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ON EXISTENCE AND UNIQUENESS OF INTEGRAL SOLUTIONS FOR A CLASS OF NONDENSELY DEFINED MIXED VOLTERRA-FREDHOLM INTEGRO FRACTIONAL NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, a class of fractional integro-differential equations of mixed type is considered and some sufficient conditions in connection with the existence of integral solutions are established with the help of the fixed point theorems due to Banach, Krasnoselskii and Darbo-Sadovskii, and also the fixed point theorem for condensing maps. Two of the results are verified by considering appropriate examples.

Keywords: Volterra-Fredholm integro-differential equation, Hille-Yosida condition, integral solution, fixed-point theorem, measure of noncompactness.

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1 Introduction

Fractional differential equations, of late, have surfaced as an immensely significant topic of investigation due to their growing number of applications in various areas of applied science and engineering (for details, see [11, 13, 14, 22]). It has come to the notice that the fractional order differential equations represent better and precise realistic scenario for describing a large number of physical phenomena in comparison with those differential equations expressed through usual integer order derivatives. There are various ways of interpolating the definition of integer order to non-integer order. Among them, the most widely known ones are Riemann-Liouville and Caputo derivatives. The main advantage of Caputo's definition in comparison to the Riemann-Liouville definition is that it allows consideration of easily interpreted initial conditions in terms of the unknown x or its derivatives such as $x(0) = x_0$, $x'(0) = x_1$, etc. (see [6]).

There are several works available in literature which have investigated the existence of mild solutions of different types of fractional differential equations. However, in many of the works such as [12, 19], the mild solution of a fractional evolution differential equation was defined by generalizing the mild solution definition of integer order evolution equations, which was not considered appropriate by Lin and Jia [15]. Zhou and his co-researchers [27, 30] came out with an appropriate concept of mild solution for fractional evolution differential equations of order $\alpha \in (0, 1)$ by using Laplace transform and probability density function $M_\alpha(\theta)$, which was defined only for $\alpha \in (0, 1)$. Subsequently, a reasonable number of researchers have used this approach to study the existence of mild solutions of fractional evolution equations of order $\alpha \in (0, 1)$. For more information on mild solutions of fractional differential equations of order $\alpha \in (1, 2)$, the readers are referred to the works in a number of articles such as [16, 17, 24].

The following fractional evolution equation has been extensively studied when the operator A is densely defined:

$$\begin{aligned} {}^C D^{\alpha}_{0+} x(t) &= Ax(t) + f(t, x(t)), \quad t \in [0, b], \ \alpha \in (0, 1), \\ x(0) &= x_0. \end{aligned}$$

Da Prato and Sinestrari [4] initiated an investigation of initial value problems with a non-dense domain wherein they introduced the concept of integral solutions of the following abstract Cauchy problem:

$$\begin{aligned} x'(t) &= Ax(t) + f(t), \quad t \in [0, b], \\ x(0) &= x_0. \end{aligned}$$

For more details, the readers are referred to some works such as [7, 8, 20]. Gu *et al.* [10] took up the following fractional semilinear equation with Caputo derivative:

$$\begin{aligned} {}^C D^{\alpha}_{0+} x(t) &= Ax(t) + f(t, x(t)), \quad t \in (0, b], \ \alpha \in (0, 1), \\ x(0) &= x_0, \end{aligned}$$

and studied the existence of integral solutions by using the measure of noncompactness.

Fu [9] studied the existence of solutions for the following semilinear neutral functional differential equations with nonlocal conditions on a general Banach space X:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[x(t) - F(t, x(h_1(t)))] &= A[x(t) - F(t, x(h_1(t)))] + G(t, x(h_2(t))), \quad 0 \le t \le a, \\ x(0) + g(x) &= x_0 \in X. \end{aligned}$$

Motivated by the works carried out by Gu *et al.* [10] and Fu [9] as mentioned above, the main objective of our present work is to study the following neutral fractional integro-differential equation of mixed type for $t \in [0, b]$:

$$\left.\begin{aligned} {}^{C}D^{\alpha}_{0+}[x(t) - u(t, x(t))] &= A[x(t) - u(t, x(t))] + f(t, x(t), (Hx)(t), (Gx)(t)),\\ x(0) &= x_0, \end{aligned}\right\} \tag{1.1}$$

where

$$(Hx)(t) = \int_0^t h(t, s, x(s))\,\mathrm{d}s \quad \text{and} \quad (Gx)(t) = \int_0^b g(t, s, x(s))\,\mathrm{d}s,$$

with $\alpha \in (0, 1)$, $J = [0, b]$ and $A : D(A) \subseteq X \to X$ as a closed linear operator on a Banach space X, which is not necessarily densely defined. The state $x(\cdot)$ assumes values in X, and $u : J \times X \to X$ is a function which satisfies some assumptions to be specified later. The functions $h : \Delta \times X \to X$, $g : J \times J \times X \to X$ and $f : J \times X \times X \times X \to X$ are given abstract functions where $\Delta = \{(t, s) \in J \times J \mid s \le t\}$.

It has been observed that establishing the compactness of the solution operator is not trivial [28]. In this work, in establishing the existence results, we use Arzelà-Ascoli theorem to show the relative compactness of the solution operator. For additional information, we refer the readers to the works in [25, 26, 28].

The present article is presented as follows: In Section 2, some definitions, theorems and lemmas relevant to our work are recalled. Section 3 presents the hypotheses, statements and derivations of the existence theorems for the integral solution of our defined problem. Section 4 presents two examples which verify two theorems. At the end, the results are precisely summarized in Section 5.

2 Preliminaries

Throughout this article, we use the following notations:

(i) X: a Banach space with norm $\|\cdot\|_X$,
(ii) $C(J, X)$: the Banach space of all continuous functions from J to X with the norm

$$\|x\|_C = \sup_{t \in J} \|x(t)\|_X,$$

(iii) $B(X)$: the Banach space of all bounded linear operators on X,
(iv) $L^p(J, X)$, $1 \le p \le \infty$: the Banach space of all measurable functions $f : J \to X$ with the following norm:

$$\|f\|_{L^p} = \begin{cases} \left(\int_J \|f\|_X^p \,\mathrm{d}t\right)^{\frac{1}{p}}, & 1 \le p < \infty,\\ \inf\limits_{\mu(\bar{J})=0} \left\{\sup\limits_{t \in J \setminus \bar{J}} \|f(t)\|_X\right\}, & p = \infty. \end{cases}$$

Theorem 2.1 *(Hölder's inequality) [28] Consider $p, q \ge 1$ with $1/p + 1/q = 1$. Then, for $f \in L^p(J, X)$ and $g \in L^q(J, X)$, $1 \le p \le \infty$,*

$$fg \in L^1(J, X) \quad \textit{and} \quad \|fg\|_{L^1} \le \|f\|_{L^p} \|g\|_{L^q}.$$

Theorem 2.2 *(Bochner's theorem) [29] A measurable function $f : J \to X$ is said to be Bochner integrable if $\|f\|$ is Lebesgue integrable.*

Definition 2.1 *[29] The following integral defines the Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f \in L^1(J)$ with lower limit 0:*

$$I_{0^+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad t > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 *[29] The following integral defines the Caputo derivative of order α of a function $f \in L^1(J)$ with lower limit 0:*

$${}^{C}D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,\mathrm{d}s = I_{0^+}^{n-\alpha} f^{(n)}(t), \quad t > 0,$$

where $n = \lceil \alpha \rceil$ denotes the least integer greater than or equal to α.

For an abstract function f with values in X, the integrals appearing in Definitions 2.1 and 2.2 are taken in Bochner's sense.

Definition 2.3 *[2] Let X be a Banach space and Ω_X be the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_X \to [0, \infty)$ defined for $B \in \Omega_X$ by*

$$\alpha(B) = \inf\{\epsilon > 0 | B \subseteq \cup_{i=1}^{n} B_i \textit{ and } diam(B_i) \leq \epsilon \textit{ for } i = 1, 2, \ldots, n\},$$

where $diam(B_i) = \sup\{\|x - y\| : x, y \in B_i\}$.

Lemma 2.1 *[2] The Kuratowski measure of noncompactness satisfies the following properties:*
(i) *$\alpha(B) = 0$ iff B is relatively compact set,*
(ii) *$B_1 \subset B_2 \implies \alpha(B_1) \leq \alpha(B_2)$,*
(iii) *$\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2)$.*

Lemma 2.2 *[3] Let X be a Banach space, and let $D \subset X$ be bounded. Then there exists a countable set $D_0 \subset D$ such that*

$$\alpha(D) \leq 2\alpha(D_0).$$

Lemma 2.3 *[1] Let X be a Banach space, and let $D \subset C(J, X)$ be equicontinuous and bounded. Then $\alpha(D(t))$ is continuous on J and*

$$\alpha_c(D) = \max_{t \in J} \alpha(D(t)).$$

Lemma 2.4 *[18] Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of Bochner integrable functions from J into X with*

$$\|x_n(t)\| \leq m(t) \textit{ for almost all } t \in J \textit{ and every } n \in \mathbb{N},$$

where $m \in L(J, \mathbb{R}^+)$. Then the function $\phi(t) = \alpha\Big(\{x_n\}_{n=1}^{\infty}\Big) \in L(J, \mathbb{R}^+)$ and it satisfies

$$\alpha\Big(\{\int_0^t x_n(s)\,\mathrm{d}s \Big| n \in \mathbb{N}\}\Big) \leq 2 \int_0^t \phi(s)\,\mathrm{d}s.$$

Theorem 2.3 *[5] Let X be a Banach space and $S \subset X$ be a bounded, closed and convex set in X and $Q : S \to S$ be a condensing map which means that $\alpha(Q(S)) \leq \alpha(S)$. Then Q has a fixed point in S.*

Theorem 2.4 *(Krasnoselskii's fixed point theorem) [23] Let X be a Banach space, and S be a bounded, closed and convex subset of X. Let P and Q map S into X such that*
(i) *$Px + Qy \in S$ whenever $x, y \in S$,*
(ii) *P is a contraction mapping,*
(iii) *Q is compact and continuous.*
Then there exists $z \in S$ such that $Pz + Qz = z$.

Theorem 2.5 *(Darbo-Sadovskii's fixed point theorem) [28] If S is a bounded, closed and convex subset of a Banach space X and the continuous mapping $Q : S \to S$ is an α-contraction, then the mapping Q has at least one fixed point in S.*

Throughout this article, it is assumed that the operator $A : D(A) \subset X \to X$ satisfies the Hille-Yosida condition, *i.e.*, there exist $\overline{M} \geq 0$ and a constant $w \in \mathbb{R}$ such that $(w, \infty) \subseteq \rho(A)$ and

$$\sup\left\{(\lambda - w)^n \|R(\lambda : A)^n\|_{B(X)} \middle| n \in \mathbb{N}, \lambda > w\right\} \leq \overline{M},$$

where $\rho(A)$ is the resolvent set of A, and $R(\lambda : A)$ denotes the resolvent of A.

Let A_0 be the part of A in $\overline{D(A)}$ defined by

$$\begin{aligned} D(A_0) &= \left\{x \in D(A) \middle| Ax \in \overline{D(A)}\right\}, \\ A_0 x &= Ax. \end{aligned}$$

Then A_0 generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\overline{D(A)}$. Assume that $\{T(t)\}_{t\geq 0}$ is uniformly bounded, *i.e.*, there exists $M > 1$ such that

$$\sup_{t\in[0,\infty)} \|T(t)\|_{B(X)} < M.$$

Problem (1.1) is now equivalent to the following integral equation:

$$\begin{aligned} x(t) &= x_0 - u(0, x(0)) + u(t, x(t)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[A\big(x(s) - u(s, x(s))\big) + f(s, x(s), (Hx)(s), (Gx)(s))\Big] \, \mathrm{d}s, \quad t \in [0, b]. \end{aligned}$$

Based on the information available in [10], we present the following definition and results: Assuming f to be continuous and $x_0 \in \overline{D(A)}$, the integral solution of problem (1.1) is defined as follows.

Definition 2.4 *A function $x : J \to X$ is said to be an integral solution of* (1.1) *if*

$$x \in C(J, X), \quad I_{0+}^{\alpha}[x(t) - u(t, x(t))] \in D(A) \ \ for\ t \in [0, b],$$

and

$$
\begin{aligned}
x(t) = x_0 - u(0, x(0)) + u(t, x(t)) + \frac{A}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [x(s) - u(s, x(s))] \, \mathrm{d}s \\
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), (Hx)(s), (Gx)(s)) \, \mathrm{d}s, \quad t \in [0, b].
\end{aligned}
$$

Remark 2.1 *If x is an integral solution of problem* (1.1)*, it can be shown that $x(t) - u(t, x(t)) \in \overline{D(A)}$ for $t \in J$.*

Now, we consider the following auxiliary problem:

$$
\left.
\begin{aligned}
{}^C D_{0^+}^{\alpha} [x(t) - u(t, x(t))] = A_0 [x(t) - u(t, x(t))] + f(t, x(t), (Hx)(t), (Gx)(t)), \ t \in [0, b], \\
x(0) = x_0.
\end{aligned}
\right\} \tag{2.1}
$$

The integral solution of (2.1) takes the following form:

$$
\begin{aligned}
x(t) = x_0 - u(0, x(0)) + u(t, x(t)) + A_0 I_{0^+}^{\alpha} [x(t) - u(t, x(t))] \\
+ I_{0^+}^{\alpha} f(t, x(t), (Hx)(t), (Gx)(t)).
\end{aligned} \tag{2.2}
$$

Let $B_\lambda = \lambda(\lambda I - A)^{-1}$. Then, since $B_\lambda x \to x$ as $\lambda \to +\infty$ for $x \in \overline{D(A)}$, we have the following lemma:

Lemma 2.5 *[10] The integral solution of* (2.2) *can be written in the following form:*

$$
\begin{aligned}
x(t) = u(t, x(t)) + S_\alpha(t) [x_0 - u(0, x(0))] \\
+ \lim_{\lambda \to \infty} \int_0^t K_\alpha(t-s) \, B_\lambda f(s, x(s), (Hx)(s), (Gx)(s)) \, \mathrm{d}s,
\end{aligned}
$$

where

$$
\begin{aligned}
S_\alpha(t) = I_{0^+}^{1-\alpha} K_\alpha(t), \ K_\alpha(t) = t^{\alpha-1} P_\alpha(t), \\
P_\alpha(t) = \int_0^\infty \alpha \, \theta \, M_\alpha(\theta) T(t^\alpha \theta) \, \mathrm{d}\theta.
\end{aligned}
$$

Here

$$
\begin{aligned}
M_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \psi_\alpha(\theta^{-\frac{1}{\alpha}}) \ \textit{and} \\
\psi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^{n-1} \theta^{(-\alpha n - 1)} \frac{\Gamma(\alpha n + 1)}{n!} \sin(n\pi\alpha), \ \theta \in (0, \infty).
\end{aligned}
$$

$M_\alpha(\theta)$ is a probability density function on $(0, \infty)$ satisfying

$$
M_\alpha(\theta) \geq 0, \ \int_0^\infty M_\alpha(\theta) \, \mathrm{d}\theta = 1, \ \int_0^\infty \theta \, M_\alpha(\theta) \, \mathrm{d}\theta = \frac{1}{\Gamma(1+\alpha)}.
$$

Lemma 2.6 *[10] For any fixed $t > 0$, $\{K_\alpha(t)\}_{t>0}$ and $\{S_\alpha(t)\}_{t>0}$ are linear operators, and for any $x \in \overline{D(A)}$,*

$$
\|K_\alpha(t)x\|_X \leq \frac{Mt^{\alpha-1}}{\Gamma(\alpha)} \|x\|_X \ \textit{and} \ \|S_\alpha(t)x\|_X \leq M\|x\|_X.
$$

Lemma 2.7 *[10] $\{K_\alpha(t)\}_{t>0}$ and $\{S_\alpha(t)\}_{t>0}$ are strongly continuous, i.e., for any $x \in \overline{D(A)}$ and $0 < t_1 < t_2 \leq b$,*

$$\|K_\alpha(t_2)x - K_\alpha(t_1)x\|_X \to 0 \ \ \text{and} \ \ \|S_\alpha(t_2)x - S_\alpha(t_1)x\|_X \to 0,$$

as $t_2 \to t_1$.

Lemma 2.8 *[29] For any fixed $t > 0$, $P_\alpha(t)$ is a linear and bounded operator, and*

$$\|P_\alpha(t)x\|_X \leq \frac{M}{\Gamma(\alpha)}\|x\|_X \ \ \text{for any } x \in \overline{D(A)}.$$

Lemma 2.9 *[21] Assume that $\{T(t)\}_{t>0}$ is compact. Then $T(t)$ is continuous in the uniform operator topology for $t > 0$, i.e., $\{T(t)\}_{t>0}$ is equicontinuous.*

3 Main Results

Our first result is based on Banach fixed point theorem. Here we use the following assumptions:
(H1)(i) there exist a constant $\alpha_1 \in (0, \alpha)$ and functions $l_1, l_2, l_3 \in L^{\frac{1}{\alpha_1}}(J, \mathbb{R}^+)$ such that

$$\|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)\|_X \leq l_1(t)\|x_1 - x_2\|_X + l_2(t)\|y_1 - y_2\|_X + l_3(t)\|z_1 - z_2\|_X,$$

for all $x_i, y_i, z_i \in X$, $i = 1, 2$ and $t \in J$.
(ii) there exist a constant $\alpha_2 \in (0, \alpha)$ and a function $l \in L^{\frac{1}{\alpha_2}}(J, \mathbb{R}^+)$ such that

$$\|f(t, x, y, z)\|_X \leq l(t), \ \ \text{for all } x, y, z \in X \ \text{and} \ \ t \in J.$$

(H2) for the function $u : J \times X \to X$, there exists a constant $L_1 > 0$ such that

$$\|u(t, x_1) - u(t, x_2)\|_X \leq L_1\|x_1 - x_2\|_X, \ \ \forall\, t \in J \text{ and } x_1, x_2 \in X.$$

(H3) there exist constants $L_2, L_3 > 0$ such that

$$\Big\| \int_0^t [h(t, s, x) - h(t, s, y)]\, \mathrm{d}s \Big\|_X \leq L_2\|x - y\|_X,$$
$$\Big\| \int_0^b [g(t, s, x) - g(t, s, y)]\, \mathrm{d}s \Big\|_X \leq L_3\|x - y\|_X,$$

for all $x, y \in X$.

Theorem 3.1 *Assume that hypotheses* **(H1)–(H3)** *hold. Then there exists a unique integral solution of* (1.1) *in $C(J, \overline{D(A)})$ provided*

$$\xi_1 = (M + 1)L_1 + \frac{M\overline{M}}{\Gamma(\alpha)}\Big\{\|l_1\|_{L^{\frac{1}{\alpha_1}}} + L_2\|l_2\|_{L^{\frac{1}{\alpha_1}}} + L_3\|l_3\|_{L^{\frac{1}{\alpha_1}}}\Big\}\frac{b^{\alpha-\alpha_1}}{\left(\frac{\alpha-\alpha_1}{1-\alpha_1}\right)^{1-\alpha_1}} < 1.$$

Proof. Define $Q : C(J, \overline{D(A)}) \to C(J, \overline{D(A)})$ by

$$Qx(t) = u(t, x(t)) + S_\alpha(t)[x_0 - u(0, x(0))] + \lim_{\lambda\to\infty} \int_0^t K_\alpha(t-s)B_\lambda f(s, x(s), (Hx)(s), (Gx)(s))\,\mathrm{d}s, \quad t \in J.$$

Then with the help of **(H1)(ii)**, Hölder's inequality and Bochner's theorem, it can be shown that Q is well-defined on $C(J, \overline{D(A)})$.

Let $x, y \in C(J, \overline{D(A)})$. Then for any $t \in [0, b]$, we have

$$\begin{aligned}
&\|(Qx)(t) - (Qy)(t)\|_X \\
&\le \|u(t, x(t)) - u(t, y(t))\|_X + \|S_\alpha(t)[u(0, x(0)) - u(0, y(0))]\|_X \\
&\quad + \|\lim_{\lambda\to\infty} \int_0^t K_\alpha(t-s)B_\lambda [f(s, x(s), (Hx)(s), (Gx)(s)) - f(s, y(s), (Hy)(s), (Gy)(s))]\,\mathrm{d}s\|_X \\
&\le L_1(1 + M)\|x - y\|_C \\
&\quad + \frac{M\overline{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s), (Hx)(s), (Gx)(s)) - f(s, y(s), (Hy)(s), (Gy)(s))]\,\mathrm{d}s\|_X \\
&\le L_1(1 + M)\|x - y\|_C \\
&\quad + \frac{M\overline{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[l_1(s)\|x(s) - y(s)\|_X + l_2(s)\|(Hx)(s) - (Hy)(s)\|_X \\
&\quad + l_3(s)\|(Gx)(s) - (Gy)(s)\|_X \Big]\,\mathrm{d}s \\
&\le \xi_1\|x - y\|_C.
\end{aligned}$$

Therefore, by using Banach fixed point theorem, we conclude that there exists a unique integral solution of our problem in $C(J, \overline{D(A)})$. This completes the proof of the theorem. □

In order to establish the next result, we introduce the following additional assumptions:

(H4) $T(t), t > 0$ is compact.

(H5)(i) for each $t \in [0, b]$, the function $f(t, \cdot, \cdot, \cdot) : X \times X \times X \to X$ is continuous and for each $(x, y, z) \in X \times X \times X$, the function $f(\cdot, x, y, z) : J \to X$ is strongly measurable.

(ii) there exists a function $l \in L(J, \mathbb{R}^+)$ such that

$$I^\alpha_{0^+} l \in C(J, \mathbb{R}^+), \quad \lim_{t\to0^+} I^\alpha_{0^+} l(t) = 0$$

and

$$\|f(t, x, y, z)\|_X \le l(t) \text{ for all } x, y, z \in X \text{ and for } t \in [0, b].$$

(H6) $u : J \times X \to X$ is continuous and there exists $L_1 > 0$ such that

$$\|u(t, x_1) - u(t, x_2)\|_X \le L_1\|x_1 - x_2\|_X, \text{ for each } t \in [0, b] \text{ and all } x_1, x_2 \in X,$$

and let $M_0 = \sup_{t\in J} \|u(t, 0)\|_X$.

(H7)(i) for each $(t, s) \in \Delta$, the function $h(t, s, \cdot) : X \to X$ is continuous, and for each $x \in X$, the function $h(\cdot, \cdot, x) : \Delta \to X$ is strongly measurable.

(ii) there exists a function $m_h(t, s) \in C(\Delta, \mathbb{R}^+)$ such that

$$\|h(t, s, x)\|_X \le m_h(t, s)\|x\|_X, \text{ for } (t, s) \in \Delta, \ x \in X$$

and $H^* = \sup_{t\in J} \int_0^t m_h(t,s)\,\mathrm{d}s < \infty$.

(H8)(i) for each $(t,s) \in J \times J$, the function $g(t,s,\cdot) : X \to X$ is continuous, and for each $x \in X$, the function $g(\cdot,\cdot,x) : J \times J \to X$ is strongly measurable.

(ii) There exists a function $m_g(t,s) \in C(J \times J, \mathbb{R}^+)$ such that

$$\|g(t,s,x)\|_X \leq m_g(t,s)\|x\|_X, \text{ for } (t,s) \in J \times J,\ x \in X$$

and $G^* = \sup_{t\in J} \int_0^b m_g(t,s)\,\mathrm{d}s < \infty$.

For our second existence result, we use Krasnoselskii's fixed point theorem.

Theorem 3.2 *Assume that hypotheses* **(H4)–(H8)** *hold. Then* (1.1) *has an integral solution in* $C(J, \overline{D(A)})$ *provided*

$$(M+1)L_1 \leq \frac{1}{2}.$$

Proof. Choose $r \geq 2\xi_2$ where $\xi_2 = M_0 + M\|x_0\|_X + M\|u(0,x(0))\|_X + \frac{M\overline{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}l(s)\,\mathrm{d}s$. Define two operators P and Q on $B_r = \{x \in C(J, \overline{D(A)}) : \|x\|_C \leq r\}$ by

$$(Px)(t) = u(t,x(t)) + S_\alpha(t)[x_0 - u(0,x(0))],$$
$$(Qx)(t) = \lim_{\lambda\to\infty} \int_0^t K_\alpha(t-s)B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))\,\mathrm{d}s.$$

***Step I*:** To show that $Px + Qy \in B_r$ whenever $x, y \in B_r$.

For any $t \in [0,b]$, we have

$$\begin{aligned}\|(Px)(t)\|_X &\leq \|u(t,x(t)) - u(t,0)\|_X + M_0 + M\|x_0 - u(0,x(0))\|_X \\ &\leq L_1\|x\|_C + M_0 + M\|x_0\|_X + M\|u(0,x(0))\|_X\end{aligned}$$

and

$$\begin{aligned}\|(Qy)(t)\|_X &\leq \frac{M\overline{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s),(Hx)(s),(Gx)(s))\|_X\,\mathrm{d}s \\ &\leq \frac{M\overline{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}l(s)\,\mathrm{d}s.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|(Px)(t) + (Qy)(t)\|_X \\ &\leq L_1 r + M_0 + M\|x_0\|_X + M\|u(0,x(0))\|_X + \frac{M\overline{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}l(s)\,\mathrm{d}s \\ &\leq r,\end{aligned}$$

which implies that $Px + Qy \in B_r$.

***Step II*:** To show that P is a contraction.

It can be easily shown that P is a contraction.

***Step III*:** To show that Q is compact and continuous. We show it in several steps.

(i) To show that $\{Qx \mid x \in B_r\}$ **is equicontinuous:**

Let $x \in B_r$ and $0 = t_1 < t_2 \leq b$. Then

$$\begin{aligned}\|(Qx)(t_2) - (Qx)(t_1)\|_X &\leq \frac{M\overline{M}}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} \|f(s, x(s), (Hx)(s), (Gx)(s))\|_X \, \mathrm{d}s \\ &\leq M\overline{M} I_{0^+}^{\alpha} l(t_2) \longrightarrow 0 \text{ as } t_2 \to 0.\end{aligned}$$

For $0 < t_1 < t_2 \leq b$, we have

$$\begin{aligned}&\|(Qx)(t_2) - (Qx)(t_1)\|_X \\ &\leq \frac{M\overline{M}}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} l(s) \, \mathrm{d}s + \frac{M\overline{M}}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] l(s) \, \mathrm{d}s \\ &\quad + \overline{M} \int_0^{t_1} (t_1 - s)^{\alpha-1} \|P_\alpha(t_2 - s) - P_\alpha(t_1 - s)\|_{B(X)} l(s) \, \mathrm{d}s \\ &\leq \frac{M\overline{M}}{\Gamma(\alpha)} \Big| \int_0^{t_2} (t_2 - s)^{\alpha-1} l(s) \, \mathrm{d}s - \int_0^{t_1} (t_1 - s)^{\alpha-1} l(s) \, \mathrm{d}s \Big| \\ &\quad + \frac{2M\overline{M}}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] l(s) \, \mathrm{d}s \\ &\quad + \overline{M} \int_0^{t_1} (t_1 - s)^{\alpha-1} \|P_\alpha(t_2 - s) - P_\alpha(t_1 - s)\|_{B(X)} l(s) \, \mathrm{d}s \\ &=: I_1 + I_2 + I_3,\end{aligned}$$

where

$$\begin{aligned}I_1 &= \frac{M\overline{M}}{\Gamma(\alpha)} \Big| \int_0^{t_2} (t_2 - s)^{\alpha-1} l(s) \, \mathrm{d}s - \int_0^{t_1} (t_1 - s)^{\alpha-1} l(s) \, \mathrm{d}s \Big|, \\ I_2 &= \frac{2M\overline{M}}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] l(s) \, \mathrm{d}s, \\ I_3 &= \overline{M} \int_0^{t_1} (t_1 - s)^{\alpha-1} \|P_\alpha(t_2 - s) - P_\alpha(t_1 - s)\|_{B(X)} l(s) \, \mathrm{d}s.\end{aligned}$$

Since $I_{0^+}^{\alpha} l \in C(J, \mathbb{R}^+)$, therefore it follows that $I_1 \to 0$ as $t_2 \to t_1$.

For I_2, we have

$$[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] l(s) \leq (t_1 - s)^{\alpha-1} l(s)$$

and $\int_0^{t_1} (t_1 - s)^{\alpha-1} l(s) \, \mathrm{d}s$ exists. Therefore, by using Lebesgue's dominated convergence theorem, we can conclude that $I_2 \to 0$ as $t_2 \to t_1$.

Now,

$$I_3 = \overline{M} \int_0^{t_1} (t_1 - s)^{\alpha-1} \|P_\alpha(t_2 - s) - P_\alpha(t_1 - s)\|_{B(X)} l(s) \, \mathrm{d}s.$$

For $\epsilon > 0$ small enough, we have

$$\begin{aligned}
I_3 &= \overline{M}\int_0^{t_1-\epsilon}(t_1-s)^{\alpha-1}\|P_\alpha(t_2-s)-P_\alpha(t_1-s)\|_{B(X)}l(s)\,\mathrm{d}s\\
&\quad+\overline{M}\int_{t_1-\epsilon}^{t_1}(t_1-s)^{\alpha-1}\|P_\alpha(t_2-s)-P_\alpha(t_1-s)\|_{B(X)}l(s)\,\mathrm{d}s\\
&\leq \overline{M}\int_0^{t_1}(t_1-s)^{\alpha-1}l(s)\,\mathrm{d}s\sup_{s\in[0,t_1-\epsilon]}\|P_\alpha(t_2-s)-P_\alpha(t_1-s)\|_{B(X)}\\
&\quad+\frac{2M\overline{M}}{\Gamma(\alpha)}\Big|\int_0^{t_1}(t_1-s)^{\alpha-1}l(s)\,\mathrm{d}s-\int_0^{t_1-\epsilon}(t_1-\epsilon-s)^{\alpha-1}l(s)\,\mathrm{d}s\Big|\\
&\quad+\frac{2M\overline{M}}{\Gamma(\alpha)}\int_0^{t_1-\epsilon}[(t_1-\epsilon-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]l(s)\,\mathrm{d}s\\
&= I_{31}+I_{32}+I_{33}.
\end{aligned}$$

From **(H4)**, it follows that $I_{31}\to 0$ as $t_2\to t_1$. It can also be shown that both I_{32} and I_{33} tend to zero as $\epsilon\to 0$ (applying similar arguments as shown for $I_1\to 0$ and $I_2\to 0$ as $t_2\to t_1$). Thus, $\|(Qx)(t_2)-(Qx)(t_1)\|_X\to 0$ independent of $x\in B_r$ as $t_2\to t_1$, which implies that $\{Qx|x\in B_r\}$ is equicontinuous.

(ii) To show that Q **is continuous:**

Let $\{x_n\}_{n=1}^{\infty}\subset B_r$ such that $x_n\to x(n\to\infty)$ in B_r. Using **(H5)(i)**, **(H7)**, **(H8)** and Lebesgue's dominated convergence theorem, it follows that

$$f(s,x_n(s),(Hx_n)(s),(Gx_n)(s))\longrightarrow f(s,x(s),(Hx)(s),(Gx)(s)) \text{ as } n\to\infty.$$

Now for each $t\in J$, we have

$$(t-s)^{\alpha-1}\|f(s,x_n(s),(Hx_n)(s),(Gx_n)(s))-f(s,x(s),(Hx)(s),(Gx)(s))\|_X\leq 2(t-s)^{\alpha-1}l(s)$$

and $2(t-s)^{\alpha-1}l(s)$ is integrable for $s\in[0,t]$ and $t\in[0,b]$. Therefore, by Lebesgue's dominated convergence theorem, as $n\to\infty$, we obtain

$$\int_0^t(t-s)^{\alpha-1}\|f(s,x_n(s),(Hx_n)(s),(Gx_n)(s))-f(s,x(s),(Hx)(s),(Gx)(s))\|_X\,\mathrm{d}s\longrightarrow 0.$$

For $t\in[0,b]$,

$$\begin{aligned}
&\|(Qx_n)(t)-(Qx)(t)\|_X\\
&\leq\frac{M\overline{M}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|f(s,x_n(s),(Hx_n)(s),(Gx_n)(s))-f(s,x(s),(Hx)(s),(Gx)(s))\|_X\,\mathrm{d}s\\
&\longrightarrow 0 \text{ as } n\to\infty.
\end{aligned}$$

Therefore, Q is continuous.

(iii) To show that Q **is uniformly bounded:**

From the inequality

$$\|(Qy)(t)\|_X\leq\frac{M\overline{M}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}l(s)\,\mathrm{d}s,$$

it follows that Q is uniformly bounded.

To show that Q **is compact:**

Here we use Arzelà-Ascoli theorem. We are required to show that for any $t \in [0,b]$, $\{(Qx)(t)|x \in B_r\}$ is relatively compact in X. For $t = 0$, it is obviously true. Therefore, by fixing $t \in (0,b]$, and for $\epsilon \in (0,t)$, $\delta > 0$ and $x \in B_r$, we define an operator $Q_{\epsilon,\delta}$ by

$$\begin{aligned}
&(Q_{\epsilon,\delta}x)(t)\\
&= \lim_{\lambda\to\infty} \alpha \int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}M_\alpha(\theta)T((t-s)^\alpha\theta)B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))\ \mathrm{d}\theta\mathrm{d}s\\
&= \alpha T(\epsilon^\alpha\delta)\lim_{\lambda\to\infty}\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}M_\alpha(\theta)T((t-s)^\alpha\theta-\epsilon^\alpha\delta)B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))\ \mathrm{d}\theta\mathrm{d}s.
\end{aligned}$$

From the compactness of $T(\epsilon^\alpha\delta)$, $(\epsilon^\alpha\delta > 0)$, we can conclude that $\{(Q_{\epsilon,\delta}x)(t)|x \in B_r\}$ is relatively compact in X for all $\epsilon \in (0,t)$ and all $\delta > 0$. Further, for any $x \in B_r$, we obtain

$$\begin{aligned}
&\|(Qx)(t) - (Q_{\epsilon,\delta}x)(t)\|_X\\
&\le \alpha M\overline{M}\int_0^t (t-s)^{\alpha-1}l(s)\,\mathrm{d}s\int_0^\delta \theta M_\alpha(\theta)\,\mathrm{d}\theta + \frac{M\overline{M}}{\Gamma(\alpha)}\int_{t-\epsilon}^t (t-s)^{\alpha-1}l(s)\,\mathrm{d}s\\
&\longrightarrow 0 \text{ as } \epsilon\to 0, \delta\to 0.
\end{aligned}$$

Therefore, $\{(Qx)(t)|x \in B_r\}$ is relatively compact in X for all $t \in (0,b]$. Consequently, by Arzelà-Ascoli theorem, Q is compact.

Now, Krasnoselskii's fixed point theorem implies that (1.1) has at least one integral solution on $C(J,\overline{D(A)})$. This completes the proof of the theorem. □

In order to establish our next result, we now assume the following additional hypotheses:
(H5)(iii) There exists a function $l \in L^\infty(J,\mathbb{R}^+)$ such that

$$\|f(t,x,y,z)\|_X \le l(t), \quad \text{for all } x,y,z \in X \text{ and for } t \in [0,b].$$

(H9) for the function $u : J\times X \to X$, there exists a constant $L_1 > 0$ such that

$$\|u(t_1,x_1) - u(t_2,x_2)\|_X \le L_1(|t_1-t_2| + \|x_1-x_2\|_X),$$

for all $t_1,t_2 \in [0,b]$ and all $x_1,x_2 \in X$. Further, let $M_0 = \sup_{t\in J}\|u(t,0)\|_X$.

The proof of the next result is based on Darbo-Sadovskii's fixed point theorem.

Theorem 3.3 *Suppose that* **(H4)**, **(H5)(i)**, **(H5)(iii)**, **(H7)–(H9)** *are satisfied. Then* (1.1) *has an integral solution in* $C(J,\overline{D(A)})$ *provided*

$$L_1 < 1.$$

Proof. Let $B_r = \{x \in C(J,\overline{D(A)}) : \|x\|_C \le r\}$ where $r = \frac{\xi_3}{1-L_1}$, $\xi_3 = M_0 + M\|x_0\|_X + M\|u(0,x(0))\|_X + \frac{M\overline{M}}{\Gamma(\alpha+1)}b^\alpha\|l\|_{L^\infty}$. Define $Q : C(J,\overline{D(A)}) \to C(J,\overline{D(A)})$ by

$$\begin{aligned}
(Qx)(t) &= u(t,x(t)) + S_\alpha(t)[x_0 - u(0,x(0))]\\
&\quad + \lim_{\lambda\to\infty}\int_0^t K_\alpha(t-s)B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))\,\mathrm{d}s\\
&= (Q_1x)(t) + (Q_2x)(t),
\end{aligned}$$

where

$$
\begin{aligned}
&(Q_1x)(t) = S_\alpha(t)[x_0 - u(0, x(0))] + \lim_{\lambda\to\infty} \int_0^t K_\alpha(t-s)B_\lambda f(s, x(s), (Hx)(s), (Gx)(s))\,\mathrm{d}s,\\
&(Q_2x)(t) = u(t, x(t)).
\end{aligned}
$$

***Step I*:** To show that $Q : B_r \to B_r$.

It follows in a straightforward manner from the fact that

$$
\begin{aligned}
\|(Qx)(t)\|_X &\le L_1\|x\|_C + M_0 + M\|x_0\|_X + M\|u(0, x(0))\|_X + \frac{M\overline{M}}{\Gamma(\alpha+1)} b^\alpha \|l\|_{L^\infty}\\
&\le r.
\end{aligned}
$$

***Step II*:** To show that Q_1 is completely continuous.

(i) Q_1 **is equicontinuous on** B_r**:**

Let $x \in B_r$ and $0 \le t_1 < t_2 \le b$. Then

$$
\begin{aligned}
\|(Q_1x)(t_2) - (Q_1x)(t_1)\|_X &\le \|S_\alpha(t_2)[x_0 - u(0, x(0))] - S_\alpha(t_1)[x_0 - u(0, x(0))]\|_X\\
&\quad + \|\lim_{\lambda\to\infty} \int_0^{t_2} K_\alpha(t_2-s)B_\lambda f(s, x(s), (Hx)(s), (Gx)(s))\,\mathrm{d}s\\
&\quad - \lim_{\lambda\to\infty} \int_0^{t_1} K_\alpha(t_1-s)B_\lambda f(s, x(s), (Hx)(s), (Gx)(s))\,\mathrm{d}s\|_X\\
&= I_1 + I_2,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \|S_\alpha(t_2)[x_0 - u(0, x(0))] - S_\alpha(t_1)[x_0 - u(0, x(0))]\|_X,\\
I_2 &= \|\lim_{\lambda\to\infty} \int_0^{t_2} K_\alpha(t_2-s)B_\lambda f(s, x(s), (Hx)(s), (Gx)(s))\,\mathrm{d}s\\
&\quad - \lim_{\lambda\to\infty} \int_0^{t_1} K_\alpha(t_1-s)B_\lambda f(s, x(s), (Hx)(s), (Gx)(s))\,\mathrm{d}s\|_X.
\end{aligned}
$$

For I_1, by Lemma 2.7, we have $I_1 \to 0$ as $t_2 \to t_1$. For $t_1 = 0$, $0 < t_2 \le b$,

$$
I_2 \le \frac{M\overline{M}}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} l(s)\,\mathrm{d}s \le \frac{M\overline{M}}{\Gamma(\alpha)} t_2^\alpha \|l\|_{L^\infty} \longrightarrow 0 \text{ as } t_2 \to 0.
$$

For $0 < t_1 < t_2 \le b$,

$$
\begin{aligned}
I_2 &\le \frac{M\overline{M}}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} l(s)\,\mathrm{d}s + \frac{M\overline{M}}{\Gamma(\alpha)} \int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}] l(s)\,\mathrm{d}s\\
&\quad + \overline{M} \int_0^{t_1} (t_1-s)^{\alpha-1} \|P_\alpha(t_2-s) - P_\alpha(t_1-s)\|_{B(X)} l(s)\,\mathrm{d}s\\
&= I_1^* + I_2^* + I_3^*,
\end{aligned}
$$

where

$$I_1^* = \frac{M\overline{M}}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} l(s)\,\mathrm{d}s,$$
$$I_2^* = \frac{M\overline{M}}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] l(s)\,\mathrm{d}s,$$
$$I_3^* = \overline{M} \int_0^{t_1} (t_1 - s)^{\alpha-1} \|P_\alpha(t_2 - s) - P_\alpha(t_1 - s)\|_{B(X)} l(s)\,\mathrm{d}s.$$

We have

$$I_1^* \le \frac{M\overline{M}}{\Gamma(\alpha)} \|l\|_{L^\infty} (t_2 - t_1)^\alpha, \text{ i.e., } I_1^* \longrightarrow 0 \text{ as } t_2 \to t_1$$

and

$$I_2^* \le \frac{M\overline{M}}{\Gamma(\alpha)} \|l\|_{L^\infty} \Big(\int_0^{t_1} (t_1 - s)^{\alpha-1}\,\mathrm{d}s - \int_0^{t_1} (t_2 - s)^{\alpha-1}\,\mathrm{d}s \Big) \le \frac{M\overline{M}}{\Gamma(\alpha)} \|l\|_{L^\infty} (t_2 - t_1)^\alpha,$$

i.e., $I_2^* \longrightarrow 0$ as $t_2 \to t_1$.

Now since

$$I_3^* = \overline{M} \int_0^{t_1} (t_1 - s)^{\alpha-1} \|P_\alpha(t_2 - s) - P_\alpha(t_1 - s)\|_{B(X)} l(s)\,\mathrm{d}s,$$

therefore, for $\epsilon > 0$ small enough, we have

$$\begin{aligned} I_3^* &\le \overline{M} \int_0^{t_1} (t_1 - s)^{\alpha-1} l(s)\,\mathrm{d}s \sup_{s\in[0,t_1-\epsilon]} \|P_\alpha(t_2 - s) - P_\alpha(t_1 - s)\|_{B(X)} \\ &\quad + \overline{M} \int_{t_1-\epsilon}^{t_1} (t_1 - s)^{\alpha-1} \|P_\alpha(t_2 - s) - P_\alpha(t_1 - s)\|_{B(X)} l(s)\,\mathrm{d}s \\ &\le \overline{M} \int_0^{t_1} (t_1 - s)^{\alpha-1} l(s)\,\mathrm{d}s \sup_{s\in[0,t_1-\epsilon]} \|P_\alpha(t_2 - s) - P_\alpha(t_1 - s)\|_{B(X)} \\ &\quad + \frac{2M\overline{M}}{\Gamma(\alpha)} \int_{t_1-\epsilon}^{t_1} (t_1 - s)^{\alpha-1} l(s)\,\mathrm{d}s \\ &\le \overline{M} \int_0^{t_1} (t_1 - s)^{\alpha-1} l(s)\,\mathrm{d}s \sup_{s\in[0,t_1-\epsilon]} \|P_\alpha(t_2 - s) - P_\alpha(t_1 - s)\|_{B(X)} + \frac{2M\overline{M}}{\Gamma(\alpha+1)} \|l\|_{L^\infty} \epsilon^\alpha \\ &= I_{31}^* + I_{32}^*. \end{aligned}$$

From **(H4)**, it follows that $I_{31}^* \to 0$ as $t_2 \to t_1$ and also $I_{32}^* \to 0$ as $\epsilon \to 0$. Thus, $\|(Q_1x)(t_2) - (Q_1x)(t_1)\|_X \to 0$ as $t_2 \to t_1$, independent of $x \in B_r$, which implies that $\{Q_1x | x \in B_r\}$ is equicontinuous.

(ii) Q_1 **is continuous on** B_r**:**

Let $\{x_n\}_{n=1}^\infty \subset B_r$ such that $x_n \to x(n \to \infty)$ in B_r. Using **(H5)(i)**, **(H7)**, **(H8)** and Lebesgue's dominated convergence theorem, it follows that

$$f(s, x_n(s), (Hx_n)(s), (Gx_n)(s)) \longrightarrow f(s, x(s), (Hx)(s), (Gx)(s)), \text{ as } n \to \infty.$$

Now for each $t \in J$, by using **(H5)(ii)**, we have

$$(t-s)^{\alpha-1}\|f(s,x_n(s),(Hx_n)(s),(Gx_n)(s)) - f(s,x(s),(Hx)(s),(Gx)(s))\|_X \\ \leq 2(t-s)^{\alpha-1}l(s) \in L^1(J,\mathbb{R}^+), \text{ for } s \in [0,t],\ t \in J.$$

Therefore, by Lebesgue's dominated convergence theorem, as $n \to \infty$, we obtain

$$\int_0^t (t-s)^{\alpha-1}\|f(s,x_n(s),(Hx_n)(s),(Gx_n)(s)) - f(s,x(s),(Hx)(s),(Gx)(s))\|_X \,\mathrm{d}s \longrightarrow 0.$$

Now, for each $t \in [0,b]$, we have

$$\begin{aligned}
&\|(Q_1x_n)(t) - (Q_1x)(t)\|_X \\
&\leq ML_1\|x_n - x\|_C \\
&\quad + \frac{M\overline{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x_n(s),(Hx_n)(s),(Gx_n)(s)) - f(s,x(s),(Hx)(s),(Gx)(s))\|_X \,\mathrm{d}s \\
&\longrightarrow 0 \text{ as } n \to \infty.
\end{aligned}$$

Therefore, Q_1 is continuous.

Further, it is obvious that Q_1 **is uniformly bounded**.

To show that **for any** $t \in J$**,** $\{Q_1x(t)|x \in B_r\}$ **is relatively compact in** X**:**

For $t = 0$, it is obvious. Therefore, we fix $t \in (0,b]$. Since

$$\begin{aligned}
S_\alpha(t)[x_0 - u(0,x(0))] &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}s^{\alpha-1}P_\alpha(s)[x_0 - u(0,x(0))]\,\mathrm{d}s \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}s^{\alpha-1}\int_0^\infty \alpha\theta M_\alpha(\theta)T(s^\alpha\theta)[x_0 - u(0,x(0))]\,\mathrm{d}\theta\mathrm{d}s,
\end{aligned}$$

then for $\epsilon \in (0,t)$ and $\delta > 0$, we define

$$\begin{aligned}
&(Q_1^{\epsilon,\delta}x)(t) \\
&= \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{t-\epsilon}\int_\delta^\infty \theta M_\alpha(\theta)(t-s)^{-\alpha}s^{\alpha-1}T(s^\alpha\theta)[x_0 - u(0,x(0))]\,\mathrm{d}\theta\mathrm{d}s \\
&\quad + \lim_{\lambda\to\infty}\alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta M_\alpha(\theta)(t-s)^{\alpha-1}T((t-s)^\alpha\theta)B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))\,\mathrm{d}\theta\mathrm{d}s \\
&= \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{t-\epsilon}\int_\delta^\infty \theta M_\alpha(\theta)(t-s)^{-\alpha}s^{\alpha-1}T(s^\alpha\theta)[x_0 - u(0,x(0))]\,\mathrm{d}\theta\mathrm{d}s + \alpha T(\epsilon^\alpha\delta) \\
&\quad \times \lim_{\lambda\to\infty}\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}M_\alpha(\theta)T((t-s)^\alpha\theta - \epsilon^\alpha\delta)B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))\,\mathrm{d}\theta\mathrm{d}s.
\end{aligned}$$

From the compactness of $T(\epsilon^\alpha\delta)$, $(\epsilon^\alpha\delta > 0)$, we obtain that $\{(Q_1^{\epsilon,\delta}x)(t)|x \in B_r\}$ is relatively compact in X for all $\epsilon \in (0,t)$ and all $\delta > 0$. Moreover, for any $x \in B_r$, we have

$$
\begin{aligned}
&\|(Q_1x)(t) - (Q_1^{\epsilon,\delta}x)(t)\|_X \\
&\leq \frac{M\alpha}{\Gamma(1-\alpha)}\Big\| \int_0^t (t-s)^{-\alpha}s^{\alpha-1}\int_0^\delta \theta M_\alpha(\theta)[x_0 - u(0,x(0))]\,\mathrm{d}\theta\mathrm{d}s\Big\|_X \\
&\quad + \frac{M\alpha}{\Gamma(1-\alpha)}\Big\| \int_{t-\epsilon}^t (t-s)^{-\alpha}s^{\alpha-1}\int_\delta^\infty \theta M_\alpha(\theta)[x_0 - u(0,x(0))]\,\mathrm{d}\theta\mathrm{d}s\Big\|_X \\
&\quad + \Big\| \lim_{\lambda\to\infty} \alpha\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}M_\alpha(\theta)T((t-s)^\alpha\theta)B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))\,\mathrm{d}\theta\mathrm{d}s\Big\|_X \\
&\quad + \Big\| \lim_{\lambda\to\infty} \alpha\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}M_\alpha(\theta)T((t-s)^\alpha\theta)B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))\,\mathrm{d}\theta\mathrm{d}s\Big\|_X \\
&\leq \frac{M\alpha}{\Gamma(1-\alpha)}B(\alpha,1-\alpha)[\|x_0\|_X + \|u(0,x(0))\|_X]\int_0^\delta \theta M_\alpha(\theta)\,\mathrm{d}\theta \\
&\quad + \frac{M}{\Gamma(1-\alpha)\Gamma(\alpha)}[\|x_0\|_X + \|u(0,x(0))\|_X]\int_{t-\epsilon}^t (t-s)^{-\alpha}s^{\alpha-1}\,\mathrm{d}s \\
&\quad + \alpha M\overline{M}\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}M_\alpha(\theta)l(s)\,\mathrm{d}\theta\mathrm{d}s \\
&\quad + \alpha M\overline{M}\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}M_\alpha(\theta)l(s)\,\mathrm{d}\theta\mathrm{d}s \\
&\leq \frac{M\alpha}{\Gamma(1-\alpha)}B(\alpha,1-\alpha)[\|x_0\|_X + \|u(0,x(0))\|_X]\int_0^\delta \theta M_\alpha(\theta)\,\mathrm{d}\theta \\
&\quad + \frac{M}{\Gamma(1-\alpha)\Gamma(\alpha)}[\|x_0\|_X + \|u(0,x(0))\|_X]\int_{t-\epsilon}^t (t-s)^{-\alpha}s^{\alpha-1}\,\mathrm{d}s \\
&\quad + \alpha M\overline{M}\int_0^t (t-s)^{\alpha-1}l(s)\,\mathrm{d}s\int_0^\delta \theta M_\alpha(\theta)\,\mathrm{d}\theta \\
&\quad + \alpha M\overline{M}\int_{t-\epsilon}^t (t-s)^{\alpha-1}l(s)\,\mathrm{d}s\int_0^\infty \theta M_\alpha(\theta)\,\mathrm{d}\theta.
\end{aligned}
$$

Therefore,

$$\|(Q_1x)(t) - (Q_1^{\epsilon,\delta}x)(t)\|_X \leq J_1 + J_2 + J_3 + J_4.$$

Using the inequality $\int_0^\infty \theta M_\alpha(\theta)\,\mathrm{d}\theta = \frac{1}{\Gamma(1+\alpha)}$, it is found that J_1, J_3 and J_4 tend to 0 as $\epsilon,\delta \to 0$. Also, upon application of the absolute continuity of the Lebesgue integral, J_2 tends to 0 as $\epsilon,\delta \to 0$. Therefore, there exist relatively compact sets arbitrarily close to the set $\{(Q_1x)(t)|x \in B_r\}$, $t > 0$ which implies that $\{(Q_1x)(t)|x \in B_r\}$, $t > 0$ is relatively compact. Consequently, $\{Q_1x|x \in B_r\}$ is a relatively compact set in X.

***Step III*:** To show that Q is continuous on B_r.

Proceeding similarly as in ***Step II***, it can be shown that Q is continuous on B_r.

***Step IV*:** To show that Q_2 is a contraction on B_r.

For any $x,y \in B_r$, we have

$$\|(Q_2x)(t) - (Q_2y)(t)\|_X \leq L_1\|x(t) - y(t)\|_X.$$

Thus

$$\|Q_2x - Q_2y\|_C \le L_1\|x-y\|_C,$$

which implies that $\alpha_c(Q_2B_r) \le L_1\alpha_c(B_r)$. Also, Q_1B_r is relatively compact in X which gives $\alpha_c(Q_1B_r) = 0$. Therefore,

$$\alpha_c(QB_r) \le \alpha_c(Q_1B_r) + \alpha_c(Q_2B_r) \le L_1\alpha_c(B_r).$$

As $L_1 < 1$, Q is an α-contraction on B_r. Hence, from Darbo-Sadovskii's fixed point theorem, it follows that Q has at least one fixed point on B_r. This completes the proof of the theorem. $\square$

Our next result for problem (1.1) is for the case where the associated C_0-semigroup is not compact. Here the following assumptions are required:

(H5)(iv) there exist a constant $\alpha_1 \in (0,\alpha)$ and a function $l \in L^{\frac{1}{\alpha_1}}(J,\mathbb{R}^+)$ such that

$$\|f(t,x,y,z)\|_X \le l(t)(\|x\|_X + \|y\|_X + \|z\|_X), \text{ for all } x, y, z \in X \text{ and for } t \in [0,b],$$

(v) there exist $l_1, l_2, l_3 \in C(J,\mathbb{R}^+)$ such that

$$\alpha(f(t,D_1,D_2,D_3)) \le l_1(t)\alpha(D_1) + l_2(t)\alpha(D_2) + l_3(t)\alpha(D_3)$$

for any bounded sets D_1, D_2, $D_3 \subset X$ and $t \in J$. Let $l_i^* = \sup_{t\in J}|l_i(t)|$, $i = 1,2,3$.

(H7)(iii) for any bounded set $D \subset X$, and $(t,s) \in \Delta$, there exists a function $\widetilde{m} : \Delta \to \mathbb{R}^+$ such that

$$\alpha(h(t,s,D)) \le \widetilde{m}(t,s)\alpha(D)$$

with $\widetilde{m}^* = \sup\limits_{t\in J}\int_0^t \widetilde{m}(t,s)\,\mathrm{d}s < \infty$.

(H8)(iii) for any bounded set $D \subset X$, and $(t,s) \in J \times J$, there exists a function $\widetilde{n} : J \times J \to \mathbb{R}^+$ such that

$$\alpha(g(t,s,D)) \le \widetilde{n}(t,s)\alpha(D)$$

with $\widetilde{n}^* = \sup\limits_{t\in J}\int_0^b \widetilde{n}(t,s)\,\mathrm{d}s < \infty$.

(H10) for each $t > 0$, $T(t)$ is equicontinuous.

Theorem 3.4 *Assume that* **(H5)(i), (iv), (v)**, **(H7)(i), (ii), (iii)**, **(H8)(i), (ii), (iii)**, **(H9)** *and* **(H10)** *hold. Then* (1.1) *has an integral solution provided*

$$\xi_4 = L_1 + \frac{M\overline{M}}{\Gamma(\alpha)}(1 + H^* + G^*)\|l\|_{L^{\frac{1}{\alpha_1}}}\frac{b^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} < 1$$

and

$$2L_1(1+M) + \frac{4M\overline{M}}{\Gamma(\alpha+1)}b^{\alpha}(l_1^* + 2l_2^*\widetilde{m}^* + 2l_3^*\widetilde{n}^*) < 1.$$

Proof. Choose $r = \frac{\phi}{1-\xi_4}$, where $\phi = M_0 + M\|x_0\|_X + M\|u(0, x(0))\|_X$ and let $B_r = \{x \in C(J, \overline{D(A)}) \mid \|x\|_C \leq r\}$. Define $Q : C(J, \overline{D(A)}) \to C(J, \overline{D(A)})$ by

$$\begin{aligned}(Qx)(t) &= u(t, x(t)) + S_\alpha(t)[x_0 - u(0, x(0))] \\ &\quad + \lim_{\lambda\to\infty} \int_0^t K_\alpha(t-s)B_\lambda f(s, x(s), (Hx)(s), (Gx)(s))\,\mathrm{d}s, \quad t \in J.\end{aligned}$$

Then proceeding similarly as in Theorem 3.3, it can be shown that $Q : B_r \to B_r$ is continuous as well as equicontinuous. Now, it remains to show that $Q : B_r \to B_r$ is a condensing operator.

For all $D \subset B_r$, $Q(D)$ is bounded and equicontinuous. Hence, by Lemma 2.2, there exists a countable set $D_0 = \{x_n\}_{n=1}^\infty \subset D$ such that

$$\alpha_c(Q(D)) \leq 2\alpha_c(Q(D_0)). \tag{3.1}$$

Since $Q(D_0) \subset Q(B_r)$ is equicontinuous, so by using Lemma 2.3, we get

$$\alpha_c(Q(D_0)) = \max_{t\in J} \alpha(Q(D_0)(t)). \tag{3.2}$$

Now, let

$$(Qx)(t) = (Q_1x)(t) + (Q_2x)(t),$$

where

$$\begin{aligned}(Q_1x)(t) &= u(t, x(t)) + S_\alpha(t)[x_0 - u(0, x(0))], \\ (Q_2x)(t) &= \lim_{\lambda\to\infty} \int_0^t K_\alpha(t-s)B_\lambda f(s, x(s), (Hx)(s), (Gx)(s))\,\mathrm{d}s.\end{aligned}$$

For $x, y \in D_0$, we have

$$\|Q_1x - Q_1y\|_C \leq L_1(1 + M)\|x - y\|_C.$$

Therefore, it follows that

$$\alpha_c(Q_1(D_0)) \leq L_1(1 + M)\alpha_c(D_0).$$

Now, for $t \in J$, we get

$$\begin{aligned}&\alpha(\{Q_2x_n(t)\}_{n=1}^\infty) \\ &= \alpha\Big(\Big\{\lim_{\lambda\to\infty} \int_0^t K_\alpha(t-s)B_\lambda f(s, x_n(s), (Hx_n)(s), (Gx_n)(s))\,\mathrm{d}s\Big\}_{n=1}^\infty\Big) \\ &\leq \frac{2M\overline{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \alpha(\{f(s, x_n(s), (Hx_n)(s), (Gx_n)(s))\}_{n=1}^\infty)\,\mathrm{d}s \\ &\leq \frac{2M\overline{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[l_1(s)\alpha(\{x_n(s)\}_{n=1}^\infty) + l_2(s)\alpha(\{Hx_n(s)\}_{n=1}^\infty) + l_3(s)\alpha(\{Gx_n(s)\}_{n=1}^\infty)\Big]\,\mathrm{d}s \\ &\leq \frac{2M\overline{M}}{\Gamma(\alpha)} \alpha_c(D)\Big[\int_0^t (t-s)^{\alpha-1} l_1(s)\,\mathrm{d}s + 2\widetilde{m}^* \int_0^t (t-s)^{\alpha-1} l_2(s)\,\mathrm{d}s + 2\widetilde{n}^* \int_0^t (t-s)^{\alpha-1} l_3(s)\,\mathrm{d}s\Big] \\ &\leq \frac{2M\overline{M}}{\Gamma(\alpha+1)} \alpha_c(D) b^\alpha (l_1^* + 2l_2^*\widetilde{m}^* + 2l_3^*\widetilde{n}^*).\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha(Q(D_0)(t)) &\le \alpha(Q_1(D_0)(t)) + \alpha(Q_2(D_0)(t))\\ &\le \Big[L_1(1+M) + \frac{2M\overline{M}}{\Gamma(\alpha+1)} b^{\alpha}(l_1^* + 2l_2^*\widetilde{m}^* + 2l_3^*\widetilde{n}^*)\Big]\alpha_c(D).\end{aligned}$$

From equations (3.1) and (3.2), we have

$$\begin{aligned}\alpha_c(Q(D)) &\le 2\Big[L_1(1+M) + \frac{2M\overline{M}}{\Gamma(\alpha+1)} b^{\alpha}(l_1^* + 2l_2^*\widetilde{m}^* + 2l_3^*\widetilde{n}^*)\Big]\alpha_c(D)\\ &< \alpha_c(D).\end{aligned}$$

Thus $Q : B_r \to B_r$ is a condensing operator and therefore from Lemma 2.3, we conclude that Q has a fixed point on B_r. This completes the proof of the theorem. □

4 Examples

Consider the following fractional partial differential system:

$$\begin{aligned}&{}^{C}D^{\alpha}_{0+}[x(t,y) - u(t,x(t,y))] = \frac{\partial^2}{\partial y^2}[x(t,y) - u(t,x(t,y))]\\ &\quad + f\big(t, x(t,y), \int_0^t h(t,s,x(s,y))\,\mathrm{d}s, \int_0^b g(t,s,x(s,y))\,\mathrm{d}s\big),\ t\in[0,b],\quad y\in\Omega=[0,\pi],\\ &x(t,0) = 0 = x(t,\pi),\quad t\in[0,b],\\ &x(0,y) = x_0(y),\quad y\in\Omega,\end{aligned}$$

where $b > 0$ is finite and $x_0 \in C(\Omega,\mathbb{R})$ with $x_0(0) = 0 = x_0(\pi)$.

Next, let $X = C(\Omega,\mathbb{R})$ and consider $A : D(A) \subset X \to X$ defined by

$$Aw = \frac{\partial^2 w}{\partial y^2}$$

with its domain of definition

$$D(A) = \Big\{w \in X : \frac{\partial^2 w}{\partial y^2} \in X \text{ and } w = 0 \text{ on } \partial\Omega\Big\}.$$

Then,

$$\overline{D(A)} = \{w \in X : w = 0 \text{ on } \partial\Omega\} \neq X.$$

Also, from [4], it is known that A satisfies Hille-Yosida condition with $\underline{(0,\infty) \subset} \rho(A)$, $\|R(\lambda : A)\| \le \lambda^{-1}$ and $\overline{M} = 1$ and generates a compact C_0-semigroup $\{T(t)\}_{t>0}$ on $\overline{D(A)}$ with $M = 1$.

Let us take

$x(t)(y) = x(t,y)$, and

$$f(t, x(t), (Hx)(t), (Gx)(t))(y) = f\big(t, x(t,y), \int_0^t h(t,s,x(s,y))\,\mathrm{d}s, \int_0^b g(t,s,x(s,y))\,\mathrm{d}s\big)$$

for $t \in [0,b]$, $y \in \Omega$. Consequently, (1.1) is the abstract formulation of the above considered problem.

For validation of Theorem 3.2, consider $u(t,x(t,y)) = \frac{1}{5}x(t,y)$. Then u satisfies **(H6)** with $L_1 = \frac{1}{5}$. Also, take

$$h(t,s,x(s,y)) = t\sin x(s,y) \ \text{ and } \ g(t,s,x(s,y)) = s\sin x(s,y).$$

Then h and g satisfy **(H7)** and **(H8)**, respectively, with $H^* = b^2$ and $G^* = \dfrac{b^2}{2}$. Consider

$$\begin{aligned} &f\big(t,x(t,y),\int_0^t h(t,s,x(s,y))\,\mathrm{d}s,\int_0^b g(t,s,x(s,y))\,\mathrm{d}s\big) \\ &= t^{\frac{1}{2}}\cos\left(\frac{|x(t,y)|}{1+|x(t,y)|} + \int_0^t h(t,s,x(s,y))\,\mathrm{d}s + \int_0^b g(t,s,x(s,y))\,\mathrm{d}s\right). \end{aligned}$$

If we choose $l(t) = t^{\frac{1}{2}}$, then f satisfies the assumptions in **(H5)**. Thus all the conditions of Theorem 3.2 are fulfilled and therefore, we can confirm the existence of an integral solution.

Next, for Theorem 3.3, consider

$$\begin{aligned} &f\big(t,x(t,y),\int_0^t h(t,s,x(s,y))\,\mathrm{d}s,\int_0^b g(t,s,x(s,y))\,\mathrm{d}s\big) \\ &= \exp(-t)\cos\left(\frac{|x(t,y)|}{1+|x(t,y)|} + \int_0^t h(t,s,x(s,y))\,\mathrm{d}s + \int_0^b g(t,s,x(s,y))\,\mathrm{d}s\right). \end{aligned}$$

Here, choose $l(t) = \exp(-t)$ and assuming u to be a suitable function satisfying **(H9)**, Theorem 3.3 implies the existence of integral solution of this problem.

5 Conclusion

This article is concerned with the existence of integral solution of a class of neutral fractional integro-differential equation of mixed type when the operator A is not dense. By using various fixed point theorems, fractional calculus and measure of noncompactness, we obtain some sufficient conditions which ensure the existence of integral solutions of the problem under consideration when the associated C_0-semigroup generated by the part of A in $\overline{D(A)}$ is compact or non-compact. Under suitable assumptions, four theorems are proposed and proved. Two theorems are verified by considering appropriate examples. These results are expected to enrich the analysis with regard to solutions for mixed Volterra-Fredholm integro fractional differential equations.

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