Journal of Nonlinear Evolution Equations and Applications
Volume 2020, Number 8, pp. 149–162 (October 2020)

ISSN 2161-3680
http://www.jneea.com

PARTIALLY OBSERVED NONLINEAR EVOLUTION EQUATIONS ON BANACH SPACES AND THEIR OPTIMAL FEEDBACK CONTROL LAWS DETERMINED BY MEASURES

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Received July 30, 2019

Accepted September 27, 2019

Communicated by Khalil Ezzinbi

Abstract. In this paper, we consider partially observed control problems for a class of nonlinear coupled evolution equations on the product of a pair of Banach spaces. The observer dynamics is subject to (possibly) impulsive perturbation. We use relaxed controls covering non-convex control problems. We prove existence of mild solutions and their regularity properties. Then we prove continuous dependence of solutions with respect to control measures. These results are then used to prove existence of optimal feedback control laws.

Keywords: Partially Observed Systems, Optimal Feedback Operator, Measure Valued Controls, Existence of solutions, Existence of optimal feedback controls.

2010 Mathematics Subject Classification: 34G20, 34H05, 49J27, 49J55, 93C20.

1 Introduction

It is known that the class of impulsive systems is contained in the broader class of systems driven by vector and operator valued measures as studied extensively by the author and his former students [2–9]. In a recent paper [2], we considered optimal control of a large class of nonlinear systems on Banach spaces driven by finitely additive measures thereby generalizing purely impulsive systems [7, 8]. In our paper [6], we considered impulsive systems which are finite dimensional version of [2] and applied to several control problems of systems related to ecology and (geosynchronous) communication satellites. For computation of optimal impulsive controls, we used some

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variants of the computational techniques developed in [12, 14]. Here, in this paper, we consider a large class of non-convex partially observed optimal control problems in Banach spaces. The system consists of a pair of coupled evolution equations on Banach spaces, one of which represents the main system to be controlled and the second represents the dynamics of the observer (monitor) which monitors the state of the main system. The observer state is fully accessible and it is this state that is fed back to the main system to be controlled. To the best of knowledge of the author, no such work seems to exist in the current literature. The results of this paper are applicable to control problems involving partial differential equations as well as ordinary differential equations.

The rest of the paper is organised as follows. In Section 2, we introduce the system dynamics and the partially observed output feedback control problem. In Section 3, we state the basic assumptions and present some results on existence and uniqueness of (mild) solutions and their regularity properties. In Section 4, we consider optimal control problems, in particular, the Bolza problem. First we prove continuous dependence of solutions with respect to control measures (weak star to strong). Using this result, we prove existence of optimal controls. We conclude the paper with some remarks on open problems.

2 System Dynamics and the Control Problem

Let the state space be given by a Banach space X and the space of observation (measured data) be given by another Banach space denoted by Y. Let U be a compact Polish space (for example, a complete separable metric space) and $I \equiv [0,T]$ a closed bounded interval. Let Σ_U denote the sigma algebra of subsets of the set U and $\mathcal{M}_0(\Sigma_U)$ denote the space of regular Borel Probability measures. Let $C(U)$ denote the Banach space of bounded continuous functions endowed with supnorm topology and let $L_1(I, C(U))$ denote the space of Bochner integrable functions defined on I and taking values in the Banach space $C(U)$. Since the Banach space $C(U)$ does not have the RNP (Radon Nikodym Property), the topological dual of $L_1(I, C(U))$ is not given by $L_\infty(I, \mathcal{M}_B(\Sigma_U))$. However, by virtue of the theory of lifting [13, Theorem 7, p. 93], it is given by the space of weak star measurable essentially bounded functions defined on I and taking values $\mathcal{M}_B(\Sigma_U)$, and it is denoted by $L^w_\infty(I, \mathcal{M}_B(\Sigma_U))$. For admissible controls, we choose the set $\mathcal{M}_{ad} \equiv M^w_\infty(I, \mathcal{M}_0(\Sigma_U)) \subset L^w_\infty(I, \mathcal{M}_B(\Sigma_U))$. By Alaoglu's theorem [11, Theorem V.4.2], the set $M^w_\infty(I, \mathcal{M}_0(\Sigma_U))$ is weak star (w^*) compact.

We consider the following system of evolution equations on Banach spaces X and Y respectively,

$$\mathrm{d}x(t) = Ax(t)\,\mathrm{d}t + F(t, x(t))\,\mathrm{d}t + \int_U G(t, y(t), \xi)\mu_t(\mathrm{d}\xi), t \in I, x(0) = x_0 \tag{2.1}$$

$$\mathrm{d}y(t) = A_0 y(t)\,\mathrm{d}t + F_0(t, y(t))\,\mathrm{d}t + G_0(t, x(t))\gamma(\mathrm{d}t), t \in I, \tag{2.2}$$

where the first equation denotes the main system to be controlled whose state is inaccessible and the second equation denotes the observer (monitor) whose state is fully accessible. The main system is to be controlled on the basis of the observation y through the feedback operator G which in turn is controlled by the measure valued function $\mu_t, t \in I$. The observer receives state information through the operator G_0 which may be occasionally cutoff by the measure γ whenever the null set $N(\gamma) \equiv \{V \in \Sigma_I : \gamma(V) = 0\} \neq \emptyset$. During this time intervals (blank periods) the system (2.1) operates on the basis of information generated by the system (2.2) in the absence of G_0. These periods can be considered as the periods of partial blackout (system failure) and the process y used to control the system (2.1) does not carry current state x information though it does carry the past information till the cutoff time. In any case, whatever information is available, the measure μ is required to adjust the controller (actuator) G so as to achieve the objective as closely as possible. The objective functional is given by

$$J(\mu) \equiv \int_{I\times U} \ell(t, x(t), y(t), \xi)\mu_t(\mathrm{d}\xi)\,\mathrm{d}t + \Phi(x(T), y(T)) \tag{2.3}$$

considered as the cost functional. The objective is to choose a control measure $\mu \in \mathcal{M}_{ad}$ that minimizes the functional (2.3). This is the problem we consider in this paper. We prove existence of optimal controls and develop necessary conditions of optimality.

3 Existence and Uniqueness of Solutions

In this section we consider the question of existence, uniqueness and regularity properties of solutions of the evolution equations (2.1)-(2.2). In the first equation, the operator A is the infinitesimal generator of a C_0 semigroup $\{S(t), t \geq 0\} \subset \mathcal{L}(X)$, and the functions $F : I \times X \longrightarrow X$ and $G : I \times Y \times U \longrightarrow X$ are Borel measurable maps and $\mu \in M^w_\infty(I, \mathcal{M}_0(\Sigma_U))$. Similarly, in the second equation which represents the observer, the operator A_0 is the infinitesimal generator of a C_0 semigroup $\{S_0(t), t \geq 0\} \subset \mathcal{L}(Y)$, and the functions $F_0 : I \times Y \longrightarrow Y$ and $G_0 : I \times X \longrightarrow Y$ are Borel measurable maps. The measure $\gamma \in \mathcal{M}_{ca}(\Sigma_I, R) \equiv \mathcal{M}_{ca}(\Sigma_I)$, where $\mathcal{M}_{ca}(\Sigma_I)$ denotes the space of countably additive real valued measures having bounded total variation. Endowed with the total variation norm, it is a Banach space.

It is well known [1] that in general these equations do not have strong solutions and hence we are interested in the mild solutions which are solutions of the associated integral equations. Using the semigroup and variation of constants formula (Duhamels formula), the differential equation (2.1) can be written as an integral equation on the Banach space X as follows:

$$x(t) = S(t)x_0 + \int_0^t S(t-s)F(s, x(s))\,\mathrm{d}s + \int_0^t\int_U S(t-s)G(s, y(s), \xi)\mu_s(\mathrm{d}\xi)\,\mathrm{d}s, t \in I. \tag{3.1}$$

Similarly, the differential equation (2.2) can be written as an integral equation on the Banach space Y as follows,

$$y(t) = S_0(t)y_0 + \int_0^t S_0(t-s)F_0(s, y(s))\,\mathrm{d}s + \int_0^t S_0(t-s)G_0(s, x(s))\gamma(\mathrm{d}s), t \in I. \tag{3.2}$$

Under suitable assumptions on the operators $\{A, A_0, F, G, F_0, G_0\}$, we will prove that these equations have solutions. The observer represented by equation (2.2) observes the state of the main system (2.1) through the operator G_0 and the measure γ. In case U consists of a finite set $\{v_i\}_{i=1}^n$, the integral equation (3.1) takes the form

$$x(t) = S(t)x_0 + \int_0^t S(t-s)F(s, x(s))\,\mathrm{d}s + \sum_{i=1}^n \int_0^t S(t-s)G(s, y(s), v_i)\mu_s(\{v_i\})\,\mathrm{d}s, t \in I. \tag{3.3}$$

Since $\mu \in M^w_\infty(I, \mathcal{M}_0(\Sigma_U))$, it is clear that $\mu_s(\{v_i\}) \geq 0$ for all $i = 1, 2, \cdots, n$ and

$$\sum_{i=1}^n \mu_s(\{v_i\}) = 1, \forall s \in I.$$

In other words, $\mu_s(\mathrm{d}\xi) = \sum \mu_s(\{v_i\})\delta_{v_i}(\mathrm{d}\xi)$. Controls of this form are called chattering controls. We consider general case containing the chattering controls as special case.

Basic Assumptions

We need the following basic assumptions.

(A1) The operators A and A_0 are the infinitesimal generators of C_0-semigroups $\{S(t), t \geq 0\}$ and $\{S_0(t), t \geq 0\}$ of bounded linear operators in the Banach spaces X and Y respectively satisfying $\sup\{\|S(t)\|_{\mathcal{L}(X)}, t \in I\} \leq M$ and $\sup\{\|S_0(t)\|_{\mathcal{L}(Y)}, t \in I\} \leq M_0$ with $\{M, M_0\}$ being finite positive numbers.

(A2) $F : I \times X \longrightarrow X$ is Borel measurable and there exists a constant $K_1 > 0$ such that

$$\begin{aligned}&(1): \ \|F(t,x)\|_X \leq K_1(1 + \|x\|_X), x \in X, t \in I,\\ &(2): \ \|F(t,x_1) - F(t,x_2)\|_X \leq K_1\|x_1 - x_2\|_X, x_1, x_2 \in X, t \in I.\end{aligned}$$

(A3) $G : I \times Y \times U \longrightarrow X$ is Borel measurable and there exists a Borel measurable function $K_2 : U \longrightarrow R_0 \equiv [0, \infty)$ such that

$$\begin{aligned}&(1): \ \|G(t,y,\xi)\|_X \leq K_2(\xi)(1 + \|y\|_Y), y \in Y, t \in I, \xi \in U,\\ &(2): \ \|G(t,y_1,\xi) - G(t,y_2,\xi)\|_X \leq K_2(\xi)\|y_1 - y_2\|_Y, y_1, y_2 \in Y, t \in I, \xi \in U,\\ &(3): \ \sup\{\int_U K_2(\xi)\mu_t(\mathrm{d}\xi), \mu \in \mathcal{M}_{ad}, t \in I\} \equiv \hat{K}_2 < \infty.\end{aligned}$$

The last condition is trivially satisfied if K_2 is continuous and U a compact Polish space.

(A4) $F_0 : I \times Y \longrightarrow Y$ is Borel measurable and there exists a constant $K_3 > 0$ such that

$$\begin{aligned}&(1): \ \|F_0(t,y)\|_Y \leq K_3(1 + \|y\|_Y), y \in Y, t \in I,\\ &(2): \ \|F_0(t,y_1) - F_0(t,y_2)\|_Y \leq K_3\|y_1 - y_2\|_Y, y_1, y_2 \in Y, t \in I.\end{aligned}$$

(A5) $G_0 : I \times X \longrightarrow Y$ is Borel measurable and there exists a constant $K_4 > 0$ such that

$$\begin{aligned}&(1): \ \|G_0(t,x)\|_Y \leq K_4(1 + \|x\|_X), x \in X, t \in I,\\ &(2): \ \|G_0(t,x_1) - G_0(t,x_2)\|_Y \leq K_4\|x_1 - x_2\|_X, x_1, x_2 \in X, t \in I.\end{aligned}$$

Let $\mathcal{Z} \equiv X \times Y$ denote the linear space of the Cartesian product of the Banach spaces X and Y. Let $z \equiv (x, y) \in X \times Y$ and define the norm topology on $\mathcal{Z}$ by

$$\|z\| \equiv \|x\|_X + \|y\|_Y.$$

The space $\mathcal{Z}$ endowed with this norm topology is a Banach space. We need the Banach spaces $B_\infty(I, X)$ and $B_\infty(I, Y)$ and their Cartesian product $B_\infty(I, X) \times B_\infty(I, Y) \equiv B_\infty(I, \mathcal{Z})$. These are function spaces defined on the interval I and taking values from the Banach spaces as indicated. Endowed with sup-norm topology, these are Banach spaces.

Theorem 3.1 *Consider the system of evolution equations* (2.1)-(2.2) *with the admissible control measure* $\mu \in \mathcal{M}_{ad} \equiv M_\infty^w(I, \mathcal{M}_0(\Sigma_U))$ *and* $\gamma \in \mathcal{M}_{ca}(\Sigma_I, R)$ *non-atomic having bounded total variation and suppose the assumptions* **(A1)***,* **(A2)***,* **(A3)***,* **(A4)** *and* **(A5)** *hold. Then, for every* $x_0 \in X$*, and* $\mu \in \mathcal{M}_{ad}$*, the system* (2.1)-(2.2) *has a unique mild solution* $z \in B_\infty(I, \mathcal{Z})$*.*

Proof. For proof we use Banach fixed point theorem. For any given $z_0 \equiv (x_0, y_0) \in X \times Y \equiv \mathcal{Z}$ and $\mu \in \mathcal{M}_{ad}$, we introduce the operator $\Gamma \equiv (\Gamma_1, \Gamma_2)$ as follows:

$$\Gamma_1(x,y)(t) \equiv S(t)x_0 + \int_0^t S(t-s)F(s,x(s))\,\mathrm{d}s + \int_0^t\int_U S(t-s)G(s,y(s),\xi)\mu_s(\mathrm{d}\xi)\,\mathrm{d}s, t \in I. \quad (3.4)$$

$$\Gamma_2(x,y)(t) \equiv S_0(t)y_0 + \int_0^t S_0(t-s)F_0(s,y(s))\,\mathrm{d}s + \int_0^t S_0(t-s)G_0(s,x(s))\gamma(\mathrm{d}s), t \in I. \quad (3.5)$$

Under the assumptions **(A1), (A2), (A3), (A4), (A5)**, we show that, Γ maps $B_\infty(I, \mathcal{Z})$ to itself. Since $\{S(t), t \geq 0\}$ and $\{S_0(t), t \geq 0\}$ are C_0-semigroups of operators on X and Y respectively and $I \equiv [0,T]$ is a finite interval, there exist finite positive numbers M and M_0 such that $\sup\{\|S(t)\|_{\mathcal{L}(X)}, t \in I\} \leq M$ and $\sup\{\|S_0(t)\|_{\mathcal{L}(Y)}, t \in I\} \leq M_0$. Computing the X norm of $\Gamma_1(x,y)(t)$ using the assumptions **(A2)** and **(A3)**, it follows from triangle inequality that for each $t \in I$,

$$\begin{aligned}\|\Gamma_1(x,y)(t)\|_X \leq{}& M\|x_0\|_X + MK_1t(1 + \sup_{0\leq s\leq t}\|x(s)\|_X) \\ &+ M(1 + \sup_{0\leq s\leq t}\|y(s)\|_Y)\int_0^t\int_U K_2(\xi)\mu_s(\mathrm{d}\xi)\,\mathrm{d}s. \quad (3.6)\end{aligned}$$

Since $\mu_t, t \in I$, is a probability measure valued function satisfying **(A3)**(3), the above inequality can be rewritten as

$$\begin{aligned}\|\Gamma_1(x,y)(t)\|_X \leq{}& M\|x_0\|_X + M(K_1t)(1 + \sup_{0\leq s\leq t}\|x(s)\|_X) \\ &+ M(\hat{K}_2t)(1 + \sup_{0\leq s\leq t}\|y(s)\|_Y), t \in I. \quad (3.7)\end{aligned}$$

It follows from this inequality and the fact that $I \equiv [0,T]$ is a finite interval, that

$$\|\Gamma_1(x,y)\|_{B_\infty(I,X)} \leq M\|x_0\|_X + MK_1T\big(1 + \|x\|_{B_\infty(I,X)}\big) + M\hat{K}_2T\big(1 + \|y\|_{B_\infty(I,Y)}\big). \quad (3.8)$$

Similarly, considering $\Gamma_2(x,y)(t)$ and computing its Y norm and using the assumptions **(A4)-(A5)**, one can easily verify that

$$\|\Gamma_2(x,y)\|_{B_\infty(I,Y)} \leq M_0\|y_0\|_Y + M_0K_3T\big(1+\|y\|_{B_\infty(I,Y)}\big) + M_0K_4|\gamma|(I)\big(1+\|x\|_{B_\infty(I,X)}\big). \quad (3.9)$$

It follows from the inequalities (3.8)-(3.9) that there exist positive constants C_0, C_1, C_2 dependent on the parameters $\{M_0, M, K_1, K_2, K_3, K_4, T, |\gamma|(I)\}$ such that

$$\|\Gamma(z)\|_{B_\infty(I,\mathcal{Z})} \leq C_0 + C_1\|z_0\|_{\mathcal{Z}} + C_1\|z\|_{B_\infty(I,\mathcal{Z})}. \quad (3.10)$$

This proves that the operator Γ maps $B_\infty(I, \mathcal{Z})$ into itself. Following similar steps while using the Lipschitz conditions in the assumptions **(A2), (A3), (A4), (A5)**, one can verify that

$$\begin{aligned}&\|\Gamma_1(z_1)(t) - \Gamma_1(z_2)(t)\|_X \equiv \|\Gamma_1(x_1,y_1)(t) - \Gamma_1(x_2,y_2)(t)\|_X \\ &\leq MK_1\int_0^t \|x_1(s) - x_2(s)\|_X\,\mathrm{d}s + M\hat{K}_2\int_0^t \|y_1(s) - y_2(s)\|_Y\,\mathrm{d}s. \quad (3.11)\end{aligned}$$

Similarly, considering the second component Γ_2 of the operator Γ we obtain the following inequality

$$\begin{aligned}&\|\Gamma_2(z_1)(t) - \Gamma_2(z_2)(t)\|_Y \equiv \|\Gamma_2(x_1,y_1)(t) - \Gamma_2(x_2,y_2)(t)\|_Y \\ &\leq M_0K_3\int_0^t \|y_1(s) - y_2(s)\|_Y\,\mathrm{d}s + M_0K_4\int_0^t \|x_1(s) - x_2(s)\|_X|\gamma|(\mathrm{d}s), \quad (3.12)\end{aligned}$$

where $|\gamma|(\cdot)$ denotes the nonnegative measure induced by the variation of the signed measure $\gamma \in \mathcal{M}_{ca}(\Sigma_I, R)$. Define the measure $\alpha \in \mathcal{M}_{ca}^{+}(\Sigma_I)$ by

$$\alpha(\sigma) \equiv \int_{\sigma} (\mathrm{d}s + |\gamma|(\mathrm{d}s)), \sigma \in \Sigma_I$$

and note that it is also a countably additive bounded positive measure. Clearly, the function β defined by $\beta(t) \equiv \int_0^t \alpha(\mathrm{d}s), t \geq 0$, is a nonnegative increasing function of bounded total variation on I. Introducing the constants $D_1 \equiv \max\{MK_1, M_0K_4\}$ and $D_2 \equiv \max\{M\hat{K}_2, M_0K_3\}$ and summing the expressions (3.11)-(3.12), we obtain the following inequality

$$\|\Gamma(z_1)(t)-\Gamma(z_2)(t)\|_{\mathcal{Z}} \leq D_1 \int_0^t \|x_1(s)-x_2(s)\|_X \,\mathrm{d}\beta(s)+D_2 \int_0^t \|y_1(s)-y_2(s)\|_Y \,\mathrm{d}\beta(s), t \in I. \tag{3.13}$$

Next defining $D \equiv \max\{D_1, D_2\}$ and using the above inequality, we obtain the following expression

$$\|\Gamma(z_1)(t) - \Gamma(z_2)(t)\|_{\mathcal{Z}} \leq D \int_0^t \|z_1(s) - z_2(s)\|_{\mathcal{Z}} \,\mathrm{d}\beta(s),\ t \in I. \tag{3.14}$$

For any pair $z_1, z_2 \in B_\infty(I, \mathcal{Z})$ and $t \in I$, define

$$\rho_t(z_1, z_2) \equiv \sup\{\|z_1(s) - z_2(s)\|_{\mathcal{Z}}, 0 \leq s \leq t\}$$

and note that $\rho_T(z_1, z_2) = \|z_1 - z_2\|_{B_\infty(I,\mathcal{Z})}$. Using this notation we observe that the inequality 3.14 is equivalent to the following inequality,

$$\rho_t(\Gamma(z_1), \Gamma(z_2)) \ \leq \int_0^t \rho_s(z_1, z_2) \ \mathrm{d}\beta(s), t \in I. \tag{3.15}$$

Considering the second iteration of the operator Γ (*i.e.* $\Gamma^2 \equiv \Gamma \circ \Gamma$), it follows from the above expression and the fact that $t \longrightarrow \rho_t(z_1, z_2)$ is a nondecreasing function of $t \geq 0$, that, for each $t \in I$, we have

$$\rho_t(\Gamma^2(z_1), \Gamma^2(z_2)) \ \leq \int_0^t \rho_s(\Gamma(z_1), \Gamma(z_2)) \,\mathrm{d}\beta(s) \leq \int_0^t \left(\int_0^s \rho_\theta(z_1, z_2) \ \mathrm{d}\beta(\theta)\right)\mathrm{d}\beta(s).$$

By assumption the measure γ is non-atomic and hence α is non-atomic and therefore $|\alpha|(\{0\}) = 0$ and consequently $\beta(0) = 0$. Thus it follows from the above inequality that

$$\rho_t(\Gamma^2(z_1), \Gamma^2(z_2)) \leq \int_0^t \rho_s(z_1, z_2)\beta(s) \,\mathrm{d}\beta(s), t \in I \tag{3.16}$$

and hence we have

$$\rho_t(\Gamma^2(z_1), \Gamma^2(z_2)) \leq \rho_t(z_1, z_2)\big(\beta^2(t)/2\big), t \in I. \tag{3.17}$$

Continuing this iterative process m times, it is easy to verify that

$$\rho_t(\Gamma^m(z_1), \Gamma^m(z_2)) \leq \big(\beta^m(t)/m!\big)\rho_t(z_1, z_2), t \in I. \tag{3.18}$$

Thus, for $t = T$, we have

$$\|\Gamma^m(z_1) - \Gamma^m(z_2)\|_{B_\infty(I,\mathcal{Z})} \ \leq \ \eta_m\|z_1 - z_2\|_{B_\infty(I,\mathcal{Z})}, \tag{3.19}$$

where $\eta_m = ((\beta(T))^m/m!)$. Since $\beta(T)$ is finite, for $m \in N$ sufficiently large, $\eta_m < 1$ and hence the m-th iteration of the operator Γ is a contraction on the Banach space $B_\infty(I, \mathcal{Z})$. Thus it follows from Banach fixed point theorem that Γ^m has a unique fixed point $z^* \in B_\infty(I, \mathcal{Z})$. Using this fact one can easily verify that z^* is also the unique fixed point of the operator Γ itself. This proves the existence of a unique mild solution $z^* = (x^*, y^*)$ of the system of evolution equations (2.1)-(2.2) in the Banach space $B_\infty(I, \mathcal{Z})$. □

Under the assumptions of Theorem 3.1, we show that the solution set is a bounded subset of the Banach space $B_\infty(I, \mathcal{Z})$. For $\mu \in \mathcal{M}_{ad} \equiv M_\infty^w(I, \mathcal{M}_0(\Sigma_U))$, let $z(\mu)$ denote the mild solution of the system of evolution equations (2.1)-(2.2).

Corollary 3.2 *Consider the system* (2.1)-(2.2) *and suppose the assumptions of Theorem 3.1 hold. Then the solution set*

$$\mathcal{S} \equiv \{z \in B_\infty(I, \mathcal{Z}) : z = z(\mu) \equiv (x(\mu), y(\mu)) \text{ for } \mu \in \mathcal{M}_{ad}\} \tag{3.20}$$

is a bounded subset of $B_\infty(I, \mathcal{Z})$.

Proof. It follows from Theorem 3.1 that, for each $\mu \in \mathcal{M}_{ad}$, the system of evolution equations (2.1)-(2.2) has a unique mild solution $z(\mu) \equiv (x(\mu), y(\mu)) \in B_\infty(I, X \times Y) \equiv B_\infty(I, \mathcal{Z})$. Thus $z(\mu)$ satisfies the following system of integral equations:

$$\begin{aligned} x(\mu)(t) = {} & S(t)x_0 + \int_0^t S(t-s)F(s, x(\mu)(s))\,\mathrm{d}s \\ & + \int_0^t \int_U S(t-s)G(s, y(\mu)(s), \xi)\mu_s(\mathrm{d}\xi)\,\mathrm{d}s, t \in I, \end{aligned} \tag{3.21}$$

$$\begin{aligned} y(\mu)(t) = {} & S_0(t)y_0 + \int_0^t S_0(t-s)F_0(s, y(\mu)(s))\,\mathrm{d}s \\ & + \int_0^t S_0(t-s)G_0(s, x(\mu)(s))\gamma(\mathrm{d}s), t \in I. \end{aligned} \tag{3.22}$$

By taking the X norm of $x(\mu)(t)$ and the Y norm of $y(\mu)(t)$ and summing these norms, it follows equations (3.21) and (3.22) and the assumptions **(A1)-(A5)** that there exist constants C_3, C_4 dependent on the parameters $\{M, M_0, K_1, K_2, \hat{K}_2, K_3, K_4, \beta(T)\}$ such that

$$\|z(\mu)(t)\|_{\mathcal{Z}} \equiv \|x(\mu)(t)\|_X + \|y(\mu)(t)\|_Y \leq C_3 + C_4 \int_0^t \|z(\mu)(s)\|_{\mathcal{Z}}\,\mathrm{d}\beta(s),\ t \in I,\ \forall\, \mu \in \mathcal{M}_{ad}. \tag{3.23}$$

By virtue of generalized Gronwall inequality [2, Lemma 5, p. 268], it follows from the above expression that

$$\sup\{\|z(\mu)(t)\|_{\mathcal{Z}}, t \in I\} \leq C_3 \exp\{C_4 \beta(T)\}, \mu \in \mathcal{M}_{ad}. \tag{3.24}$$

In view of the assumption **(A3)**(3), this inequality holds uniformly with respect to $\mu \in \mathcal{M}_{ad}$. Hence we have

$$\sup\{\|z(\mu)\|_{B_\infty(I,\mathcal{Z})}, \mu \in \mathcal{M}_{ad}\} \leq C_3 \exp\{C_4 \beta(T)\} < \infty. \tag{3.25}$$

Thus the solution set $\mathcal{S}$ is a bounded subset of $B_\infty(I, \mathcal{Z})$. □

4 Optimal Control

In this section we consider the question of existence of optimal controls. For this purpose we use the continuity of the control to solution map $\mu \longrightarrow z(\mu)$ with respect to the weak star topology on $M_\infty^w(I,\Sigma_U)$ and the strong norm topology on $B_\infty(I,\mathcal{Z})$.

Theorem 4.1 *Consider the partially observed system* (2.1)-(2.2) *with the operator A being the generator of a compact C_0-semigroup $\{S(t), t>0\}$ in X, and the assumptions of Theorem 3.1 and Corollary 3.2 hold with $\gamma \in \mathcal{M}_{ca}(\Sigma_I)$ being non-atomic. Then the map $\mu \longrightarrow z(\mu)$ from $\mathcal{M}_{ad} = M_\infty^w(I,\mathcal{M}_0(\Sigma_U))$ to $B_\infty(I,\mathcal{Z})$ is continuous with respect to the relative weak* topology on $\mathcal{M}_{ad}$ and the norm topology on $B_\infty(I,\mathcal{Z})$.*

Proof. Let $\{\mu^n,\mu^o\} \in \mathcal{M}_{ad}$ and suppose $\mu^n \overset{w*}{\longrightarrow} \mu^o$. Let $z^n \equiv (x(\mu^n), y(\mu^n))$ and $z^o \equiv (x(\mu^o), y(\mu^o))$ denote the unique mild solution of the system of evolution equations (2.1)-(2.2) corresponding to the same initial states, $(x(\mu^n)(0), y(\mu^n)(0)) = (x(\mu^o)(0), y(\mu^o)(0)) = (x_0, y_0)$ and control measures μ^n and μ^o respectively. This means that $\{z^n, z^o\}$ satisfy the following system of integral equations:

$$x^n(t) = S(t)x_0 + \int_0^t S(t-s)F(s,x^n(s))\,\mathrm{d}s + \int_0^t\int_U S(t-s)G(s,y^n(s),\xi)\mu_s^n(\mathrm{d}\xi)\,\mathrm{d}s, t\in I, \quad (4.1)$$

$$y^n(t) = S_0(t)y_0 + \int_0^t S_0(t-s)F_0(s,y^n(s))\,\mathrm{d}s + \int_0^t S_0(t-s)G_0(s,x^n(s))\gamma(\mathrm{d}s), t\in I, \quad (4.2)$$

$$x^o(t) = S(t)x_0 + \int_0^t S(t-s)F(s,x^o(s))\,\mathrm{d}s + \int_0^t\int_U S(t-s)G(s,y^o(s),\xi)\mu_s^o(\mathrm{d}\xi)\,\mathrm{d}s, t\in I, \quad (4.3)$$

$$y^o(t) = S_0(t)y_0 + \int_0^t S_0(t-s)F_0(s,y^o(s))\,\mathrm{d}s + \int_0^t S_0(t-s)G_0(s,x^o(s))\gamma(\mathrm{d}s), t\in I, \quad (4.4)$$

where $x^n(t) \equiv x(\mu^n)(t), x^o(t) \equiv x(\mu^o)(t)$, $y^n(t) \equiv y(\mu^n)(t)$ and $y^o(t) \equiv y(\mu^o)(t), t\in I$. Subtracting the expression (4.3) from (4.1) term by term and suitably rearranging terms, we obtain the following identity

$$\begin{aligned} x^n(t) - x^o(t) = &\int_0^t S(t-s)[F(s,x^n(s)) - F(s,x^o(s))]\,\mathrm{d}s \\ &+ \int_0^t\int_U S(t-s)[G(s,y^n(s),\xi) - G(s,y^o(s),\xi)]\mu_s^n(\mathrm{d}\xi)\,\mathrm{d}s \\ &+ \int_0^t\int_U S(t-s)G(s,y^o(s),\xi)\big(\mu_s^n - \mu_s^o)(\mathrm{d}\xi)\,\mathrm{d}s, t\in I. \end{aligned} \quad (4.5)$$

For convenience we denote the last term of the above expression by e_n giving

$$e_n(t) \equiv \int_0^t\int_U S(t-s)G(s,y^o(s),\xi)\big(\mu_s^n - \mu_s^o)(\mathrm{d}\xi)\,\mathrm{d}s, t\in I. \quad (4.6)$$

Evaluating the X norm on either side of the expression (4.5) and using the assumptions **(A2)** and **(A3)** and triangle inequality, we obtain the following inequality

$$\begin{aligned} \|x^n(t) - x^o(t)\|_X \le &\int_0^t MK_1\|x^n(s) - x^o(s)\|_X\,\mathrm{d}s \\ &+ \int_0^t\int_U MK_2(\xi)\|y^n(s) - y^o(s)\|_Y\mu_s^n(\mathrm{d}\xi)\,\mathrm{d}s + \|e_n(t)\|_X, t\in I. \end{aligned} \quad (4.7)$$

Next, using the assumption **(A3)**(3) related to the function K_2, it follows from the above inequality that

$$\|x^n(t)-x^o(t)\|_X \le \int_0^t MK_1\|x^n(s)-x^o(s)\|_X\,\mathrm{d}s + M\hat{K}_2\int_0^t \|y^n(s)-y^o(s)\|_Y\,\mathrm{d}s + \|e_n(t)\|_X, t\in I. \tag{4.8}$$

Similarly, subtracting the expression (4.4) from (4.2) term by term, we obtain the following expression

$$y^n(t)-y^o(t) = \int_0^t S_0(t-s)[F_0(s,y^n(s))-F_0(s,y^o(s))]\,\mathrm{d}s + \int_0^t S_0(t-s)[G_0(s,x^n(s))-G_0(s,x^o(s))]\gamma(\mathrm{d}s), t\in I. \tag{4.9}$$

Computing the Y norm on either side of the above expression and using the assumptions **(A4)-(A5)** and triangle inequality, we obtain the following inequality

$$\|y^n(t)-y^o(t)\|_Y \le M_0K_3\int_0^t \|y^n(s)-y^o(s)\|_Y\,\mathrm{d}s + M_0K_4\int_0^t \|x^n(s)-x^o(s)\|_X|\gamma|(\mathrm{d}s), t\in I. \tag{4.10}$$

Again, without loss of generality we can add up (4.8) and (4.10) term by term and using the (augmented) measure $\alpha(\sigma)\equiv\int_\sigma[\mathrm{d}s+|\gamma|(\mathrm{d}s)], \sigma\subset I, \sigma\in\Sigma_I$, we obtain the following inequality

$$\begin{aligned}&\|x^n(t)-x^o(t)\|_X+\|y^n(t)-y^o(t)\|_Y\\ &\le \|e_n(t)\|_X + (MK_1+M_0K_4)\int_0^t \|x^n(s)-x^o(s)\|_X\alpha(\mathrm{d}s)\\ &+(M\hat{K}_2+M_0K_3)\int_0^t \|y^n(s)-y^o(s)\|_Y\alpha(\mathrm{d}s), t\in I.\end{aligned} \tag{4.11}$$

Defining the constant

$$C\equiv\max\{(MK_1+M_0K_4),(M\hat{K}_2+M_0K_3)\}$$

and recalling that $\|z^n(t)-z^o(t)\|_{\mathcal{Z}}=\|x^n(t)-x^o(t)\|_X+\|y^n(t)-y^o(t)\|_Y$, we can rewrite the above inequality in the following compact form

$$\|z^n(t)-z^o(t)\|_{\mathcal{Z}} \le \|e_n(t)\|_X + C\int_0^t \|z^n(s)-z^o(s)\|_{\mathcal{Z}}\alpha(\mathrm{d}s), t\in I. \tag{4.12}$$

It follows from generalized Gronwall inequality [3, Lemma 5, p. 268], applied to the above expression, that

$$\|z^n(t)-z^o(t)\|_{\mathcal{Z}} \le \|e_n(t)\|_X + \big(C\exp(C\alpha(I))\big)\int_0^t \|e_n(s)\|_X\alpha(\mathrm{d}s), t\in I. \tag{4.13}$$

It suffices to show that $e_n(t)$, given by the expression (4.6), converges to zero strongly in X uniformly on I. Here, we use the compactness of the semigroup $\{S(t), t>0\}$ and the weak* convergence of μ^n to μ^o. For any $\varepsilon>0$, we can rewrite the expression (4.6) as

$$e_n(t)=e_n^{(1)}(t)+e_n^{(2)}(t), t\in I,$$

where

$$e_n^{(1)}(t) \equiv S(\varepsilon)\bigg(\int_0^{t-\varepsilon}\int_U S(t-\varepsilon-s)G(s,y^o(s),\xi)\big(\mu_s^n-\mu_s^o\big)(\mathrm{d}\xi)\,\mathrm{d}s\bigg), t\in[\varepsilon,T],$$
$$e_n^{(2)}(t) = \int_{t-\varepsilon}^t\int_U S(t-s)G(s,y^o(s),\xi)(\mu_s^n-\mu_s^o)(\mathrm{d}\xi)\,\mathrm{d}s, t\in[\varepsilon,T].$$

Dealing with the first term $e_n^{(1)}$, it follows from weak* convergence of μ^n to μ^o that the integral within the round bracket converges weakly to zero in the Banach space X. Since by assumption the semigroup is compact, the operator $S(\varepsilon)$ is compact and hence the first term converges strongly to zero uniformly with respect to $t\in[\varepsilon,T]$ for any $\varepsilon>0$. In other words, $\lim_{n\to\infty}\sup\{\|e_n^{(1)}(t)\|_X, t\in[\varepsilon,T]\}=0$ for any $\varepsilon>0$. Considering the second term $e_n^{(2)}$ and computing its norm and recalling the assumption **(A3)**(3), we obtain the following estimate

$$\|e_n^{(2)}(t)\|_X \le 2M\hat{K}_2\int_{t-\varepsilon}^t\big(1+\|y^o(s)\|_Y\big)\,\mathrm{d}s, t\in[\varepsilon,T].$$

Since $y^o\in B_\infty(I,Y)$, it is clear that

$$\|e_n^{(2)}(t)\|_X \le 2M\hat{K}_2(1+\|y^o\|_{B_\infty(I,Y)})\,\varepsilon$$

for all $t\in I$. Hence the above integral converges to zero as $\varepsilon\downarrow 0$ uniformly on I. Thus, $\|e_n\|_{B_\infty(I,X)}\xrightarrow{s}0$, and hence it follows from Lebesgue bounded convergence theorem that the expression on the righthand side of the inequality (4.13) converges to zero uniformly with respect to $t\in I$. Hence $z^n\longrightarrow z^o$ in the norm topology of $B_\infty(I,\mathcal{Z})$. This proves the continuity of the map $\mu\longrightarrow z(\mu)$ in the sense as stated in the theorem. □

We make use of the above result to prove the existence of optimal controls. This is presented in the following theorem.

Theorem 4.2 *Consider the partially observed system* (2.1)-(2.2) *with the cost functional given by*

$$J(\mu)\equiv\int_{I\times U}\ell(t,z(t),\xi)\mu_t(\mathrm{d}\xi)\,\mathrm{d}t+\Phi(z(T))\equiv J_1(\mu)+J_2(\mu), \tag{4.14}$$

where $z\in B_\infty(I,\mathcal{Z})\equiv B_\infty(I,X)\times B_\infty(I,Y)$ is the mild solution of the system (2.1)-(2.2) *corresponding to the control $\mu\in\mathcal{M}_{ad}$. Suppose the assumptions of Theorem 4.1 hold and the functions $\ell: I\times\mathcal{Z}\times U\longrightarrow R$ and $\Phi:\mathcal{Z}\longrightarrow R$ are all Borel measurable and satisfy the following properties:*

(1) *$\ell(t,z,\xi)$ is a real valued Borel measurable function defined on $I\times\mathcal{Z}\times U$, continuous in the last two arguments, and for each $t\in I$ it is continuous on $\mathcal{Z}$ uniformly with respect to the third argument $\xi\in U$, and there exist $a\in L_1^+(I)$ and $b\ge 0$ and $p\in[1,\infty)$ such that*

$$|\ell(t,z,\xi)|\le a(t)+b\|z\|_{\mathcal{Z}}^p.$$

(2) *$\Phi:\mathcal{Z}\longrightarrow R$ is lower semicontinuous and there exist $c,d\ge 0$ such that*

$$|\Phi(z)|\le c+d\|z\|_{\mathcal{Z}}^p.$$

Then, there exists an optimal control measure at which J attains its minimum.

Proof. Since by Alaoglu's theorem [11, Theorem V.4.2, p. 424], $\mathcal{M}_{ad} \equiv M_\infty^w(I, \mathcal{M}_0(U))$ is compact in the weak star topology, it suffices to verify that the map $\mu \longrightarrow J(\mu)$ is weak star lower semicontinuous on $\mathcal{M}_{ad}$. Let $\mu^n \xrightarrow{w*} \mu^o$ in $\mathcal{M}_{ad}$. Then it follows from Theorem 4.1 that, (along a subsequence if necessary) $z^n \equiv z(\mu^n) \xrightarrow{s} z(\mu^o) \equiv z^o$ in the norm topology of the Banach space $B_\infty(I, Z)$. First we verify that J_1 is weak star lower semicontinuous. Define the functions $\ell_n(t,\xi) \equiv \ell(t, z^n(t), \xi)$, $\ell_o(t,\xi) \equiv \ell(t, z^o(t), \xi)$ for $(t,\xi) \in I \times U$. By Corollary 3.2, the solution set $\mathcal{S}$ is a bounded subset of $B_\infty(I, \mathcal{Z})$. Thus it follows from the property (1) (of ℓ) that $\ell_n, \ell_o \in L_1(I, C(U))$. Now note that for each $h \in L_1(I, C(U))$ and $\mu \in \mathcal{M}_{ad} \subset L_\infty^w(I, \mathcal{M}_B(U))$, we have the natural duality pairing as follows:

$$< h, \mu >\equiv \int_{I\times U} h(t,\xi)\mu_t(\mathrm{d}\xi)\,\mathrm{d}t.$$

Since $t \longrightarrow \mu_t$ is a probability measure valued function defined on I, it is clear that

$$| < h, \mu > | \leq \|h\|_{L_1(I,C(U))}.$$

Returning to our problem, note that for $\mu^o \in \mathcal{M}_{ad}$ and $\ell_o \in L_1(I, C(U))$, as defined above, we have

$$J_1(\mu^o) =< \ell_o, \mu^o >= \int_{I\times U} \ell_o(t,\xi)\mu_t^o(\mathrm{d}\xi)\,\mathrm{d}t. \tag{4.15}$$

Clearly, by definition of the functional J_1, we have

$$\begin{aligned} J_1(\mu^o) &=< \ell_o, \mu^o >=< \ell_o, \mu^o - \mu^n > + < \ell_o - \ell_n, \mu^n > + < \ell_n, \mu^n > \\ &=< \ell_o, \mu^o - \mu^n > + < \ell_o - \ell_n, \mu^n > + J_1(\mu^n). \end{aligned} \tag{4.16}$$

Since $\mu^n \xrightarrow{w*} \mu^o$, it is clear that the first term on the righthand side of the above expression converges to zero. Thus for any $\varepsilon > 0$, there exists an integer $n_1(\varepsilon)$ such that

$$| < \ell_o, \mu^o - \mu^n > | < \frac{\varepsilon}{2},\ \forall\, n > n_1(\varepsilon). \tag{4.17}$$

Considering the second term, $< \ell_o - \ell_n, \mu^n >$, we show that this also converges to zero as $n \to \infty$. By definition

$$< \ell_o - \ell_n, \mu^n >\equiv \int_{I\times U} \big(\ell_o(t,\xi) - \ell_n(t,\xi)\big)\mu_t^n(\mathrm{d}\xi)\,\mathrm{d}t. \tag{4.18}$$

Since $z^n \xrightarrow{s} z^o$ in $B_\infty(I, \mathcal{Z})$, $z^n(t) \xrightarrow{s} z^o(t)$ in $\mathcal{Z}$ for almost all $t \in I$. Hence it follows from condition (1) as stated in the theorem, $\ell_n \xrightarrow{s} \ell_o$ in the norm topology of $L_1(I, C(U))$. Since $\{\mu^n\}$ are probability measure valued functions, it is clear that

$$| < \ell_o - \ell_n, \mu^n > | \leq \int_{I\times U} |\big(\ell_o(t,\xi) - \ell_n(t,\xi)\big)|\mu_t^n(\mathrm{d}\xi)\,\mathrm{d}t \leq \int_I \|\ell_o - \ell_n\|_{C(U)}(t)\,\mathrm{d}t. \tag{4.19}$$

It follows from continuity of ℓ on $\mathcal{Z}$ uniformly with respect to $\xi \in U$, that

$$\|\ell_o - \ell_n\|_{C(U)}(t) \longrightarrow 0 \text{ for almost all } t \in I.$$

Next, it follows from the growth property (1) of ℓ and the fact that the solution set $\mathcal{S}$ (see Corollary 3.2) is a bounded subset of $B_\infty(I, \mathcal{Z})$, that the function given by the integrand $\|\ell_o - \ell_n\|_{C(U)}(t)$

is dominated by an integrable function on I. Hence it follows from Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty}\int_I \|\ell_o-\ell_n\|_{C(U)}(t)\,\mathrm{d}t \longrightarrow 0. \tag{4.20}$$

Thus for every $\varepsilon>0$, there exists an integer $n_2(\varepsilon)$ such that

$$|<\ell_o-\ell_n,\mu^n>|<\frac{\varepsilon}{2},\ \forall\, n>n_2(\varepsilon). \tag{4.21}$$

Define $n(\varepsilon)\equiv\max\{n_1(\varepsilon),n_2(\varepsilon)\}$. Using the inequalities (4.17) and (4.21) in the expression (4.16), we conclude that

$$J_1(\mu^o)\le\varepsilon+J_1(\mu^n),\ \forall\, n>n(\varepsilon). \tag{4.22}$$

Since $\varepsilon>0$ is arbitrary, it follows from the above expression that

$$J_1(\mu^o)\le\underline{\lim}\,J_1(\mu^n). \tag{4.23}$$

Hence J_1 is weak star lower semicontinuous on $\mathcal{M}_{ad}$. Next, we consider the second term $J_2(\mu^n)\equiv\Phi(z^n(T))$. Since $z^n(T)\xrightarrow{s}z^o(T)$ in $\mathcal{Z}$ as $\mu^n\xrightarrow{w*}\mu^o$ and Φ is lower semicontinuous on $\mathcal{Z}$, it is clear that

$$J_2(\mu^o)\equiv\Phi(z^o(T))\le\underline{\lim}\,\Phi(z^n(T))\equiv\underline{\lim}\,J_2(\mu^n). \tag{4.24}$$

In other words, J_2 is also weak star lower semicontinuous on $\mathcal{M}_{ad}$. Since finite sum of lower semicontinuous functions is lower semicontinuous, it follows from (4.23) and (4.24) that $J(\mu^o)\le\underline{\lim}\,J(\mu^n)$. This proves that J is weak star lower semicontinuous on $\mathcal{M}_{ad}$ and since $\mathcal{M}_{ad}$ is weak star compact, we conclude that there exists a $\mu^o\in\mathcal{M}_{ad}$ at which J attains its minimum. This completes the proof. □

Remark 4.3 *Since $\mu\longrightarrow x(\mu)$ is rarely a convex map, it is clear from the expression for $J(\mu)$ given by* (4.14) *that it is not convex. Hence we cannot expect uniqueness of the control measure. However, using the fact that J is weak star lower semi-continuous on $\mathcal{M}_{ad}$, one can prove that the set of minimizers,*

$$\mathcal{O}_p\equiv\Big\{\mu\in\mathcal{M}_{ad}: J(\mu)=\inf\{J(\varrho),\varrho\in\mathcal{M}_{ad}\}\Big\},$$

is a weak star closed subset of $\mathcal{M}_{ad}$ and hence a weak star compact subset of $\mathcal{M}_{ad}$.

Remark 4.4 *Let $M(I,U)$ denote the class of Borel measurable functions defined on I and taking values in the compact Polish space U. For each $u\in M(I,U)$, define the Dirac measure $\nu_u\equiv\delta_{u(t)}(\mathrm{d}\xi), t\in I$. Then for any $h\in L_1(I,C(U))$, we have*

$$\nu_u(h)=\int_{I\times U}h(t,\xi)\delta_{u(t)}(\mathrm{d}\xi)\,\mathrm{d}t=\int_I h(t,u(t))\,\mathrm{d}t$$

and it follows from this expression that

$$|\nu_u(h)|\le\int_I\sup\{|h(t,\xi)|,\xi\in U\}\,\mathrm{d}t\equiv\int_I\|h(t)\|_{C(U)}\,\mathrm{d}t<\infty.$$

This shows that $M(I,U)$ can be embedded in the topological space $M_\infty^w(I,M_0(U))$. Let i denote the embedding operator: $M(I,U)\overset{i}{\hookrightarrow}M_\infty^w(I,M_0(U))$. In fact $i(M(I,U))$ is the set of extreme points of $M_\infty^w(I,M_0(U))$ and one has

$$\overline{co(i(M(I,U)))}^{w*}=M_\infty^w(I,M_0(U)).$$

This follows from Krein-Milman theorem [11, Theorem V.8.4, p. 440] which, in this particular case, states that a weak star compact convex set coincides with the weak star closed convex hull of its extreme points. Thus if $\mu^o \in M_\infty^w(I, \mathcal{M}_0(U))$ is optimal, then, for every $\varepsilon > 0$, there exists a sequence $\{u_k^o \in M(I,U),\ \alpha_k \geq 0, \sum \alpha_k = 1\}$ such that for every $h \in L_1(I, C(U))$

$$\int_I \Big| \int_U h(t,\xi)\mu_t^o(\mathrm{d}\xi) - \sum_{k\geq 1} \alpha_k h(t, u_k^o(t))\Big| \, \mathrm{d}t < \varepsilon.$$

Some Open Problems:

(P1): In Theorem 3.1 we assumed that $\gamma \in \mathcal{M}_{ca}(\Sigma_I)$ is non-atomic. For a wider scope of applications, in particular where it may be necessary to include measures containing both regular and singular components, it is important to relax this assumption.

(P2): We have not developed the necessary (and or sufficient) conditions of optimality. For computation of optimal control laws this is of great importance.

References

[1] N. U. Ahmed, *Semigroup Theory with Applications to Systems and Control*, Pitman Research Notes in Mathematics Series, Vol. 246, John Wiley & Sons, Inc., New York, 1991.

[2] N. U. Ahmed, *A class of nonlinear evolution equations on Banach spaces driven by finitely additive measures and its optimal control*, Nonlinear Functional Analysis and Applications **24**, no. 4 (2019), pp. 837–864.

[3] N. U. Ahmed, *Some remarks on the dynamics of impulsive systems in Banach spaces*, Dynamics of Continuous, Discrete and Impulsive Systems Series A **8** (2001), pp. 261–274.

[4] N. U. Ahmed, *Impulsive evolution equations in infinite dimensional spaces*, Dynamics of Continuous, Discrete and Impulsive Systems Series A **10** (2003), pp. 11–24.

[5] N. U. Ahmed, *Some recent developments in systems and control theory on infinite dimensional Banach spaces: Part 1 and Part 2, pp. 3–44*, Springer Proceedings in Mathematics and Statistics, Vol. 86(Optimization and Control Techniques and Applications, Edited by H. Xu, K. L. Teo, Y. Zhang), Springer, Berlin, 2014.

[6] N. U. Ahmed, Shi'an Wang, *Measure driven nonlinear dynamic systems with applications to optimal impulsive controls*, Submitted.

[7] N. U. Ahmed, *Existence of optimal controls for a general class of impulsive systems on Banach spaces*, SIAM Journal on Control and Optimization **42** (2003), pp. 669–685.

[8] N. U. Ahmed, *Optimal relaxed controls for systems governed by impulsive differenti inclusions*, Nonlinear Functional Analysis and Applications **10**, no. 3 (2005), pp. 427–460.

[9] N. U. Ahmed, *Optimal structural control of systems determined by strongly nonlinear operator valued measures*, Pure and Applied Functional Analysis **3**, no. 1 (2018), pp. 1–25.

[10] J. Diestel, J. J. Uhl, Jr., *Vector Measures*, American Mathematical Society, Providence, Rhode Island, 1977.

[11] N. Dunford, J. T. Schwartz, *Linear Operators, Part 1: General Theory*, Interscience Publishers Inc., New York, 1964.

[12] Q. Lin, R. Loxton, K. L. Teo, Y. H. Wu, *A new computational method for optimizing nonlinear impulsive systems*, Dynamics of Continuous, Discrete and Impulsive Systems Series B **18** (2011), pp. 59–76.

[13] A. Ionescu Tulcea, C. Ionescu Tulcea, *Topics in the Theory of Lifting*, Springer-Verlag, Berlin, 1969.

[14] C. Z. Wu, K. L. Teo, *Global impulsive optimal control computation*, Journal of Industrial and Management Optimization **2** (2006), pp. 435–450.