X, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}_X})$*-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \cdot\cdot\cdot, b_k^n); x_k)) \in \mathrm{R}_X$ there exist a subsequence $((\mathbf{b}; \mathbf{x})_{k_l})$ of $((\mathbf{b}; \mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that* (1.6)-(1.7) *hold pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ as well as that the convergence in* (1.6)-(1.7) *is uniform in $(\mathbf{t}; x)$ for any set of the collection $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})}$.*

It is clear that the assumption $X \in \mathcal{B}$ implies that a continuous function $F : \mathbb{R}^n \times X \to Y$ is (compactly) $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic if and only if the above requirements hold for any sequence $((\mathbf{b}; \mathbf{x})_k) \in \mathrm{R}_X$ and the set $B = X$.

We also need the following definition from [3]:

Definition 1.5 *Suppose that $\mathbb{D} \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded. By $C_{0,\mathbb{D},\mathcal{B}}(\mathbb{R}^n \times X : Y)$ we denote the vector space consisting of all continuous functions $Q : \mathbb{R}^n \times X \to Y$ such that, for every $B \in \mathcal{B}$, we have $\lim_{\mathbf{t}\in\mathbb{D},|\mathbf{t}|\to+\infty} Q(\mathbf{t}; x) = 0$, uniformly for $x \in B$.*

1.3 Stepanov multi-dimensional almost periodic functions in Lebesgue spaces with variable exponents

In this subsection, we will repeat the basic definitions about Stepanov multi-dimensional almost periodic functions in Lebesgue spaces with variable exponents ([4]). Let us recall that our standing assumptions are that Ω is a fixed compact subset of $\mathbb{R}^n$ with positive Lebesgue measure and $p \in \mathcal{P}(\Omega)$.

We need the following notion:

Definition 1.6 *Suppose that $F : \mathbb{R}^n \times X \to Y$ satisfies that for each $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$, the function $F(\mathbf{t} + \mathbf{u}; x)$ belongs to the space $L^{p(\mathbf{u})}(\Omega : Y)$. Then we say that $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-bounded on $\mathcal{B}$ if and only if for each $B \in \mathcal{B}$ we have*

$$\sup_{\mathbf{t}\in\mathbb{R}^n;x\in B}\Big\|\big[\hat{F}_\Omega(\mathbf{t};x)\big](u)\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} = \sup_{\mathbf{t}\in\mathbb{R}^n;x\in B}\Big\|F(\mathbf{t}+\mathbf{u};x)\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} < \infty.$$

Definition 1.7 *Suppose that $F : \mathbb{R}^n \times X \to Y$, and the function $\hat{F} : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined and continuous. Then we say that the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic if and only if the function $\hat{F}_\Omega : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, i.e., for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \cdots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \cdots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ such that*

$$\lim_{l\to+\infty}\Big\|F\big(\mathbf{t}+\mathbf{u}+(b_{k_l}^1,\cdots,b_{k_l}^n);x\big) - \Big[F^*(\mathbf{t};x)\Big](\mathbf{u})\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0,$$

uniformly for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$.

The notion introduced in the last definition is a special case of the following notion (this can be seen using the collection R_X defined in (1.5)):

Definition 1.8 *Suppose that $F : \mathbb{R}^n \times X \to Y$, and the function $\hat{F} : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined and continuous. Then we say that the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic if and only if the function $\hat{F}_\Omega : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic, i.e., for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b};\mathbf{x})_k = ((b_k^1, b_k^2, \cdots, b_k^n); x_k)) \in \mathrm{R}_X$ there exist a subsequence $((\mathbf{b};\mathbf{x})_{k_l} = ((b_{k_l}^1, b_{k_l}^2, \cdots, b_{k_l}^n); x_{k_l}))$ of $((\mathbf{b};\mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ such that*

$$\lim_{l\to+\infty}\Big\|F\big(\mathbf{t}+\mathbf{u}+(b_{k_l}^1,\cdots,b_{k_l}^n);x+x_{k_l}\big) - \Big[F^*(\mathbf{t};x)\Big](\mathbf{u})\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0,$$

uniformly for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$.

We will also use the conclusions from the following remark:

Remark 1.9 *(i) Suppose that $p \in D_+(\Omega)$, there exists a finite constant $L \geq 1$ such that*

$$\|F(\mathbf{t};x) - F(\mathbf{t};y)\|_Y \leq L\|x-y\|, \quad \mathbf{t} \in \mathbb{R}^n, \quad x,\ y \in X \tag{1.8}$$

and the mapping $\hat{F}_\Omega : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined. Then it is continuous.

(ii) Suppose that $F : \mathbb{R}^n \times X \to Y$ is continuous and $p \in D_+(\Omega)$. Then the continuity of mapping $\hat{F}_\Omega : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ follows directly by applying the dominated convergence theorem (see Lemma 1.2(iv)).

2 Stepanov multi-dimensional almost automorphic type functions

This section investigates Stepanov multi-dimensional almost automorphic type functions in Lebesgue spaces with variable exponents. Besides our standing assumptions, we will occasionally use the following one:

(ST) The function $F : \mathbb{R}^n \times X \to Y$ satisfies that the Bochner transform $\hat{F} : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined and continuous. For each $B \in \mathcal{B}$ and $\mathbf{b} = (\mathbf{b}_k) \in \mathrm{R}$ we have that $W_{B,\mathbf{b}} : B \to P(P(\mathbb{R}^n))$ and $\mathrm{P}_{B,\mathbf{b}} \in P(P(\mathbb{R}^n \times B))$; for each $B \in \mathcal{B}$ and $(\mathbf{b}; \mathbf{x}) = ((\mathbf{b}; \mathbf{x})_k) \in \mathrm{R}_X$ we have $W_{B,(\mathbf{b};\mathbf{x})} : B \to P(P(\mathbb{R}^n))$ and $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})} \in P(P(\mathbb{R}^n \times B))$.

If this condition holds, then we can simply introduce the following classes of Stepanov multi-dimensional almost automorphic type functions:

Definition 2.1 *Suppose that (ST) holds. Then we say that the function $F(\cdot;\cdot)$ is:*

(i) Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic if and only if the function $\hat{F} : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic;

(ii) Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic [Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic] if and only if the function $\hat{F} : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic [$(\mathrm{R}, \mathcal{B}, \mathrm{R}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic];

(iii) Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic if and only if the function $\hat{F} : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic;

(iv) Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic [Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic] if and only if the function $\hat{F} : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathrm{R}_X, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic [$(\mathrm{R}_X, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic].

For the functions of the form $F : \mathbb{R}^n \to Y$, we will omit the term "$\mathcal{B}$" from the notation henceforth.

Without any doubt, the most important case in our analysis is that one in which we have that R (R_X) is a collection of all sequences $\mathbf{b}(\cdot)$ in $\mathbb{R}^n$ ($(\mathbf{b}; \mathbf{x})$ in $\mathbb{R}^n \times X$). If this is the case and $\Omega = [0, 1]^n$, then we will simply say that the function $F : \mathbb{R}^n \to Y$ is Stepanov $p(\cdot)$-almost automorphic.

Let $k \in \mathbb{N}$ and $F_i : \mathbb{R}^n \times X \to Y_i$ $(1 \leq i \leq k)$. Then we define the function $(F_1, \cdots, F_k) : \mathbb{R}^n \times X \to Y_1 \times \cdots \times Y_k$ by

$$(F_1, \cdots, F_k)(\mathbf{t}; x) := (F_1(\mathbf{t}; x), \cdots, F_k(\mathbf{t}; x)), \quad \mathbf{t} \in \mathbb{R}^n, \ x \in X.$$

Keeping in mind the introduced notion, we immediately get:

Proposition 2.2 *Suppose that $k \in \mathbb{N}$,* (ST) *and the following condition hold:*

(C1) for each set $B \in \mathcal{B}$, for each sequence $(\mathbf{b};\mathbf{x}) = ((\mathbf{b};\mathbf{x})_k) \in \mathrm{R}_X$ and for every subsequence $(\mathbf{b};\mathbf{x})'$ of $(\mathbf{b};\mathbf{x})$ we have $W_{B,(\mathbf{b};\mathbf{x})}(x) \subseteq W_{B,(\mathbf{b};\mathbf{x})'}(x)$ for all $x \in B$ and $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})} \subseteq \mathrm{P}_{B,(\mathbf{b};\mathbf{x})'}$.

If the function $F_i : \mathbb{R}^n \times X \to Y_i$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic [Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B}, W_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic; Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic] for $1 \leq i \leq k$, then the function $(F_1, \cdot \cdot \cdot, F_k)(\cdot;\cdot)$ has the same property.

Clearly, we also have the following:

Proposition 2.3 *Suppose that* (ST) *holds.*

(i) If the function $F : \mathbb{R}^n \times X \to Y$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic, then $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic, where for each $B \in \mathcal{B}$ and $\mathbf{b} \in \mathrm{R}$ we have $\mathrm{P}_{B,\mathbf{b}} = \{\{\mathbb{R}^n \times B\}\}$.

(ii) Suppose that for each set $B \in \mathcal{B}$ and sequence $(\mathbf{b}_k; x_k) \in \mathrm{R}_X$ we have that there exists an integer $k_0 \in \mathbb{N}$ such that, for every integer $k \geq k_0$, we have $B - x_k \subseteq B$. If the function $F : \mathbb{R}^n \times X \to Y$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic, then $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic, where for each $B \in \mathcal{B}$ and $(\mathbf{b};\mathbf{x}) \in \mathrm{R}_X$ we have $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})} = \{\{\mathbb{R}^n \times B\}\}$.

We continue by providing two illustrative examples; the first one is a slight modification of [5, Example 3] and the second one is a slight modification of [5, Example 5]:

Example 2.4 *Suppose that $\varphi : \mathbb{R} \to \mathbb{C}$ is an almost periodic function, $\Omega := [0,1]^2$ and $(T(t))_{t\in\mathbb{R}} \subseteq L(X,Y)$ is an operator family which is strongly locally integrable and not strongly continuous at zero. Suppose, further, that there exist a finite real number $M \geq 1$ and a real number $\gamma \in (0,1)$ such that*

$$\big\|T(t)\big\|_{L(X,Y)} \leq \frac{M}{|t|^{\gamma}}, \quad t \in \mathbb{R} \setminus \{0\}, \tag{2.1}$$

as well as that R is the collection of all sequences in $\Delta_2 \equiv \{(t,t) : t \in \mathbb{R}\}$ and $\mathcal{B}$ is the collection of all bounded subsets of X. Define a function $F : \mathbb{R}^2 \times X \to Y$ by

$$F(t,s;x) := e^{\int_s^t \varphi(\tau)\, \mathrm{d}\tau} T(t-s)x, \quad (t,s) \in \mathbb{R}^2,\ x \in X.$$

Let for each bounded subset B of X and for each sequence $(\mathbf{b}_k = (b_k, b_k))$ in R the collection $P_{B,(\mathbf{b}_k)}$ be constituted of all sets of form $\{(t,s) \in \mathbb{R}^2 : |t-s| \leq L\} \times B$, where $L > 0$. Then the function $F(\cdot,\cdot;\cdot)$ is Stepanov $(\Omega, 1)$-$(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic, which can be deduced as follows. First of all, it can be simply shown with the help of Fubini theorem and our assumption (2.1) *that for each real numbers $s,\ t \in \mathbb{R}$ and for each element $x \in X$ we have ($\mathbf{u} = (u_1, u_2)$):*

$$\big(u_1, u_2\big) \mapsto F\big(t+u_1, s+u_2; x\big) \equiv e^{\int_{s+u_2}^{t+u_1} \varphi(r)\, \mathrm{d}r} T\big(t-s+(u_1-u_2)\big)x \in L^1([0,1]^2 : Y).$$

Furthermore, it can be simply shown with the help of the Fubini theorem, the dominated convergence theorem and an elementary argumentation that the function $\hat{F}_\Omega : \mathbb{R}^2 \times X \to Y$ *is continuous. Let a real number* $L > 0$ *and a bounded subset* B *of* X *be fixed, and let* $(t, s) \in \mathbb{R}^2$ *satisfy* $|t - s| \leq L$. *By Bochner's criterion, there exist a subsequence* (b_{k_l}, b_{k_l}) *of* (b_k, b_k) *and a function* $\varphi^* : \mathbb{R} \to \mathbb{C}$ *such that* $\lim_{l\to+\infty} \varphi(r + b_{k_l}) = \varphi^*(r)$, *uniformly in* $r \in \mathbb{R}$. *Set*

$$\Big[F^*(t, s; x)\Big]\big(u_1, u_2\big) := e^{\int_{s+u_2}^{t+u_1} \varphi^*(r)\, \mathrm{d}r} T\big(t - s + (u_1 - u_2)\big)x, \quad (t, s) \in \mathbb{R}^2,\ x \in X.$$

Arguing as in [5], we can simply shown that this is the right choice of required limit function as well as that the function $F(\cdot, \cdot; \cdot)$ *is Stepanov* $(\Omega, 1)$-$(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic, as claimed. Observe, finally, that the function* $F(\cdot, \cdot; \cdot)$ *is not* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic in general since it is not necessarily continuous in general as well as that the higher-dimensional analogue of this example can be constructed in the same way as in [5]. The interested reader may try to reconsider the conclusions established in [5, Example 4], provided that the functions* $f_j(\cdot)$ *from this example are Stepanov* p*-almost automorphic (Stepanov* p*-almost periodic) for some finite real exponent* $p \geq 1$.

Example 2.5 *Suppose that* R *is any collection of sequences in* $\mathbb{R}^n$ *such that each subsequence of a sequence* $(\mathbf{b}_k) \in \mathrm{R}$ *also belongs to* R, *as well as that* R' *is any collection of sequences in* $\mathbb{R}^m$ *such that each subsequence of a sequence* $(\mathbf{b}'_k) \in \mathrm{R}'$ *also belongs to* R'. *Let* $f_i : \mathbb{R}^n \to \mathbb{R}$ *be a Stepanov* 1*-bounded, Stepanov* $(\Omega_1, 1)$-R*-almost automorphic function (*$1 \leq i \leq p$*), and* $g_j : \mathbb{R}^m \to \mathbb{R}$ *be a Stepanov* 1*-bounded, Stepanov* $(\Omega_1, 1)$-R'*-almost automorphic function (*$1 \leq j \leq q$*). Define the functions* $F : \mathbb{R}^n \to \mathbb{R}^q$ *by* $F(\mathbf{t}) := \sum_{i=1}^p f_i(\mathbf{t})e_i$ *and* $G : \mathbb{R}^m \to \mathbb{R}^q$ *by* $G(\mathbf{s}) := \sum_{j=1}^q g_j(\mathbf{s})e_j$. *Define also the function* $F \bigotimes G : \mathbb{R}^n \times \mathbb{R}^m \to M_{p\times q}(\mathbb{R})$ *by (*$\mathbf{t} \in \mathbb{R}^n,\ \mathbf{s} \in \mathbb{R}^m$*)*

$$F \bigotimes G\,(\mathbf{t}, \mathbf{s}) := \begin{pmatrix} f_1(\mathbf{t})g_1(\mathbf{s}) & f_1(\mathbf{t})g_2(\mathbf{s}) & \cdots & f_1(\mathbf{t})g_q(\mathbf{s}) \\ f_2(\mathbf{t})g_1(\mathbf{s}) & f_2(\mathbf{t})g_2(\mathbf{s}) & \cdots & f_2(\mathbf{t})g_q(\mathbf{s}) \\ \vdots & \vdots & \ddots & \vdots \\ f_p(\mathbf{t})g_1(\mathbf{s}) & f_p(\mathbf{t})g_2(\mathbf{s}) & \cdots & f_p(\mathbf{t})g_q(\mathbf{s}) \end{pmatrix},$$

where $M_{p\times q}(\mathbb{R})$ *denotes the set of all real matrices of format* $p \times q$. *Set, for every* $\mathbf{u} \in \Omega_1$, $\mathbf{v} \in \Omega_2$, $\mathbf{t} \in \mathbb{R}^n$ *and* $\mathbf{s} \in \mathbb{R}^m$,

$$\Big[F \bigotimes G\,(\mathbf{t}, \mathbf{s})\Big](\mathbf{u}, \mathbf{v})$$
$$:= \begin{pmatrix} \big[f_1^*(\mathbf{t})\big](\mathbf{u})\big[g_1^*(\mathbf{s})\big](\mathbf{v}) & \big[f_1^*(\mathbf{t})\big](\mathbf{u})\big[g_2^*(\mathbf{s})\big](\mathbf{v}) & \cdots & \big[f_1^*(\mathbf{t})\big](\mathbf{u})\big[g_q^*(\mathbf{s})\big](\mathbf{v}) \\ \big[f_2^*(\mathbf{t})\big](\mathbf{u})\big[g_1^*(\mathbf{s})\big](\mathbf{v}) & \big[f_2^*(\mathbf{t})\big](\mathbf{u})\big[g_2^*(\mathbf{s})\big](\mathbf{v}) & \cdots & \big[f_2^*(\mathbf{t})\big](\mathbf{u})\big[g_q^*(\mathbf{s})\big](\mathbf{v}) \\ \vdots & \vdots & \ddots & \vdots \\ \big[f_p^*(\mathbf{t})\big](\mathbf{u})\big[g_1^*(\mathbf{s})\big](\mathbf{v}) & \big[f_p^*(\mathbf{t})\big](\mathbf{u})\big[g_2^*(\mathbf{s})\big](\mathbf{v}) & \cdots & \big[f_p^*(\mathbf{t})\big](\mathbf{u})\big[g_q^*(\mathbf{s})\big](\mathbf{v}) \end{pmatrix}.$$

Then it is not difficult to prove that $F \bigotimes G$ *is Stepanov* $(\Omega_1 \times \Omega_2, 1)$-$(\mathrm{R} \times \mathrm{R}')$*-almost automorphic, where* $\mathrm{R} \times \mathrm{R}' := \{(\mathbf{b}, \mathbf{b}') : \mathbf{b} \in \mathrm{R},\ \mathbf{b}' \in \mathrm{R}'\}$. *If, in addition to the above, for each* $i \in \mathbb{N}_p$ *we have that* $f_i : \mathbb{R}^n \to \mathbb{R}$ *is a Stepanov* $(\Omega_1, 1)$-$(\mathrm{R}, \mathrm{P}_\mathrm{R})$*-almost automorphic function as well as that for each* $j \in \mathbb{N}_q$ *we have that* $g_j : \mathbb{R}^n \to \mathbb{R}$ *is a Stepanov* $(\Omega_2, 1)$-$(\mathrm{R}', \mathrm{P}'_{\mathrm{R}'})$*-almost automorphic function, then the function* $F \bigotimes G$ *is a Stepanov* $(\Omega_1 \times \Omega_2, 1)$-$(\mathrm{R} \times \mathrm{R}', \mathrm{P}''_{\mathrm{R}\times\mathrm{R}'})$*-almost automorphic function, provided that for each sequence* $\mathbf{b}$ *from* R *(*$\mathbf{c}$ *from* R'*) each set of the collection* $\mathrm{P}_\mathbf{b}$ *(*$\mathrm{P}_\mathbf{c}$*) belongs to the collection* $\mathrm{P}_{\mathbf{b}'}$ *(*$\mathrm{P}_{\mathbf{c}'}$*) for any subsequence* $\mathbf{b}'$ *of* $\mathbf{b}$ *(*$\mathbf{c}'$ *of* $\mathbf{c}$*) and for each sequence* $(\mathbf{b}; \mathbf{c})$ *belonging to* $\mathrm{R} \times \mathrm{R}'$ *the collection* $\mathrm{P}''_{(\mathbf{b};\mathbf{c})}$ *consists of all direct products of sets from the collections* $\mathrm{P}_\mathbf{b}$ *and* $\mathrm{P}'_\mathbf{c}$.

Further on, let us consider the notion introduced in Definition 2.1(iii). If the function $\hat{F} : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic, then for each set $B \in \mathcal{B}$ and sequence $(\mathbf{b}_k; x_k) \in \mathrm{R}_X$ there exist a subsequence $(\mathbf{b}_{k_l}; x_{k_l})$ of $(\mathbf{b}_k; x_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that

$$\lim_{l\to+\infty}\Big\|F\big(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_l}; x+x_{k_l}\big) - \Big[F^*(\mathbf{t};x)\Big](\mathbf{u})\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0 \tag{2.2}$$

and

$$\lim_{l\to+\infty}\Big\|F^*\big(\mathbf{t}+\mathbf{u}-\mathbf{b}_{k_l}; x-x_{k_l}\big) - F(\mathbf{t}+\mathbf{u};x)\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0$$

pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$. In general case, it is very difficult to deduce the existence of a function $G : \mathbb{R}^n \times X \to Y$ such that $G(\mathbf{t}+\mathbf{u};x) = [F^*(\mathbf{t};x)](\mathbf{u})$ for all $x \in B$ and a.e. $\mathbf{t} \in \mathbb{R}^n,\ u \in \Omega$. But, this can be always done provided that $\Omega = [0,1]^n$, which can be simply deduced by using the first limit equality (2.2) and the proof of [9, Proposition 3.1] with appropriate modifications (we can write down the set $\mathbb{R}^n$ as the union of sets $\mathbf{k}+\Omega$ when $\mathbf{k} \in \mathbb{Z}^n$ and define after that $G(\mathbf{t};x) := [F^*(\mathbf{k};x)](\mathbf{t}-\mathbf{k})$ if $\mathbf{t} \in \mathbf{k}+\Omega$ for some $\mathbf{k} \in \mathbb{Z}^n$).

Further on, in [4, Proposition 2.23, Proposition 2.25] (see also [9, Theorem 3.3]), we have clarified several embedding type results for the spaces of Stepanov multi-dimensional almost periodic functions. These results can be reformulated for the corresponding spaces of Stepanov multi-dimensional almost automorphic functions since their proofs simply follow by applying Lemma 1.2. For example, if $p \in D_+(\Omega)$ and $1 \leq p^- \leq p(\mathbf{u}) \leq p^+ < +\infty$ for a.e. $\mathbf{u} \in \Omega$, then any Stepanov (Ω, p^+)-$(\mathrm{R}_X, \mathcal{B})$-almost automorphic function is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-almost automorphic and any Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-almost automorphic function is Stepanov (Ω, p^-)-$(\mathrm{R}_X, \mathcal{B})$-almost automorphic; in particular, any Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-almost automorphic function is Stepanov $(\Omega, 1)$-$(\mathrm{R}_X, \mathcal{B})$-almost automorphic (this statement actually holds for any $p \in \mathcal{P}(\Omega)$).

Concerning the statements of [4, Proposition 2.24] and [9, Proposition 3.4], we will clarify the following result, only:

Proposition 2.6 *Suppose that the function $F : \mathbb{R}^n \times X \to Y$ is $(\mathrm{R}, \mathcal{B})$-multi almost automorphic, $p \in D_+(\Omega)$ and for each set $B \in \mathcal{B}$ we have $\sup_{\mathbf{t}\in\mathbb{R}^n; x\in B} \|F(\mathbf{t};x)\|_Y < +\infty$. Then the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-almost automorphic.*

Proof. Due to our conclusion from Remark 1.9(ii), the function $\hat{F}_\Omega : \mathbb{R}^n \times X \to Y$ is continuous. Since $p \in D_+(\Omega)$, we have $1 \in E^{p(\mathbf{u})}(\Omega)$ so that the final conclusion simply follows by applying the dominated convergence theorem and our assumption that for each set $B \in \mathcal{B}$ we have $\sup_{\mathbf{t}\in\mathbb{R}^n; x\in B} \|F(\mathbf{t};x)\|_Y < +\infty$. □

We continue by stating the following illustrative examples:

Example 2.7 *Let $\mathcal{F} := \{F \in L^1_{loc}(\mathbb{R}^n : Y)\ ;\ supp(F)\ \text{ is compact}\}$, let $p \in \mathcal{P}(\Omega)$ and let R denote any collection of sequences in $\mathbb{R}^n$ such that there exists a sequence $(\mathbf{b}_k) \in \mathrm{R}$ whose any subsequence is unbounded. Then a non-trivial function $F \in \mathcal{F}$ cannot be Stepanov $(\Omega, p(\mathbf{u}))$-R-multi-almost automorphic, which can be shown arguing as in [21, Example 1, Example 2]. On the other hand, a non-trivial function $F \in \mathcal{F}$ can belong to certain classes of equi-Weyl multi-dimensional almost automorphic functions (see [23, Example 3.5] for more details).*

Example 2.8 *(see the second part of [4, Example 2.9]) Suppose that* $\Omega := [0,1]^n$ *and* $p(\mathbf{u}) := 1-\ln(u_1 \cdot u_2 \cdots u_n)$*, where* $\mathbf{u}=(u_1,u_2,\cdots,u_n)\in\Omega$*, and* $F(x_1,x_2,\cdots,x_n) := \sin(x_1+x_2+\cdots+x_n)+\sin(\sqrt{2}(x_1+x_2+\cdots+x_n))$, $(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n$*. Set* $H(\mathbf{t}) := sign(F(\mathbf{t}))$, $\mathbf{t}\in\mathbb{R}^n$*. Then the function* $H(\cdot)$ *is essentially bounded and therefore Stepanov* $(\Omega, p(\mathbf{u}))$*-bounded. On the other hand, we know that the function* $H(\cdot)$ *cannot be Stepanov* $(\Omega, p(\mathbf{u}))$*-almost periodic; the argumentation used for proving this fact in combination with the argumentation used in [21, Example 2] shows that the function* $H(\cdot)$ *cannot be Stepanov* $(\Omega, p(\mathbf{u}))$*-*R*-almost automorphic, where* R *denotes the collection of all sequences in* $\mathbb{R}^n$*.*

In [4, Proposition 2.18, Proposition 2.19], we have analyzed the pointwise products of Stepanov multi-dimensional almost periodic type functions. The following result is an analogue of the above-mentioned results for Stepanov multi-dimensional almost automorphic functions (for simplicity, we assume that the function $f(\cdot)$ depends only on the argument $\mathbf{t}\in\mathbb{R}^n$):

Proposition 2.9 *Suppose that* (ST) *and* (C1) *hold with* $\mathrm{R}_X = \mathrm{R}$ *as well as that* $p,\ q,\ r \in \mathcal{P}(\Omega)$ *and* $1/p(\mathbf{u}) + 1/r(\mathbf{u}) = 1/q(\mathbf{u})$*. Suppose, further, that:*

(i) $f : \mathbb{R}^n \to \mathbb{C}$ *is Stepanov-*$(\Omega, r(\mathbf{u}))$*-bounded and Stepanov* $(\Omega, r(\mathbf{u}))$*-*R*-multi-almost automorphic [Stepanov* $(\Omega, r(\mathbf{u}))$*-*$(\mathrm{R}, W^f_{\mathrm{R}})$*-multi-almost automorphic];*

(ii) $F : \mathbb{R}^n \times X \to Y$ *is a Stepanov* $(\Omega, p(\mathbf{u}))$*-*$(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic [Stepanov* $(\Omega, p(\mathbf{u}))$*-*$(\mathrm{R}, \mathcal{B}, W^F_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic; Stepanov* $(\Omega, p(\mathbf{u}))$*-*$(\mathrm{R}, \mathcal{B}, \mathrm{P}^F_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic] function satisfying that*

$$\sup_{\mathbf{t}\in\mathbb{R}^n; x\in B} \left\| \hat{F}_\Omega(\mathbf{t};x) \right\|_{L^{p(\mathbf{u})}(\Omega)} < \infty. \tag{2.3}$$

Define

$$F_1(\mathbf{t};x) := f(\mathbf{t})F(\mathbf{t};x), \quad \mathbf{t}\in\mathbb{R}^n,\ x\in X$$

and

(iii) $W^{F_1}_{\mathcal{B},(\mathbf{b}_k)}(x)$ *to be the collection of all sets of the form* $D\cap D'$*, where* $D \in W^F_{\mathcal{B},(\mathbf{b}_k)}(x)$ *and* $D' \in W^f_{(\mathbf{b}_k)}$ *for all* $B\in\mathcal{B}$, $(\mathbf{b}_k)\in\mathrm{R}$ *and* $x \in B$ *[*$\mathrm{P}^{F_1}_{\mathcal{B},(\mathbf{b}_k)}$ *to be the collection of all sets of the form* $D\cap D'$*, where* $D\in \mathrm{R}^F_{\mathcal{B},(\mathbf{b}_k)}$ *and* $D' \in W^f_{(\mathbf{b}_k)}$ *for all* $B\in\mathcal{B}$ *and* $(\mathbf{b}_k)\in\mathrm{R}$*].*

Then $F_1(\cdot;\cdot)$ *is Stepanov-*$(\Omega, q(\mathbf{u}))$*-*$(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic [Stepanov* $(\Omega, q(\mathbf{u}))$*-*$(\mathrm{R}, \mathcal{B}, W^{F_1}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic; Stepanov* $(\Omega, q(\mathbf{u}))$*-*$(\mathrm{R}, \mathcal{B}, \mathrm{P}^{F_1}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic].*

Proof. Let $(\mathbf{b}_k)\in\mathrm{R}$ and $B\in\mathcal{B}$ be given. Then we have

$$\begin{aligned}&\hat{F}_{1\Omega}(\mathbf{t}';x') - \hat{F}_{1\Omega}(\mathbf{t};x)\\ &\quad = \hat{f}_\Omega(\mathbf{t}')\cdot\left[\hat{F}_\Omega(\mathbf{t}';x') - \hat{F}_\Omega(\mathbf{t};x)\right] + \left[\hat{f}_\Omega(\mathbf{t}') - \hat{f}_\Omega(\mathbf{t})\right]\cdot \hat{F}_\Omega(\mathbf{t};x)\end{aligned}$$

for every $\mathbf{t},\ \mathbf{t}'\in\mathbb{R}^n$ and $x,\ x'\in X$. Since the mapping $\hat{f}_\Omega(\cdot)\in L^{r(\mathbf{u})}(\Omega : \mathbb{C})$ is continuous and the mapping $\hat{F}_\Omega(\cdot;\cdot)$ is continuous, the above equality in combination with the Hölder inequality (see Lemma 1.2(i)) shows that the mapping $\hat{F}_{1\Omega}(\cdot;\cdot) \in L^{p(\mathbf{u})}(\Omega : \mathbb{C})$ is continuous, as well. In the remainder of proof, we will consider only the general class of Stepanov-$(\Omega, q(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions. Since (C1) holds, we know that there exist a subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ and two functions $f^* : \mathbb{R}^n \to \mathbb{C}$ and $F^* : \mathbb{R}^n \times X \to Y$ such that, for every $\mathbf{t} \in \mathbb{R}^n$ and $x \in B$, we have

$$\lim_{l\to+\infty} f\big(\mathbf{t}+\mathbf{b}_{k_l}+\mathbf{u}\big) = f^*(\mathbf{t}+\mathbf{u}), \text{ and } \lim_{l\to+\infty} f^*\big(\mathbf{t}+\mathbf{u}-\mathbf{b}_{k_l}\big) = f(\mathbf{t}+\mathbf{u}),$$

for the topology of $L^{r(\mathbf{u})}(\mathbb{R}^n : \mathbb{C})$,

$$\lim_{l\to+\infty} F\big(\mathbf{t}+\mathbf{b}_{k_l}+\mathbf{u};x\big) = \big[F^*(\mathbf{t};x)\big](\mathbf{u}),$$

and

$$\lim_{l\to+\infty} \big[F^*\big(\mathbf{t}-\mathbf{b}_{k_l};x\big)\big](\mathbf{u}) = F(\mathbf{t}+\mathbf{u};x),$$

for the topology of $L^{p(\mathbf{u})}(\mathbb{R}^n : Y)$. Define

$$\big[F_1^*(\mathbf{t};x)\big](\mathbf{u}) := \big[f^*(\mathbf{t})\big](\mathbf{u}) \cdot \big[F^*(\mathbf{t};x)\big](\mathbf{u}), \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X,\ u \in \Omega.$$

Since

$$\begin{aligned} &F_1(\mathbf{t}+\mathbf{b}_{k_l}+\mathbf{u};x) - \big[F_1^*(\mathbf{t};x)\big](\mathbf{u}) \\ &\quad = \big[f(\mathbf{t}+\mathbf{b}_{k_l}+\mathbf{u}) - f^*(\mathbf{t}+\mathbf{u})\big] \cdot F(\mathbf{t}+\mathbf{b}_{k_l}+\mathbf{u};x) \\ &\quad + f^*(\mathbf{t}+\mathbf{u}) \cdot \Big[F(\mathbf{t}+\mathbf{b}_{k_l}+\mathbf{u};x) - \big[F^*(\mathbf{t};x)\big](\mathbf{u})\Big] \end{aligned}$$

for every $\mathbf{t} \in \mathbb{R}^n$, $\mathbf{u} \in \Omega$ and $x \in B$, the first limit equality follows from the Hölder inequality, the obvious estimate $\sup_{\mathbf{t}\in\mathbb{R}^n} \|f^*(\mathbf{t})\|_{L^{r(\mathbf{u})}(\Omega)} < \infty$ and (2.3). The second limit equality can be proved similarly. □

Now we would like to present the following illustrative example:

Example 2.10 *Suppose that $\Omega = [0,1]^n$, R denotes the collection of all sequences in $\mathbb{R}^n$ and, for every $i \in \mathbb{N}_n$, the function $f_i(\cdot)$ is Stepanov (Ω, p)-R-almost automorphic for every finite exponent $p \in [1,\infty)$. Set*

$$F\big(t_1, t_2, \cdot\cdot\cdot, t_n\big) := f_1(t_1) f_2(t_2) \cdot\cdot\cdot f_n(t_n), \quad \mathbf{t} = \big(t_1, t_2, \cdot\cdot\cdot, t_n\big) \in \mathbb{R}^n.$$

Applying Proposition 2.9, we have that the function $\mathbf{t} \mapsto F(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^n$ is Stepanov-$(\Omega, p(\mathbf{u}))$-R-almost automorphic for any $p \in D_+(\Omega)$.

Concerning the convolution invariance of Stepanov multi-dimensional almost automorphy, we will state and prove the following result (cf. also [4, Proposition 2.10], where we have used slightly different assumptions):

Proposition 2.11 *Suppose that $h \in L^1(\mathbb{R}^n)$, $p \in D_+(\Omega)$ and $F : \mathbb{R}^n \times X \to Y$ is a Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic function satisfying that for each $B \in \mathcal{B}$ there exists a finite real number $\epsilon_B > 0$ such that*

$$\sup_{\mathbf{t}\in\mathbb{R}^n, x\in B^{\cdot}} \big\|F(\mathbf{t},x)\big\|_Y < +\infty,$$

where $B^{\cdot} \equiv B^{\circ} \cup \bigcup_{x\in\partial B} B(x,\epsilon_B)$. Let condition (CI) *hold, where:*

(CI) $\mathrm{R}_X = \mathrm{R}$, *or* $X \in \mathcal{B}$ *and* R_X *is general.*

Then the function

$$(h * F)(\mathbf{t}; x) := \int_{\mathbb{R}^n} h(\sigma) F(\mathbf{t} - \sigma; x)\, \mathrm{d}\sigma, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X$$

is well defined, Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic, and for each* $B \in \mathcal{B}$ *we have* $\sup_{\mathbf{t} \in \mathbb{R}^n, x \in B^{\cdot}} \|(h * F)(\mathbf{t}; x)\|_Y < +\infty$.

Proof. Arguing similarly as in the proof of [5, Proposition 5], we have that the function $(h*F)(\cdot;\cdot)$ is well defined as well as that $\sup_{\mathbf{t} \in \mathbb{R}^n, x \in B^{\cdot}} \|(h * F)(\mathbf{t}; x)\|_Y < +\infty$ for all $B \in \mathcal{B}$. By definition, the function $\hat{F}_\Omega(\cdot;\cdot)$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic; furthermore, since we have assumed that $p \in D_+(\Omega)$, an application of Lemma 1.2(ii) shows that for each set $B \in \mathcal{B}$ we have

$$\sup_{\mathbf{t} \in \mathbb{R}^n, x \in B^{\cdot}} \big\| \hat{F}_\Omega(\mathbf{t}; x) \big\|_{L^{p(\mathbf{u})}(\Omega:Y)} < +\infty.$$

Applying [5, Proposition 5], we get that the function $h * \hat{F}_\Omega(\cdot;\cdot)$ is well defined and $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic. Now the final conclusion follows from the equality

$$h * \hat{F}_\Omega = h \hat{*} F_\Omega$$

and a corresponding definition of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphy. ☐

In order to transfer the statement of [4, Proposition 2.22] to multi-dimensional almost automorphic type functions, we need to clarify the following lemma:

Lemma 2.12 *Suppose that* (ST) *and* (C1) *hold. If for every* $\epsilon > 0$, $B \in \mathcal{B}$ *and* $(\mathbf{b}_k) \in \mathrm{R}$, *there exist a subsequence* $(\mathbf{b}_{k_l}^{\epsilon})$ *of* $(\mathbf{b}_k)$ *and a function* $F^{*,\epsilon} : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ *such that, for every* $x \in B$ *and* $\mathbf{t} \in \mathbb{R}^n$ *[for every* $x \in B$, *for every* $D \in W_{B,(\mathbf{b}_k)}(x)$ *and for every* $\mathbf{t} \in D$; *for every* $D \in \mathrm{P}_{B,(\mathbf{b}_k)}$ *and for every* $(\mathbf{t}; x) \in D$*], there exists* $l_0 \in \mathbb{N}$ *such that, for every* $l \geq l_0$, *we have*

$$\Big\| F\big(\mathbf{t} + \mathbf{u} + (b_{k_l}^1, \cdot\cdot\cdot, b_{k_l}^n)^{\epsilon}; x\big) - \Big[F^{*,\epsilon}(\mathbf{t}; x)\Big](\mathbf{u}) \Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq \epsilon/2$$

and

$$\Big\| \Big[F^{*,\epsilon}\big(\mathbf{t} - (b_{k_l}^1, \cdot\cdot\cdot, b_{k_l}^n)^{\epsilon}; x\big)\Big](\mathbf{u}) - F(\mathbf{t} + \mathbf{u}; x) \Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq \epsilon/2,$$

then $F(\cdot;\cdot)$ *is Stepanov* $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic [Stepanov* $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic; Stepanov* $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic].*

Proof. Let $B \in \mathcal{B}$ and $(\mathbf{b}_k) \in \mathrm{R}$ be fixed. Suppose that $s \in \mathbb{N}$. Using our assumption, we have that there exist a subsequence $(\mathbf{b}_k^s)$ of $(\mathbf{b}_k)$ and a function $F_s^* : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ such that, for every $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ [for every $x \in B$, for every $D \in W_{B,(\mathbf{b}_k)}(x)$ and for every $\mathbf{t} \in D$; for every $D \in \mathrm{P}_{B,(\mathbf{b}_k)}$ and for every $(\mathbf{t};x) \in D$], there exists $k_0 \in \mathbb{N}$ such that, for every $k \geq k_0$, we have

$$\Big\| F\big(\mathbf{t}+\mathbf{u}+\mathbf{b}_k^s;x\big) - \Big[F_s^*(\mathbf{t};x)\Big](\mathbf{u})\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq 1/s$$

and

$$\Big\| \Big[F_s^*\big(\mathbf{t}-\mathbf{b}_k^s;x\big)\Big](\mathbf{u}) - F(\mathbf{t}+\mathbf{u};x)\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq 1/s;$$

furthermore, since (C1) holds, we may assume that $(\mathbf{b}_k^{s+1})$ is a subsequence of $(\mathbf{b}_k^s)$ for all $s \in \mathbb{N}$. It is not difficult to prove that, for every fixed point $\mathbf{t} \in \mathbb{R}^n$ and element $x \in X$, the sequence $(F_s^*(\mathbf{t};x))$ is a Cauchy sequence in $L^{p(\mathbf{u})}(\Omega : Y)$ and therefore convergent; indeed, let $\epsilon > 0$ be given and let $s_0 \in \mathbb{N}$ satisfy $2/s < \epsilon$. Suppose that $s_1 \geq s$ and $s_2 \geq s$. Then there exist two sufficiently large integers $k,\ k' \in \mathbb{N}$ such that

$$\begin{aligned}
&\Big\| \Big[F_{s_1}^*(\mathbf{t};x)\Big](\mathbf{u}) - \Big[F_{s_2}^*(\mathbf{t};x)\Big](\mathbf{u})\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \\
&\quad\leq \Big\| \Big[F_{s_1}^*(\mathbf{t};x)\Big](\mathbf{u}) - F\big(\mathbf{t}+\mathbf{u}+\mathbf{b}_k^{s_1};x\big)\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \\
&\quad+ \Big\| F\big(\mathbf{t}+\mathbf{u}+\mathbf{b}_k^{s_1};x\big) - F\big(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k'}^{s_2};x\big)\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \\
&\quad+ \Big\| F\big(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k'}^{s_2};x\big) - \Big[F_{s_2}^*(\mathbf{t};x)\Big](\mathbf{u})\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \\
&\quad= \Big\| \Big[F_{s_1}^*(\mathbf{t};x)\Big](\mathbf{u}) - F\big(\mathbf{t}+\mathbf{u}+\mathbf{b}_k^{s_1};x\big)\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \\
&\quad+ \Big\| F\big(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k'}^{s_2};x\big) - \Big[F_{s_2}^*(\mathbf{t};x)\Big](\mathbf{u})\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \\
&\quad\leq (1/s_1) + (1/s_2) \leq 2/s < \epsilon.
\end{aligned}$$

Set $F^*(\mathbf{t};x) := \lim_{s\to+\infty} F_s^*(\mathbf{t};x)$, $\mathbf{t} \in \mathbb{R}^n$, $x \in B$ and $\mathbf{c}_k := \mathbf{b}_k^k$, $k \in \mathbb{N}$ (with a little loss of generality; we can always use here the well known diagonal procedure). Observe that, in the case of consideration of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphy, for every $x \in B$ and for every $D \in W_{B,(\mathbf{b}_k)}(x)$, the above limit is uniform in $\mathbf{t} \in D$, as well as that, in the case of consideration of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphy, for every $D \in \mathrm{P}_{B,(\mathbf{b}_k)}$, the above limit is uniform in $(\mathbf{t};x) \in D$. Furthermore, for every $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ [for every $x \in B$, for every $D \in W_{B,(\mathbf{b}_k)}(x)$ and for every $\mathbf{t} \in D$; for every $D \in \mathrm{P}_{B,(\mathbf{b}_k)}$ and for every $(\mathbf{t};x) \in D$], we have the existence of a sufficiently large integer $k \in \mathbb{N}$ such that

$$\begin{aligned}
&\Big\| F(\mathbf{t}+\mathbf{c}_k+\mathbf{u}) - \big[F^*(\mathbf{t};x)\big](\mathbf{u})\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \\
&\leq \Big\| F(\mathbf{t}+\mathbf{c}_k+\mathbf{u}) - \big[F_k^*(\mathbf{t};x)\big](\mathbf{u})\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \\
&+ \Big\| \big[F_k^*(\mathbf{t};x)\big](\mathbf{u}) - \big[F^*(\mathbf{t};x)\big](\mathbf{u})\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq (\epsilon/2) + (\epsilon/2) = \epsilon.
\end{aligned}$$

This completes the proof of lemma. □

Now we are able to state the following result:

Theorem 2.13 *Suppose that $k \in \mathbb{N}$, (ST) and (C1) hold. Let $\mathcal{B}$ be any family of compact subsets of X, let $\mathcal{B}_f$ be the collection of all finite subsets of X, and let $F : \mathbb{R}^n \times X \to Y$ satisfy the following conditions:*

(i) The function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}_f)$-multi-almost automorphic [Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}_f, W^f_{\mathcal{B}_f,\mathrm{R}})$-multi-almost automorphic; Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}_f, \mathrm{P}^f_{\mathcal{B}_f,\mathrm{R}})$-multi-almost automorphic];

(ii) For every $B \in \mathcal{B}$, $(\mathbf{b}) = (\mathbf{b}_k) \in \mathrm{R}$ and $\epsilon > 0$, there exist a subsequence $(\mathbf{b}_{k_l}) \in \mathrm{R}$ of $(\mathbf{b}_k)$ and a real number $\delta > 0$ such that, for every point $\mathbf{t} \in \mathbb{R}^n$ and for every two elements $x',\ x'' \in B$, there exist two integers $m_0,\ l_0 \in \mathbb{N}$ such that, for every integer $m \geq m_0$, we have

$$\|x' - x''\| \leq \delta \Longrightarrow \Big\| F\big(\mathbf{t} + \mathbf{u} + \mathbf{b}_{k_m}; x'\big) - F\big(\mathbf{t} + \mathbf{u} + \mathbf{b}_{k_m}; x''\big) \Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq \epsilon/2 \tag{2.4}$$

and, for every integer $m \geq m_0$ and $l \geq l_0$, we have

$$\|x' - x''\| \leq \delta \Longrightarrow \Big\| F\big(\mathbf{t} + \mathbf{u} - \mathbf{b}_{k_l} + \mathbf{b}_{k_m}; x'\big) - F\big(\mathbf{t} + \mathbf{u} - \mathbf{b}_{k_l} + \mathbf{b}_{k_m}; x''\big) \Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq \epsilon/2 \tag{2.5}$$

[for each $B \in \mathcal{B}$, $(\mathbf{b}_k) \in \mathrm{R}$ and $\epsilon > 0$, there exist a subsequence $(\mathbf{b}_{k_l}) \in \mathrm{R}$ of $(\mathbf{b}_k)$ and a real number $\delta > 0$ such that, for every two elements $x',\ x'' \in B$, set $D \in W_{B;(\mathbf{b}_k)}(x')$ and point $\mathbf{t} \in D$, there exist two integers $m_0,\ l_0 \in \mathbb{N}$ such that, for every integer $m \geq m_0$, the implication (2.4) *holds as well as that, for every integer $l \geq l_0$ we have $D - \mathbf{b}_{k_l} \subseteq W_{B,(\mathbf{b}_k)}(x') \cap W_{B,(\mathbf{b}_k)}(x'')$ and* (2.5)*; for each $B \in \mathcal{B}$, $(\mathbf{b}_k) \in \mathrm{R}$ and $\epsilon > 0$, there exist a subsequence $(\mathbf{b}_{k_l}) \in \mathrm{R}$ of $(\mathbf{b}_k)$ and a real number $\delta > 0$ such that, for every $D \in \mathrm{P}_{(\mathbf{b};\mathbf{x})}$, $(\mathbf{t}; x') \in D$ and $x'' \in B$, there exist two integers $m_0,\ l_0 \in \mathbb{N}$ such that, for every integer $m \geq m_0$, the implication* (2.4) *holds as well as that, for every integer $l \geq l_0$ we have $D - (\mathbf{b}_{k_l}, 0) \subseteq \mathrm{P}_{B,(\mathbf{b}_k)}$ and* (2.5)*];*

(iii) For each set $B \in \mathcal{B}$ and for each finite subset B' of B, we have $W^f_{B';(\mathbf{b}_k)}(x) \supseteq W_{B';(\mathbf{b}_k)}(x)$ for all $x \in B,\ x' \in B'$ and $\mathrm{P}^f_{B';(\mathbf{b}_k)} \supseteq \mathrm{P}_{B;(\mathbf{b})}$.

Then the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic [Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic; Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic].

Proof. We may assume that $p(\mathbf{u}) \equiv p \in [1, \infty)$; the proof in general case can be deduced similarly. Let $\epsilon > 0$, $B \in \mathcal{B}$ and $(\mathbf{b}; \mathbf{x}) = ((\mathbf{b}_k; x_k)) \in \mathrm{R}_X$ be given. Then there exist a subsequence $(\mathbf{b}_{k_l}) \in \mathrm{R}$ of $(\mathbf{b}_k)$ and a real number $\delta > 0$ such that (ii) holds, which implies that there exists a finite subset $\{x'_1, \cdot \cdot \cdot, x'_s\} \subseteq B$ $(s \in \mathbb{N})$ such that $B \subseteq \bigcup_{i=1}^{l} B(x'_i, \delta)$. Due to (i) and (C1), we have the existence of a subsequence of $(\mathbf{b}_{k_l})$ [w.l.o.g. we may assume that this subsequence is equal to the initial sequence $(\mathbf{b}_{k_l})$] and a function $F^* : \mathbb{R}^n \times X \to L^p(\Omega : Y)$ such that, for every $\mathbf{t} \in \mathbb{R}^n$, there exists an integer $m_0 \in \mathbb{N}$ such that, for every $m \geq m_0$, we have:

$$\left(\int_\Omega \left\| F(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x_i') - \left[F^*(\mathbf{t};x_i')\right](\mathbf{u})\right\|_Y^p \, d\mathbf{u}\right) \leq \epsilon/2, \quad m \geq m_0,\ i \in \mathbb{N}_s \tag{2.6}$$

and

$$\left(\int_\Omega \left\| \left[F^*(\mathbf{t}-\mathbf{b}_{k_m};x_i')\right](\mathbf{u}) - F(\mathbf{t}+\mathbf{u};x_i')\right\|_Y^p \, d\mathbf{u}\right) \leq \epsilon/2, \quad m \geq m_0,\ i \in \mathbb{N}_s. \tag{2.7}$$

Let $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ be fixed [let $x \in B$, $D \in W_{B,(\mathbf{b}_k)}(x)$ and $\mathbf{t} \in D$ be fixed; let $D \in \mathrm{P}_{B,(\mathbf{b}_k)}$ and $(\mathbf{t};x) \in D$ be fixed]. By the foregoing, there exists $i \in \mathbb{N}_s$ such that $\|x - x_i'\| \leq \delta$. By (ii), we have the existence of an integer $m_1 \in \mathbb{N}$ such that, for every integer $m \geq m_1$, one has

$$\left\| F\big(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x\big) - F\big(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x_i'\big)\right\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq \epsilon/2. \tag{2.8}$$

Assume first that the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}_f)$-multi-almost automorphic. Then we have

$$\begin{aligned}
&\left(\int_\Omega \left\| F(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x) - \left[F^*(\mathbf{t};x_i')\right](\mathbf{u})\right\|_Y^p \, d\mathbf{u}\right)^{\frac{1}{p}} \\
\leq &\left(\int_\Omega \left\| F(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x) - F(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x_i')\right\|_Y^p \, d\mathbf{u}\right)^{\frac{1}{p}} \\
&+ \left(\int_\Omega \left\| F(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k_m};x_i') - \left[F^*(\mathbf{t};x_i')\right](\mathbf{u})\right\|_Y^p \, d\mathbf{u}\right)^{\frac{1}{p}} \\
\leq &(\epsilon/2) + (\epsilon/2) = \epsilon, \quad m \geq m_0 + m_1,
\end{aligned} \tag{2.9}$$

where (2.9) follows by applying (2.6) and (2.8); in the case of consideration of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphy [Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphy], we also need to apply condition (iii) and the corresponding assumptions from the issue (ii). For the second limit equation, we use the estimates

$$\begin{aligned}
&\left(\int_\Omega \left\| \left[F^*(\mathbf{t}-\mathbf{b}_{k_l};x)\right](\mathbf{u}) - F(\mathbf{t}+\mathbf{u};x)\right\|_Y^p \, d\mathbf{u}\right)^{\frac{1}{p}} \\
\leq &\left(\int_\Omega \left\| \left[F^*(\mathbf{t}-\mathbf{b}_{k_l};x)\right](\mathbf{u}) - \left[F^*(\mathbf{t}-\mathbf{b}_{k_l};x_i')\right](\mathbf{u})\right\|_Y^p \, d\mathbf{u}\right)^{\frac{1}{p}} \\
&+ \left(\int_\Omega \left\| \left[F^*(\mathbf{t}-\mathbf{b}_{k_l};x_i')\right](\mathbf{u}) - F(\mathbf{t}+\mathbf{u};x)\right\|_Y^p \, d\mathbf{u}\right)^{\frac{1}{p}}, \quad m \geq m_0 + m_1,\ l \geq l_0,
\end{aligned}$$

where we have applied (2.5), (2.7), the limit equality

$$\lim_{l \to +\infty} F\big(\mathbf{t}+\mathbf{u}-\mathbf{b}_{k_l}+\mathbf{b}_{k_m};x'\big) = \left[F^*(\mathbf{t}-\mathbf{b}_{k_l};x_i')\right](\mathbf{u}),$$

and the corresponding limit equality with the element x_i' replaced therein with the element x. Then the final conclusion follows from Lemma 2.12. □

It is clear that some known statements for multi-dimensional almost automorphic functions can be straightforwardly extended to the corresponding Stepanov classes by using the properties of the Bochner transform. For example, suppose that $F : \mathbb{R}^n \times X \to Y$ is a Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi almost automorphic function, where R denotes the collection of all sequences in $\mathbb{R}^n$ and $\mathcal{B}$ denotes any collection of compact subsets of X. If there exists a finite real constant $L > 0$ such that (1.8) holds, then, for every set $B \in \mathcal{B}$, we have that the set $\{\hat{F}_\Omega(\mathbf{t}, x) : \mathbf{t} \in \mathbb{R}^n,\ x \in B\}$ is relatively compact in $L^{p(\mathbf{u})}(\Omega : Y)$. This can be deduced with the help of our conclusion from Remark 1.9(i) and [5, Proposition 1]. Similarly, we can clarify the supremum formula for Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi almost automorphic functions using [5, Proposition 3] and some sufficient conditions ensuring the invariance of various types of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorpy under the composition with continuous functions (see [5, Proposition 2]).

Suppose now that $\mathbb{D} \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded. Any of the function spaces from Definition 2.1 can be extended by introducing the corresponding space of $\mathbb{D}$-asymptotically Stepanov multi-almost automorphic functions; for example, we can introduce the following notion:

Definition 2.14 *We say that the function $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic if and only if there exist a Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic function $H : \mathbb{R}^n \times X \to Y$ and a function $Q \in S^{\Omega, p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\mathbb{R}^n \times X : Y)$ such that $F(\mathbf{t}; x) = H(\mathbf{t}; x) + Q(\mathbf{t}; x)$ for a.e. $\mathbf{t} \in \mathbb{R}^n$ and all $x \in X$. If $X = \{0\}$ and $\mathcal{B} = \{X\}$, then we also say that the function $F(\cdot)$ is $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-R-multi-almost automorphic.*

Using [5, Proposition 4] (cf. also [5, Proposition 8] and [4, Proposition 2.11] for the corresponding statement concerning the Stepanov multi-dimensional almost periodic functions), we can formulate a great number of corresponding statements for ($\mathbb{D}$-asymptotically) Stepanov multi-dimensional almost automorphic functions. For example, we can prove the following:

Proposition 2.15 *Suppose that for each integer $j \in \mathbb{N}$ the function $F_j(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic. If for each $B \in \mathcal{B}$ there exists $\epsilon_B > 0$ such that*

$$\lim_{j \to +\infty} \sup_{\mathbf{t} \in \Lambda; x \in B^{\cdot}} \Bigl\| F_j(\mathbf{t} + \mathbf{u}; x) - F(\mathbf{t} + \mathbf{u}; x) \Bigr\|_{L^{p(\mathbf{u})}(\Omega : Y)} = 0,$$

where $B^{\cdot} \equiv B^\circ \cup \bigcup_{x \in \partial B} B(x, \epsilon_B)$, then the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic.

It is worth noting that any such space of $\mathbb{D}$-asymptotically Stepanov multi-dimensional almost automorphic functions has the linear vector structure provided that the collection R (R_X) has the property that, for every sequence which belongs to R (R_X), any its subsequence belongs to R (R_X); in [5, Proposition 6], we have also analyzed the translation invariance and the homogeneity of $\mathbb{D}$-asymptotically Stepanov multi-almost automorphic functions (under certain conditions, the decomposition of a $\mathbb{D}$-asymptotically Stepanov multi-almost automorphic function into its Stepanov multi-almost automorphic part and the corrective part is unique, which simply follows from an application of the deduced supremum formula; cf. also [5, Proposition 7]). We can simply transfer the corresponding parts of the above-mentioned proposition to $\mathbb{D}$-asymptotically Stepanov multi-almost automorphic functions. Details can be left to the interested readers.

Before we switch to the next subsection, we would like to clarify the following extension of [11, Theorem 3.3] and [20, Lemma 2.3.4] for Stepanov multi-dimensional almost automorphic type functions depending on two parameters:

Theorem 2.16 *(i) Suppose that $\Omega = [0,1]^n$, the collection R_X (R) has the property that, for every sequence which belongs to R_X (R), any its subsequence belongs to R_X (R) and the function $F : \mathbb{R}^n \times X \to Y$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic). If the function $F(\cdot;\cdot)$ is uniformly convergent on $\mathbb{R}^n \times X$ (if for each $B \in \mathcal{B}$ there exists $\epsilon_B > 0$ such that the function $F(\cdot;\cdot)$ is uniformly convergent on $\mathbb{R}^n \times B^{\cdot}$, where $B^{\cdot} := B^{\circ} \cup \bigcup_{x \in \partial B} B(x, \epsilon_B)$), then the function $F(\cdot;\cdot)$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic ($(\mathrm{R}, \mathcal{B})$-multi-almost automorphic).*

(ii) Suppose that $\Omega = [0,1]^n$ and the function $F : \mathbb{R}^n \to Y$ is Stepanov $\mathbb{R}^n$-asymptotically $(\Omega, p(\mathbf{u}))$-R-multi-almost automorphic, where R denotes the collection of all sequences in $\mathbb{R}^n$. If the function $F(\cdot)$ is uniformly continuous, then $F(\cdot)$ is asymptotically almost automorphic.

Proof. We will prove the part (i) only for $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions. Define, for every $s \in \mathbb{N}$, $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$, $F_s(\mathbf{t}; x) := \int_\Omega F(\mathbf{t} + (\mathbf{u}/s); x)\, d\mathbf{u}$. Suppose that an integer $s \in \mathbb{N}$ is fixed. In order to prove that the function $F_s(\cdot;\cdot)$ is continuous at the fixed point $(\mathbf{t}; x) \in \mathbb{R}^n \times X$, let us take arbitrary real number $\epsilon > 0$ and choose after that a set $B \in \mathcal{B}$ such that $x \in B$. Then there exists a real number $\epsilon_B > 0$ such that the function $F(\cdot;\cdot)$ is uniformly convergent on $\mathbb{R}^n \times B^{\cdot}$. Using this fact, the required continuity of function $F_s(\cdot;\cdot)$ at $(\mathbf{t}; x)$ follows from the equality

$$F_s(\mathbf{t}; x) - F_s(\mathbf{t}'; x') = \int_\Omega \Big[F(\mathbf{t} + (\mathbf{u}/s); x) - F(\mathbf{t}' + (\mathbf{u}/s); x') \Big]\, d\mathbf{u}$$

and the fact that for each sufficiently small real number $\delta > 0$ we have that $B(x', \delta) \subseteq B^{\cdot}$. Similarly we can prove that for each set $B \in \mathcal{B}$ the sequence $(F_s(\cdot;\cdot))$ converges uniformly to the function $F(\cdot;\cdot)$ on the set $\mathbb{R}^n \times B^{\circ}$. In the remainder of the proof of (i) we may assume without loss of generality (see Lemma 1.2(ii)) that $p(\mathbf{u}) \equiv 1$. Due to [5, Proposition 4], it suffices to show that the function $F_s(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic for all $s \in \mathbb{N}$. Let a sequence $(\mathbf{b}_k) \in \mathrm{R}$ and a set $B \in \mathcal{B}$ be given. By our assumption, we have the existence of a subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to L^1(\Omega : Y)$ such that $\lim_{l \to +\infty} F(\mathbf{t} + \cdot + \mathbf{b}_{k_l}; x) = [F^*(\mathbf{t}; x)](\cdot)$ and $\lim_{l \to +\infty} [F^*(\mathbf{t} - \mathbf{b}_{k_l}; x)](\cdot) = F(\mathbf{t} + \cdot; x)$ for the topology of $L^1(\Omega : Y)$. Define, for fixed integer $s \in \mathbb{N}$,

$$F_s^*(\mathbf{t}; x) := s^n \int_{\Omega/s} \Big[F^*(\mathbf{t}; x) \Big](\mathbf{u})\, d\mathbf{u}, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X.$$

Then, for every $s \in \mathbb{N}$, $\mathbf{t} \in \mathbb{R}^n$ and $x \in B$, we have

$$\begin{aligned}
&\big\| F_s(\mathbf{t} + \mathbf{b}_{k_l}) - F_s^*(\mathbf{t}; x) \big\|_Y \\
&= \left\| s^n \int_{\Omega/s} \Big[F(\mathbf{t} + \mathbf{u} + \mathbf{b}_{k_l}; x)\, d\mathbf{u} - s^n \int_{\Omega/s} \Big[F^*(\mathbf{t}; x) \Big](\mathbf{u})\, d\mathbf{u} \right\|_Y \\
&\leq s^n \int_\Omega \Big\| F(\mathbf{t} + \mathbf{u} + \mathbf{b}_{k_l}; x) - \Big[F^*(\mathbf{t}; x) \Big](\mathbf{u}) \Big\|_Y\, d\mathbf{u} \to 0, \quad l \to +\infty.
\end{aligned}$$

We can similarly prove the second limit equation. The proof of (ii) follows by applying [5, Lemma 1] and the argumentation contained in the proof of [18, Proposition 3.1]. $\square$

We close this section by observing that the composition principles for one-dimensional Stepanov p-almost automorphic type functions ($1 \leq p < \infty$), established by Z. Fan, J. Liang, T.-J. Xiao [14] and H.-S. Ding, J. Liang, T.-J. Xiao [11, 12], have recently been reconsidered and slightly generalized by T. Diagana and M. Kostić [9, Section 4] for one-dimensional Stepanov $p(x)$-almost automorphic type functions (see also the research article [22]). The above-mentioned results admit straightforward reformulations in the multi-dimensional setting and, because of that, we will not reconsider these results here (cf. also [4, Section 4] for more details given in the almost periodic case). For simplicity, we will not consider here various questions about the invariance of Stepanov multi-dimensional almost automorphic properties under the actions of convolution products, as well (cf. [4, Section 5], [5, Subsection 2.5] and [9, Section 5] for more details).

2.1 Applications to the heat equation and the wave equation

In this subsection, we analyze the Stepanov almost automorphic solutions of the heat equation and the wave equation with respect to the space variable (see also Example 1.1).

In [5, Subsection 3.2], we have revisited the classical theory of partial differential equations of second order and provided some new applications in the qualitative analysis of solutions of the wave equations in $\mathbb{R}^3$:

$$u_{tt}(t,x) = d^2\Delta_x u(t,x), \quad x \in \mathbb{R}^3,\ t > 0;\ \ u(0,x) = g(x),\ \ u_t(0,x) = h(x), \tag{2.10}$$

where $d > 0$, $g \in C^3(\mathbb{R}^3 : \mathbb{R})$ and $h \in C^2(\mathbb{R}^3 : \mathbb{R})$. By the famous Kirchhoff formula (see *e.g.*, [29, Theorem 5.4, pp. 277-278]), the function

$$\begin{aligned} u(t,x) &:= \frac{\partial}{\partial t}\left[\frac{1}{4\pi d^2 t}\int_{\partial B_{dt}(x)} g(\sigma)\ \mathrm{d}\sigma\right] + \frac{1}{4\pi d^2 t}\int_{\partial B_{dt}(x)} h(\sigma)\ \mathrm{d}\sigma \\ &= \frac{1}{4\pi}\int_{\partial B_1(0)} g(x+dt\omega)\ \mathrm{d}\omega + \frac{dt}{4\pi}\int_{\partial B_1(0)} \nabla g(x+dt\omega)\cdot\omega\ \mathrm{d}\omega \\ &+ \frac{t}{4\pi}\int_{\partial B_1(0)} h(x+dt\omega)\ \mathrm{d}\omega, \quad t \geq 0,\ x \in \mathbb{R}^3, \end{aligned} \tag{2.11}$$

is a unique solution of problem (2.10) which belongs to the class $C^2([0,\infty)\times\mathbb{R}^3)$. Suppose now that a number $t_0 > 0$ is fixed. Let us also assume that the functions $g(\cdot)$, $\nabla g(\cdot)$ and $h(\cdot)$ are bounded and Stepanov $([0,1]^3, 1)$-R-multi-almost automorphic, where R is any collection of sequences in $\mathbb{R}^3$ such that, for every sequence $(\mathbf{b}_k) \in \mathrm{R}$, any subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ also belongs to R. Using the dominated convergence theorem and the Fubini theorem, we can simply conclude that the function $x \mapsto u(t_0,x)$, $x \in \mathbb{R}^3$ is likewise bounded and Stepanov $([0,1]^3, 1)$-R-multi-almost automorphic. We can similarly consider the following wave equation in $\mathbb{R}^2$:

$$u_{tt}(t,x) = d^2\Delta_x u(t,x), \quad x \in \mathbb{R}^2,\ t > 0;\ \ u(0,x) = g(x),\ \ u_t(0,x) = h(x), \tag{2.12}$$

where $d > 0$, $g \in C^3(\mathbb{R}^2 : \mathbb{R})$ and $h \in C^2(\mathbb{R}^2 : \mathbb{R})$. By the Poisson formula ([29, Theorem 5.5, pp. 280-281]), we have that the function

$$\begin{aligned} u(t,x) &:= \frac{\partial}{\partial t}\left[\frac{1}{2\pi d}\int_{\partial B_{dt}(x)}\frac{g(\sigma)}{\sqrt{d^2t^2-|x-y|^2}}\,\mathrm{d}\sigma\right]+\frac{1}{2\pi d}\int_{\partial B_{dt}(x)}\frac{h(\sigma)}{\sqrt{d^2t^2-|x-y|^2}}\,\mathrm{d}\sigma \\ &= d\int_{B_1(0)}\frac{g(x+dt\sigma)}{\sqrt{1-|\sigma|^2}}\,\mathrm{d}\sigma+d^2t\int_{B_1(0)}\frac{\nabla g(x+dt\sigma)\cdot\sigma}{\sqrt{1-|\sigma|^2}}\,\mathrm{d}\sigma \\ &+dt\int_{B_1(0)}\frac{h(x+dt\sigma)}{\sqrt{1-|\sigma|^2}}\,\mathrm{d}\sigma,\quad t\geq 0,\ x\in\mathbb{R}^2, \end{aligned}$$

is a unique solution of problem (2.12) which belongs to the class $C^2([0,\infty)\times\mathbb{R}^3)$. The consideration is then similar to the consideration already given in the three-dimensional case.

Let us finally consider the one-dimensional case. Then the unique regular solution of wave equation

$$u_{tt}(t,x)=d^2\Delta_x u(t,x),\quad x\in\mathbb{R},\ t>0;\ \ u(0,x)=g(x),\ \ u_t(0,x)=h(x),$$

where $d>0$, $g\in C^2(\mathbb{R}:\mathbb{R})$ and $h\in C^1(\mathbb{R}:\mathbb{R})$, is given by the d'Alembert formula

$$u(x,t)=\frac{1}{2}\big[g(x-dt)+g(x+dt)\big]+\frac{1}{2d}\int_{x-dt}^{x+dt}h(s)\,\mathrm{d}s,\quad x\in\mathbb{R},\ t>0.$$

Suppose that the functions $g(\cdot)$ and $h^{[1]}(\cdot)\equiv\int_0^{\cdot}h(s)\,\mathrm{d}s$ are Stepanov 1-almost automorphic. Then we can simply prove with the help of the dominated convergence theorem and the Fubini theorem that the solution $u(x,t)$ is Stepanov 1-almost automorphic in the variable $(x,t)\in\mathbb{R}^2$.

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