Journal of Nonlinear Evolution Equations and Applications ISSN 2161-3680
Volume 2021, Number 5, pp. 95–118 (May 2021) http://www.jneea.com

STACKELBERG CONTROL IN AN UNBOUNDED DOMAIN FOR A PARABOLIC EQUATION

LIONEL LANDRY DJOMEGNE NJOUKOUE*, **GABRIEL DEUGOUE**†
University of Dschang, BP 67 Dschang, Cameroon, West Region

Received on January 28, 2021, revised version on February 25, 2021

Accepted on March 5, 2021

Communicated by Gaston M. N'Guérékata

Abstract. In this paper, we investigate the Stackelberg strategy for a parabolic equation in an unbounded domain of $\mathbb{R}^N$, where $N \in \mathbb{N} \setminus \{0\}$. We assume that we can act on the system through two hierarchical controls. One control – called follower – solves an optimal control problem, while the other one – named leader – solves a null controllability problem. The results are obtained using an appropriate observability inequality of Carleman and under the assumption that the uncontrolled region is bounded.

Keywords: Carleman inequality, null controllability, observability inequality, optimal control, parabolic equation, Stackelberg control, unbounded domains.

2010 Mathematics Subject Classification: 35K05, 35K55, 49J20, 93B05.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$, where $N \in \mathbb{N} \setminus \{0\}$, be an unbounded connected open set at least of class $\mathcal{C}^2$ uniformly with boundary Γ (see [2] for a precise definition). Let $\mathcal{O}$ and ω be two non-empty subsets of Ω such that $\mathcal{O} \cap \omega = \emptyset$. For the time $T > 0$, we set $Q = (0,T) \times \Omega$, $\omega_T = (0,T) \times \omega$, $\mathcal{O}_T = (0,T) \times \mathcal{O}$ and $\Sigma = (0,T) \times \Gamma$. Then, we consider the following heat equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + a_0 y = v\chi_{\mathcal{O}} + k\chi_{\omega} & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0,.) = y^0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

*e-mail address: landry.djomegne@yahoo.fr
†e-mail address: agdeugoue@yahoo.fr

where $y^0 \in L^2(\Omega)$ and the controls v and k belong to $L^2(\mathcal{O}_T)$ and $L^2(\omega_T)$, respectively. The functions $\chi_{\mathcal{O}}$ and χ_ω are, respectively, the characteristic functions of the control sets $\mathcal{O}$ and ω. The function a_0 is such that $a_0\chi_{\Omega\setminus\omega} \in L^\infty(Q)$. We also assume that the unbounded sets Ω and ω are such that

$$\Omega \setminus \omega \text{ is bounded.} \tag{1.2}$$

Under the assumptions on the data we prove as in [18, Theorem 9.1, p. 341] that the system (1.1) has a unique solution

$$y := y(v,k) \in \mathcal{C}\big([0,T]; L^2(\Omega)\big) \cap L^2\big((0,T); H_0^1(\Omega)\big).$$

Let $\mathcal{O}_d \subset \Omega$ be an open set representing an observation domain. We define the following functional

$$J(v,k) = \|y(v,k) - z_d\|^2_{L^2(\mathcal{O}_d^T)} + \alpha\|v\|^2_{L^2(\mathcal{O}_T)}, \tag{1.3}$$

where $\alpha > 0$ is a constant, $z_d \in L^2(\mathcal{O}_d^T)$ is a desired state and $\mathcal{O}_d^T = (0,T) \times \mathcal{O}_d$.

The motivation of our problem comes for example from environmental sciences. The system (1.1) can describe the diffusion of a pollutant in an unbounded domain (*e.g.*, a river). We can view the state variable y as the concentration of the pollutant. Then, ω can be regarded as a part of the domain where we can apply a control k which intends to clear the pollutant in the river at a given final time T. Additionally, we want to reduce the concentration of the pollutant to a desired state z_d in $\mathcal{O}_d$ and to this purpose we apply the control v in $\mathcal{O}$. In other words, we want to control the concentration of the pollutant in the Stackelberg sense.

In this paper, we apply the Stackelberg control strategy which combines the optimal control problem and controllability problem. In order to explain the methodology, we consider the following problems.

Problem 1 (Optimal control problem) *Let ω and $\mathcal{O}$ be two non-empty subsets of Ω such that $\mathcal{O} \cap \omega = \emptyset$. Given $k \in L^2(\omega_T)$ and $y^0 \in L^2(\Omega)$, find the control $\hat{v} := \hat{v}(k) \in L^2(\mathcal{O}_T)$ such that*

$$J(\hat{v},k) = \inf_{v \in L^2(\mathcal{O}_T)} J(v,k), \tag{1.4}$$

where the functional J is given by (1.3)*.*

Problem 2 (Null controllability problem) *Let ω, $\mathcal{O}$ and $\mathcal{O}_d$ be three non-empty subsets of Ω such that $\mathcal{O}\cap\omega = \emptyset$ and $\mathcal{O}_d\cap\omega \neq \emptyset$. Assume that* (1.2) *holds true. Let $\hat{v}(k)$ be the optimal control obtained in the Problem* 1*. Given $y^0 \in L^2(\Omega)$, find a control $\hat{k} \in L^2(\omega_T)$ such that if $\hat{y} = y(t,x;\hat{v}(\hat{k}),\hat{k})$ is a solution of* (1.1)*, then*

$$\hat{y}(T,x) = y(T,x;\hat{v}(\hat{k}),\hat{k}) = 0 \text{ for } x \in \Omega. \tag{1.5}$$

There are several works on optimal control problem as well as on approximate or/and null controllability in bounded domains for parabolic equations. We, respectively, refer for instance to [19, 21, 29] and to [9, 10, 11, 12, 30] and the reference therein. Concerning the study of null controllability of parabolic equations in unbounded domains, few results are available in the literature. S. B. De Menezes and V. R. Cabanillas [4] studied the null controllability of a semilinear heat equation in an unbounded domain with nonlinearities of the form $f(y)$, the real function f being of class $\mathcal{C}^1$ and globally Lipschitz. The results were achieved by means of a fixed-point theorem and under the assumption that the uncontrolled domain is bounded. V. R. Cabanillas, S. B. De Menezes and E. Zuazua [2] considered the null controllability of a nonlinear parabolic equation with the nonlinearities of the form $f(y, \nabla y)$. Still under the assumption that the uncontrolled region is bounded, M. González-Burgos and L. de Teresa [14] proved the null controllability of a semilinear heat equation with the nonlinearities of the form $f(y, \nabla y)$ which grows slower than $|y| \log^{3/2}(1 + |y| + |\nabla y|) + |\nabla y| \log^{1/2}(1 + |y| + |\nabla y|)$ at infinity. L. de Teresa [5] proved the approximate controllability of a semilinear heat equation in $\mathbb{R}^N$ by an alternative method that consists in writing the heat equation in the similarity variables and using the weighted Sobolev spaces. L. de Teresa and E. Zuazua [6] proved the approximate controllability of a semilinear heat equation in an unbounded domain with the nonlinearities of the form $f(y)$, f being globally Lipschitz. The results are achieved by means of an approximation method which consists in approximating the domain Ω by a sequence of bounded domains $\Omega_R = \Omega \cap B_R$, where B_R is the ball centered at zero with radius R.

If we succeed in solving Problem 1 and Problem 2, then system (1.1) is null controllable in the sense of Stackelberg. Hence, the control v, which solves the optimal control problem, is called follower, while the control k, which solves the null controllability problem, is named leader. The Stackelberg leadership model was introduced by H. von Stackelberg [28]. This model is a strategic game in economics in which two firms compete on the market with the same product. The first to act must integrate the reaction of the other company in the choices it makes in the amount of product that it decides to put on the market. There are few works in the literature on Stackelberg's control of partial differential equations. J. L. Lions used this concept for a linear parabolic equation with two controls (see [20]). The follower aimed to bring the state of the system to a desired state, while the leader solved a problem of approximate controllability. O. Nakoulima [27] used the concept of Stackelberg for a linear and backward parabolic equation with two controls to be determined. In his work, the follower solved a null controllability problem with constraints on control while the other control solved an optimal control problem. M. Mercan [22, 23] revisited the notion of controllability in the sense of Stackelberg given by O. Nakoulima [27] by choosing the follower of minimal norm. This new notion is then applied by M. Mercan and O. Nakoulima [24] on the controllability of a two-stroke problem with constraints on the states. The results were obtained by means of a Carleman inequality adapted to the constraints. M. Kéré, M. Mercan and G. Mophou considered the Stackelberg strategy for coupled parabolic equations with a finite number of constraints on one of the state (see [17]). The follower control was supposed to bring the solution of the coupled system subject to a finite number of constraints at rest at final time $T > 0$ while the leader control expressed that the state of the coupled system do not move too far from a given state. Recently, L. L. Djomegne Njoukoué, G. Mophou and G. Deugoué [8] proved the hierarchic control for a linear backward heat equation with two controls. The follower solved an optimal control problem while the leader solved a null controllability problem. The results were obtained by means of a Carleman inequality associated to a non-homogeneous Dirichlet boundary condition. The authors in [15, 16, 25] used the Stackelberg control to combine the concepts of controllability with robustness. In their work the leader is responsible for controllability to trajectories objective and the follower solves a robust control problem. Recently, in [13, 26], the authors used Stackelberg control to solve problems with incomplete data. In their work, the leader solves a null controllability problem and the follower solves an inf-sup optimization problem.

The works cited above on Stackelberg strategy for partial differential equations were considered in bounded domains. As far as we know Stackelberg null controllability has not yet been considered in an unbounded domain.

In this paper we consider a Stackelberg null controllability of a heat equation in an unbounded domain. Using the fact that the control which brings the system to rest at final time acts on an open unbounded set such that the uncontrolled domain is bounded, we prove using an appropriate inequality of observability that the system (1.1) is Stackelberg null controllable. More precisely, we prove the following result.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N$, where $N \in \mathbb{N}\backslash\{0\}$, be an unbounded connected open set with boundary Γ at least of class $\mathcal{C}^2$ uniformly. Let $\mathcal{O}$, ω and $\mathcal{O}_d$ be three non-empty subsets of Ω such that $\mathcal{O}\cap\omega = \emptyset$ and $\mathcal{O}_d \cap \omega \neq \emptyset$. Assume that the assumption* (1.2) *holds true and that the parameter α is large enough. Then, there exists a positive weight function θ* (*the definition of θ will be given later*) *such that for any $y^0 \in L^2(\Omega)$ and for any $z_d \in L^2(\mathcal{O}_d^T)$ with $\theta z_d \in L^2(\mathcal{O}_d^T)$ there exists a unique control $\hat{k} \in L^2(\omega_T)$ and an associated optimal control $\hat{v}$ such that the corresponding solution to system* (1.1) *satisfies* (1.5).

Remark 1.1

(a) *In this paper, we are supposing that $\mathcal{O} \cap \omega = \emptyset$; this means that the domain of the follower control and the leader control are disjoint. Note that in a realistic situation the leader control cannot decide what to do at the points in the domain of the follower. Indeed, if $\mathcal{O} \cap \omega \neq \emptyset$ once the leader has been chosen, the follower is modifying the leader at those points.*

(b) *In Theorem* 1.1, *we assumed that α must be large enough. This assumption on α is needed at the level where we want to establish an observability inequality of Carleman.*

(c) *The assumption* (1.2) *is important to solve the Stackeberg control associated to system* (1.1). *Indeed, the potential in* (1.1) *is not bounded since our domain is unbounded. However, using assumption* (1.2), *the problem can be reduced to the case where the potential is now supported in a bounded set.*

The rest of this paper is organized as follows. In Section 2, we study the Stackelberg null controllability problem for an auxiliary heat equation. In Section 3, we give the proof of Theorem 1.1 using the results obtained in the previous section. A conclusion is given in Section 4.

2 Solution of Stackelberg null controllability problem for an auxiliary heat equation

We study Problem 1 and then Problem 2 for the following initial-boundary value problem for the linear system

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + ay = v\chi_{\mathcal{O}} + h\chi_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0,.) = y^0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $v\chi_{\mathcal{O}}, h\chi_\omega \in L^2(Q)$ and $y^0 \in L^2(\Omega)$. We assume that there exists a constant $K > 0$ such that

$$\|a\|_{L^\infty(Q)} \leq K. \tag{2.2}$$

Under the above assumptions on the data the system (2.1) has a unique solution $y := y(v,h) \in \mathcal{C}([0,T]; L^2(\Omega)) \cap L^2((0,T); H_0^1(\Omega))$. Moreover, there exists a positive constant $C = C(K,T)$, which depends on K and T, such that

$$\|y(T,\cdot)\|^2_{L^2(\Omega)} + \|y\|^2_{L^2((0,T);H_0^1(\Omega))} \leq C\big(\|y^0\|^2_{L^2(\Omega)} + \|(v\chi_{\mathcal{O}} + h\chi_\omega)\|^2_{L^2(Q)}\big). \tag{2.3}$$

From now on, we write $C(X)$ to denote a positive constant whose value varies from line to line but depends on X.

2.1 Solution of Problem 1 for system (2.1)

We are interested in Problem 1 for the linear system (2.1), that is, for any $h \in L^2(\omega_T)$ we look for $\hat{v} := \hat{v}(h) \in L^2(\mathcal{O}_T)$ such that

$$J(\hat{v},h) = \inf_{v \in L^2(\mathcal{O}_T)} J(v,h), \tag{2.4}$$

where

$$J(v,h) = \|y(v,h) - z_d\|^2_{L^2(\mathcal{O}_d^T)} + \alpha\|v\|^2_{L^2(\mathcal{O}_T)}, \tag{2.5}$$

with $z_d \in L^2(\mathcal{O}_d^T)$, $\alpha > 0$ and $y = y(v,h)$ being the solution of the linear system (2.1).

Proposition 2.1 *Assume that* (2.2) *holds true. Assume also that there exists a constant* $C = C(K,T) > 0$ *such that*

$$\alpha > C. \tag{2.6}$$

Then, for any $h \in L^2(\omega_T)$ *there exists a unique optimal control* $\hat{v} := \hat{v}(h) \in L^2(\mathcal{O}_T)$ *such that* (2.4) *holds true.*

Proof. Let $z = z(v)$ and $l = l(h)$ be, respectively, solutions of

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z + az = v\chi_{\mathcal{O}} & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0,.) = 0 & \text{in } \Omega \end{cases} \tag{2.7}$$

and

$$\begin{cases} \dfrac{\partial l}{\partial t} - \Delta l + al = h\chi_\omega & \text{in } Q, \\ l = 0 & \text{on } \Sigma, \\ l(0,.) = y^0 & \text{in } \Omega. \end{cases} \tag{2.8}$$

Then, it is clear that z and l belong to $\mathcal{C}([0,T]; L^2(\Omega)) \cap L^2((0,T); H_0^1(\Omega))$. Moreover, there exists $C = C(K,T) > 0$ such that

$$\|z(T,\cdot)\|^2_{L^2(\Omega)} + \|z\|^2_{L^2((0,T);H_0^1(\Omega))} \leq C\|v\|^2_{L^2(\mathcal{O}_T)} \tag{2.9}$$

and

$$\|l(T,\cdot)\|^2_{L^2(\Omega)} + \|l\|^2_{L^2((0,T);H_0^1(\Omega))} \leq C\big(\|y^0\|^2_{L^2(\Omega)} + \|h\|^2_{L^2(\omega_T)}\big). \tag{2.10}$$

Define on $L^2(\mathcal{O}_T) \times L^2(\mathcal{O}_T)$ the symmetric bilinear functional $a(.,.)$ by

$$a(u,v) = \int_{\mathcal{O}_d^T} z(u)\, z(v)\, \mathrm{d}x\, \mathrm{d}t + \alpha \int_{\mathcal{O}_T} u\, v\, \mathrm{d}x\, \mathrm{d}t \quad \text{for all } u, v \in L^2(\mathcal{O}_T), \tag{2.11}$$

and on $L^2(\mathcal{O}_T)$ the linear functional

$$\mathcal{L}(v) = \int_{\mathcal{O}_d^T} z(v)\, (z_d - l(h))\, \mathrm{d}x\, \mathrm{d}t \quad \text{for all } v \in L^2(\mathcal{O}_T).$$

Then, for any $v \in L^2(\mathcal{O}_T)$, using (2.9) and (2.6), we have that the bilinear functional $a(.,.)$ is coercive on $L^2(\mathcal{O}_T)$, that is,

$$a(v,v) = \|z(v)\|^2_{L^2(\mathcal{O}_d^T)} + \alpha \|v\|^2_{L^2(\mathcal{O}_T)} \geq \beta \|v\|^2_{L^2(\mathcal{O}_T)}$$

with $\beta = \alpha - C > 0$. Using (2.9), we infer that $a(.,.)$ is continuous on $L^2(\mathcal{O}_T) \times L^2(\mathcal{O}_T)$ and

$$|a(u,v)| \leq C(K,T,\alpha) \|u\|_{L^2(\mathcal{O}_T)} \|v\|_{L^2(\mathcal{O}_T)}.$$

Moreover, by (2.9) and (2.10), we get that the linear functional $\mathcal{L}$ is continuous, that is,

$$|\mathcal{L}(v)| \leq C\Big(\|z_d\|_{L^2(\mathcal{O}_d^T)} + \|y^0\|_{L^2(\Omega)} + \|h\|_{L^2(\omega_T)} \Big) \|v\|_{L^2(\mathcal{O}_T)},$$

where $C = C(K,T) > 0$.

Now, observing that y (*i.e.*, the solution of the linear system (2.1)) can be decomposed as $y = z + l$, we have that the cost function defined by (2.5) can be rewritten as

$$J(v) = a(v,v) - 2\mathcal{L}(v) + \|l(h) - z_d\|^2_{L^2(Q)}.$$

It then follows from the fact that the symmetric bilinear functional $a(.,.)$ is continuous and coercive and the linear functional $\mathcal{L}$ is continuous that there exists a unique optimal control $\hat{v} := \hat{v}(h) \in L^2(\mathcal{O}_T)$ such that (2.4) holds. □

Proposition 2.2 *Assume that* (2.2) *and* (2.6) *hold true. Let* $\hat{v} := \hat{v}(h)$ *be the solution of* (2.4) *and let* $\hat{y} := y(\hat{v}, h)$ *be the associated state. Then, there exists* $\hat{p} \in L^2((0,T); H_0^1(\Omega)) \cap \mathcal{C}((0,T); L^2(\Omega))$ *such that* $\{\hat{v}, \hat{y}, \hat{p}\}$ *satisfies the following optimality system*

$$\begin{cases} \dfrac{\partial \hat{y}}{\partial t} - \Delta \hat{y} + a\hat{y} = \hat{v}\chi_{\mathcal{O}} + h\chi_\omega & \text{in } Q, \\ \hat{y} = 0 & \text{on } \Sigma, \\ \hat{y}(0,.) = y^0 & \text{in } \Omega, \end{cases} \tag{2.12}$$

$$\begin{cases} -\dfrac{\partial \hat{p}}{\partial t} - \Delta \hat{p} + a\hat{p} = (\hat{y} - z_d)\, \chi_{\mathcal{O}_d} & \text{in } Q, \\ \hat{p} = 0 & \text{on } \Sigma, \\ \hat{p}(T,.) = 0 & \text{in } \Omega \end{cases} \tag{2.13}$$

and

$$\hat{v} = -\frac{\hat{p}}{\alpha} \text{ in } \mathcal{O}_T. \tag{2.14}$$

Proof. From Proposition 2.1, we already have (2.12). To complete the proof of Proposition 2.2, we express the Euler–Lagrange optimality conditions which characterize the optimal control $\hat{v}$:

$$\lim_{\lambda\to 0}\frac{J(\hat{v}+\lambda v,h)-J(\hat{v},h)}{\lambda}=0 \text{ for every } v\in L^2(\mathcal{O}_T),$$

which after calculations gives

$$\int_{\mathcal{O}_d^T} z(v)(\hat{y}-z_d)\,\mathrm{d}x\,\mathrm{d}t+\alpha\int_{\mathcal{O}_T}\hat{v}v\,\mathrm{d}x\,\mathrm{d}t=0 \text{ for every } v\in L^2(\mathcal{O}_T), \tag{2.15}$$

where $z(v)$ is the solution of

$$\begin{cases}\dfrac{\partial z}{\partial t}-\Delta z+az=v\chi_{\mathcal{O}} & \text{in } Q,\\ z=0 & \text{on } \Sigma,\\ z(0,.)=0 & \text{in } \Omega.\end{cases} \tag{2.16}$$

To interpret the relation (2.15), we consider the adjoint state solution of (2.13). Since $(\hat{y}-z_d)\chi_{\mathcal{O}_d}\in L^2(Q)$, we deduce that the system given by (2.13) has a unique solution in $L^2((0,T);H_0^1(\Omega))\cap\mathcal{C}([0,T];L^2(\Omega))$. Thus, multiplying (2.16) by $\hat{p}$, that is the solution of (2.13), and integrating by parts over Q, we obtain

$$\int_Q z(v)(\hat{y}-z_d)\chi_{\mathcal{O}_d}\,\mathrm{d}x\,\mathrm{d}t=\int_{\mathcal{O}_T}v\,\hat{p}\,\mathrm{d}x\,\mathrm{d}t=0 \text{ for every } v\in L^2(\mathcal{O}_T). \tag{2.17}$$

This, together with (2.15), gives

$$\int_{\mathcal{O}_T}(\hat{p}+\alpha\hat{v})v\,\mathrm{d}x\,\mathrm{d}t=0 \text{ for every } v\in L^2(\mathcal{O}_T),$$

from which we deduce (2.14). □

Remark 2.1 *Note that the follower $\hat{v}$, the adjoint state $\hat{p}$ and the state $\hat{y}$ depend on h. Moreover, in view of* (2.15)*, for every $v\in L^2(\mathcal{O}_T)$ we have*

$$\begin{aligned}0&=\int_{\mathcal{O}_d^T} z(v)(\hat{y}-z_d)\,\mathrm{d}x\,\mathrm{d}t+\alpha\int_{\mathcal{O}_T}\hat{v}v\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_{\mathcal{O}_d^T} z(v)\,z(\hat{v})\,\mathrm{d}x\,\mathrm{d}t-\int_Q z(v)(z_d-l)\,\mathrm{d}x\,\mathrm{d}t+\alpha\int_{\mathcal{O}_T}\hat{v}v\,\mathrm{d}x\,\mathrm{d}t\\ &=a(v,\hat{v})-\int_{\mathcal{O}_d^T} z(v)(z_d-l)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where the bilinear functional $a(.,.)$ is given by (2.11) *and the function $l=l(h)$ is a solution to* (2.8)*. Hence, taking $v=\hat{v}$ in the latter identity and using the coerciveness of $a(.,.)$, we deduce that*

$$\beta\|\hat{v}\|^2_{L^2(\mathcal{O}_T)}\le\|z(\hat{v})\|_{L^2(\mathcal{O}_d^T)}\|z_d-l\|_{L^2(\mathcal{O}_d^T)},$$

which in view of (2.9) *and* (2.10) *gives*

$$\|\hat{v}\|_{L^2(\mathcal{O}_T)}\le\frac{1}{\beta}C\Big(\|z_d\|_{L^2(\mathcal{O}_d^T)}+\|y^0\|_{L^2(\Omega)}+\|h\|_{L^2(\omega_T)}\Big), \tag{2.18}$$

where $C=C(K,T)>0$ and $\beta=\alpha-C>0$ is the coerciveness coefficient.

2.2 Carleman inequalities

In order to solve Problem 2 for system (2.1), that is, the null controllability of a cascade linear system (2.12)–(2.14), we need to establish appropriate inequalities of observability. So, for any $\rho^T \in L^2(\Omega)$ we consider the following adjoint systems of (2.12)–(2.14):

$$\begin{cases} -\dfrac{\partial \rho}{\partial t} - \Delta\rho + a\rho = \Psi\chi_{\mathcal{O}_d} & \text{in } Q, \\ \rho = 0 & \text{on } \Sigma, \\ \rho(T,.) = \rho^T & \text{in } \Omega, \end{cases} \tag{2.19}$$

$$\begin{cases} \dfrac{\partial \Psi}{\partial t} - \Delta\Psi + a\Psi = -\dfrac{1}{\alpha}\rho\chi_{\mathcal{O}} & \text{in } Q, \\ \Psi = 0 & \text{on } \Sigma, \\ \Psi(0,.) = 0 & \text{in } \Omega, \end{cases} \tag{2.20}$$

where the function a satisfies (2.2). We assume that there exists an unbounded set $\omega_1 \subset \omega$ such that

$$\Omega \setminus \omega_1 \text{ is bounded.} \tag{2.21}$$

Remark 2.2 *Note that with* (2.21) *we have that assumption* (1.2) *holds true.*

Since $\mathcal{O}_d \cap \omega \neq \emptyset$, there exist an open set ω_0 such that

$$\omega_0 \subset \omega_1 \subset \mathcal{O}_d \cap \omega \subset \omega \text{ with } d_0 = \operatorname{dist}(\omega_0, \Omega \setminus \bar{\omega}_1) > 0 \tag{2.22}$$

and a function ψ such that

$$\begin{cases} \psi \in \mathcal{C}^2(\bar{\Omega}),\ \psi \geq 0 \text{ in } \Omega, \\ |\nabla\psi| \geq \tau_0 > 0 \text{ in } \bar{\Omega} \setminus \omega_0, \\ \dfrac{\partial \psi}{\partial \nu} \leq 0 \text{ on } \partial\Omega,\ \displaystyle\sum_{|\beta| \leq 2} |D^\beta \psi| \leq \tau_1 \text{ in } \Omega, \end{cases} \tag{2.23}$$

where τ_0 and τ_1 are two positive constants. For the existence of such a function ψ in the case when Ω is unbounded, we refer to [14, Example 1, p. 7].

Let λ be a positive real number. For any $(t,x) \in Q$, we set

$$\varphi(t,x) = \frac{e^{\lambda(m_1 + \psi(x))}}{t(T-t)}, \tag{2.24}$$

$$\eta(t,x) = \frac{e^{\lambda(\|\psi\|_{L^\infty(\Omega)} + m_2)} - e^{\lambda(m_1 + \psi(x))}}{t(T-t)}, \tag{2.25}$$

with $m_2 > m_1$. Then, there exists a positive constant (still denoted by $C(T)$) such that

$$\left|\frac{\partial \eta}{\partial t}\right| \leq C(T)\varphi^2, \tag{2.26a}$$

$$\left|\frac{\partial \varphi}{\partial t}\right| \leq C(T)\varphi^2. \tag{2.26b}$$

For any $F_0 \in L^2(Q)$ and $z_0 \in L^2(\Omega)$, consider the following system

$$\begin{cases} -\dfrac{\partial z}{\partial t} - \Delta z = F_0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T,.) = z_0 & \text{in } \Omega. \end{cases} \tag{2.27}$$

Now, we state a Carleman estimate for solutions to the heat equation (2.27).

Proposition 2.3 ([14]) *Suppose that assumptions* (2.22)–(2.23) *hold true. Let φ and η be defined by* (2.24) *and* (2.25)*, respectively. Then, there exist positive constants $\sigma_1(\tau_0, \tau_1, d_0) \geq 1$, $\lambda_1(\tau_0, \tau_1, d_0) \geq 1$ and $C(\tau_0, \tau_1, d_0) > 0$ such that for all $\lambda \geq \lambda_1$, $s \geq s_1 = \sigma_1(T + T^2)$ and for any solution of* (2.27)*, which we denote by z, we have*

$$\begin{aligned} &s\lambda^2 \int_Q e^{-2s\eta}\varphi|\nabla z|^2 \,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^4 \int_Q e^{-2s\eta}\varphi^3|z|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq C(\tau_0, \tau_1, d_0)\left(s^3\lambda^4 \int_0^T\int_{\omega_1} e^{-2s\eta}\varphi^3|z|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_Q e^{-2s\eta}|F_0|^2 \,\mathrm{d}x\,\mathrm{d}t\right). \end{aligned} \tag{2.28}$$

Now, consider the following system

$$\begin{cases} -\dfrac{\partial z}{\partial t} - \Delta z + az = f & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T,.) = z_0 & \text{in } \Omega, \end{cases} \tag{2.29}$$

with $f \in L^2(Q)$, $z_0 \in L^2(\Omega)$. Then, we have the following result for (2.29).

Proposition 2.4 *Suppose that assumptions* (2.2) *and* (2.22)–(2.23) *hold true. Let φ and η be defined by* (2.24) *and* (2.25)*, respectively. Then, there exist positive constants $s_2 \geq 1$, $\lambda_1 \geq 1$ and $C = C(\tau_0, \tau_1, d_0) > 0$ such that for all $\lambda \geq \lambda_1$, $s \geq s_2$ and for any solution of* (2.29)*, which we denote by z, we have*

$$\begin{aligned} &s\lambda^2 \int_Q e^{-2s\eta}\varphi|\nabla z|^2 \,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^4 \int_Q e^{-2s\eta}\varphi^3|z|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq C\left(s^3\lambda^4 \int_0^T\int_{\omega_1} e^{-2s\eta}\varphi^3|z|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_Q e^{-2s\eta}|f|^2\mathrm{d}x\,\mathrm{d}t\right). \end{aligned} \tag{2.30}$$

Proof. System (2.29) can be rewritten as

$$\begin{cases} -\dfrac{\partial z}{\partial t} - \Delta z = -az + f & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T,.) = z_0 & \text{in } \Omega. \end{cases}$$

Hence, z is the solution of (2.27) corresponding to $F_0 = -az + f \in L^2(Q)$. Thus, applying Proposition 2.3 to z and using (2.2), we deduce that there exists $C = C(\tau_0, \tau_1, d_0)$ such that

$$
\begin{aligned}
&s\lambda^2 \int_Q e^{-2s\eta}\varphi|\nabla z|^2 \,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^4 \int_Q e^{-2s\eta}\varphi^3|z|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le C\bigg(s^3\lambda^4 \int_0^T\int_{\omega_1} e^{-2s\eta}\varphi^3|z|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_Q e^{-2s\eta}|f|^2 \,\mathrm{d}x\,\mathrm{d}t\bigg) \\
&\qquad + C\bigg(s^2\lambda^4 K \int_Q e^{-2s\eta}\varphi^3|z|^2 \,\mathrm{d}x\,\mathrm{d}t\bigg),
\end{aligned}
$$

because $s, \lambda > 1$ and $\varphi^{-1} \in L^\infty(Q)$. Choosing $s \ge s_2 = \max(s_1, 2C(\tau_0, \tau_1, d_0, K))$ in the latter inequality, we obtain (2.30). □

Remark 2.3 *Note that if we make a change of variable t for $T - t$ in* (2.29) *we obtain*

$$
\begin{cases}
\dfrac{\partial \tilde z}{\partial t} - \Delta \tilde z + a\tilde z = \tilde f & \text{in } Q, \\
\tilde z = 0 & \text{on } \Sigma, \\
\tilde z(0, .) = z_0 & \text{in } \Omega,
\end{cases}
\tag{2.31}
$$

where $\tilde z(t, x) = z(T - t, x)$ and $\tilde f(t, x) = f(T - t, x)$. Then, the global Carleman inequality (2.30) *is also valid for any $\tilde z$. This means that there exist positive constants $s_2 \ge 0$, $\lambda_1 \ge 1$ and $C(\tau_0, \tau_1, d_0) > 0$ such that for all $\lambda \ge \lambda_1$, $s \ge s_2$ and for any solution of* (2.31)*, which we denote by $\tilde z$, we have*

$$
\begin{aligned}
&s\lambda^2 \int_Q e^{-2s\eta}\varphi|\nabla \tilde z|^2 \,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^4 \int_Q e^{-2s\eta}\varphi^3|\tilde z|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le C\bigg(s^3\lambda^4 \int_0^T\int_{\omega_1} e^{-2s\eta}\varphi^3|\tilde z|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_Q e^{-2s\eta}|\tilde f|^2 \,\mathrm{d}x\,\mathrm{d}t\bigg).
\end{aligned}
\tag{2.32}
$$

From now on, we will adopt for a suitable function z the following notation

$$
\mathcal{K}(z) = s\lambda^2 \int_Q e^{-2s\eta}\varphi|\nabla z|^2 \,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^4 \int_Q e^{-2s\eta}\varphi^3|z|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{2.33}
$$

In the following result, we present the Carleman inequality for the solutions to systems (2.19) and (2.20).

Proposition 2.5 *Assume that α is large enough. Then, under the assumptions of Proposition* 2.4*, there exist positive constants $s_3 \ge 1$, $\lambda_2 \ge 1$ and $C = C(K, T, \tau_0, \tau_1, d_0) > 0$ such that for all $\lambda \ge \lambda_2$, $s \ge s_3$ the following estimate holds true for any solution (φ, Ψ) of* (2.19)–(2.20)

$$
\mathcal{K}(\rho) + \mathcal{K}(\Psi) \le C s^7\lambda^9 \int_{\omega_T} e^{-2s\eta}\varphi^7|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{2.34}
$$

Proof. We proceed in two steps.

Step 1. We prove that there exists $C = C(\tau_0, \tau_1, d_0) > 0$ such that

$$
\mathcal{K}(\rho) + \mathcal{K}(\Psi) \le C\bigg(s^3\lambda^4 \int_0^T\int_{\omega_1} e^{-2s\eta}\varphi^3\left(|\rho|^2 + |\Psi|^2\right) \,\mathrm{d}x\,\mathrm{d}t\bigg). \tag{2.35}
$$

Applying (2.30) to the solution ρ of (2.19) and (2.32) to the solution Ψ of (2.20), and then using the notation (2.33), we, respectively, obtain

$$\mathcal{K}(\rho) \le C(\tau_0,\tau_1,d_0)\left(s^3\lambda^4\int_0^T\int_{\omega_1} e^{-2s\eta}\varphi^3|\rho|^2\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{O}_d^T} e^{-2s\eta}|\Psi|^2\,\mathrm{d}x\,\mathrm{d}t\right)$$

and

$$\mathcal{K}(\Psi) \le C(\tau_0,\tau_1,d_0)\left(s^3\lambda^4\int_0^T\int_{\omega_1} e^{-2s\eta}\varphi^3|\Psi|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\alpha^2}\int_{\mathcal{O}_T} e^{-2s\eta}|\rho|^2\,\mathrm{d}x\,\mathrm{d}t\right).$$

Since $s,\lambda > 1$ and $\varphi^{-1}\in L^\infty(Q)$, we can write

$$\begin{aligned}\mathcal{K}(\rho)+\mathcal{K}(\Psi) \le C&\left(s^3\lambda^4\int_0^T\int_{\omega_1} e^{-2s\eta}\varphi^3\left(|\rho|^2+|\Psi|^2\right)\,\mathrm{d}x\,\mathrm{d}t\right)\\ &+C\left(1+\frac{1}{\alpha^2}\right)\cdot\left(s^2\lambda^4\int_Q e^{-2s\eta}\varphi^3\left(|\Psi|^2+|\rho|^2\right)\,\mathrm{d}x\,\mathrm{d}t\right),\end{aligned}$$

where $C = C(\tau_0,\tau_1,d_0) > 0$. Set $s_3 = \max\left(s_2, 2\left(1+\frac{1}{\alpha^2}\right)C\right)$. Choosing in the latter inequality $s \ge s_3$, we deduce (2.35).

Step 2. Now, we want to eliminate the local term on the right-hand side corresponding to Ψ in the estimate (2.35).

So, let ω_2 be a non-empty open set such that $\omega_1\subset\omega_2\subset\mathcal{O}_d\cap\omega$. Introduce as in [1, 7] the cut off function $\xi\in C_0^\infty(\Omega)$ such that

$$0\le\xi\le 1,\ \ \xi = 1 \text{ in } \omega_1,\ \ \xi = 0 \text{ in } \Omega\setminus\omega_2, \tag{2.36a}$$

$$\frac{\Delta\xi}{\xi^{1/2}}\in L^\infty(\omega_2),\ \ \frac{\nabla\xi}{\xi^{1/2}}\in[L^\infty(\omega_2)]^N. \tag{2.36b}$$

Set $u = s^3\lambda^4\varphi^3e^{-2s\eta}$. Then, $u(T) = u(0) = 0$ and we have

$$\frac{\partial u}{\partial t} = u\left[3\varphi^{-1}\frac{\partial\varphi}{\partial t} - 2s\frac{\partial\eta}{\partial t}\right], \tag{2.37a}$$

$$\nabla(u\xi) = u\left[(3\lambda+2s\lambda\varphi)\xi\nabla\Psi+\nabla\xi\right] \tag{2.37b}$$

and

$$\begin{aligned}\Delta(u\xi) = {}& u\xi(14s\lambda^2\varphi+4s^2\lambda^2\varphi^2+9\lambda^2)|\nabla\psi|^2 + u\xi\Delta\psi(3\lambda+2s\lambda\varphi)\\ &+2u(3\lambda+2s\lambda\varphi)\nabla\psi.\nabla\xi+u\Delta\xi.\end{aligned} \tag{2.38}$$

Now, if we multiply the first equation of (2.19) by $u\xi\Psi$ and integrate by parts over Q, we obtain

$$\begin{aligned}&-\frac{1}{\alpha}\int_Q\rho^2u\xi\,\mathrm{d}x\,\mathrm{d}t + \int_Q\rho\xi\Psi\frac{\partial u}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - 2\int_Q\rho\nabla\Psi.\nabla(u\xi)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad-\int_Q\rho\Psi\Delta(u\xi)\,\mathrm{d}x\,\mathrm{d}t = \int_Q u\xi|\Psi|^2\chi_{\mathcal{O}_d}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.39}$$

If we set

$$J_1 = -\frac{1}{\alpha}\int_Q \rho^2 u\xi \, \mathrm{d}x\,\mathrm{d}t, \qquad J_2 = \int_Q \rho\xi\Psi\frac{\partial u}{\partial t}\,\mathrm{d}x\,\mathrm{d}t,$$

$$J_3 = -2\int_Q \rho\nabla\Psi.\nabla(u\xi)\,\mathrm{d}x\,\mathrm{d}t, \qquad J_4 = -\int_Q \rho\Psi\Delta(u\xi)\,\mathrm{d}x\,\mathrm{d}t,$$

the formula (2.39) can be rewritten as

$$J_1 + J_2 + J_3 + J_4 = \int_Q u\xi|\Psi|^2\chi_{\mathcal{O}_d}\,\mathrm{d}x\,\mathrm{d}t. \tag{2.40}$$

Let us estimate J_i, $i = 1, \ldots, 4$. We have

$$\begin{aligned} J_1 &= -\frac{1}{\alpha}\int_Q \rho^2 u\xi\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{1}{\alpha}\int_Q s^3\lambda^4\varphi^3 e^{-2s\eta}|\rho|^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Using the Young inequality, (2.26a), (2.26b), (2.36b), (2.37a), (2.37b) and (2.38), we obtain

$$\begin{aligned} J_2 &= \int_Q \rho\xi\Psi\frac{\partial u}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{\gamma_1}{2}\int_Q \xi u|\Psi|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2\gamma_1}\int_Q \xi u|\rho|^2\left[18\varphi^{-2}\left(\frac{\partial\varphi}{\partial t}\right)^2 + 8s^2\left(\frac{\partial\eta}{\partial t}\right)^2\right]\mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{\gamma_1}{2}\int_0^T\int_{\omega_1} u|\Psi|^2\,\mathrm{d}x\,\mathrm{d}t + C(T)\int_0^T\int_{\omega_2} s^5\lambda^4\varphi^7 e^{-2s\eta}|\rho|^2\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

for some $\gamma_1 > 0$. Furthermore,

$$\begin{aligned} J_3 &= -2\int_Q \rho\nabla\Psi.\nabla(u\xi)\,\mathrm{d}x\,\mathrm{d}t \\ &= -2\int_Q \rho\xi u(3\lambda + 2s\lambda\varphi)\nabla\psi.\nabla\Psi\,\mathrm{d}x\,\mathrm{d}t - 2\int_Q \rho u\nabla\Psi.\nabla\xi\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{1}{2}\int_Q s\lambda\varphi e^{-2s\eta}|\nabla\Psi|^2\,\mathrm{d}x\,\mathrm{d}t + C(\tau_1)\int_0^T\int_{\omega_2} s^7\lambda^9\varphi^7 e^{-2s\eta}|\rho|^2\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

and

$$\begin{aligned} J_4 &= -\int_Q \rho\Psi\Delta(u\xi)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_Q \rho u\xi\Psi(14s\lambda^2\varphi + 4s^2\lambda^2\varphi^2 + 9\lambda^2)|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_Q \rho u\xi\Psi\Delta\psi(3\lambda + 2s\lambda\varphi)|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - 2\int_Q \rho u\Psi(3\lambda + 2s\lambda\varphi)\nabla\psi.\nabla\xi\,\mathrm{d}x\,\mathrm{d}t - \int_Q \rho u\Psi\Delta\xi\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

which after some calculations gives

$$J_4 \leq \sum_{i=2}^{5} \frac{\gamma_i}{2} \int_0^T \int_{\omega_1} u|\Psi|^2 \,\mathrm{d}x\,\mathrm{d}t + C(\tau_1) \int_0^T \int_{\omega_2} s^7\lambda^8\varphi^7 e^{-2s\eta}|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t$$

for some $\gamma_i > 0$, $i = 2, \ldots, 5$. Finally, choosing the γ_i such that $\sum_{i=1}^{5} \frac{\gamma_i}{2} = \frac{1}{2}$, it follows from (2.40) that

$$\begin{aligned}
&\int_0^T \int_{\omega_1} s^3\lambda^4\varphi^3 e^{-2s\eta}|\Psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq \int_Q s\lambda\varphi e^{-2s\eta}|\nabla\Psi|^2 \,\mathrm{d}x\,\mathrm{d}t + C\left(K, \tau_1, T\right) \int_0^T \int_{\omega_2} s^7\lambda^9\varphi^7 e^{-2s\eta}|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \frac{1}{\alpha} \int_Q s^3\lambda^4\varphi^3 e^{-2s\eta}|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.41}$$

Combining (2.35) with (2.41), we deduce that

$$\begin{aligned}
\mathcal{K}(\rho) + \mathcal{K}(\Psi) \leq C(\tau_0, \tau_1, d_0) &\int_Q s\lambda\varphi e^{-2s\eta}|\nabla\Psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&+ C(K, T, \tau_0, \tau_1, d_0) \int_0^T \int_{\omega_2} s^7\lambda^9\varphi^7 e^{-2s\eta}|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&+ \frac{1}{\alpha} C(\tau_0, \tau_1, d_0) \int_Q s^3\lambda^4\varphi^3 e^{-2s\eta}|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Choosing in the latter inequality $\lambda \geq \lambda_2 = \max(\lambda_1, 2C(\tau_0, \tau_1, d_0))$, we obtain

$$\begin{aligned}
\mathcal{K}(\rho) + \mathcal{K}(\Psi) \leq C(K, T, \tau_0, \tau_1, d_0) &\int_0^T \int_{\omega_2} s^7\lambda^9\varphi^7 e^{-2s\eta}|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&+ \frac{1}{\alpha} C(\tau_0, \tau_1, d_0) \int_Q s^3\lambda^4\varphi^3 e^{-2s\eta}|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

If we take α large enough, we can absorb the last term of the latter relation on the left-hand side. Using the fact that $\omega_2 \subset \omega$, we deduce (2.34). Therefore, the proof is complete. $\square$

Now, we are going to establish the observability inequality of Carleman in the sense that the weight functions do not vanish at $t = 0$. We define the functions $\tilde{\varphi}$ and $\tilde{\eta}$ as follows:

$$\tilde{\varphi}(t, x) = \begin{cases} \varphi(\frac{T}{2}, x), & \text{if } t \in [0, \frac{T}{2}], \\ \varphi(t, x), & \text{if } t \in [\frac{T}{2}, T] \end{cases} \tag{2.42}$$

and

$$\tilde{\eta}(t, x) = \begin{cases} \eta(\frac{T}{2}, x), & \text{if } t \in [0, \frac{T}{2}], \\ \eta(t, x), & \text{if } t \in [\frac{T}{2}, T], \end{cases} \tag{2.43}$$

where the functions φ and η are given by (2.24) and (2.25), respectively. From now on, we fix $\lambda = \lambda_2$ and $s = s_3$. We have the following result.

Proposition 2.6 *Assume that the assumptions of Proposition* 2.5 *hold true. Then, there exist a weight function* θ, *positive constants* $s_3 \geq 1$, $\lambda_2 \geq 1$ *and* $C = C(K, T, \tau_0, \tau_1, d_0) > 0$ *such that the following estimate holds true for any solution* (φ, Ψ) *of* (2.19)–(2.20)

$$\|\rho(0,.)\|^2_{L^2(\Omega)} + \int_Q \frac{1}{\theta^2}|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_Q \frac{1}{\theta^2}|\Psi|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C \int_{\omega_T} |\rho|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{2.44}$$

Proof. Let us introduce a function $\beta \in \mathcal{C}^1([0,T])$ such that

$$0 \leq \beta \leq 1,\ \beta(t) = 1 \text{ for } t \in [0, T/2],\ \beta(t) = 0 \text{ for } t \in [3T/4, T],\ |\beta'(t)| \leq C/T. \tag{2.45}$$

For any $(t,x) \in Q$, we set $\zeta(t,x) = \beta(t)e^{-r(T-t)}\rho(t,x)$, where $r > 0$. Then, in view of (2.19), the function ζ is a solution of

$$\begin{cases} -\dfrac{\partial \zeta}{\partial t} - \Delta\zeta + a\zeta + r\zeta = \beta e^{-r(T-t)}\Psi\chi_{\mathcal{O}_d} - \beta' e^{-r(T-t)}\rho & \text{in } Q, \\ \zeta = 0 & \text{on } \Sigma, \\ \zeta(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{2.46}$$

If we multiply the first equation in (2.46) by ζ and integrate by parts over Q, we get

$$\begin{aligned} &\frac{1}{2}\|\zeta(0,.)\|^2_{L^2(\Omega)} + \|\nabla\zeta\|^2_{L^2(Q)} + r\|\zeta\|^2_{L^2(Q)} \\ &\quad \leq (K+1)\,\|\zeta\|^2_{L^2(Q)} + \frac{1}{2}\int_0^{3T/4}\int_\Omega |\Psi|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_{T/2}^{3T/4}\int_\Omega |\rho|^2 \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Hence, if we choose in the latter identity r such that $r = K + \frac{3}{2}$, using the definition of ζ, we deduce that

$$\begin{aligned} &\int_\Omega |\rho(0,x)|^2 \,\mathrm{d}x + \int_0^{T/2}\int_\Omega |\nabla\rho|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^{T/2}\int_\Omega |\rho|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq C(K,T)\left(\int_0^{3T/4}\int_\Omega |\Psi|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{T/2}^{3T/4}\int_\Omega |\rho|^2 \,\mathrm{d}x\,\mathrm{d}t\right). \end{aligned}$$

Now, using the fact that the functions $\tilde{\varphi}$ and $\tilde{\eta}$, defined by (2.42) and (2.43), respectively, have lower and upper bounds for $(t,x) \in [0, T/2] \times \Omega$, we get

$$\begin{aligned} &\int_\Omega |\rho(0,x)|^2 \mathrm{d}x + \widetilde{\mathcal{K}}_{[0,T/2]}(\rho) \\ &\quad \leq C(K,T)\left(\int_0^{3T/4}\int_\Omega |\Psi|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{T/2}^{3T/4}\int_\Omega |\rho|^2 \,\mathrm{d}x\,\mathrm{d}t\right), \end{aligned} \tag{2.47}$$

where

$$\widetilde{\mathcal{K}}_{[a,b]}(z) = \int_a^b\int_\Omega e^{-2s_3\tilde{\eta}}\tilde{\varphi}|\nabla z|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_a^b\int_\Omega e^{-2s_3\tilde{\eta}}\tilde{\varphi}^3|z|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{2.48}$$

Adding the term $\widetilde{\mathcal{K}}_{[0,T/2]}(\Psi)$ to both sides of inequality (2.47), we obtain

$$\begin{aligned} &\int_\Omega |\rho(0,x)|^2 \,\mathrm{d}x + \widetilde{\mathcal{K}}_{[0,T/2]}(\rho) + \widetilde{\mathcal{K}}_{[0,T/2]}(\Psi) \\ &\quad \leq C(K,T)\left(\int_0^{3T/4}\int_\Omega |\Psi|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{T/2}^{3T/4}\int_\Omega |\rho|^2 \,\mathrm{d}x\,\mathrm{d}t\right) + \widetilde{\mathcal{K}}_{[0,T/2]}(\Psi). \end{aligned} \tag{2.49}$$

In order to eliminate the term $\widetilde{\mathcal{K}}_{[0,T/2]}(\Psi)$ on the right-hand side of (2.49), we use the standard energy estimates for the first equation of (2.20) and we obtain

$$\int_0^{T/2}\int_\Omega |\nabla\Psi|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T/2}\int_\Omega |\Psi|^2\,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{\alpha^2}C(K,T)\int_0^{T/2}\int_\Omega |\rho|^2\,\mathrm{d}x\,\mathrm{d}t,$$

where C is independent of α. Since the functions $\tilde{\varphi}$ and $\tilde{\eta}$ have lower and upper bounds for $(t,x)\in[0,T/2]\times\Omega$, from the previous inequality we obtain

$$\widetilde{\mathcal{K}}_{[0,T/2]}(\Psi) \le \frac{1}{\alpha^2}C(K,T)\int_0^{T/2}\int_\Omega e^{-2s_3\tilde{\eta}}\tilde{\varphi}^3\,|\rho|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{2.50}$$

Replacing (2.50) in (2.49) and taking α large enough, we obtain

$$\begin{aligned}&\int_\Omega |\rho(0,x)|^2\,\mathrm{d}x + \widetilde{\mathcal{K}}_{[0,T/2]}(\rho) + \widetilde{\mathcal{K}}_{[0,T/2]}(\Psi)\\ &\qquad \le C(K,T)\left(\int_{T/2}^{3T/4}\int_\Omega (|\rho|^2+|\Psi|^2)\,\mathrm{d}x\,\mathrm{d}t\right).\end{aligned} \tag{2.51}$$

Since the functions φ and η defined by (2.24) and (2.25), respectively, have the lower and upper bounds for $(t,x)\in[T/2,3T/4]\times\Omega$, using (2.34) we obtain

$$\begin{aligned}&\int_\Omega |\rho(0,x)|^2\,\mathrm{d}x + \widetilde{\mathcal{K}}_{[0,T/2]}(\rho) + \widetilde{\mathcal{K}}_{[0,T/2]}(\Psi)\\ &\quad \le C(K,T)\left(\mathcal{K}(\rho)+\mathcal{K}(\Psi)\right)\\ &\quad \le C(K,T,\tau_0,\tau_1,d_0)\int_{\omega_T} e^{-2s_3\eta}\varphi^7|\rho|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.52}$$

On the other hand, since $\eta=\tilde{\eta}$ and $\varphi=\tilde{\varphi}$ in $[T/2,T]\times\Omega$, we use again estimate (2.34) and we obtain

$$\begin{aligned}\widetilde{\mathcal{K}}_{[T/2,T]}(\rho) + \widetilde{\mathcal{K}}_{[T/2,T]}(\Psi) &= \mathcal{K}(\rho)+\mathcal{K}(\Psi)\\ &\le C(K,T,\tau_0,\tau_1,d_0)\int_{\omega_T} e^{-2s_3\eta}\varphi^7|\rho|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.53}$$

Adding (2.52) and (2.53) and using the fact that $e^{-2s_3\eta}\varphi^7\in L^\infty(Q)$, we deduce that

$$\|\rho(0,\cdot)\|^2_{L^2(\Omega)} + \widetilde{\mathcal{K}}_{[0,T]}(\rho) + \widetilde{\mathcal{K}}_{[0,T]}(\Psi) \le C(K,T,\tau_0,\tau_1,d_0)\int_{\omega_T}|\rho|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{2.54}$$

Using the definition of $\widetilde{\mathcal{K}}_{[a,b]}$ given by (2.48), we can rewrite the inequality (2.54) as

$$\begin{aligned}&\|\rho(0,\cdot)\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega e^{-2s_3\tilde{\eta}}\tilde{\varphi}^3|\rho|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega e^{-2s_3\tilde{\eta}}\tilde{\varphi}^3|\Psi|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\quad \le C(K,T,\tau_0,\tau_1,d_0)\int_{\omega_T}|\rho|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.55}$$

If we set

$$\frac{1}{\theta^2} = e^{-2s_3\tilde{\eta}}\tilde{\varphi}^3, \tag{2.56}$$

then, in view of (2.55) and (2.56), we deduce the estimation (2.44). □

2.3 Solution of Problem 2 for system (2.1)

In this subsection, we are interested in the following null controllability problem. Assume that (2.2) holds. Given $z_d \in L^2(\mathcal{O}_d^T)$ and $y^0 \in L^2(\Omega)$, find a control $\hat{h} \in L^2(\omega_T)$ such that if $\hat{y}(\hat{h}) = \hat{y}(t,x;\hat{v}(\hat{h}),\hat{h}) \in L^2((0,T);H_0^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega))$ and $\hat{p} \in L^2((0,T);H_0^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega))$ are solutions of (2.12)–(2.14), then

$$\hat{y}(T,.;\hat{h}) = 0 \text{ in } \Omega. \tag{2.57}$$

Proposition 2.7 *Let Ω be an unbounded open subset of $\mathbb{R}^N$ with boundary Γ of class $\mathcal{C}^2$. Let also $\mathcal{O}$, ω, ω_1 and $\mathcal{O}_d$ be four non-empty subsets of Ω such that $\omega_1 \subset \omega$, $\mathcal{O} \cap \omega = \emptyset$ and $\mathcal{O}_d \cap \omega \neq \emptyset$. Assume that* (2.2) *holds true. Then, there exists a positive real weight function θ given by* (2.56) *such that for any function $y^0 \in L^2(\Omega)$ and $z_d \in L^2(\mathcal{O}_d^T)$ with $\theta z_d \in L^2(\mathcal{O}_d^T)$ there exists a unique control $\hat{h} \in L^2(\omega_T)$ such that the null controllability problem* (2.12)–(2.14) *and* (2.57) *holds. Moreover,*

$$\hat{h} = \hat{\rho} \ \ in \ \ \omega_T, \tag{2.58}$$

where $\hat{\rho}$ satisfies

$$-\frac{\partial \hat{\rho}}{\partial t} - \Delta \hat{\rho} + a\hat{\rho} = \hat{\Psi}\chi_{\mathcal{O}_d} \ \ in \ \ Q \tag{2.59}$$

and $\hat{\Psi}$ is a solution of

$$\frac{\partial \hat{\Psi}}{\partial t} - \Delta \hat{\Psi} + a\hat{\Psi} = -\frac{1}{\alpha}\hat{\rho}\chi_{\mathcal{O}} \ \ in \ \ Q. \tag{2.60}$$

In addition, there exists a constant $C = C(K,T,\tau_0,\tau_1,d_0) > 0$ such that

$$\|\hat{h}\|_{L^2(\omega_T)} \leq C\left(\|y^0\|_{L^2(\Omega)} + \|\theta z_d\|_{L^2(\mathcal{O}_d^T)}\right). \tag{2.61}$$

Proof. To prove the null controllability (2.12)–(2.14) and (2.57), we proceed in three steps using a penalization method.

Step 1. For any $\varepsilon > 0$ we define the cost function

$$J_\varepsilon(h) = \frac{1}{2\varepsilon}\|\hat{y}(T,.;h)\|^2_{L^2(\Omega)} + \frac{1}{2}\|h\|^2_{L^2(\omega_T)}. \tag{2.62}$$

Then, we consider the optimal control problem: find $h_\varepsilon \in L^2(\omega_T)$ such that

$$J_\varepsilon(h_\varepsilon) = \min_{h \in L^2(\omega_T)} J_\varepsilon(h). \tag{2.63}$$

Using classical arguments we can prove that there exists a unique solution h_ε to (2.63) (see [19] for example). If we denote by $(\hat{y}_\varepsilon, \hat{p}_\varepsilon) = (\hat{y}_\varepsilon(h_\varepsilon), \hat{p}_\varepsilon(h_\varepsilon))$ the solution to (2.12)–(2.14) corresponding to h_ε, using an Euler–Lagrange first order optimality condition, we can prove that there exist ρ_ε and Ψ_ε such that $(\hat{y}_\varepsilon, \hat{p}_\varepsilon, \rho_\varepsilon, \Psi_\varepsilon, h_\varepsilon)$ is a solution of the following optimality system

$$\begin{cases} \dfrac{\partial \hat{y}_\varepsilon}{\partial t} - \Delta \hat{y}_\varepsilon + a\hat{y}_\varepsilon = \hat{v}_\varepsilon\chi_{\mathcal{O}} + h_\varepsilon\chi_\omega & \text{in } Q, \\ \hat{y}_\varepsilon = 0 & \text{on } \Sigma, \\ \hat{y}_\varepsilon(0,.) = y^0 & \text{in } \Omega, \end{cases} \tag{2.64}$$

$$\hat{v}_\varepsilon = -\frac{\hat{p}_\varepsilon}{\alpha} \ \text{ in } \ \mathcal{O}_T, \tag{2.65}$$

$$\begin{cases} -\dfrac{\partial \hat{p}_\varepsilon}{\partial t} - \Delta \hat{p}_\varepsilon + a\hat{p}_\varepsilon = (\hat{y}_\varepsilon - z_d)\chi_{\mathcal{O}_d} & \text{in } Q, \\ \hat{p}_\varepsilon = 0 & \text{on } \Sigma, \\ \hat{p}_\varepsilon(T,.) = 0 & \text{in } \Omega, \end{cases} \tag{2.66}$$

$$\begin{cases} -\dfrac{\partial \rho_\varepsilon}{\partial t} - \Delta \rho_\varepsilon + a\rho_\varepsilon = \Psi_\varepsilon \chi_{\mathcal{O}_d} & \text{in } Q, \\ \rho_\varepsilon = 0 & \text{on } \Sigma, \\ \rho_\varepsilon(T,.) = -\dfrac{1}{\varepsilon}\hat{y}_\varepsilon(T,\cdot) & \text{in } \Omega, \end{cases} \tag{2.67}$$

$$\begin{cases} \dfrac{\partial \Psi_\varepsilon}{\partial t} - \Delta \Psi_\varepsilon + a\Psi_\varepsilon = -\dfrac{\rho_\varepsilon}{\alpha}\chi_{\mathcal{O}} & \text{in } Q, \\ \Psi_\varepsilon = 0 & \text{on } \Sigma, \\ \Psi_\varepsilon(0,.) = 0 & \text{in } \Omega \end{cases} \tag{2.68}$$

and

$$h_\varepsilon = \rho_\varepsilon \ \text{ in } \ \omega_T. \tag{2.69}$$

Step 2. We give estimates on $\hat{y}_\varepsilon, \hat{v}_\varepsilon, \hat{p}_\varepsilon$ and h_ε independent on ε.

If we multiply the first line in (2.64) by ρ_ε (*i.e.*, the solution of (2.67)) and the first line in (2.66) by Ψ_ε (*i.e.*, the solution of (2.68)), and integrate by parts over Q and use (2.65), we obtain the following equations

$$\begin{aligned} -\frac{1}{\varepsilon}\|\hat{y}_\varepsilon(T,\cdot)\|^2_{L^2(\Omega)} - \int_\Omega y^0 \rho_\varepsilon(0,x)\,\mathrm{d}x + \int_{\mathcal{O}_d^T} \hat{y}_\varepsilon \Psi_\varepsilon \mathrm{d}x\,\mathrm{d}t \\ = -\frac{1}{\alpha}\int_{\mathcal{O}_T} \hat{p}_\varepsilon \rho_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \|h_\varepsilon\|^2_{L^2(\omega_T)} \end{aligned} \tag{2.70}$$

and

$$-\frac{1}{\alpha}\int_{\mathcal{O}_T} \hat{p}_\varepsilon \rho_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{O}_d^T} \hat{y}_\varepsilon \Psi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathcal{O}_d^T} z_d \Psi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t. \tag{2.71}$$

Combining (2.70) and (2.71), we obtain

$$\|h_\varepsilon\|^2_{L^2(\omega_T)} + \frac{1}{\varepsilon}\|\hat{y}_\varepsilon(T,\cdot)\|^2_{L^2(\Omega)} = \int_{\mathcal{O}_d^T} z_d \Psi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega y^0 \rho_\varepsilon(0,x)\,\mathrm{d}x,$$

which using the Cauchy–Schwarz inequality and the fact that $\theta z_d \in L^2(\mathcal{O}_d^T)$ gives

$$\|h_\varepsilon\|^2_{L^2(\omega_T)} + \frac{1}{\varepsilon}\|\hat{y}_\varepsilon(T,\cdot)\|^2_{L^2(\Omega)} \leq \|\theta z_d\|_{L^2(\mathcal{O}_d^T)} \left\|\frac{1}{\theta}\Psi_\varepsilon\right\|_{L^2(Q)} + \|y^0\|_{L^2(\Omega)}\|\rho_\varepsilon(0,\cdot)\|_{L^2(\Omega)}.$$

This implies that

$$\begin{aligned} &\|h_\varepsilon\|^2_{L^2(\omega_T)} + \frac{1}{\varepsilon}\|\hat{y}_\varepsilon(T,\cdot)\|^2_{L^2(\Omega)} \\ &\quad \leq \left(\|\theta z_d\|^2_{L^2(\mathcal{O}_d^T)} + \|y^0\|^2_{L^2(\Omega)}\right)^{1/2} \times \left(\left\|\frac{1}{\theta}\Psi_\varepsilon\right\|^2_{L^2(Q)} + \|\rho_\varepsilon(0,\cdot)\|^2_{L^2(\Omega)}\right)^{1/2}. \end{aligned} \tag{2.72}$$

Now, if we apply the Carleman inequality (2.44) to ρ_ε and Ψ_ε, then there exists $C = C(K, T, \tau_0, \tau_1, d_0) > 0$ such that

$$\int_Q \frac{1}{\theta^2}|\rho_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_Q \frac{1}{\theta^2}|\Psi_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t + \|\rho_\varepsilon(0,.)\|^2_{L^2(\Omega)} \le C\int_{\omega_T} |\rho_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{2.73}$$

Using (2.72), (2.73) and (2.69), we obtain

$$\frac{1}{\varepsilon}\|\hat{y}_\varepsilon(T,\cdot)\|^2_{L^2(\Omega)} + \|h_\varepsilon\|^2_{L^2(\omega_T)} \le C\left(\|\theta z_d\|_{L^2(\mathcal{O}_d^T)} + \|y^0\|_{L^2(\Omega)}\right)\|h_\varepsilon\|_{L^2(\omega_T)}.$$

It follows that

$$\|h_\varepsilon\|_{L^2(\omega_T)} \le C\left(\|\theta z_d\|_{L^2(\mathcal{O}_d^T)} + \|y^0\|_{L^2(\Omega)}\right) \tag{2.74}$$

and

$$\|\hat{y}_\varepsilon(T,\cdot)\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}\left(\|\theta z_d\|_{L^2(\mathcal{O}_d^T)} + \|y^0\|_{L^2(\Omega)}\right), \tag{2.75}$$

where $C = C(K, T, \tau_0, \tau_1, d_0) > 0$. In view of Remark 2.1, (2.18) and (2.74), we have that

$$\|\hat{v}_\varepsilon\|_{L^2(\mathcal{O}_T)} \le C\left(\|z_d\|_{L^2(\mathcal{O}_d^T)} + \|\theta z_d\|_{L^2(\mathcal{O}_d^T)} + \|y^0\|_{L^2(\Omega)}\right), \tag{2.76}$$

where $C = C(K, T, \tau_0, \tau_1, d_0) > 0$. Since $\hat{y}_\varepsilon$ and $\hat{p}_\varepsilon$ satisfy (2.64)–(2.66), using (2.69), (2.74), (2.76), we prove that

$$\|\hat{y}_\varepsilon\|_{L^2((0,T);H_0^1(\Omega))} \le C\left(\|z_d\|_{L^2(\mathcal{O}_d^T)} + \|\theta z_d\|_{L^2(\mathcal{O}_d^T)} + \|y^0\|_{L^2(\Omega)}\right), \tag{2.77a}$$

$$\|\hat{p}_\varepsilon\|_{L^2((0,T);H_0^1(\Omega))} \le C\left(\|z_d\|_{L^2(\mathcal{O}_d^T)} + \|\theta z_d\|_{L^2(\mathcal{O}_d^T)} + \|y^0\|_{L^2(\Omega)}\right), \tag{2.77b}$$

where $C = C(K, T, \tau_0, \tau_1, d_0) > 0$.

Step 3. We study the convergence when $\varepsilon \to 0$ of sequences h_ε, $\hat{v}_\varepsilon$, $\hat{y}_\varepsilon$, $\hat{p}_\varepsilon$, Ψ_ε and ρ_ε.

In view of (2.74), (2.75), (2.76) and (2.77), we can extract subsequences of h_ε, $\hat{v}_\varepsilon$, $\hat{y}_\varepsilon$ and $\hat{p}_\varepsilon$ (still denoted h_ε, $\hat{v}_\varepsilon$, $\hat{y}_\varepsilon$ and $\hat{p}_\varepsilon$, respectively) such that when $\varepsilon \to 0$, we have

$$h_\varepsilon \rightharpoonup \hat{h} \text{ weakly in } L^2(\omega_T), \tag{2.78a}$$

$$\hat{v}_\varepsilon \rightharpoonup \hat{v} \text{ weakly in } L^2(\mathcal{O}_T), \tag{2.78b}$$

$$\hat{y}_\varepsilon \rightharpoonup \hat{y} \text{ weakly in } L^2((0,T);H_0^1(\Omega)), \tag{2.78c}$$

$$\hat{p}_\varepsilon \rightharpoonup \hat{p} \text{ weakly in } L^2((0,T);H_0^1(\Omega)), \tag{2.78d}$$

$$\hat{y}_\varepsilon(T,.) \to 0 \text{ strongly in } L^2(\Omega). \tag{2.78e}$$

From (2.65), (2.78b) and (2.78d) we obtain

$$\hat{v} = -\frac{\hat{p}}{\alpha} \text{ in } \mathcal{O}_T. \tag{2.79}$$

Moreover, using the weak lower semi-continuity of the norm, we deduce from (2.76) and (2.78b) that

$$\|\hat{v}\|_{L^2(\mathcal{O}_T)} \le C\left(\|z_d\|_{L^2(\mathcal{O}_d^T)} + \|\theta z_d\|_{L^2(\mathcal{O}_d^T)} + \|y^0\|_{L^2(\Omega)}\right), \tag{2.80}$$

where $C = C(K, T, \tau_0, \tau_1, d_0) > 0$.

Let $\mathcal{D}(Q)$ be the set of functions of class C^∞ on Q with compact support. If we multiply the first equation in (2.64) by $\Phi \in \mathcal{D}(Q)$ and the first equation in (2.66) by $\xi \in \mathcal{D}(Q)$, integrate by parts over Q, and then take the limit when $\varepsilon \to 0$ while using (2.78), we, respectively, deduce that

$$\int_Q \hat{y}\left(-\frac{\partial \Phi}{\partial t} - \Delta\Phi + a\Phi\right) \mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{O}_T} \hat{v}\Phi \,\mathrm{d}x\,\mathrm{d}t + \int_{\omega_T} \hat{h}\Phi \,\mathrm{d}x\,\mathrm{d}t$$

and

$$\int_Q \hat{p}\left(\frac{\partial \xi}{\partial t} - \Delta\xi + a\xi\right) \mathrm{d}x\,\mathrm{d}t = \int_Q (\hat{y} - z_d)\chi_{\mathcal{O}_d}\xi \,\mathrm{d}x\,\mathrm{d}t.$$

This, after an integration by parts over Q, gives, respectively,

$$\int_Q \left(\frac{\partial \hat{y}}{\partial t} - \Delta\hat{y} + a\hat{y}\right) \Phi \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{O}_T} \hat{v}\Phi \,\mathrm{d}x\,\mathrm{d}t + \int_{\omega_T} \hat{h}\Phi \,\mathrm{d}x\,\mathrm{d}t \text{ for every } \Phi \in \mathcal{D}(Q)$$

and

$$\int_Q \left(-\frac{\partial \hat{p}}{\partial t} - \Delta\hat{p} + a\hat{p}\right) \xi \,\mathrm{d}x\,\mathrm{d}t = \int_Q (\hat{y} - z_d)\chi_{\mathcal{O}_d}\xi \,\mathrm{d}x\,\mathrm{d}t \text{ for every } \xi \in \mathcal{D}(Q).$$

Hence, we deduce that

$$\frac{\partial \hat{y}}{\partial t} - \Delta\hat{y} + a\hat{y} = \hat{v}\chi_{\mathcal{O}} + \hat{h}\chi_\omega \text{ in } Q, \tag{2.81a}$$

$$-\frac{\partial \hat{p}}{\partial t} - \Delta\hat{p} + a\hat{p} = (\hat{y} - z_d)\chi_{\mathcal{O}_d} \text{ in } Q. \tag{2.81b}$$

Observing that $\hat{y}$, $\hat{p} \in L^2((0,T); H_0^1(\Omega))$ and $\frac{\partial \hat{y}}{\partial t}$, $\frac{\partial \hat{p}}{\partial t} \in L^2((0,T); H^{-1}(\Omega))$ we deduce that $(\hat{y}(0), \hat{y}(T), \hat{p}(T))$ exists in $(L^2(\Omega))^3$. The traces of $\hat{y}(t)$ and $\hat{p}(t)$ exist in $L^2(\Gamma)$ for almost every $t \in (0,T)$. It then follows from (2.78c) and (2.78d) that

$$\hat{y} = 0 \text{ on } \Sigma, \tag{2.82a}$$

$$\hat{p} = 0 \text{ on } \Sigma. \tag{2.82b}$$

If we multiply the first equation in (2.64) by $\Phi \in \mathcal{C}^\infty(\bar{Q})$ such that $\Phi|_\Sigma = 0$ and the first equation in (2.66) by $\xi \in \mathcal{C}^\infty(\bar{Q})$ such that $\xi|_\Sigma = 0$, $\xi(0) = 0$ in Ω, and then integrate by parts over Q, we, respectively, get

$$\begin{aligned}\int_{\mathcal{O}_T} \hat{v}_\varepsilon\Phi \,\mathrm{d}x\,\mathrm{d}t &+ \int_{\omega_T} h_\varepsilon\Phi \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_\Omega y^0\,\Phi(0)\,\mathrm{d}x + \int_\Omega \hat{y}_\varepsilon(T)\,\Phi(T)\,\mathrm{d}x + \int_Q \hat{y}_\varepsilon\left(-\frac{\partial \Phi}{\partial t} - \Delta\Phi + a\Phi\right) \mathrm{d}x\,\mathrm{d}t \\ &\qquad \text{for every } \Phi \in \mathcal{C}^\infty(\bar{Q}) \text{ such that } \Phi|_\Sigma = 0\end{aligned}$$

and

$$\int_Q (\hat{y}_\varepsilon - z_d)\chi_{\mathcal{O}_d}\xi \,\mathrm{d}x\,\mathrm{d}t = \int_Q \hat{p}_\varepsilon\left(\frac{\partial \xi}{\partial t} - \Delta\xi + a\xi\right) \mathrm{d}x\,\mathrm{d}t,$$
$$\text{for every } \xi \in \mathcal{C}^\infty(\bar{Q}) \text{ such that } \xi|_\Sigma = 0,\ \xi(0) = 0 \text{ in } \Omega.$$

Passing in these latter identities to the limit when ε tends toward zero, while using (2.78), then integrating by parts over Q and using (2.82), we obtain

$$\int_{\mathcal{O}_T} \hat{v}\Phi \,\mathrm{d}x\,\mathrm{d}t + \int_{\omega_T} \hat{h}\Phi \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega (\hat{y}(0) - y^0)\,\Phi(0)\,\mathrm{d}x - \int_\Omega \hat{y}(T)\,\Phi(T)\,\mathrm{d}x + \int_Q \Phi\left(\frac{\partial \hat{y}}{\partial t} - \Delta\hat{y} + a\hat{y}\right)\mathrm{d}x\,\mathrm{d}t$$
$$\text{for every } \Phi \in \mathcal{C}^\infty(\bar{Q}) \text{ such that } \Phi|_\Sigma = 0$$

and

$$\int_Q (\hat{y} - z_d)\chi_{\mathcal{O}_d}\xi \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \hat{p}(T)\,\xi(T)\,\mathrm{d}x + \int_Q \xi\left(-\frac{\partial \hat{p}}{\partial t} - \Delta\hat{p} + a\hat{p}\right)\mathrm{d}x\,\mathrm{d}t$$
$$\text{for every } \xi \in \mathcal{C}^\infty(\bar{Q}) \text{ such that } \xi|_\Sigma = 0,\ \xi(0) = 0 \text{ in } \Omega.$$

This, in view of (2.81), gives, respectively,

$$0 = \int_\Omega (\hat{y}(0) - y^0)\Phi(0)\,\mathrm{d}x - \int_\Omega \hat{y}(T)\Phi(T)\,\mathrm{d}x \ \text{ for every } \Phi \in \mathcal{C}^\infty(\bar{Q}) \text{ such that } \Phi|_\Sigma = 0 \tag{2.83}$$

and

$$0 = \int_\Omega \hat{p}(T)\,\xi(T)\,\mathrm{d}x \text{ for every } \xi \in \mathcal{C}^\infty(\bar{Q}) \text{ such that } \xi|_\Sigma = 0,\ \xi(0) = 0 \text{ in } \Omega. \tag{2.84}$$

From (2.84) we deduce that

$$\hat{p}(T,.) = \ \text{ on } \Omega. \tag{2.85}$$

Taking Φ in (2.83) such that $\Phi(T) = 0$ in Ω, we deduce that

$$\hat{y}(0,.) = y^0 \text{ on } \Omega. \tag{2.86}$$

And it finally follows from (2.83) that

$$\hat{y}(T,.) = 0 \text{ on } \Omega. \tag{2.87}$$

If we apply the Carleman inequality (2.44) to ρ_ε and Ψ_ε, then there exists $C = C(K, T, \tau_0, \tau_1, d_0) > 0$ such that

$$\left\|\frac{1}{\theta}\rho_\varepsilon\right\|_{L^2(Q)} + \left\|\frac{1}{\theta}\Psi_\varepsilon\right\|_{L^2(Q)} \le C\big(\|y^0\|_{L^2(\Omega)} + \|\theta z_d\|_{L^2(\mathcal{O}_d^T)}\big),$$

because (2.69) and (2.74) hold true. Hence, in view of the definition of θ given by (2.56), we obtain

$$\|\rho_\varepsilon\|_{L^2((\tau,T-\tau)\times\Omega)} + \|\Psi_\varepsilon\|_{L^2((\tau,T-\tau)\times\Omega)} \le C\big(\|y^0\|_{L^2(\Omega)} + \|\theta z_d\|_{L^2(\mathcal{O}_d^T)}\big),$$

where $\tau > 0$. Consequently, there exist $\hat{\rho}$ and $\hat{\Psi}$ in $L^2\left((\tau, T-\tau)\times\Omega\right)$ such that

$$\rho_\varepsilon \rightharpoonup \hat{\rho} \text{ weakly in } L^2((\tau, T-\tau)\times\Omega),$$
$$\Psi_\varepsilon \rightharpoonup \hat{\Psi} \text{ weakly in } L^2((\tau, T-\tau)\times\Omega).$$

Therefore,

$$\begin{aligned} &\rho_\varepsilon \rightharpoonup \hat{\rho} \text{ weakly in } \mathcal{D}'(Q),\\ &\Psi_\varepsilon \rightharpoonup \hat{\Psi} \text{ weakly in } \mathcal{D}'(Q), \end{aligned} \tag{2.88}$$

where $\mathcal{D}'(Q)$ denotes the dual of $\mathcal{D}(Q)$, and it follows from (2.69) and (2.74) that

$$\rho_\varepsilon \rightharpoonup \hat{\rho} \text{ weakly in } L^2(\omega_T). \tag{2.89}$$

From (2.69), (2.78a) and (2.89), we have that $\hat{h} = \hat{\rho}\chi_\omega$.

Proceeding as for the convergence of $\hat{y}_\varepsilon$ on pages 112–114 while passing to the limit in (2.68), we prove using the second convergence of (2.88) that $\hat{\Psi}$ satisfies (2.60). Passing to the limit in (2.67) while using the first convergence of (2.88), we have that $\hat{\rho}$ satisfies (2.59).

It then follows from (2.79), (2.81), (2.82), (2.85), (2.86) and (2.87) that ($\hat{h}$, $\hat{y}$, $\hat{p}$) is a solution of the null controllability problems (2.12)–(2.14) and (2.57). Finally, using the weak lower semi-continuity of the norm and (2.78a), we deduce from (2.74) the estimate (2.61). □

Remark 2.4 *By Proposition* 2.7 *we proved that there exists a positive real weight function θ given by* (2.56) *such that for any function $y^0 \in L^2(\Omega)$ and $z_d \in L^2(\mathcal{O}_d^T)$ with $\theta z_d \in L^2(\mathcal{O}_d^T)$, there exists a unique control $\hat{h} \in L^2(\omega_T)$ such that if $\hat{y} = \hat{y}(\hat{v}, \hat{h})$ is a solution of* (2.1)*, then $\hat{y}$ satisfies* (2.57)*. Moreover, $\hat{v}$ is given by* (2.14) *and $\hat{h} = \hat{\rho}$ in ω_T, where $\hat{\rho}$ satisfies* (2.59)–(2.60) *and the estimation* (2.61)*.*

3 Proof of Theorem 1.1

We rewrite the system (1.1) as follows

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + a_0\chi_{\Omega\setminus\omega} y = v\chi_{\mathcal{O}} + h\chi_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0,.) = y^0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $k = (a_0 y + h)$. Then, h is a control of system (3.1) if and only if $k = a_0 y + h$ is a control of system (1.1). The advantage of writing the system (1.1) in the form (3.1) is that the potential is now bounded and we can use the Carleman inequality (2.44).

Now, taking $a = a_0\chi_{\Omega\setminus\omega} \in L^\infty(Q)$ in (2.1), it follows from Proposition 2.7 and Remark 2.4 that there exists a unique control $\hat{h} \in L^2(\omega_T)$ such that if $\hat{y} = \hat{y}(\hat{v}, \hat{h})$ is the solution of (2.1), then $\hat{y}$ satisfies (2.57). Moreover, $\hat{v}$ is given by (2.14) and

$$\hat{h} = \hat{\rho} \text{ in } \omega_T,$$

where $\hat{\rho}$ satisfies (2.59)–(2.60) and the estimation (2.61). Now, observing that $\hat{k} = (a_0\hat{y} + \hat{h})$, we have that there exists a unique control $\hat{k} \in L^2(\omega_T)$ such that if $\hat{y} = \hat{y}(\hat{v}, \hat{h})$ is the solution of (1.1), then $\hat{y}$ satisfies (1.5). Moreover, $\hat{v}$ is given by (2.14) and

$$\hat{k} = (a_0\hat{y} + \hat{h}) \text{ in } \omega_T,$$

where $\hat{\rho}$ satisfies (2.59)–(2.60). Moreover, using (2.61) we have that

$$\|\hat{k}\|_{L^2(\omega_T)} \leq C\left(\|y^0\|_{L^2(\Omega)} + \|\theta z_d\|_{L^2(\mathcal{O}_d^T)}\right), \tag{3.2}$$

where $C = C(\|a_0\|_{L^\infty(\Omega\setminus\omega)}, T, \tau_0, \tau_1, d_0) > 0$. The proof of Theorem 1.1 is then complete.

4 Conclusion

In this work, we proved the null controllability through Stackelberg control associated to system (1.1) in an unbounded domain. We obtained the result under the assumptions that on the one hand $\mathcal{O} \cap \omega = \emptyset$ and the other hand that the set $\Omega \setminus \omega$ is bounded.

Note that the first positive result on null controllability in an unbounded domain was obtained in [4] under the assumption that the set $\Omega \setminus \omega$ is bounded. The fact that Ω and ω are such that $\Omega \setminus \omega$ is bounded is just one example of unbounded domains. We refer the reader to [14] where he/she will find other examples of unbounded domains where the Carleman inequality (2.28) that we have established remains true.

We can extend this work to the case when the unbounded sets Ω and ω are such that $\Omega \setminus \omega$ is an unbounded set with infinite measure. We refer to [14] for more detail.

It is also possible to extend this work using arguments given in [3] to control the linear system (1.1) with a control acting on an unbounded region ω of finite measure. This can allows us to remove the hypothesis (1.2).

Acknowledgments

The first author was supported by the German Academic Exchange Service (D.A.A.D) under the Scholarship Program PhD AIMS-Cameroon. The authors would like to express their gratitude to the unknown referees for helpful advice.

References

[1] O. Bodart, C. Fabre *Controls insensitizing the norm of the solution of a semilinear heat equation*, Journal of Mathematical Analysis and Applications **195** (1995), 658–683.

[2] V. R. Cabanillas, S. B. De Menezes, E. Zuazua, *Null controllability in unbounded domains for the semilinear heat equation with nonlinearities involving gradient terms*, Journal of Optimization Theory and Application **110** (2001), no. 2, 245–264.

[3] P. Cannarsa, P. Martinez, J. Vancostenoble, *Null controllability of the heat equation in unbounded domains by a finite measure control region*, ESAIM: Control, Optimisation and Calculus of Variations **28** (2004), no. 3, 381–408.

[4] S. B. De Menezes, V. R. Cabanillas, *Null controllability for the semilinear heat equation in unbounded domains*, Pesquimat **4** (2001), no. 2, 35–54.

[5] L. de Teresa, *Approximate controllability of a semilinear heat equation in* $\mathbb{R}^N$, SIAM Journal on Control and Optimization **36** (1998), no. 6, 2128–2147.

[6] L. de Teresa, E. Zuazua, *Approximate controllability of the semilinear heat equation in unbounded domains*, Nonlinear Analysis. Theory, Methods & Applications **37** (1999), no. 8, 1059–1090.

[7] L. de Teresa, *Insensitizing controls for a semilinear heat equation*: *Semilinear heat equation*, Communications in Partial Differential Equations **25** (2000), no. 1-2, 39–72.

[8] L. L. Djomegne Njoukoue, G. Mophou, G. Deugoue, *Stackelberg control of a backward linear heat equation*, Advance in Evolution Equations: Evolutionary Processes and Applications **10** (2019), 127–149.

[9] A. Doubova, E. Fernández-Cara, M. González-Burgos, E. Zuazua, *On the controllability of parabolic systems with nonlinear term involving the state and the gradient*, SIAM Journal on Control and Optimization **41** (2002), no. 3, 798–819.

[10] C. Fabre, J-P. Puel, E. Zuazua, *Approximate controllability for the semilinear heat equation*, Proceedings of the Royal Society of Edinburgh **125A** (1995), no. 1, 31–61.

[11] E. Fernández-Cara, *Null controllability of the semilinear heat equation*, ESAIM: Control, Optimisation and Calculus of Variations **2** (1997), 87–107.

[12] E. Fernández-Cara, E. Zuazua, *Null and approximate controllability for weakly blowing up semilinear heat equations*, Annales de l'institut Henri Poincaré (C), Analyse Non Linéaire **17** (2000), no. 5, 583–616.

[13] R. G. Foko Tiomela, G. Mophou, G. N'Guérékata, *Hierarchic control of a linear heat equation with missing data*. Mathematical Methods in the Applied Sciences **43** (2020), no. 10, 1- 22.

[14] M. González-Burgos, L. de Teresa, *Some results on controllability for linear and nonlinear heat equations in unbounded domains*, Advances in Differential Equations **12** (2007), no. 11, 1201–1240.

[15] V. Hernández-Santamaría, L. de Teresa, *Robust Stackelberg controllability for linear and semilinear heat equations*, Evolution Equation and Control Theory **7** (2018), no. 2, 247–273.

[16] V. Hernández-Santamaría, L. Peralta, *Some remarks on the Robust Stackelberg controllability for the heat equation with controls on the boundary*, Discrete and Continuous Dynamical Sysstems Series B **25** (2020), no. 1, 161–190.

[17] M. Kéré, M. Mercan, G. Mophou, *Control of Stackelberg for a coupled parabolic equations*, Journal of Dynamical and Control Systems **23** (2017), 709–733.

[18] O. A. Ladyzenskaja, V. A. Solonnikov, N. N. Ural'ceva , *Linear and quasilinear equations of parabolic type*, American Mathematical Society, Rhode Island, 1968.

[19] J. L. Lions, *Optimal control of systems governed by partial differential equations*, Springer, New York, 1971.

[20] J. L. Lions, *Some remarks on Stackelberg's optimization*, Mathematical Model and Methods in Applied Sciences **4** (1994), no. 4, 477–487.

[21] J. Macki, A. Strauss, *Introduction to optimal control theory*, Springer-Verlag, 1995.

[22] M. Mercan, *Optimal control for distributed linear systems subjected to null-controllability*, Applicable Analysis **92** (2013), no. 9, 1928–1943.

[23] M. Mercan, *Optimal control for distributed linear systems subjected to null controllability with constraints on the state*, Advances in interdisciplinary mathematical research: Applications to engineering, physical and life sciences (B. Toni, ed.), Springer Proc. Math. Stat., vol. 37, Springer, New York, 2013, pp. 213–232.

[24] M. Mercan, O. Nakoulima, *Control of Stackelberg for a two stroke problem*, Dynamics of continuous, Discrete and Impulsive Systems: Applications & Algorithms **22** (2015), 441–463.

[25] C. Montoya, L. de Teresa, *Robust Stackelberg controllability for the Navier–Stokes equations*, Nonlinear Differential Equations and Applications **25** (2018), no. 46, https://doi.org/10.1007/s00030-018-0537-3.

[26] G. Mophou, M. Kéré, L. L. Djomegne Njoukoué, *Robust hierarchic control for a population dynamics model with missing birth rate*, Mathematics of Control, Signals, and Systems **32** (2020), 209–239.

[27] O. Nakoulima, *Optimal control for distributed systems subject to null controllability. Application to discriminating sentinels*, ESAIM: Control, Optimisation and Calculus of Variations **13** (2007), no. 4, 623–638.

[28] H. von Stackelberg, *Marktform und Gleichgewicht*, Verlag von Julius Springer, Wien und Berlin, 1934.

[29] F. Tröltzsch, *Optimal control of partial differential equations*: *theory, methods and applications*, American Mathematical Society, Providence, Rhode Island, 2005.

[30] E. Zuazua, *Approximate controllability for the semilinear heat equation with globally Lipschitz nonlinearities*, Control and Cybernetics **28** (1999), 665–683.