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EXISTENCE AND UNIQUENESS OF THE WEAK SOLUTION FOR THE MOORE-GIBSON-THOMPSON EQUATION WITH TERM VISCOELASTIC MEMORY AND NON CLASSICAL CONDITION

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Abstract. This work deals with the existence and uniqueness of a nonlocal mixed boundary value problem for the Moore-Gibson-Thompson equation with term viscoelastic memory. Galerkin's method was the main used tool for proving the existence of the given nonlocal problem.

Keywords: Galerkin method, Exponential decay, Polynomial decay, viscoelastic damping, intrinsic decay rates.

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1 Introduction

In this works, we study the existence and uniqueness of solutions of the following for the Moore-Gibson-Thompson equation with term viscoelastic memory

$$a\,u_{ttt} + \beta\,u_{tt} - c^2\Delta u - b\,\Delta u_t + \int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s = 0\,, \quad (x\,,\,t) \in Q_T\,, \tag{1.1}$$

with initial data

$$u(x\,,\,0) = u_0(x)\,,\; u_t(x\,,\,0) = u_1(x)\,,\; u_{tt}(x\,,\,0) = u_2(x)\,, \quad x \in \Omega\,, \tag{1.2}$$

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and non-classical boundary conditions

$$\frac{\partial u}{\partial \eta} = \int_0^t \int_\Omega u(\xi\,,\,\tau)\;\mathrm{d}\xi \mathrm{d}\tau\,, \quad (x\,,\,t) \in \partial\Omega \times (0\,,\,T)\,, \tag{1.3}$$

where $Q_T := \Omega \times (0\,,\,T)$, $\Omega \in \mathbb{R}^n$, $T < \infty$, $g(\cdot) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are given functions which will be specified later, and $u_0(x)$, $u_1(x)$ and $u_2(x)$ are given functions and $\dfrac{\partial u}{\partial \eta}$ designates the normal derivative.

All the parameters a and b are assumed to be positive constants. In a physical context of the acoustic waves, the variable u denotes a scalar acoustic velocity potential $\overrightarrow{v} = -\nabla u$ with $\overrightarrow{v}$ denoting the acoustic particle velocity, c^2 denotes the speed of sound, a denotes thermal relaxation resulting from replacing Fourier law by the Maxwell-Cattaneo law, the coefficient $b \approxeq \delta + a\,c^2$, where δ is the diffusibility of the sound and the coefficient $\beta > 0$ describes natural damping effects associated with an acoustic environment, see Lebon and Cloot [10]. The convolution term $\int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s$ reflects the memory effects of materials due to viscoelasticity. Here the convolution kernel g satisfies proper conditions exhibiting "memory character" which will be explained later.

This model of (1.1) arises in high-frequency ultrasound applications accounting for thermal flux and molecular relaxation times. According to revisited extended irreversible thermodynamics, thermal flux relaxation leads to the third-order derivative in time while molecular relaxation leads to non-local effects governed by memory terms.

By applying mathematical modeling to various phenomena of physics, ecology and biology, there often arise problems with non-classical boundary conditions, which connect the values of the unknown function on the boundary and inside of the given domain. Sometimes the physical phenomena are modeled by non classical boundary value problems which involve a boundary condition as an integral condition over the spatial domain of a function of the desired solution. The nonlocal boundary condition arises mainly when the data on the boundary cannot be measured directly, but their average values are known. In the very recent years, nonlocal problems, particularly those with integral constraints have received great attention. The physical significance of nonlocal conditions such as a mean, total mass, moments, etc., has served as a fundamental cause for the considerably increasing interest to this kind of boundary value problems. Nonlocal problems are generally encountered in thermoelasticity, plasma physics, chemical engineering, heat transmission and underground water flow. See in this regard the papers by Shi and Shilor [23], Ewing and Lin [5], Choi and Chan [4].

As a special application see Bouziani [3], where the author has considered a nonlocal problem which is proposed in the mathematical modeling of technologic process of external elimination of gas, practices in the refining of impurities of Silicon lamina. The nonlocal condition appearing in this mathematical model represents the total mass of impurities in the lamina. For some hyperbolic non local mixed problems, the reader should consult the works done by Nukushev [21], and Pulkina [22], Muravei and Philinovskii [20], Mesloub and Messaoudi [12, 13], Mesloub and Bouziani [14], Mesloub and Lekrine [15], Beilin [1]. Recent works dealing with nonlinear nonlocal mixed problems can be found in Mesloub [16, 17, 18].

The motivation of our work is due to some results regarding the following research papers:

Boulaaras, Zaraï, Draifia [2] studied the Galerkin method for nonlocal mixed boundary value problem for the Moore-Gibson-Thompson equation with integral condition

$$\begin{cases} a\,u_{ttt} + \beta\,u_{tt} - c^2\Delta u - b\,\Delta u_t = 0\,, & (x\,,\,t) \in Q_T\,, \\ u(x\,,\,0) = u_0(x)\,,\, u_t(x\,,\,0) = u_1(x)\,,\, u_{tt}(x\,,\,0) = u_2(x)\,, & x \in \Omega\,, \\ \dfrac{\partial u}{\partial \eta} = \int_0^t \int_\Omega u(\xi\,,\,\tau)\,\mathrm{d}\xi\mathrm{d}\tau\,, & (x\,,\,t) \in \partial\Omega \times (0\,,\,T)\,, \end{cases}$$

where $Q_T := \Omega \times (0\,,\,T),\ \Omega \in \mathbb{R}^n,\ T < \infty$, and $u_0(x),\ u_1(x)$ and $u_2(x)$ are given functions and $\dfrac{\partial u}{\partial \eta}$ designates the normal derivative.

However, Boulaaras, Zaraï, Draifia [2] did not study the existence and uniqueness of problem (1.1)–(1.3) for $g \neq 0$. Motivated by the above research, we will consider the existence and uniqueness for $g \neq 0$ of the model (1.1)–(1.3) in this paper.

The outline of the paper is as follows. In the second section we present some spaces, and we supply an appropriate definition of weak solution of the posed problem and establish some useful inequalities. Section 3 is devoted to the study of existence of the weak solution of the posed problem by applying Galerkin's method. Finally, in section 4, we prove the uniqueness of the generalized solution of the posed problem.

2 Preliminaries

In this section, we introduce some functional spaces, a definition of generalized solution of problem (1.1)–(1.3) and establish some useful inequalities, which will be used for the remaining of the present paper.

Let $W_2^1(Q_T) := \{u \in L^2(Q_T) : D^\alpha u \in L^2(Q_T)$ for all $|\alpha| \leq 1\}$ be the usual Sobolev space, whose elements u are in $L^2(Q_T)$ along with u_t and u_x, and have in Q_T generalized derivatives u_{xx} and the finite norm

$$\|u\|_{W_2^1(Q_T)} := \left(\int_0^T \int_\Omega [u^2(t) + |\nabla u(t)|^2 + u_t^2(t)]\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}.$$

The scalar product in $W_2^1(Q_T)$ is defined by

$$(u(t)\,,\,v(t))_{W_2^1(Q_T)} := \int_0^T \int_\Omega (u(t)v(t) + \nabla u(t) \cdot \nabla v(t) + u_t(t)v_t(t))\,\mathrm{d}x\mathrm{d}t\,.$$

Now let $V(Q_T)$ and $W(Q_T)$ be the set spaces defined respectively by

$$V(Q_T) := \{u \in W_2^1(Q_T) : u_t \in W_2^1(Q_T)\}\,,$$

and

$$W(Q_T) := \{u \in V(Q_T) : u(x\,,\,T) = 0\}\,.$$

Consider the equation

$$\begin{aligned} &a(u_{ttt}(t)\,,\,v(t))_{L^2(Q_T)} + \beta(u_{tt}(t)\,,\,v(t))_{L^2(Q_T)} - c^2(\Delta u(t)\,,\,v(t))_{L^2(Q_T)} \\ &- b(\Delta u_t(t)\,,\,v(t))_{L^2(Q_T)} + (\int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s\,,\,v(t))_{L^2(Q_T)} = 0\,, \end{aligned} \tag{2.1}$$

where $(\cdot,\cdot)_{L^2(Q_T)}$ stands for the inner product in $L^2(Q_T)$, u is supposed to be a solution of (1.1)–(1.3) and $v \in W(Q_T)$. Evaluation of the inner product in (2.1) and use of boundary condition in (1.1)–(1.3) leads to

$$\begin{aligned}
&- a(u_{tt}(t)\,,\, v_t(t))_{L^2(Q_T)} - \beta(u_t(t)\,,\, v_t(t))_{L^2(Q_T)} + c^2(\nabla u(t)\,,\, \nabla v(t))_{L^2(Q_T)} \\
&+ b(\nabla u_t(t)\,,\, \nabla v(t))_{L^2(Q_T)} - (\int_0^t g(t-s)\nabla u(s)\,\mathrm{d}s\,,\, \nabla v(t))_{L^2(Q_T)} \\
&= c^2 \int_0^T \int_{\partial\Omega} [v(t)(\int_0^t \int_\Omega u(\xi\,,\,\tau)\ \ \mathrm{d}\xi \mathrm{d}\tau)] \mathrm{d}s_x \mathrm{d}t + b\int_0^T \int_{\partial\Omega} [v(t)(\int_\Omega u(\xi\,,\,t)\ \ \mathrm{d}\xi)] \mathrm{d}s_x \mathrm{d}t \\
&- b\int_0^T \int_{\partial\Omega} [v(t)(\int_\Omega u_0(\xi)\ \ \mathrm{d}\xi)] \mathrm{d}s_x \mathrm{d}t + \beta(u_1(x)\,,\, v(x\,,\,0))_{L^2(\Omega)} + a(u_2(x)\,,\, v(x,0))_{L^2(\Omega)} \\
&- \int_0^T \int_{\partial\Omega} [v(t)(\int_0^t g(t-s)\{\int_0^s \int_\Omega u(\xi\,,\,\tau)\ \ \ \mathrm{d}\xi \mathrm{d}\tau\} \mathrm{d}s)] \mathrm{d}s_x \mathrm{d}t\,. \qquad (2.2)
\end{aligned}$$

Definition 2.1 *A function $u \in V(Q_T)$ is called a generalized solution of problem* (1.1)–(1.3) *if it satisfies equation* (2.2) *for each $v \in W(Q_T)$.*

We recall the binary notation

$$(h\Box w)(t) := \int_0^t h(t-s)\|w(x,s) - w(x,t)\|^2_{L^2(\Omega)}\,\mathrm{d}s\,.$$

We give some useful inequalities:

- **Gronwall inequality**. Let $h(t)$ and $y(t)$ be two nonnegative integrable functions on the interval I with $h(t)$ non decreasing. If for any $t \in I$, we have

  $$y(t) \le h(t) + c\int_0^t y(s)\,\mathrm{d}s\,,$$

  where c is a positive constant, then

  $$y(t) \le h(t)\exp(c\,t)\,.$$

- **Cauchy–Schwarz inequality**. If $f\,,\ g \in L^2(\Omega)$, then

  $$(\int_I f(t)g(t)\,\mathrm{d}t)^2 \le \|f(t)\|^2_{L^2(\Omega)} \times \|g(t)\|^2_{L^2(\Omega)}\,.$$

- ε**–Cauchy inequality**. For all $\alpha\,,\ \beta \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}^*_+$, we have

  $$|\alpha\,\beta| \le \frac{\varepsilon}{2}\alpha^2 + \frac{1}{2\,\varepsilon}\beta^2\,.$$

- **Trace inequality**. If $w \in W^2_1(\Omega)$ where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, then

  $$(\mathbf{A}) \qquad \|w(t)\|^2_{L^2(\partial\Omega)} \le \varepsilon\|\nabla w(t)\|^2_{L^2(\Omega)} + l(\varepsilon)\|w(t)\|^2_{L^2(\Omega)}\,, \quad l(\varepsilon) > 0\,,$$

  where $l(\varepsilon)$ is a positive constant that depends only on ε and on the domain Ω.

3 Existence of the generalized solution

We suppose that the kernel $g(t)$ is a $C^1(\mathbb{R}_+, \mathbb{R}_+)$ function satisfying

$(\mathbf{A1})$ $\int_0^\infty g(s)\,\mathrm{d}s < c^2$.

$(\mathbf{A2})$ $g'(t) \le 0$ for all $t \ge 0$.

$(\mathbf{A3})$ $g''(t) \ge 0$ for all $t \ge 0$.

Notation we denote by $\bar{g}$ the following expression

$$\bar{g} := \int_0^\infty g(s)\,\mathrm{d}s\ .$$

We now give the main result on the existence of solution of problem (1.1)–(1.3) and prove it by using the Galerkin method.

Theorem 3.1 *Assume that the hypotheses* $(\mathbf{A1}) - (\mathbf{A3})$ *hold, the initial data* $u_0 \in W_2^1(\Omega),\ u_1 \in W_2^1(\Omega)$ *and* $u_2 \in L^2(\Omega)$, *then there is at least one generalized solution in* $V(Q_T)$ *to problem* (1.1)–(1.3).

Proof. Let $\{Z_k(x)\}_{k\ge1}$ be a fundamental system in $W_2^1(\Omega)$ and assume for convenience that it has been orthonormalized in $L^2(\Omega)$, that is $(Z_k(x)\,,\,Z_l(x))_\Omega = \delta_{k\,,\,l}$. We seek an approximate solution $u^N(x)$ in the form

$$u^N(x\,,\,t) = \sum_{k=1}^N C_k(t) Z_k(x)\ , \tag{3.1}$$

where the constants $C_k(t)$ are defined by the conditions

$$C_k(t) = (u^N(x\,,\,t)\,,\,Z_k(x))_{L^2(\Omega)}\ , \quad k = 1, \cdots, N\ ,$$

and can be determined from the relations for all $l = 1, \cdots, N$

$$\begin{aligned}
&a(u^N_{ttt}\,,\,Z_l(x))_{L^2(\Omega)} + \beta(u^N_{tt}\,,\,Z_l(x))_{L^2(\Omega)} + c^2(\nabla u^N\,,\,\nabla Z_l(x))_{L^2(\Omega)}\\
&+ b(\nabla u^N_t\,,\,\nabla Z_l(x))_{L^2(\Omega)} - (\int_0^t g(t-s)\nabla u^N(s)\,\mathrm{d}s\,,\,\nabla Z_l(x))_{L^2(\Omega)}\\
&= c^2\int_{\partial\Omega} Z_l(x)(\int_0^t\int_\Omega u^N(\xi\,,\,\tau)\ \mathrm{d}\xi\mathrm{d}\tau)\mathrm{d}s_x + b\int_{\partial\Omega} Z_l(x)(\int_0^t\int_\Omega u^N_\tau(\xi\,,\,\tau)\ \mathrm{d}\xi\mathrm{d}\tau)\mathrm{d}s_x\\
&- \int_{\partial\Omega}\left[Z_l(x)(\int_0^t g(t-s)\left\{\int_0^s\int_\Omega u^N(\xi\,,\,\tau)\ \mathrm{d}\xi\mathrm{d}\tau\right\}\mathrm{d}s)\right]\mathrm{d}s_x\ .
\end{aligned} \tag{3.2}$$

Substitution of (3.1) into (3.2) gives for all $l = 1, \cdots, N$

$$\begin{aligned}
&\int_\Omega \sum_{k=1}^{N} \{a\, C_k'''(t) Z_k(x) Z_l(x) + \beta\, C_k''(t) Z_k(x) Z_l(x) + c^2 C_k(t) \nabla Z_k(x) \cdot \nabla Z_l(x) \\
&+ b\, C_k'(t) \nabla Z_k(x) \cdot \nabla Z_l(x) - (\int_0^t g(t-s) C_k(s)\ \mathrm{d}s) \nabla Z_k(x) \cdot \nabla Z_l(x)\} \mathrm{d}x \\
&= c^2 \sum_{k=1}^{N} \int_0^t C_k(\tau) (\int_{\partial\Omega} Z_l(x) (\int_\Omega Z_k(\xi)\ \ \mathrm{d}\xi) \mathrm{d}s_x) \mathrm{d}\tau \\
&+ b \sum_{k=1}^{N} \int_0^t C_k'(\tau) (\int_{\partial\Omega} Z_l(x) (\int_\Omega Z_k(\xi)\ \ \mathrm{d}\xi) \mathrm{d}s_x) \mathrm{d}\tau \\
&- \sum_{k=1}^{N} \int_0^t \left\{ g(t-\tau) \int_0^\tau C_k(\eta)\ \mathrm{d}\eta \right\} (\int_{\partial\Omega} Z_l(x) (\int_\Omega Z_k(\xi)\ \mathrm{d}\xi) \mathrm{d}s_x) \mathrm{d}\tau\ . \qquad (3.3)
\end{aligned}$$

From (3.3) it follows that for all $l = 1, \cdots, N$

$$\begin{aligned}
&\sum_{k=1}^{N} \{a\, C_k'''(t) (Z_k(x)\,,\, Z_l(x))_{L^2(\Omega)} + \beta\, C_k''(t) (Z_k(x)\,,\, Z_l(x))_{L^2(\Omega)} \\
&+ c^2 C_k(t) (\nabla Z_k(x)\,,\, \nabla Z_l(x))_{L^2(\Omega)} + b\, C_k'(t) (\nabla Z_k(x)\,,\, \nabla Z_l(x))_{L^2(\Omega)} \\
&- (\int_0^t g(t-s) C_k(s)\, \mathrm{d}s) (\nabla Z_k(x)\,,\, \nabla Z_l(x))_{L^2(\Omega)}\} \\
&= c^2 \sum_{k=1}^{N} \int_0^t C_k(\tau) (\int_{\partial\Omega} Z_l(x) (\int_\Omega Z_k(\xi)\ \ \mathrm{d}\xi) \mathrm{d}s_x) \mathrm{d}\tau \\
&+ b \sum_{k=1}^{N} \int_0^t C_k'(\tau) (\int_{\partial\Omega} Z_l(x) (\int_\Omega Z_k(\xi)\ \ \mathrm{d}\xi) \mathrm{d}s_x) \mathrm{d}\tau \\
&- \sum_{k=1}^{N} \int_0^t \left\{ g(t-\tau) \int_0^\tau C_k(\eta)\ \mathrm{d}\eta \right\} (\int_{\partial\Omega} Z_l(x) (\int_\Omega Z_k(\xi)\ \mathrm{d}\xi) \mathrm{d}s_x) \mathrm{d}\tau\ .
\end{aligned}$$

If in this last equality, we make the notations

$$(Z_k(x)\,,\, Z_l(x))_{L^2(\Omega)} := \delta_{kl} = \begin{cases} 1\,, & k = l\,, \\ 0\,, & k \neq l\,, \end{cases}$$

and

$$(\nabla Z_k(x)\,,\, \nabla Z_l(x))_{L^2(\Omega)} := \gamma_{kl}\,,$$

and

$$\int_{\partial\Omega} Z_l(x) \int_\Omega Z_k(\xi)\ \mathrm{d}\xi \mathrm{d}s := \chi_{kl}\,,$$

and

$$\int_0^t g(t-s) C_k(s)\, \mathrm{d}s := \zeta(C_k)(t)\,,$$

and

$$g(t-\tau) \int_0^\tau C_k(\eta)\, \mathrm{d}\eta := h(C_k)(\tau)\,,$$

then we have

$$\sum_{k=1}^{N}\{a\,C_k'''(t)\delta_{kl}+\beta\,C_k''(t)\delta_{kl}+c^2C_k(t)\gamma_{kl}+b\,C_k'(t)\gamma_{kl}-\zeta(C_k)(t)\gamma_{kl}$$
$$-\int_0^t\left[c^2C_k(\tau)+b\,C_k'(\tau)-h(C_k)(\tau)\right]\chi_{kl}\,\mathrm{d}\tau\}=0\,. \tag{3.4}$$

Differentiation with respect to t in (3.4) leads to the following system

$$\sum_{k=1}^{N}\left\{a\,C_k''''(t)\delta_{kl}+\beta\,C_k'''(t)\delta_{kl}+c^2C_k'(t)\gamma_{kl}+b\,C_k''(t)\gamma_{kl}-\zeta'(C_k)(t)\gamma_{kl}\right.$$
$$\left.-\,c^2C_k(t)\chi_{kl}-b\,C_k'(t)\chi_{kl}+h(C_k)(t)\chi_{kl}\right\}d\tau=0\,, \tag{3.5}$$

and initial conditions

$$\begin{cases}\sum\limits_{k=1}^{N}\left\{a\,C_k'''(0)\delta_{kl}+\beta\,C_k''(0)\delta_{kl}+c^2C_k(0)\gamma_{kl}+b\,C_k'(0)\gamma_{kl}\right\}=0\,,\\ C_k(0)=(Z_k\,,\,u_0(x))_{L^2(\Omega)},\ C_k'(0)=(Z_k\,,\,u_1(x))_{L^2(\Omega)},\ C_k''(t)=(Z_k\,,\,u_2(x))_{L^2(\Omega)}\,.\end{cases} \tag{3.6}$$

The equation (3.5) form a system of linear ordinary differential equations with constant coefficients of four order in t for the unknown's $C_k(t),\ k=1,\cdot,N$. By a well-known theorem on the solvability of such systems, problem (3.5) is inequality solvable for the initial conditions (3.6). Thus for every n there exists a function $u^N(x)$ satisfying (3.2). Let us obtain bounds for u^N which do not depend on N. To do this, we multiply each equation of (3.2) by the appropriate $C_k'(t)$, add them up from 1 to N and then integrate with respect to t from 0 to τ with $\tau\le T$, we obtain

$$a(u_{ttt}^N(t)\,,\,u_t^N(t))_{L^2(Q_\tau)}+\beta(u_{tt}^N(t)\,,\,u_t^N(t))_{L^2(Q_\tau)}+c^2(\nabla u^N(t)\,,\,\nabla u_t^N(t))_{L^2(Q_\tau)}$$
$$+\,b(\nabla u_t^N(t)\,,\,\nabla u_t^N(t))_{L^2(Q_\tau)}-(\int_0^t g(t-s)\nabla u^N(s)\,\mathrm{d}s\,,\,\nabla u_t^N(t))_{L^2(Q_\tau)}$$
$$=c^2\int_0^\tau\int_{\partial\Omega}u_t^N(x\,,\,t)(\int_0^t\int_\Omega u^N(\xi\,,\,\eta)\quad\mathrm{d}\xi\mathrm{d}\eta)\mathrm{d}s_x\mathrm{d}t$$
$$+\,b\int_0^\tau\int_{\partial\Omega}u_t^N(x\,,\,t)(\int_0^t\int_\Omega u_\eta^N(\xi\,,\,\eta)\quad\mathrm{d}\xi\mathrm{d}\eta)\mathrm{d}s_x\mathrm{d}t$$
$$-\int_0^\tau\int_{\partial\Omega}\left[u_t^N(x\,,\,t)(\int_0^t g(t-s)\left\{\int_0^s\int_\Omega u^N(\xi\,,\,\eta)\quad\mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\mathrm{d}t\,. \tag{3.7}$$

By using direct calculation, we get evaluations of each term of (3.7).

$$a(u_{ttt}^N(t)\,,\,u_t^N(t))_{L^2(Q_\tau)} \tag{3.8}$$
$$=a(u_{\tau\tau}^N(x\,,\,\tau)\,,\,u_\tau^N(x\,,\,\tau))_{L^2(\Omega)}-a(u_{tt}^N(x\,,\,0)\,,\,u_t^N(x\,,\,0))_{L^2(\Omega)}-a\int_0^\tau\|u_{tt}^N(x\,,\,t)\|_{L^2(\Omega)}^2\,\mathrm{d}t\,,$$

and

$$\beta(u_{tt}^N(t)\,,\,u_t^N(t))_{L^2(Q_\tau)}=\frac{\beta}{2}\|u_\tau^N(x\,,\,\tau)\|_{L^2(\Omega)}^2-\frac{\beta}{2}\|u_t^N(x\,,\,0)\|_{L^2(\Omega)}^2\,, \tag{3.9}$$

and

$$c^2(\nabla u^N(t)\,,\,\nabla u_t^N(t))_{L^2(Q_\tau)}=\frac{c^2}{2}\|\nabla u^N(x\,,\,\tau)\|_{L^2(\Omega)}^2-\frac{c^2}{2}\|\nabla u^N(x\,,\,0)\|_{L^2(\Omega)}^2\,, \tag{3.10}$$

and

$$b(\nabla u_t^N(t)\,,\,\nabla u_t^N(t))_{L^2(Q_\tau)} = b\int_0^\tau \|\nabla u_t^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,, \tag{3.11}$$

and

$$\begin{aligned}
&-(\int_0^t g(t-s)\nabla u^N(s)\,\mathrm{d}s\,,\,\nabla u_t^N(t))_{L^2(Q_\tau)}\\
&=\frac{1}{2}(g\Box\nabla u^N)(\tau)-\frac{1}{2}(\int_0^\tau g(s)\,\mathrm{d}s)\|\nabla u^N(\tau)\|^2_{L^2(\Omega)}\\
&-\frac{1}{2}\int_0^\tau (g'\Box\nabla u^N)(t)\,\mathrm{d}t+\frac{1}{2}\int_0^\tau g(t)\|\nabla u^N(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,,
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
&c^2\int_0^\tau\int_{\partial\Omega} u_t^N(t)(\int_0^t\int_\Omega u^N(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta)\mathrm{d}s_x\mathrm{d}t\\
&=c^2\int_{\partial\Omega} u^N(x\,,\,\tau)(\int_0^\tau\int_\Omega u^N(\xi\,,\,t)\ \ \mathrm{d}\xi\mathrm{d}t)\mathrm{d}s_x - c^2\int_{\partial\Omega}\int_0^\tau u^N(x\,,\,t)(\int_\Omega u^N(\xi\,,\,t)\ \ \mathrm{d}\xi)\mathrm{d}t\mathrm{d}s_x\,,
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
&b\int_0^\tau\int_{\partial\Omega} u_t^N(t)(\int_0^t\int_\Omega u_\eta^N(\xi,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta)\mathrm{d}s_x\mathrm{d}t\\
&=b\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)(\int_\Omega u^N(\xi,t)\ \ \mathrm{d}\xi)\mathrm{d}t\mathrm{d}s_x - b\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)(\int_\Omega u^N(\xi,0)\ \ \mathrm{d}\xi)\mathrm{d}t\mathrm{d}s_x\,.
\end{aligned} \tag{3.14}$$

Substitution of equations (3.8)–(3.14) into (3.7), we get

$$\begin{aligned}
&a(u_{\tau\tau}^N(x\,,\,\tau)\,,\,u_\tau^N(x\,,\,\tau))_{L^2(\Omega)}+\frac{\beta}{2}\|u_\tau^N(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&+\frac{1}{2}(c^2-\int_0^\tau g(s)\,\mathrm{d}s)\|\nabla u^N(x\,,\,\tau)\|^2_{L^2(\Omega)}+\frac{1}{2}(g\Box\nabla u^N)(\tau)\\
&=a(u_{tt}^N(x\,,\,0)\,,\,u_t^N(x\,,\,0))_{L^2(\Omega)}+\frac{\beta}{2}\|u_t^N(x\,,\,0)\|^2_{L^2(\Omega)}\\
&+\frac{c^2}{2}\|\nabla u^N(x\,,\,0)\|^2_{L^2(\Omega)}+a\int_0^\tau \|u_{tt}(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\
&-b\int_0^\tau \|\nabla u_t^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t+c^2\int_{\partial\Omega} u^N(x\,,\,\tau)(\int_0^\tau\int_\Omega u^N(\xi\,,\,t)\ \ \mathrm{d}\xi\mathrm{d}t)\mathrm{d}s_x\\
&-c^2\int_{\partial\Omega}\int_0^\tau u^N(x\,,\,t)\int_\Omega u^N(\xi\,,\,t)\ \ \mathrm{d}\xi\mathrm{d}t\mathrm{d}s_x+b\int_{\partial\Omega}\int_0^\tau u_t^N(x\,,\,t)(\int_\Omega u^N(\xi\,,\,t)\ \ \mathrm{d}\xi)\mathrm{d}t\mathrm{d}s_x\\
&-b\int_{\partial\Omega}\int_0^\tau u_t^N(x\,,\,t)(\int_\Omega u^N(\xi\,,\,0)\ \ \mathrm{d}\xi)\mathrm{d}t\mathrm{d}s_x\\
&+\frac{1}{2}\int_0^\tau (g'\Box\nabla u^N)(t)\,\mathrm{d}t-\frac{1}{2}\int_0^\tau g(t)\|\nabla u^N(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\
&-\int_{\partial\Omega}\int_0^\tau\left[u_t^N(x\,,\,t)(\int_0^t g(t-s)\left\{\int_0^s\int_\Omega u^N(\xi\,,\,\eta)\ \ \ \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}t\mathrm{d}s_x\,.
\end{aligned} \tag{3.15}$$

To do this, we multiply each equation of (3.2) by the appropriate $C_k''(t)$, add them up from 1 to

N and then integrate with respect to t from 0 to τ, with $\tau \le T$, we obtain

$$\begin{aligned}
&a(u^N_{ttt}(t)\,,\,u^N_{tt}(t))_{L^2(Q_\tau)} + \beta(u^N_{tt}(t)\,,\,u^N_{tt}(t))_{L^2(Q_\tau)}\\
&+ c^2(\nabla u^N(t)\,,\,\nabla u^N_{tt}(t))_{L^2(Q_\tau)} + b(\nabla u^N_t(t)\,,\,\nabla u^N_{tt}(t))_{L^2(Q_\tau)}\\
&-(\int_0^t g(t-s)\nabla u^N(s)\,\mathrm{d}s\,,\,\nabla u^N_{tt}(t))_{L^2(Q_\tau)}\\
&= c^2\int_0^\tau\int_{\partial\Omega} u^N_{tt}(x\,,\,t)(\int_0^t\int_\Omega u^N(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta)\mathrm{d}s_x\mathrm{d}t\\
&+ b\int_0^\tau\int_{\partial\Omega} u^N_{tt}(x\,,\,t)(\int_0^t\int_\Omega u^N_\eta(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta)\mathrm{d}s_x\mathrm{d}t\\
&-\int_0^\tau\int_{\partial\Omega}\left[u^N_{tt}(x,t)(\int_0^t g(t-s)\left\{\int_0^s\int_\Omega u^N(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\mathrm{d}t\,. \qquad (3.16)
\end{aligned}$$

By using direct calculation, we get evaluations of each term of (3.16).

$$a(u^N_{ttt}(t)\,,\,u^N_{tt}(t))_{L^2(Q_\tau)} = \frac{a}{2}\|u^N_{\tau\tau}(x\,,\,\tau)\|^2_{L^2(\Omega)} - \frac{a}{2}\|u^N_{tt}(x\,,\,0)\|^2_{L^2(\Omega)}\,, \qquad (3.17)$$

and

$$\beta(u^N_{tt}(t)\,,\,u^N_{tt}(t))_{L^2(Q_\tau)} = \beta\int_0^\tau \|u^N_{tt}(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,, \qquad (3.18)$$

and

$$\begin{aligned}
c^2(\nabla u^N(t)\,,\,\nabla u^N_{tt}(t))_{L^2(Q_\tau)} &= c^2(\nabla u^N(x\,,\,\tau)\,,\,\nabla u^N_\tau(x\,,\,\tau))_{L^2(\Omega)}\\
&\quad - c^2(\nabla u^N(x\,,\,0)\,,\,\nabla u^N_t(x\,,\,0))_{L^2(\Omega)}\\
&\quad - c^2\int_0^\tau \|\nabla u^N_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,, \qquad (3.19)
\end{aligned}$$

and

$$b(\nabla u^N_t(t)\,,\,\nabla u^N_{tt}(t))_{L^2(Q_\tau)} = \frac{b}{2}\|\nabla u^N_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} - \frac{b}{2}\|\nabla u^N_t(x\,,\,0)\|^2_{L^2(\Omega)}\,, \qquad (3.20)$$

and

$$\begin{aligned}
&-(\int_0^t g(t-s)\nabla u^N(s)\,\mathrm{d}s\,,\,\nabla u^N_{tt}(t))_{L^2(Q_\tau)}\\
&= \frac{1}{2}(-g'\Box\nabla u^N)(\tau) + \frac{1}{2}g(\tau)\|\nabla u^N(\tau)\|^2_{L^2(\Omega)}\\
&- \int_0^\tau g(\tau-s)(\nabla u^N(s)\,,\,\nabla u^N_\tau(\tau))_{L^2(\Omega)}\,\mathrm{d}s - \frac{g(0)}{2}\|\nabla u^N(0)\|^2_{L^2(\Omega)}\\
&+ \frac{1}{2}\int_0^\tau (g''\Box\nabla u^N)(t)\,\mathrm{d}t - \frac{1}{2}\int_0^\tau g'(t)\|\nabla u^N(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,, \qquad (3.21)
\end{aligned}$$

and

$$\begin{aligned}
&c^2\int_0^\tau\int_{\partial\Omega} u^N_{tt}(t)(\int_0^t\int_\Omega u^N(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta)\mathrm{d}s_x\mathrm{d}t \qquad (3.22)\\
&= c^2\int_{\partial\Omega} u^N_\tau(x\,,\,\tau)(\int_0^\tau\int_\Omega u^N(\xi\,,\,t)\ \ \mathrm{d}\xi\mathrm{d}t)\mathrm{d}s_x - c^2\int_{\partial\Omega}\int_0^\tau u^N_t(x\,,\,t)(\int_\Omega u^N(\xi\,,\,t)\ \ \mathrm{d}\xi)\mathrm{d}t\mathrm{d}s_x\,,
\end{aligned}$$

and

$$b\int_0^\tau\int_{\partial\Omega}u_{tt}^N(x\,,\,t)(\int_0^t\int_\Omega u_\eta^N(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta)\mathrm{d}s_x\mathrm{d}t = b\int_{\partial\Omega}u_\tau^N(x\,,\,\tau)(\int_\Omega u^N(\xi\,,\,\tau)\ \mathrm{d}\xi)\mathrm{d}s_x$$
$$-\,b\int_{\partial\Omega}u_\tau^N(x\,,\,\tau)(\int_\Omega u^N(\xi\,,\,0)\ \mathrm{d}\xi)\mathrm{d}s_x - b\int_{\partial\Omega}\int_0^\tau u_t^N(x\,,\,t)(\int_\Omega u_t^N(\xi\,,\,t)\ \ \mathrm{d}\xi)\mathrm{d}t\mathrm{d}s_x\,, \quad (3.23)$$

and

$$\begin{aligned}
&-\int_0^\tau\int_{\partial\Omega}\left[u_{tt}^N(x\,,\,t)(\int_0^t g(t-s)\left\{\int_0^s\int_\Omega u^N(\xi\,,\,\eta)\quad \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\mathrm{d}t\\
&= -\int_{\partial\Omega}\left[u_\tau^N(x\,,\,\tau)(\int_0^\tau g(\tau-s)\left\{\int_0^s\int_\Omega u^N(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\\
&+\int_{\partial\Omega}\int_0^\tau\left[u_t^N(x\,,\,t)(\int_0^t g'(t-s)\left\{\int_0^s\int_\Omega u^N(\xi\,,\,\eta)\quad \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}t\mathrm{d}s_x\\
&+g(0)\int_{\partial\Omega}\int_0^\tau\left[u_t^N(x\,,\,t)(\int_0^t\int_\Omega u^N(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta)\right]\mathrm{d}t\mathrm{d}s_x\,. \qquad (3.24)
\end{aligned}$$

Substitution of equalities (3.17)–(3.24) into (3.16) results in

$$\begin{aligned}
&\frac{a}{2}\|u_{\tau\tau}^N(x,\tau)\|_{L^2(\Omega)}^2 + c^2(\nabla u^N(x\,,\,\tau)\,,\,\nabla u_\tau^N(x\,,\,\tau))_{L^2(\Omega)}\\
&+\frac{b}{2}\|\nabla u_\tau^N(x\,,\,\tau)\|_{L^2(\Omega)}^2 + \frac{1}{2}(-g'\Box\nabla u^N)(\tau) + \frac{1}{2}g(\tau)\|\nabla u^N(\tau)\|_{L^2(\Omega)}^2\\
&-\int_0^\tau g(\tau-s)(\nabla u^N(s)\,,\,\nabla u_\tau^N(\tau))_{L^2(\Omega)}\,\mathrm{d}s\\
&=\frac{a}{2}\|u_{tt}^N(x\,,\,0)\|_{L^2(\Omega)}^2 + c^2(\nabla u^N(x\,,\,0)\,,\,\nabla u_t^N(x\,,\,0))_{L^2(\Omega)}\\
&+\frac{b}{2}\|\nabla u_t^N(x\,,\,0)\|_{L^2(\Omega)}^2 + \frac{g(0)}{2}\|\nabla u^N(x\,,\,0)\|_{L^2(\Omega)}^2\\
&-\beta\int_0^\tau\|u_{tt}^N(x\,,\,t)\|_{L^2(\Omega)}^2\,\mathrm{d}t + c^2\int_0^\tau\|\nabla u_t^N(x\,,\,t)\|_{L^2(\Omega)}^2\,\mathrm{d}t\\
&+\frac{1}{2}\int_0^\tau g'(t)\|\nabla u^N(t)\|_{L^2(\Omega)}^2\,\mathrm{d}t - \frac{1}{2}\int_0^\tau(g''\Box\nabla u^N)(t)\,\mathrm{d}t\\
&+c^2\int_{\partial\Omega}u_\tau^N(x\,,\,\tau)(\int_0^\tau\int_\Omega u^N(\xi\,,\,t)\ \ \mathrm{d}\xi\mathrm{d}t)\mathrm{d}s_x\\
&-c^2\int_{\partial\Omega}\int_0^\tau u_t^N(x\,,\,t)(\int_\Omega u^N(\xi\,,\,t)\ \ \mathrm{d}\xi)\mathrm{d}t\mathrm{d}s_x\\
&+b\int_{\partial\Omega}u_\tau^N(x\,,\,\tau)(\int_\Omega u^N(\xi\,,\,\tau)\ \mathrm{d}\xi)\mathrm{d}s_x - b\int_{\partial\Omega}u_\tau^N(x\,,\,\tau)(\int_\Omega u^N(\xi\,,\,0)\ \mathrm{d}\xi)\mathrm{d}s_x\\
&-b\int_{\partial\Omega}\int_0^\tau u_t^N(x\,,\,t)(\int_\Omega u_t^N(\xi\,,\,t)\ \ \mathrm{d}\xi)\mathrm{d}t\mathrm{d}s_x\\
&-\int_{\partial\Omega}\left[u_\tau^N(x\,,\,\tau)(\int_0^\tau g(\tau-s)\left\{\int_0^s\int_\Omega u^N(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\\
&+\int_{\partial\Omega}\int_0^\tau\left[u_t^N(x\,,\,t)(\int_0^t g'(t-s)\left\{\int_0^s\int_\Omega u^N(\xi\,,\,\eta)\quad \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}t\mathrm{d}s_x\\
&+g(0)\int_{\partial\Omega}\int_0^\tau\left[u_t^N(x\,,\,t)(\int_0^t\int_\Omega u^N(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta)\right]\mathrm{d}t\mathrm{d}s_x\,. \qquad (3.25)
\end{aligned}$$

On multiply (3.15) by k and summing in (3.25), we get

$$\begin{aligned}
&\frac{k}{2}(c^2-\int_0^\tau g(s)\,\mathrm{d}s)\|\nabla u^N(x\,,\,\tau)\|^2_{L^2(\Omega)}+\frac{1}{2}g(\tau)\|\nabla u^N(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&+c^2(\nabla u^N(x\,,\,\tau)\,,\,\nabla u^N_\tau(x\,,\,\tau))_{L^2(\Omega)}+\frac{k\,\beta}{2}\|u^N_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&+\frac{b}{2}\|\nabla u^N_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}+\frac{a}{2}\|u^N_{\tau\tau}(x\,,\,\tau)\|^2_{L^2(\Omega)}+k\,a(u^N_{\tau\tau}(x\,,\,\tau)\,,\,u^N_\tau(x\,,\,\tau))_{L^2(\Omega)}\\
&+\frac{k}{2}(g\Box\nabla u^N)(\tau)+\frac{1}{2}(-g'\Box\nabla u^N)(\tau)-\int_0^\tau g(\tau-s)(\nabla u^N(s)\,,\,\nabla u^N_\tau(\tau))_{L^2(\Omega)}\,\mathrm{d}s\\
&=\frac{k\,c^2+g(0)}{2}\|\nabla u^N(x\,,\,0)\|^2_{L^2(\Omega)}+c^2(\nabla u^N(x\,,\,0)\,,\,\nabla u^N_t(x\,,\,0))_{L^2(\Omega)}+\frac{k\,\beta}{2}\|u^N_t(x\,,\,0)\|^2_{L^2(\Omega)}\\
&+\frac{b}{2}\|\nabla u^N_t(x\,,\,0)\|^2_{L^2(\Omega)}+\frac{a}{2}\|u^N_{tt}(x\,,\,0)\|^2_{L^2(\Omega)}+k\,a(u^N_{tt}(x\,,\,0)\,,\,u^N_t(x\,,\,0))_{L^2(\Omega)}\\
&+\frac{1}{2}\int_0^\tau g'(t)\|\nabla u^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t-\frac{k}{2}\int_0^\tau g(t)\|\nabla u^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\
&+(c^2-k\,b)\int_0^\tau\|\nabla u^N_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t+(k\,a-\beta)\int_0^\tau\|u_{tt}(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\
&-\frac{1}{2}\int_0^\tau(g''\Box\nabla u^N)(t)\,\mathrm{d}t+\frac{k}{2}\int_0^\tau(g'\Box\nabla u^N)(t)\,\mathrm{d}t\\
&+k\,c^2\int_{\partial\Omega}u^N(x\,,\,\tau)(\int_0^\tau\int_\Omega u^N(\xi\,,\,t)\;\,\mathrm{d}\xi\mathrm{d}t)\mathrm{d}s_x+c^2\int_{\partial\Omega}u^N_\tau(x\,,\,\tau)(\int_0^\tau\int_\Omega u^N(\xi\,,\,t)\;\,\mathrm{d}\xi\mathrm{d}t)\mathrm{d}s_x\\
&+b\int_{\partial\Omega}u^N_\tau(x\,,\,\tau)(\int_\Omega u^N(\xi\,,\,\tau)\;\mathrm{d}\xi)\mathrm{d}s_x-b\int_{\partial\Omega}u^N_\tau(x\,,\,\tau)(\int_\Omega u^N(\xi\,,\,0)\;\mathrm{d}\xi)\mathrm{d}s_x\\
&-\int_{\partial\Omega}\left[u^N_\tau(x\,,\,\tau)(\int_0^\tau g(\tau-s)\left\{\int_0^s\int_\Omega u^N(\xi\,,\,\eta)\;\;\mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\\
&-k\,c^2\int_{\partial\Omega}\int_0^\tau u^N(x\,,\,t)\int_\Omega u^N(\xi\,,\,t)\;\,\mathrm{d}\xi\mathrm{d}t\mathrm{d}s_x+(k\,b-c^2)\int_{\partial\Omega}\int_0^\tau u^N_t(x\,,\,t)(\int_\Omega u^N(\xi\,,\,t)\;\,\mathrm{d}\xi)\mathrm{d}t\mathrm{d}s_x\\
&-k\,b\int_{\partial\Omega}\int_0^\tau u^N_t(x\,,\,t)(\int_\Omega u^N(\xi\,,\,0)\;\,\mathrm{d}\xi)\mathrm{d}t\mathrm{d}s_x\\
&-k\int_{\partial\Omega}\int_0^\tau\left[u^N_t(x\,,\,t)(\int_0^t g(t-s)\left\{\int_0^s\int_\Omega u^N(\xi\,,\,\eta)\;\;\mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}t\mathrm{d}s_x\\
&-b\int_{\partial\Omega}\int_0^\tau u^N_t(x\,,\,t)(\int_\Omega u^N_t(\xi\,,\,t)\;\,\mathrm{d}\xi)\mathrm{d}t\mathrm{d}s_x\\
&+\int_{\partial\Omega}\int_0^\tau\left[u^N_t(x\,,\,t)(\int_0^t g'(t-s)\left\{\int_0^s\int_\Omega u^N(\xi\,,\,\eta)\;\;\mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}t\mathrm{d}s_x\\
&+g(0)\int_{\partial\Omega}\int_0^\tau\left[u^N_t(x\,,\,t)(\int_0^t\int_\Omega u^N(\xi\,,\,\eta)\;\;\mathrm{d}\xi\mathrm{d}\eta)\right]\mathrm{d}t\mathrm{d}s_x\,, \qquad (3.26)
\end{aligned}$$

where $0<k<\dfrac{1}{2}$. By using Young's inequality (for $\varepsilon=\varepsilon_1$), ($\mathbf{A}$) and Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&k\,c^2\int_{\partial\Omega}u^N(x\,,\,\tau)(\int_0^\tau\int_\Omega u^N(\xi\,,\,t)\;\,\mathrm{d}\xi\mathrm{d}t)\mathrm{d}s_x \qquad (3.27)\\
&\le\frac{k\,c^2}{2\,\varepsilon_1}(\varepsilon\|\nabla u^N(x\,,\,\tau)\|^2_{L^2(\Omega)}+l(\varepsilon)\|u^N(x\,,\,\tau)\|^2_{L^2(\Omega)})+\frac{k\,c^2\,\varepsilon_1\,T\|\Omega\|\|\partial\Omega\|}{2}\int_0^\tau\|u^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,.
\end{aligned}$$

By using Young's inequality (for $\varepsilon = \varepsilon_2$), (**A**) and Cauchy-Schwarz inequality, we get

$$\begin{aligned} & c^2 \int_{\partial\Omega} u_\tau^N(x\,,\,\tau)(\int_0^\tau \int_\Omega u^N(\xi\,,\,t)\ \mathrm{d}\xi \mathrm{d}t)\mathrm{d}s_x \\ & \leq \frac{c^2}{2\,\varepsilon_2}(\varepsilon \|\nabla u_\tau^N(x\,,\,\tau)\|^2_{L^2(\Omega)} + l(\varepsilon)\|u_\tau^N(x\,,\,\tau)\|^2_{L^2(\Omega)}) \\ & + \frac{c^2\,\varepsilon_2 \|\Omega\|\|\partial\Omega\| T}{2} \int_0^\tau \|u^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,. \end{aligned} \tag{3.28}$$

By using Young's inequality (for $\varepsilon = \varepsilon_3$), (**A**) and Cauchy-Schwarz inequality, we get

$$\begin{aligned} & b \int_{\partial\Omega} u_\tau^N(x\,,\,\tau)(\int_\Omega u^N(\xi\,,\,\tau)\ \mathrm{d}\xi)\mathrm{d}s_x \\ & \leq \frac{b}{2\,\varepsilon_3}(\varepsilon \|\nabla u_\tau^N(x\,,\,\tau)\|^2_{L^2(\Omega)} + l(\varepsilon)\|u_\tau^N(x\,,\,\tau)\|^2_{L^2(\Omega)}) + \frac{b\,\varepsilon_3 \|\Omega\|\|\partial\Omega\|}{2}\|u^N(x\,,\,\tau)\|^2_{L^2(\Omega)}\,. \end{aligned} \tag{3.29}$$

By using Young's inequality (for $\varepsilon = \varepsilon_4$), (**A**) and Cauchy-Schwarz inequality, we get

$$\begin{aligned} & -b \int_{\partial\Omega} u_\tau^N(x\,,\,\tau)(\int_\Omega u^N(\xi\,,\,0)\ \mathrm{d}\xi)\mathrm{d}s_x \\ & \leq \frac{b}{2\,\varepsilon_4}(\varepsilon \|\nabla u_\tau^N(x\,,\,\tau)\|^2_{L^2(\Omega)} + l(\varepsilon)\|u_\tau^N(x\,,\,\tau)\|^2_{L^2(\Omega)}) + \frac{b\,\varepsilon_4 \|\Omega\|\|\partial\Omega\|}{2}\|u^N(x\,,\,0)\|^2_{L^2(\Omega)}\,. \end{aligned} \tag{3.30}$$

By using Young's inequality (for $\varepsilon = \varepsilon_5$), (**A**), Cauchy-Schwarz inequality and integration by parts, we get

$$\begin{aligned} & -\int_{\partial\Omega}\left[u_\tau^N(x\,,\,\tau)(\int_0^\tau g(\tau-s)\left\{\int_0^s\int_\Omega u^N(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x \\ & \leq \frac{1}{2\,\varepsilon_5}(\varepsilon \|\nabla u_\tau^N(x\,,\,\tau)\|^2_{L^2(\Omega)} + l(\varepsilon)\|u_\tau^N(x\,,\,\tau)\|^2_{L^2(\Omega)}) \\ & + \frac{\varepsilon_5(\sup g(t))^2\,T^3\|\Omega\|\|\partial\Omega\|}{4}\int_0^\tau \|u^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,. \end{aligned} \tag{3.31}$$

By using Young's inequality (for $\varepsilon = 1$), (**A**) and Cauchy-Schwarz inequality, we get

$$\begin{aligned} & -k\,c^2 \int_{\partial\Omega}\int_0^\tau u^N(x\,,\,t)(\int_\Omega u^N(\xi\,,\,t)\ \mathrm{d}\xi)\mathrm{d}t\mathrm{d}s_x \\ & \leq \frac{k\,c^2\,\varepsilon}{2}\int_0^\tau \|\nabla u^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{k\,c^2(l(\varepsilon)+\|\Omega\|\|\partial\Omega\|)}{2}\int_0^\tau \|u^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,. \end{aligned} \tag{3.32}$$

By using Young's inequality (for $\varepsilon = 1$), (**A**) and Cauchy-Schwarz inequality, we get

$$\begin{aligned} & (k\,b - c^2)\int_{\partial\Omega}\int_0^\tau u_t^N(x\,,\,t)(\int_\Omega u^N(\xi\,,\,t)\ \mathrm{d}\xi)\mathrm{d}t\mathrm{d}s_x \\ & \leq \frac{(k\,b + c^2)}{2}(\varepsilon \int_0^\tau \|\nabla u_t^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + l(\varepsilon)\int_0^\tau \|u_t^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t) \\ & + \frac{(k\,b + c^2)\|\Omega\|\|\partial\Omega\|}{2}\int_0^\tau \|u^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,. \end{aligned} \tag{3.33}$$

By using Young's inequality (for $\varepsilon = 1$), $(\mathbf{A})$ and Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&- k\, b \int_{\partial\Omega} \int_0^\tau u_t^N(x\,,\,t)(\int_\Omega u^N(\xi\,,\,0)\ \ \mathrm{d}\xi)\mathrm{d}t\mathrm{d}s_x \\
&\leq \frac{k\, b}{2}(\varepsilon \int_0^\tau \|\nabla u_t^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + l(\varepsilon)\int_0^\tau \|u_t^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t) \\
&+ \frac{k\, b\|\Omega\|\|\partial\Omega\| T}{2}\|u^N(x\,,\,0)\|^2_{L^2(\Omega)}\,.
\end{aligned} \tag{3.34}$$

By using Young's inequality (for $\varepsilon = 1$), $(\mathbf{A})$, Cauchy-Schwarz inequality and integration by parts, we get

$$\begin{aligned}
&- k \int_{\partial\Omega} \int_0^\tau \left[u_t^N(x\,,\,t)(\int_0^t g(t-s)\left\{\int_0^s \int_\Omega u^N(\xi\,,\,\eta)\quad \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}t\mathrm{d}s_x \\
&\leq \frac{k}{2}(\varepsilon \int_0^\tau \|\nabla u_t^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + l(\varepsilon)\int_0^\tau \|u_t^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t) \\
&+ \frac{k(\sup g(t))^2\, T^4 \|\Omega\|\|\partial\Omega\|}{4}\int_0^\tau \|u^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,.
\end{aligned} \tag{3.35}$$

By using Young's inequality (for $\varepsilon = 1$), $(\mathbf{A})$ and Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&- b \int_{\partial\Omega} \int_0^\tau u_t^N(x\,,\,t)(\int_\Omega u^N(\xi\,,\,t)\ \ \mathrm{d}\xi)\mathrm{d}t\mathrm{d}s_x \\
&\leq \frac{b\,\varepsilon}{2}\int_0^\tau \|\nabla u_t^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{b(l(\varepsilon) + \|\Omega\|\|\partial\Omega\|)}{2}\int_0^\tau \|u_t^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,.
\end{aligned} \tag{3.36}$$

By using Young's inequality (for $\varepsilon = 1$), $(\mathbf{A})$, Cauchy-Schwarz inequality and integration by parts, we get

$$\begin{aligned}
&\int_{\partial\Omega} \int_0^\tau \left[u_t^N(x\,,\,t)(\int_0^t g'(t-s)\left\{\int_0^s \int_\Omega u^N(\xi\,,\,\eta)\quad \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}t\mathrm{d}s_x \\
&\leq \frac{1}{2}(\varepsilon \int_0^\tau \|\nabla u_t^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + l(\varepsilon)\int_0^\tau \|u_t^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t) \\
&+ \frac{(\sup \|g'(t)\|)^2\, T^4 \|\Omega\|\|\partial\Omega\|}{4}\int_0^\tau \|u^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,.
\end{aligned} \tag{3.37}$$

By using Young's inequality (for $\varepsilon = 1$), $(\mathbf{A})$, Cauchy-Schwarz inequality and integration by parts, we get

$$\begin{aligned}
&g(0) \int_{\partial\Omega} \int_0^\tau \left[u_t^N(x\,,\,t)(\int_0^t \int_\Omega u^N(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta)\right]\mathrm{d}t\mathrm{d}s_x \\
&\leq \frac{g(0)}{2}(\varepsilon \int_0^\tau \|\nabla u_t^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + l(\varepsilon)\int_0^\tau \|u_t^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t) \\
&+ \frac{g(0)\|\partial\Omega\|\|\Omega\| T^2}{4}\int_0^\tau \|u^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,.
\end{aligned} \tag{3.38}$$

By using Young's inequality (for $\varepsilon = \varepsilon_6$), we get

$$- \frac{c^2\,\varepsilon_6}{2}\|\nabla u^N(x\,,\,\tau)\|^2_{L^2(\Omega)} - \frac{c^2}{2\,\varepsilon_6}\|\nabla u_\tau^N(x\,,\,\tau)\|^2_{L^2(\Omega)} \leq c^2(\nabla u^N(x\,,\,\tau)\,,\,\nabla u_\tau^N(x\,,\,\tau))_{L^2(\Omega)}\,. \tag{3.39}$$

By using Young's inequality (for $\varepsilon = 1$), we get

$$-\frac{k\,a}{2}\|u^N_{\tau\tau}(x\,,\,\tau)\|^2_{L^2(\Omega)} - \frac{k\,a}{2}\|u^N_{\tau}(x\,,\,\tau)\|^2_{L^2(\Omega)} \leq k\,a(u^N_{\tau\tau}(x\,,\,\tau)\,,\,u^N_{\tau}(x\,,\,\tau))_{L^2(\Omega)}\,. \tag{3.40}$$

By using Young's inequality (for $\varepsilon = \varepsilon_7$ and $\varepsilon = \varepsilon_8$) and using $\int_0^\tau g(s)\,\mathrm{d}s \leq \bar{g}$, we get

$$\begin{aligned}&-\int_0^\tau g(\tau - s)(\nabla u^N(s)\,,\,\nabla u^N_\tau(\tau))_{L^2(\Omega)}\,\mathrm{d}s\\ &\leq \frac{\varepsilon_7}{2}(g\Box\nabla u^N)(\tau) + \left\{\frac{1}{2\,\varepsilon_7} + \frac{1}{2\,\varepsilon_8}\right\}\bar{g}\|\nabla u^N_\tau(\tau)\|^2_{L^2(\Omega)} + \frac{\varepsilon_8\,\bar{g}}{2}\|\nabla u^N(\tau)\|^2_{L^2(\Omega)}\,.\end{aligned} \tag{3.41}$$

By using Young's inequality (for $\varepsilon = 1$), we get

$$c^2(\nabla u^N(x\,,\,0)\,,\,\nabla u^N_t(x\,,\,0))_{L^2(\Omega)} \leq \frac{c^2}{2}\|\nabla u^N(x\,,\,0)\|^2_{L^2(\Omega)} + \frac{c^2}{2}\|\nabla u^N_t(x\,,\,0)\|^2_{L^2(\Omega)}\,. \tag{3.42}$$

By using Young's inequality (for $\varepsilon = 1$), we get

$$k\,a(u^N_{tt}(x\,,\,0)\,,\,u^N_t(x\,,\,0))_{L^2(\Omega)} \leq \frac{k\,a}{2}\|u^N_{tt}(x\,,\,0)\|^2_{L^2(\Omega)} + \frac{k\,a}{2}\|u^N_t(x\,,\,0)\|^2_{L^2(\Omega)}\,. \tag{3.43}$$

Combining inequalities (3.27)–(3.43) into (3.26) and make use of the following inequalities

$$\begin{aligned}m_1\|u^N(x\,,\,\tau)\|^2_{L^2(\Omega)} &\leq m_1\|u^N(x\,,\,t)\|^2_{L^2(Q_\tau)} + m_1\|u^N_t(x\,,\,t)\|^2_{L^2(Q_\tau)} + m_1\|u^N(x\,,\,0)\|^2_{L^2(\Omega)}\,,\\ m_2\|u^N_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} &\leq m_2\|u^N_t(x\,,\,t)\|^2_{L^2(Q_\tau)} + m_2\|u^N_{tt}(x\,,\,t)\|^2_{L^2(Q_\tau)} + m_2\|u^N_t(x\,,\,0)\|^2_{L^2(\Omega)}\,,\end{aligned}$$

and

$$\begin{aligned}&m_3\|\nabla u^N(x\,,\,\tau)\|^2_{L^2(\Omega)}\\ &\leq m_3\|\nabla u^N(x\,,\,t)\|^2_{L^2(Q_\tau)} + m_3\|\nabla u^N_t(x\,,\,t)\|^2_{L^2(Q_\tau)} + m_3\|\nabla u^N(x\,,\,0)\|^2_{L^2(\Omega)}\,,\end{aligned}$$

we have

$$\begin{aligned}
&\left\{m_1 - \frac{k\,c^2\,l(\varepsilon)}{2\,\varepsilon_1} - \frac{b\,\varepsilon_3\|\Omega\|\|\partial\Omega\|}{2}\right\}\|u^N(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&+\left\{m_3 + \frac{k(c^2-\bar g)}{2} - \frac{c^2\,\varepsilon_6}{2} - \frac{\varepsilon_8\,\bar g}{2} - \frac{k\,c^2\,\varepsilon}{2\,\varepsilon_1}\right\}\|\nabla u^N(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&+\left\{m_2 + \frac{k\,\beta}{2} - \frac{k\,a}{2} - \frac{c^2\,l(\varepsilon)}{2\,\varepsilon_2} - \frac{b\,l(\varepsilon)}{2\,\varepsilon_3} - \frac{b\,l(\varepsilon)}{2\,\varepsilon_4} - \frac{l(\varepsilon)}{2\,\varepsilon_5}\right\}\|u^N_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&+\left\{\frac{b}{2} - \frac{c^2}{2\,\varepsilon_6} - \frac{\bar g}{2\,\varepsilon_7} - \frac{\bar g}{2\,\varepsilon_8} - \frac{c^2\,\varepsilon}{2\,\varepsilon_2} - \frac{b\,\varepsilon}{2\,\varepsilon_3} - \frac{b\,\varepsilon}{2\,\varepsilon_4} - \frac{\varepsilon}{2\,\varepsilon_5}\right\}\|\nabla u^N_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&+\frac{a(1-k)}{2}\|u^N_{\tau\tau}(x\,,\,\tau)\|^2_{L^2(\Omega)} + \frac{k-\varepsilon_7}{2}(g\Box\nabla u^N)(\tau)\\
&\le\left\{m_1 + \frac{b\,\varepsilon_4\|\Omega\|\|\partial\Omega\|}{2} + \frac{k\,b\|\Omega\|\|\partial\Omega\|T}{2}\right\}\|u^N(x\,,\,0)\|^2_{L^2(\Omega)}\\
&+\left\{m_3 + \frac{k\,c^2+g(0)}{2} + \frac{c^2}{2}\right\}\|\nabla u^N(x\,,\,0)\|^2_{L^2(\Omega)}\\
&+\left\{m_2 + \frac{k\,\beta}{2} + \frac{b}{2} + \frac{k\,a}{2}\right\}\|u^N_t(x\,,\,0)\|^2_{L^2(\Omega)} + \frac{c^2}{2}\|\nabla u^N_t(x\,,\,0)\|^2_{L^2(\Omega)}\\
&+\frac{a(1+k)}{2}\|u^N_{tt}(x\,,\,0)\|^2_{L^2(\Omega)} + \gamma_1\int_0^\tau\|u^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\
&+\left\{\frac{k\,c^2\,\varepsilon}{2} + m_3\right\}\int_0^\tau\|\nabla u^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + \gamma_2\int_0^\tau\|u^N_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\
&+\gamma_3\int_0^\tau\|\nabla u^N_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + (k\,a+\beta+m_2)\int_0^\tau\|u_{tt}(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,,
\end{aligned}\tag{3.44}$$

where

$$\begin{aligned}
\gamma_1 :=&\ \frac{k\,c^2\,\varepsilon_1\,T\|\Omega\|\|\partial\Omega\|}{2} + \frac{c^2\,\varepsilon_2\|\Omega\|\|\partial\Omega\|T}{2} + \frac{\varepsilon_5(\sup g(t))^2\,T^3\|\Omega\|\|\partial\Omega\|}{4}\\
&+\frac{k\,c^2(l(\varepsilon)+\|\Omega\|\|\partial\Omega\|)}{2} + \frac{(k\,b+c^2)\|\Omega\|\|\partial\Omega\|}{2} + \frac{k(\sup g(t))^2\,T^4\|\Omega\|\|\partial\Omega\|}{4}\\
&+\frac{(\sup\|g'(t)\|)^2\,T^4\|\Omega\|\|\partial\Omega\|}{4} + \frac{g(0)\|\partial\Omega\|\|\Omega\|T^2}{4} + m_1\,,\\
\gamma_2 :=&\ \frac{(k\,b+c^2)l(\varepsilon)}{2} + \frac{k\,b\,l(\varepsilon)}{2} + \frac{k\,l(\varepsilon)}{2} + \frac{b(l(\varepsilon)+\|\Omega\|\|\partial\Omega\|)}{2} + \frac{l(\varepsilon)}{2} + \frac{g(0)l(\varepsilon)}{2} + m_1 + m_2\,,\\
\gamma_3 :=&\ c^2 + k\,b + \frac{(k\,b+c^2)\varepsilon}{2} + \frac{k\,b\,\varepsilon}{2} + \frac{k\,\varepsilon}{2} + \frac{b\,\varepsilon}{2} + \frac{\varepsilon}{2} + \frac{g(0)\varepsilon}{2} + m_3\,.
\end{aligned}$$

Let

$$\begin{cases}
m_1 := \dfrac{k\,c^2\,l(\varepsilon)}{\varepsilon_1} + b\,\varepsilon_3\|\Omega\|\|\partial\Omega\|\,,\\
m_2 := \dfrac{k\,a}{2} + \dfrac{c^2\,l(\varepsilon)}{2\,\varepsilon_2} + \dfrac{b\,l(\varepsilon)}{2\,\varepsilon_3} + \dfrac{b\,l(\varepsilon)}{2\,\varepsilon_4} + \dfrac{l(\varepsilon)}{2\,\varepsilon_5}\,,\\
m_3 := \dfrac{c^2\,\varepsilon_6}{2} + \dfrac{\varepsilon_8\,\bar g}{2} + \dfrac{k\,c^2\,\varepsilon}{2\,\varepsilon_1}\,,
\end{cases}\tag{3.45}$$

choosing $\varepsilon_2,\ \varepsilon_3,\ \varepsilon_4,\ \varepsilon_5,\ \varepsilon_6$ and ε_8 sufficiently large

$$\frac{c^2}{2\,\varepsilon_6} + \frac{\bar g}{2\,\varepsilon_8} + \frac{c^2\,\varepsilon}{2\,\varepsilon_2} + \frac{b\,\varepsilon}{2\,\varepsilon_3} + \frac{b\,\varepsilon}{2\,\varepsilon_4} + \frac{\varepsilon}{2\,\varepsilon_5} < \frac{b}{6}\,,\tag{3.46}$$

and ε_7 satisfying

$$\frac{3\,\bar{g}}{b} < \varepsilon_7 < k < \frac{1}{2}\,. \tag{3.47}$$

By using (3.45)–(3.47) into (3.44), we get

$$\begin{aligned}
&\frac{m_1}{2}\|u^N(x\,,\,\tau)\|^2_{L^2(\Omega)} + \frac{k(c^2-\bar{g})}{2}\|\nabla u^N(x\,,\,\tau)\|^2_{L^2(\Omega)} \\
&+ \frac{k\,\beta}{2}\|u^N_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} + \frac{b}{6}\|\nabla u^N_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} + \frac{a}{4}\|u^N_{\tau\tau}(x\,,\,\tau)\|^2_{L^2(\Omega)} \\
&\le \left\{m_1 + \frac{b\,\varepsilon_4\|\Omega\|\|\partial\Omega\|}{2} + \frac{k\,b\|\Omega\|\|\partial\Omega\|T}{2}\right\}\|u^N(x\,,\,0)\|^2_{L^2(\Omega)} \\
&+ \left\{m_3 + \frac{k\,c^2+g(0)}{2} + \frac{c^2}{2}\right\}\|\nabla u^N(x\,,\,0)\|^2_{L^2(\Omega)} \\
&+ \left\{m_2 + \frac{k\,\beta}{2} + \frac{b}{2} + \frac{k\,a}{2}\right\}\|u^N_t(x\,,\,0)\|^2_{L^2(\Omega)} + \frac{c^2}{2}\|\nabla u^N_t(x\,,\,0)\|^2_{L^2(\Omega)} \\
&+ \frac{a(1+k)}{2}\|u^N_{tt}(x\,,\,0)\|^2_{L^2(\Omega)} + \gamma_1\int_0^\tau \|u^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\
&+ \left\{\frac{k\,c^2\,\varepsilon}{2} + m_3\right\}\int_0^\tau \|\nabla u^N(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + \gamma_2\int_0^\tau \|u^N_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\
&+ \gamma_3\int_0^\tau \|\nabla u^N_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + (k\,a+\beta+m_2)\int_0^\tau \|u_{tt}(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,,
\end{aligned} \tag{3.48}$$

we can put (3.48) into the simple form

$$\begin{aligned}
&\|u^N(x\,,\,\tau)\|^2_{L^2(\Omega)} + \|\nabla u^N(x\,,\,\tau)\|^2_{L^2(\Omega)} + \|u^N_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} \\
&+ \|\nabla u^N_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} + \|u^N_{\tau\tau}(x\,,\,\tau)\|^2_{L^2(\Omega)} \\
&\le D\int_0^\tau \Big\{\|u^N(x\,,\,t)\|^2_{L^2(\Omega)} + \|\nabla u^N(x\,,\,t)\|^2_{L^2(\Omega)} + \|u^N_t(x\,,\,t)\|^2_{L^2(\Omega)} \\
&+\|\nabla u^N_t(x\,,\,t)\|^2_{L^2(\Omega)} + \|u_{tt}(x\,,\,t)\|^2_{L^2(\Omega)}\Big\}\,\mathrm{d}t \\
&+ D\left\{\|u^N(x\,,\,0)\|^2_{W^1_2(\Omega)} + \|u^N_t(x\,,\,0)\|^2_{W^1_2(\Omega)} + \|u^N_{tt}(x\,,\,0)\|^2_{L^2(\Omega)}\right\}\,,
\end{aligned} \tag{3.49}$$

where

$$D := \frac{\begin{array}{c}\max\Big\{m_1 + \frac{b\,\varepsilon_4\|\Omega\|\|\partial\Omega\|}{2} + \frac{k\,b\|\Omega\|\|\partial\Omega\|T}{2}\,,\ m_3 + \frac{k\,c^2+g(0)}{2} + \frac{c^2}{2}\,, \\ m_2 + \frac{k\,\beta}{2} + \frac{b}{2} + \frac{k\,a}{2}\,,\ \frac{c^2}{2}\,,\ \frac{a(1+k)}{2},\ \gamma_1\,,\ \frac{k\,c^2\,\varepsilon}{2} + m_3\,,\ \gamma_2\,,\ \gamma_3\,,\ k\,a+\beta+m_2\Big\}\end{array}}{\min\left\{\frac{m_1}{2}\,,\ \frac{k(c^2-\bar{g})}{2}\,,\ \frac{k\,\beta}{2}\,,\ \frac{b}{6}\,,\ \frac{a}{4}\right\}}\,.$$

To inequality (3.49) we apply **Gronwall's lemma** and then integrate from 0 to τ, we obtain

$$\begin{aligned}
&\|u^N(x\,,\,t)\|^2_{W^1_2(Q_\tau)} + \|u^N_t(x\,,\,t)\|^2_{W^1_2(Q_\tau)} \\
&\le De^{DT}\left\{\|u_0(x)\|^2_{W^1_2(\Omega)} + \|u_1(x)\|^2_{W^1_2(\Omega)} + \|u_2(x)\|^2_{L^2(\Omega)}\right\}\,.
\end{aligned} \tag{3.50}$$

We deduce from (3.50) that

$$\|u^N(x\,,\,t)\|^2_{W^1_2(Q_\tau)} + \|u^N_t(x\,,\,t)\|^2_{W^1_2(Q_\tau)} \le A\,.$$

Therefore the sequence $\left\{u^N\right\}_{N\geq 1}$ is bounded in $V(Q_T)$, and we can extract from it a subsequence for which we use the same notation which converges weakly in $V(Q_T)$ to a limit function $u(x\,,\,t)$. We have to show that $u(x\,,\,t)$ is a generalized solution of (1.1)–(1.3). Since $u^N(x\,,\,t)\longrightarrow u(x\,,\,t)$ in $L^2(Q_T)$ and $u^N(x\,,\,0)\longrightarrow \zeta(x)$ in $L^2(\Omega)$, then $u(x\,,\,0)=\zeta(x)$. Now to prove that (3.2) holds. We multiply each of the relations (3.2) by a function $p_l(t)\in W_2^1(0\,,\,T)\,,\, p_l(T)=0$, then add up the obtained equalities ranging from $l=1$ to $l=N$, and integrate over t on $(0\,,\,T)$. If we let $\eta^N=\sum\limits_{k=1}^{N}p_k(t)Z_k(x)$, then we have

$$\begin{aligned}
&-a(u^N_{tt}(t)\,,\,\eta^N_t(t))_{L^2(Q_T)}-\beta(u^N_t(t)\,,\,\eta^N_t(t))_{L^2(Q_T)}\\
&+c^2(\nabla u^N(t)\,,\,\nabla\eta^N(t))_{L^2(Q_T)}+b(\nabla u^N_t(t)\,,\,\nabla\eta^N(t))_{L^2(Q_T)}\\
&-(\int_0^t g(t-s)\nabla u^N(s)\,\mathrm{d}s\,,\,\nabla\eta^N(t))_{L^2(Q_T)}\\
&=a(u^N_{tt}(x\,,\,0)\,,\,\eta^N(0))_{L^2(\Omega)}+\beta(u^N_t(x\,,\,0)\,,\,\eta^N(0))_{L^2(\Omega)}\\
&+c^2\int_{\partial\Omega}\int_0^T\eta^N(x\,,\,t)(\int_0^t\int_\Omega u^N(\xi\,,\,\tau)\ \ \mathrm{d}\xi\mathrm{d}\tau)\mathrm{d}t\mathrm{d}s_x\\
&+b\int_{\partial\Omega}\int_0^T\eta^N(x\,,\,t)\int_\Omega u^N(\xi\,,\,t)\ \ \mathrm{d}\xi\mathrm{d}t\mathrm{d}s_x\\
&-b\int_{\partial\Omega}\int_0^T\eta^N(x\,,\,t)\int_\Omega u^N(\xi\,,\,0)\ \ \mathrm{d}\xi\mathrm{d}t\mathrm{d}s_x\\
&-\int_{\partial\Omega}\int_0^T\left[\eta^N(t)(\int_0^t g(t-s)\left\{\int_0^s\int_\Omega u^N(\xi\,,\,\tau)\quad \mathrm{d}\xi\mathrm{d}\tau\right\}\mathrm{d}s)\right]\mathrm{d}t\mathrm{d}s_x\,,
\end{aligned}$$

for all η^N of the form $\sum\limits_{k=1}^{N}p_l(t)Z_k(x)$.

Since

$$\begin{aligned}
&-(\int_0^t g(t-s)\left\{\nabla u^N(s)-\nabla u(s)\right\}\,\mathrm{d}s\,,\,\nabla\eta^N(t))_{L^2(Q_T)}\\
&\leq\frac{(\sup g(t))T}{\sqrt{2}}\|\nabla\eta^N(t)\|_{L^2(Q_T)}\|\nabla u^N(t)-\nabla u(t)\|_{L^2(Q_T)}\,,
\end{aligned}$$

and

$$\int_0^t\int_\Omega(u^N(\xi\,,\,\tau)-u(\xi\,,\,\tau))\ \mathrm{d}\xi\mathrm{d}\tau\leq\sqrt{T\|\Omega\|}\|u^N(t)-u(t)\|_{L^2(Q_T)}\,,$$

and

$$\begin{aligned}
&\int_0^T\eta^N(x\,,\,t)\int_\Omega(u^N_t(\xi\,,\,t)-u_t(\xi\,,\,t))\ \mathrm{d}\xi\mathrm{d}t\\
&\leq\sqrt{\|\Omega\|}(\int_0^T(\eta^N(x\,,\,t))^2\,\mathrm{d}t)^{1/2}\|u^N_t(t)-u_t(t)\|_{L^2(Q_T)}\,,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T\eta^N(x\,,\,t)\int_\Omega(u^N(\xi\,,\,0)-u(\xi\,,\,0))\ \mathrm{d}\xi\mathrm{d}t\\
&\leq\sqrt{\|\Omega\|}(\int_0^T(\eta^N(x\,,\,t))^2\,\mathrm{d}t)^{1/2}\|u^N(x\,,\,0)-u(x\,,\,0)\|_{L^2(Q_T)}\,,
\end{aligned}$$

and

$$- \int_0^T \left[\eta^N(t) (\int_0^t g(t-s) \left\{ \int_0^s \int_\Omega \left\{ u^N(\xi\,,\,\tau) - u(\xi\,,\,\tau) \right\} \quad \mathrm{d}\xi \mathrm{d}\tau \right\} \mathrm{d}s) \right] \mathrm{d}t$$
$$\leq \frac{(\sup g(t)) T^2 \sqrt{\|\Omega\|}}{2\sqrt{2}} (\int_0^T (\eta^N(x\,,\,t))^2 \,\mathrm{d}t)^{1/2} \|u^N(t) - u(t)\|_{L^2(Q_T)}\,,$$

and

$$\|u^N(t) - u(t)\|_{W_2^1(Q_T)} \longrightarrow 0\,, \text{ as } N \longrightarrow \infty\,,$$

therefore we have

$$- (\int_0^t g(t-s) \nabla u^N(s) \,\mathrm{d}s\,,\, \nabla \eta^N(t))_{L^2(Q_T)}$$
$$\longrightarrow -(\int_0^t g(t-s) \nabla u(s) \,\mathrm{d}s\,,\, \nabla \eta(t))_{L^2(Q_T)}\,, \text{ as } N \longrightarrow \infty\,,$$

and

$$\int_{\partial\Omega} \int_0^T \eta^N(x\,,\,t) \int_0^t \int_\Omega u^N(\xi\,,\,\tau) \quad \mathrm{d}\xi \mathrm{d}\tau \mathrm{d}t \mathrm{d}s$$
$$\longrightarrow \int_{\partial\Omega} \int_0^T \eta(x\,,\,t) \int_0^t \int_\Omega u(\xi\,,\,\tau) \quad \mathrm{d}\xi \mathrm{d}\tau \mathrm{d}t \mathrm{d}s\,, \text{ as } N \longrightarrow \infty\,,$$

and

$$\int_{\partial\Omega} \int_0^T \eta^N(x\,,\,t) \int_\Omega u^N(\xi\,,\,t) \quad \mathrm{d}\xi \mathrm{d}t \mathrm{d}s$$
$$\longrightarrow \int_{\partial\Omega} \int_0^T \eta(x\,,\,t) \int_\Omega u(\xi\,,\,t) \quad \mathrm{d}\xi \mathrm{d}t \mathrm{d}s\,, \text{ as } N \longrightarrow \infty\,,$$

and

$$\int_{\partial\Omega} \int_0^T \eta^N(x\,,\,t) \int_0^t \int_\Omega u^N(\xi\,,\,0) \quad \mathrm{d}\xi \mathrm{d}\tau \mathrm{d}t \mathrm{d}s$$
$$\longrightarrow \int_{\partial\Omega} \int_0^T \eta(x\,,\,t) \int_0^t \int_\Omega u(\xi\,,\,0) \quad \mathrm{d}\xi \mathrm{d}\tau \mathrm{d}t \mathrm{d}s\,, \text{ as } N \longrightarrow \infty\,,$$

and

$$- \int_{\partial\Omega} \int_0^T \left[\eta^N(t) (\int_0^t g(t-s) \left\{ \int_0^s \int_\Omega u^N(\xi\,,\,\tau) \quad \mathrm{d}\xi \mathrm{d}\tau \right\} \mathrm{d}s) \right] \mathrm{d}t \mathrm{d}s_x$$
$$\longrightarrow - \int_{\partial\Omega} \int_0^T \left[\eta(t) (\int_0^t g(t-s) \left\{ \int_0^s \int_\Omega u(\xi\,,\,\tau) \quad \mathrm{d}\xi \mathrm{d}\tau \right\} \mathrm{d}s) \right] \mathrm{d}t \mathrm{d}s_x\,, \text{ as } N \longrightarrow \infty\,.$$

Thus, the limit function u satisfies (3.2) for every $\eta^N = \sum_{k=1}^{N} p_l(t) Z_k(x)$. We denote by $\mathbb{Q}_N$ the totality of all functions of the form $\eta^N = \sum_{k=1}^{N} p_l(t) Z_k(x)$, with $p_l(t) \in W_2^1(0,T),\ p_l(T) = 0$. But $\cup_{l=1}^N \mathbb{Q}_N$ is dense in $W(Q_T)$, then relation (3.2) holds for all $u \in W(Q_T)$. Thus we have shown that the limit function $u(x\,,\,t)$ is a generalized solution of problem (1.1)–(1.3) in $V(Q_T)$. □

4 Unicity of the generalized solution

We now give the main result on the unicity of generalized solution of the problem (1.1)–(1.3).

Theorem 4.1 *Assume that the hypotheses* $(\mathbf{A1})-(\mathbf{A3})$ *hold, then the problem* (1.1)–(1.3) *cannot have more than one generalized solution in* $V(Q_T)$.

Proof. Suppose that $u_1 \in V(Q_T)$ and $u_2 \in V(Q_T)$ are two solutions of problem (1.1)–(1.3) such that u_1 is different from u_2. Then $U := u_1 - u_2$ solves

$$\begin{cases} a\, U_{ttt} + \beta\, U_{tt} - c^2\, \Delta U - b\, \Delta U_t + \int_0^t g(t-s)\Delta U(s)\, \mathrm{d}s = 0\,, \\ U(x\,,\,0) = U_t(x\,,\,0) = U_{tt}(x\,,\,0) = 0\,, \\ \dfrac{\partial U}{\partial n} = \int_0^t \int_\Omega U(\varepsilon\,,\,\tau)\ \mathrm{d}\varepsilon \mathrm{d}\tau\,, \quad x \in \partial\Omega\,, \end{cases} \tag{4.1}$$

and (2.2) gives

$$\begin{aligned} &- a(U_{tt}(t)\,,\, v_t(t))_{L^2(Q_T)} - \beta(U_t(t)\,,\, v_t(t))_{L^2(Q_T)} \\ &+ c^2(\nabla U(t)\,,\, \nabla v(t))_{L^2(Q_T)} + b(\nabla U_t(t)\,,\, \nabla v(t))_{L^2(Q_T)} \\ &- (\int_0^t g(t-s)\nabla U(x\,,\,s)\,\mathrm{d}s\,,\, \nabla v(x\,,\,t))_{L^2(Q_T)} \\ &= c^2 \int_0^T \int_{\partial\Omega} \left[v(x\,,\,t)(\int_0^t \int_\Omega U(\xi\,,\,\tau)\ \ \mathrm{d}\xi \mathrm{d}\tau) \right] \mathrm{d}s_x \mathrm{d}t \\ &+ b \int_0^T \int_{\partial\Omega} \left[v(x\,,\,t) \int_\Omega U(\xi\,,\,t)\ \ \mathrm{d}\xi \right] \mathrm{d}s_x \mathrm{d}t \\ &- \int_0^T \int_{\partial\Omega} \left[v(x\,,\,t)(\int_0^t g(t-s) \left\{ \int_0^s \int_\Omega U(\xi\,,\,\tau)\ \ \mathrm{d}\xi \mathrm{d}\tau \right\} \mathrm{d}s) \right] \mathrm{d}s_x \mathrm{d}t\,. \end{aligned} \tag{4.2}$$

Definite the function $v(x\,,\,t)$ by

$$v(x\,,\,t) := \begin{cases} \int_t^\tau U(x\,,\,s)\,\mathrm{d}s\,, & 0 \le t \le \tau\,, \\ 0\,, & \tau \le t \le T\,. \end{cases} \tag{4.3}$$

It is obvious that $v \in W(Q_T)$ and $v_t(x\,,\,t) = -U(x\,,\,t)$ for all $t \in [0\,,\,\tau]$. Integration by parts in the **LHS** of (4.2) gives

$$-a(U_{tt}(t)\,,\, v_t(t))_{L^2(Q_T)} = a(U_\tau(x\,,\,\tau)\,,\, U(x\,,\,\tau))_{L^2(\Omega)} - a\int_0^\tau \|U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,, \tag{4.4}$$

and

$$-\beta(U_t(t)\,,\, v_t(t))_{L^2(Q_T)} = \frac{\beta}{2}\|U(x\,,\,\tau)\|^2_{L^2(\Omega)}\,, \tag{4.5}$$

and

$$c^2(\nabla U(t)\,,\, \nabla v(t))_{L^2(Q_T)} = \frac{c^2}{2}\|\nabla v(x\,,\,0)\|^2_{L^2(\Omega)}\,, \tag{4.6}$$

and

$$b(\nabla U_t(t)\,,\, \nabla v(t))_{L^2(Q_T)} = b\int_0^\tau \|\nabla v_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,. \tag{4.7}$$

Substitution of (4.4)–(4.7) into (4.2) leads to the equality

$$\begin{aligned}
&a(U_\tau(x\,,\,\tau)\,,\,U(x\,,\,\tau))_{L^2(\Omega)}+\frac{\beta}{2}\|U(x\,,\,\tau)\|^2_{L^2(\Omega)}+\frac{c^2}{2}\|\nabla v(x\,,\,0)\|^2_{L^2(\Omega)}\\
&=a\int_0^\tau\|U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t-b\int_0^\tau\|\nabla v_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\
&+(\int_0^t g(t-s)\nabla U(x\,,\,s)\,\mathrm{d}s\,,\,\nabla v(x\,,\,t))_{L^2(Q_T)}\\
&+c^2\int_0^T\int_{\partial\Omega}\left[v(x\,,\,t)(\int_0^t\int_\Omega U(\xi\,,\,\tau)\ \ \mathrm{d}\xi\mathrm{d}\tau)\right]\mathrm{d}s_x\mathrm{d}t\\
&+b\int_0^T\int_{\partial\Omega}\left[v(x\,,\,t)(\int_\Omega U(\xi\,,\,t)\ \ \mathrm{d}\xi)\right]\mathrm{d}s_x\mathrm{d}t\\
&-\int_0^T\int_{\partial\Omega}\left[v(x\,,\,t)(\int_0^t g(t-s)\left\{\int_0^s\int_\Omega U(\xi\,,\,\tau)\quad\mathrm{d}\xi\mathrm{d}\tau\right\}\mathrm{d}s)\right]\mathrm{d}s_x\mathrm{d}t\,.
\end{aligned}\tag{4.8}$$

Now since

$$v^2(x\,,\,t)=(\int_t^\tau U(x\,,\,s)\,\mathrm{d}s)^2\le\tau\int_0^\tau U^2(x\,,\,s)\,\mathrm{d}s=\tau\|U(x\,,\,t)\|^2_{L^2(0\,,\,\tau)}\,,$$

then

$$\|v(x\,,\,t)\|^2_{L^2(Q_\tau)}\le\tau^2\|U(x\,,\,t)\|^2_{L^2(Q_\tau)}\le T^2\|U(x\,,\,t)\|^2_{L^2(Q_\tau)}\,.\tag{4.9}$$

The $\mathbf{RHS}$ of (4.8) can be bounded, using the inequality $(\mathbf{A})$ and (4.9) we get

$$\begin{aligned}
&(\int_0^t g(t-s)\nabla U(x\,,\,s)\,\mathrm{d}s\,,\,\nabla v(x\,,\,t))_{L^2(Q_T)}\\
&\le\frac{(\sup g(t))^2\,T^2}{4}\int_0^\tau\|\nabla U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t+\frac{1}{2}\int_0^\tau\|\nabla v(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,,
\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}
&c^2\int_0^T\int_{\partial\Omega}\left[v(x\,,\,t)(\int_0^t\int_\Omega U(\xi\,,\,\tau)\ \ \mathrm{d}\xi\mathrm{d}\tau)\right]\mathrm{d}s_x\mathrm{d}t\\
&\le\frac{c^2\,T^2(2\,l(\varepsilon)+\|\Omega\|\|\partial\Omega\|)}{4}\int_0^\tau\|U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t+\frac{c^2\,\varepsilon}{2}\int_0^\tau\|\nabla v(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,,
\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}
&b\int_0^T\int_{\partial\Omega}\left[v(x\,,\,t)(\int_\Omega U(\xi\,,\,t)\ \ \mathrm{d}\xi)\right]\mathrm{d}s_x\mathrm{d}t\\
&\le\frac{b(T^2\,l(\varepsilon)+\|\Omega\|\|\partial\Omega\|)}{2}\int_0^\tau\|U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t+\frac{b\,\varepsilon}{2}\int_0^\tau\|\nabla v(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,,
\end{aligned}\tag{4.12}$$

and

$$\begin{aligned}
&-\int_0^T\int_{\partial\Omega}\left[v(x\,,\,t)(\int_0^t g(t-s)\left\{\int_0^s\int_\Omega U(\xi\,,\,\tau)\quad\mathrm{d}\xi\mathrm{d}\tau\right\}\mathrm{d}s)\right]\mathrm{d}s_x\mathrm{d}t\\
&\le\frac{T^2(8\,l(\varepsilon)+(\sup g(t))^2\,T^2\|\Omega\|\|\partial\Omega\|)}{16}\int_0^\tau\|U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t+\frac{\varepsilon}{2}\int_0^\tau\|\nabla v(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,,
\end{aligned}\tag{4.13}$$

and

$$-\frac{a}{2}\|U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}-\frac{a}{2}\|U(x\,,\,\tau)\|^2_{L^2(\Omega)}\le a(U_\tau(x\,,\,\tau)\,,\,U(x\,,\,\tau))_{L^2(\Omega)}\,. \tag{4.14}$$

Combination of (4.10)–(4.14) and (4.8) results in

$$\begin{aligned}&\left\{\frac{\beta}{2}-\frac{a}{2}\right\}\|U(x\,,\,\tau)\|^2_{L^2(\Omega)}-\frac{a}{2}\|U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}+\frac{c^2}{2}\|\nabla v(x\,,\,0)\|^2_{L^2(\Omega)}\\&\le\gamma_4\int_0^\tau\|U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t+\frac{(\sup g(t))^2\,T^2}{4}\int_0^\tau\|\nabla U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\&+a\int_0^\tau\|U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t+\frac{1+c^2\,\varepsilon+b\,\varepsilon+\varepsilon}{2}\int_0^\tau\|\nabla v(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,,\end{aligned} \tag{4.15}$$

where

$$\begin{aligned}\gamma_4:=&\frac{c^2\,T^2(2\,l(\varepsilon)+\|\Omega\|\|\partial\Omega\|)}{4}+\frac{b(T^2\,l(\varepsilon)+\|\Omega\|\|\partial\Omega\|)}{2}\\&+\frac{T^2(8\,l(\varepsilon)+(\sup g(t))^2\,T^2\|\Omega\|\|\partial\Omega\|)}{16}\,.\end{aligned}$$

Now multiply the differential equation in (4.1) by U_t and integrate over $Q_\tau:=\Omega\times(0\,,\,\tau)$, we obtain

$$\begin{aligned}&a(U_{ttt}(t)\,,\,U_t(t))_{L^2(Q_\tau)}+\beta(U_{tt}(t)\,,\,U_t(t))_{L^2(Q_\tau)}\\&-c^2(\Delta U(t)\,,\,U_t(t))_{L^2(Q_\tau)}-b(\Delta U_t(t)\,,\,U_t(t))_{L^2(Q_\tau)}\\&+(\int_0^t g(t-s)\Delta U(s)\,\mathrm{d}s\,,\,U_t(t))_{L^2(Q_\tau)}=0\,.\end{aligned} \tag{4.16}$$

By using direct calculation, we get

$$a(U_{ttt}(t)\,,\,U_t(t))_{L^2(Q_\tau)}=a(U_{\tau\tau}(x\,,\,\tau)\,,\,U_\tau(x\,,\,\tau))_{L^2(\Omega)}-a\int_0^\tau\|U_{tt}(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,, \tag{4.17}$$

and

$$\beta(U_{tt}(t)\,,\,U_t(t))_{L^2(Q_\tau)}=\frac{\beta}{2}\|U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}\,, \tag{4.18}$$

and

$$\begin{aligned}&-c^2(\Delta U(t)\,,\,U_t(t))_{L^2(Q_\tau)}\\&=-c^2\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_0^t\int_\Omega U(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta)\right]\mathrm{d}s_x\mathrm{d}t+\frac{c^2}{2}\|\nabla U(x\,,\,\tau)\|^2_{L^2(\Omega)}\,,\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}&-b(\Delta U_t(t)\,,\,U_t(t))_{L^2(Q_\tau)}\\&=-b\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_\Omega U(\xi\,,\,t)\ \ \mathrm{d}\xi)\right]\mathrm{d}s_x\mathrm{d}t+b\int_0^\tau\|\nabla U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,,\end{aligned} \tag{4.20}$$

and by using integration by parts and direct calculation, we get

$$
\begin{aligned}
&(\int_0^t g(t-s)\Delta U(s)\,\mathrm{d}s\,,\,U_t(t))_{L^2(Q_\tau)}\\
&=\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_0^t g(t-s)\left\{\int_0^s\int_\Omega U(\xi\,,\,\eta)\quad \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\mathrm{d}t\\
&-\int_0^\tau\int_0^t g(t-s)(\nabla U(s)\,,\,\nabla U_t(t))_{L^2(\Omega)}\,\mathrm{d}s\mathrm{d}t\\
&=\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_0^t g(t-s)\left\{\int_0^s\int_\Omega U(\xi\,,\,\eta)\quad \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\mathrm{d}t\\
&+\frac{1}{2}(g\Box\nabla U)(\tau)-\frac{1}{2}(\int_0^\tau g(s)\,\mathrm{d}s)\|\nabla U(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&-\frac{1}{2}\int_0^\tau(g'\Box\nabla U)(t)\,\mathrm{d}t+\frac{1}{2}\int_0^\tau g(t)\|\nabla U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,. \qquad (4.21)
\end{aligned}
$$

Substitution of equations (4.17)–(4.21) into(4.16) results in

$$
\begin{aligned}
&a(U_{\tau\tau}(x\,,\,\tau)\,,\,U_\tau(x\,,\,\tau))_{L^2(\Omega)}+\frac{\beta}{2}\|U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&+\frac{1}{2}(c^2-\int_0^\tau g(s)\,\mathrm{d}s)\|\nabla U(x\,,\,\tau)\|^2_{L^2(\Omega)}+\frac{1}{2}(g\Box\nabla U)(\tau)\\
&=a\int_0^\tau\|U_{tt}(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t-b\int_0^\tau\|\nabla U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\
&+c^2\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_0^t\int_\Omega U(\xi\,,\,\eta)\quad \mathrm{d}\xi\mathrm{d}\eta)\right]\mathrm{d}s_x\mathrm{d}t\\
&+b\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_\Omega U(\xi,t)\quad \mathrm{d}\xi)\right]\mathrm{d}s_x\mathrm{d}t\\
&-\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_0^t g(t-s)\left\{\int_0^s\int_\Omega U(\xi\,,\,\eta)\quad \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\mathrm{d}t\\
&+\frac{1}{2}\int_0^\tau(g'\Box\nabla U)(t)\,\mathrm{d}t-\frac{1}{2}\int_0^\tau g(t)\|\nabla U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,. \qquad (4.22)
\end{aligned}
$$

By using Young's inequality, $(\mathbf{A})$ and Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
&c^2\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_0^t\int_\Omega U(\xi\,,\,\eta)\quad \mathrm{d}\xi\mathrm{d}\eta)\right]\mathrm{d}s_x\mathrm{d}t\\
&\leq\frac{c^2}{2}\int_0^\tau(\varepsilon\|\nabla U_t(x\,,\,t)\|^2_{L^2(\Omega)}+l(\varepsilon)\|U_t(x\,,\,t)\|^2_{L^2(\Omega)})\,\mathrm{d}t\\
&+\frac{c^2\,T^2\|\Omega\|\|\partial\Omega\|}{4}\int_0^\tau\|U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,, \qquad (4.23)
\end{aligned}
$$

and

$$
\begin{aligned}
&b\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_\Omega U(\xi\,,\,t)\quad \mathrm{d}\xi)\right]\mathrm{d}s_x\mathrm{d}t \qquad (4.24)\\
&\leq\frac{b}{2}\int_0^\tau(\varepsilon\|\nabla U_t(x\,,\,t)\|^2_{L^2(\Omega)}+l(\varepsilon)\|U_t(x\,,\,t)\|^2_{L^2(\Omega)})\,\mathrm{d}t+\frac{b\|\Omega\|\|\partial\Omega\|}{2}\int_0^\tau\|U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,,
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_0^\tau \int_{\partial\Omega} \left[U_t(x\,,\,t)(\int_0^t g(t-s)\left\{ \int_0^s \int_\Omega U(\xi\,,\,\eta) \quad \mathrm{d}\xi \mathrm{d}\eta \right\} \mathrm{d}s) \right] \mathrm{d}s_x \mathrm{d}t \\
&\leq \frac{1}{2} \int_0^\tau (\varepsilon \|\nabla U_t(x\,,\,t)\|^2_{L^2(\Omega)} + l(\varepsilon)\|U_t(x\,,\,t)\|^2_{L^2(\Omega)})\,\mathrm{d}t \\
&+ \frac{(\sup g(t))^2\, T^4 \|\Omega\| \|\partial\Omega\|}{16} \int_0^\tau \|U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,. \qquad (4.25)
\end{aligned}
$$

By using Young's inequality (for $\varepsilon = \delta'$), we get

$$
-\frac{a\,\delta'}{2}\|U_{\tau\tau}(x\,,\,\tau)\|^2_{L^2(\Omega)} - \frac{a}{2\,\delta'}\|U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} \leq a(U_{\tau\tau}(x\,,\,\tau)\,,\,U_\tau(x\,,\,\tau))_{L^2(\Omega)}\,. \qquad (4.26)
$$

By replacement (4.23)–(4.26) into (4.22), we get

$$
\begin{aligned}
&\frac{c^2-\bar{g}}{2}\|\nabla U(x\,,\,\tau)\|^2_{L^2(\Omega)} + \left\{ \frac{\beta}{2} - \frac{a}{2\,\delta'} \right\} \|U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} \\
&- \frac{a\,\delta'}{2}\|U_{\tau\tau}(x\,,\,\tau)\|^2_{L^2(\Omega)} + \frac{1}{2}(g\Box\nabla U)(\tau) \\
&\leq \|\Omega\|\|\partial\Omega\| \left\{ \frac{c^2\,T^2}{4} + \frac{b}{2} + \frac{(\sup g(t))^2\,T^4}{16} \right\} \int_0^\tau \|U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\
&+ \frac{l(\varepsilon)(c^2+b+1)}{2} \int_0^\tau \|U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{\varepsilon(c^2+b+1)}{2} \int_0^\tau \|\nabla U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\
&+ a \int_0^\tau \|U_{tt}(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,. \qquad (4.27)
\end{aligned}
$$

Now multiply the differential equation in (4.1) by $U_{tt}(t)$ and integrate over $Q_\tau := \Omega \times (0\,,\,\tau)$, we obtain

$$
\begin{aligned}
&a(U_{ttt}(t)\,,\,U_{tt}(t))_{L^2(Q_\tau)} + \beta(U_{tt}(t)\,,\,U_{tt}(t))_{L^2(Q_\tau)} \\
&- c^2(\Delta U(t)\,,\,U_{tt}(t))_{L^2(Q_\tau)} - b(\Delta U_t(t)\,,\,U_{tt}(t))_{L^2(Q_\tau)} \\
&+ (\int_0^t g(t-s)\Delta U(s)\,\mathrm{d}s\,,\,U_{tt}(t))_{L^2(Q_\tau)} = 0\,. \qquad (4.28)
\end{aligned}
$$

Boundary and initial conditions in (4.1) and standard integration by parts in (4.28) leads to

$$
a(U_{ttt}(t)\,,\,U_{tt}(t))_{L^2(Q_\tau)} = \frac{a}{2}\|U_{\tau\tau}(x\,,\,\tau)\|^2_{L^2(\Omega)}\,, \qquad (4.29)
$$

and

$$
\beta(U_{tt}(t)\,,\,U_{tt}(t))_{L^2(Q_\tau)} = \beta \int_0^\tau \|U_{tt}(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,, \qquad (4.30)
$$

and

$$
\begin{aligned}
-c^2(\Delta U(t)\,,\,U_{tt}(t))_{L^2(Q_\tau)} &= c^2(\nabla U_\tau(x\,,\,\tau)\,,\,\nabla U(x\,,\,\tau))_{L^2(\Omega)} - c^2 \int_0^\tau \|\nabla U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\
&- c^2 \int_{\partial\Omega} \left[U_\tau(x\,,\,\tau)(\int_0^\tau \int_\Omega U(\xi\,,\,\eta)\ \mathrm{d}\xi\mathrm{d}\eta) \right] \mathrm{d}s_x \\
&+ c^2 \int_0^\tau \int_{\partial\Omega} \left[U_t(x\,,\,t)(\int_\Omega U(\xi\,,\,t)\ \mathrm{d}\xi) \right] \mathrm{d}s_x \mathrm{d}t\,, \qquad (4.31)
\end{aligned}
$$

and

$$-b(\Delta U_t(t)\,,\,U_{tt}(t))_{L^2(Q_\tau)} = \frac{b}{2}\|\nabla U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} - b\int_{\partial\Omega}\left[U_\tau(x\,,\,\tau)(\int_\Omega U(\xi\,,\,\tau)\ \mathrm{d}\xi)\right]\mathrm{d}s_x + b\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_\Omega U_t(\xi\,,\,t)\ \ \mathrm{d}\xi)\right]\mathrm{d}s_x\mathrm{d}t\,, \tag{4.32}$$

and

$$\begin{aligned}&(\int_0^t g(t-s)\Delta U(s)\,\mathrm{d}s\,,\,U_{tt}(t))_{L^2(Q_\tau)}\\&= \int_0^\tau\int_{\partial\Omega}\left[U_{tt}(x\,,\,t)(\int_0^t g(t-s)\left\{\int_0^s\int_\Omega U(\xi\,,\,\eta)\quad \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\mathrm{d}t\\&-(\int_0^t g(t-s)\nabla U(s)\,\mathrm{d}s\,,\,\nabla U_{tt}(t))_{L^2(Q_\tau)}\,,\end{aligned}$$

by using

$$\begin{aligned}&\int_0^\tau\int_{\partial\Omega}\left[U_{tt}(x\,,\,t)(\int_0^t g(t-s)\left\{\int_0^s\int_\Omega U(\xi\,,\,\eta)\quad \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\mathrm{d}t\\&= \int_{\partial\Omega}\left[U_\tau(x\,,\,\tau)(\int_0^\tau g(\tau-s)\left\{\int_0^s\int_\Omega U(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\\&-\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_0^t g'(t-s)\left\{\int_0^s\int_\Omega U(\xi\,,\,\eta)\quad \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\mathrm{d}t\\&-g(0)\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_0^t\int_\Omega U(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta)\right]\mathrm{d}s_x\mathrm{d}t\,,\end{aligned}$$

and

$$\begin{aligned}&-(\int_0^t g(t-s)\nabla U(s)\,\mathrm{d}s\,,\,\nabla U_{tt}(t))_{L^2(Q_\tau)}\\&= \frac{1}{2}(-g'\Box\nabla U)(\tau) + \frac{1}{2}g(\tau)\|\nabla U(x\,,\,\tau)\|^2_{L^2(\Omega)} - \int_0^\tau g(\tau-s)(\nabla U(s)\,,\,\nabla U_\tau(\tau))_{L^2(\Omega)}\,\mathrm{d}s\\&+\frac{1}{2}\int_0^\tau(g''\Box\nabla U)(t)\,\mathrm{d}t - \frac{1}{2}\int_0^\tau g'(t)\|\nabla U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,,\end{aligned}$$

we get

$$\begin{aligned}&(\int_0^t g(t-s)\Delta U(s)\,\mathrm{d}s\,,\,U_{tt}(t))_{L^2(Q_\tau)}\\&= \int_{\partial\Omega}\left[U_\tau(x\,,\,\tau)(\int_0^\tau g(\tau-s)\left\{\int_0^s\int_\Omega U(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\\&-\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_0^t g'(t-s)\left\{\int_0^s\int_\Omega U(\xi\,,\,\eta)\quad \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\mathrm{d}t\\&-g(0)\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_0^t\int_\Omega U(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta)\right]\mathrm{d}s_x\mathrm{d}t\\&+\frac{1}{2}(-g'\Box\nabla U)(\tau) + \frac{1}{2}g(\tau)\|\nabla U(x\,,\,\tau)\|^2_{L^2(\Omega)} - \int_0^\tau g(\tau-s)(\nabla U(s)\,,\,\nabla U_\tau(\tau))_{L^2(\Omega)}\,\mathrm{d}s\\&+\frac{1}{2}\int_0^\tau(g''\Box\nabla U)(t)\,\mathrm{d}t - \frac{1}{2}\int_0^\tau g'(t)\|\nabla U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,.\end{aligned} \tag{4.33}$$

Substitution of (4.29)–(4.33) into (4.28) leads to the equality

$$\begin{aligned}
&\frac{1}{2}g(\tau)\|\nabla U(x\,,\,\tau)\|^2_{L^2(\Omega)}+\frac{b}{2}\|\nabla U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&+c^2(\nabla U_\tau(x\,,\,\tau)\,,\,\nabla U(x\,,\,\tau))_{L^2(\Omega)}+\frac{a}{2}\|U_{\tau\tau}(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&=c^2\int_0^\tau\|\nabla U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t-\beta\int_0^\tau\|U_{tt}(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\
&+c^2\int_{\partial\Omega}\left[U_\tau(x\,,\,\tau)(\int_0^\tau\int_\Omega U(\xi\,,\,\eta)\ \mathrm{d}\xi\mathrm{d}\eta)\right]\mathrm{d}s_x+b\int_{\partial\Omega}\left[U_\tau(x\,,\,\tau)(\int_\Omega U(\xi\,,\,\tau)\ \mathrm{d}\xi)\right]\mathrm{d}s_x\\
&-\int_{\partial\Omega}\left[U_\tau(x\,,\,\tau)(\int_0^\tau g(\tau-s)\left\{\int_0^s\int_\Omega U(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\\
&+\int_0^\tau g(\tau-s)(\nabla U(s)\,,\,\nabla U_\tau(\tau))_{L^2(\Omega)}\,\mathrm{d}s-c^2\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_\Omega U(\xi\,,\,t)\ \ \mathrm{d}\xi)\right]\mathrm{d}s_x\mathrm{d}t\\
&-b\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_\Omega U_t(\xi\,,\,t)\ \ \mathrm{d}\xi)\right]\mathrm{d}s_x\mathrm{d}t\\
&+\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_0^t g'(t-s)\left\{\int_0^s\int_\Omega U(\xi\,,\,\eta)\quad\mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\mathrm{d}t\\
&+g(0)\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_0^t\int_\Omega U(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta)\right]\mathrm{d}s_x\mathrm{d}t\\
&+\frac{1}{2}(g'\Box\nabla U)(\tau)+\frac{1}{2}\int_0^\tau g'(t)\|\nabla U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t-\frac{1}{2}\int_0^\tau(g''\Box\nabla U)(t)\,\mathrm{d}t\,. \qquad (4.34)
\end{aligned}$$

By using Young's inequality (for $\varepsilon=1$), $(\mathbf{A})$, Cauchy-Schwarz inequality and integration by parts, we get

$$\begin{aligned}
&c^2\int_{\partial\Omega}\left[U_\tau(x\,,\,\tau)(\int_0^\tau\int_\Omega U(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta)\right]\mathrm{d}s_x \qquad (4.35)\\
&\le\frac{c^2}{2\,\delta_1}(\varepsilon\|\nabla U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}+l(\varepsilon)\|U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)})+\frac{c^2\,\delta_1\,T\|\partial\Omega\|\|\Omega\|}{2}\int_0^\tau\|U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,,
\end{aligned}$$

and

$$\begin{aligned}
&b\int_{\partial\Omega}\left[U_\tau(x\,,\,\tau)(\int_\Omega U(\xi\,,\,\tau)\ \mathrm{d}\xi)\right]\mathrm{d}s_x\\
&\le\frac{b}{2\,\delta_2}(\varepsilon\|\nabla U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}+l(\varepsilon)\|U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)})+\frac{b\,\delta_2\|\Omega\|\|\partial\Omega\|}{2}\|U(x\,,\,\tau)\|^2_{L^2(\Omega)}\,. \qquad (4.36)
\end{aligned}$$

By using Young's inequality (for $\varepsilon=\delta_3$), $(\mathbf{A})$, Cauchy-Schwarz inequality and integration by parts, we get

$$\begin{aligned}
&-\int_{\partial\Omega}\left[U_\tau(x\,,\,\tau)(\int_0^\tau g(\tau-s)\left\{\int_0^s\int_\Omega U(\xi\,,\,\eta)\ \ \mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\\
&\le\frac{1}{2\,\delta_3}(\varepsilon\|\nabla U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}+l(\varepsilon)\|U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)})\\
&+\frac{\delta_3(\sup g(t))^2\,T^3\|\Omega\|\|\partial\Omega\|}{4}\int_0^\tau\|U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,. \qquad (4.37)
\end{aligned}$$

By using Young's inequality (for $\varepsilon = \delta_4$ and $\varepsilon = \delta_5$) and using $\int_0^\tau g(s)\,\mathrm{d}s \le \bar{g}$, we get

$$\begin{aligned}&\int_0^\tau g(\tau - s)(\nabla U(s)\,,\,\nabla U_\tau(\tau))_{L^2(\Omega)}\,\mathrm{d}s\\ &\le \frac{\delta_4}{2}(g\Box\nabla U)(\tau) + \left\{\frac{1}{2\,\delta_4} + \frac{1}{2\,\delta_5}\right\}\bar{g}\|\nabla U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} + \frac{\delta_5\,\bar{g}}{2}\|\nabla U(x\,,\,\tau)\|^2_{L^2(\Omega)}\,, \end{aligned}\tag{4.38}$$

and

$$\begin{aligned}&-c^2\int_{\partial\Omega}\left[\int_0^\tau U_t(x\,,\,t)(\int_\Omega U(\xi\,,\,t)\ \ \mathrm{d}\xi)\mathrm{d}t\right]\mathrm{d}s_x\\ &\le \frac{c^2}{2}\int_0^\tau\left\{\varepsilon\|\nabla U_t(x\,,\,t)\|^2_{L^2(\Omega)} + l(\varepsilon)\|U_t(x\,,\,t)\|^2_{L^2(\Omega)}\right\}\,\mathrm{d}t\\ &+\frac{c^2\,\|\Omega\|\|\partial\Omega\|}{2}\int_0^\tau\|U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,,\end{aligned}\tag{4.39}$$

and

$$\begin{aligned}&-b\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_\Omega U_t(\xi\,,\,t)\ \ \mathrm{d}\xi)\right]\mathrm{d}s_x\mathrm{d}t\\ &\le \frac{b(l(\varepsilon) + \|\Omega\|\|\partial\Omega\|)}{2}\int_0^\tau\|U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{b\,\varepsilon}{2}\int_0^\tau\|\nabla U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,.\end{aligned}\tag{4.40}$$

By using Young's inequality (for $\varepsilon = 1$), $(\mathbf{A})$, Cauchy-Schwarz inequality and integration by parts, we get

$$\begin{aligned}&\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_0^t g'(t-s)\left\{\int_0^s\int_\Omega U(\xi\,,\,\eta)\quad\mathrm{d}\xi\mathrm{d}\eta\right\}\mathrm{d}s)\right]\mathrm{d}s_x\mathrm{d}t\\ &\le \frac{1}{2}\int_0^\tau(\varepsilon\|\nabla U_t(x\,,\,t)\|^2_{L^2(\Omega)} + l(\varepsilon)\|U_t(x\,,\,t)\|^2_{L^2(\Omega)})\,\mathrm{d}t\\ &+\frac{(\sup\|g'(t)\|)^2\,T^4\,\|\Omega\|\|\partial\Omega\|}{16}\int_0^\tau\|U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,.\end{aligned}\tag{4.41}$$

By using Young's inequality (for $\varepsilon = 1$), $(\mathbf{A})$, Cauchy-Schwarz inequality and integration by parts, we get

$$\begin{aligned}&g(0)\int_0^\tau\int_{\partial\Omega}\left[U_t(x\,,\,t)(\int_0^t\int_\Omega U(\xi\,,\,\eta)\quad\mathrm{d}\xi\mathrm{d}\eta)\right]\mathrm{d}s_x\mathrm{d}t\\ &\le \frac{g(0)}{2}\int_0^\tau(\varepsilon\|\nabla U_t(x\,,\,t)\|^2_{L^2(\Omega)} + l(\varepsilon)\|U_t(x\,,\,t)\|^2_{L^2(\Omega)})\,\mathrm{d}t\\ &+\frac{g(0)\|\partial\Omega\|\|\Omega\|T^2}{4}\int_0^\tau\|U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,.\end{aligned}\tag{4.42}$$

By using Young's inequality (for $\varepsilon = \delta_6$), $(\mathbf{A})$ and Cauchy-Schwarz inequality, we get

$$-\frac{c^2}{2\,\delta_6}\|\nabla U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} - \frac{c^2\,\delta_6}{2}\|\nabla U(x\,,\,\tau)\|^2_{L^2(\Omega)} \le c^2(\nabla U_\tau(x\,,\,\tau)\,,\,\nabla U(x\,,\,\tau))_{L^2(\Omega)}\,.\tag{4.43}$$

Hence combination of inequalities (4.35)–(4.43) and equality (4.34) yields

$$
\begin{aligned}
&-\frac{b\,\delta_2\|\Omega\|\|\partial\Omega\|}{2}\|U(x\,,\,\tau)\|^2_{L^2(\Omega)} - \left\{\frac{c^2\,\delta_6}{2}+\frac{\delta_5\,\bar g}{2}\right\}\|\nabla U(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&-l(\varepsilon)\left\{\frac{c^2}{2\,\delta_1}+\frac{b}{2\,\delta_2}+\frac{1}{2\,\delta_3}\right\}\|U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&+\left\{\frac{b}{2}-\frac{c^2}{2\,\delta_6}-\frac{c^2\,\varepsilon}{2\,\delta_1}-\frac{b\,\varepsilon}{2\,\delta_2}-\frac{\varepsilon}{2\,\delta_3}-\frac{\bar g}{2\,\delta_4}-\frac{\bar g}{2\,\delta_5}\right\}\|\nabla U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&+\frac{a}{2}\|U_{\tau\tau}(x\,,\,\tau)\|^2_{L^2(\Omega)}-\frac{\delta_4}{2}(g\Box\nabla U)(\tau)\\
&\le \gamma_5\int_0^\tau\|U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\
&+\frac{\left[(c^2+(1+g(0)))l(\varepsilon)+b(l(\varepsilon)+\|\Omega\|\|\partial\Omega\|)\right]}{2}\int_0^\tau\|U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\
&+\frac{(2\,c^2+(c^2+b+1+g(0))\varepsilon)}{2}\int_0^\tau\|\nabla U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,, \qquad (4.44)
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma_5 :=& \frac{c^2\,\delta_1\,T\|\partial\Omega\|\|\Omega\|}{2}+\frac{\delta_3(\sup g(t))^2\,T^3\|\Omega\|\|\partial\Omega\|}{4}\\
&+\frac{c^2\|\Omega\|\|\partial\Omega\|}{2}+\frac{(\sup\|g'(t)\|)^2\,T^4\|\Omega\|\|\partial\Omega\|}{16}+\frac{g(0)\|\partial\Omega\|\|\Omega\|T^2}{4}\,.
\end{aligned}
$$

Adding side to side (4.15), (4.27) and (4.44), we obtain

$$
\begin{aligned}
&\left\{\frac{\beta}{2}-\frac{a}{2}-\frac{b\,\delta_2\|\Omega\|\|\partial\Omega\|}{2}\right\}\|U(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&+\left\{\frac{c^2-\bar g}{2}-\frac{c^2\,\delta_6}{2}-\frac{\delta_5\,\bar g}{2}\right\}\|\nabla U(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&+\left\{\frac{\beta}{2}-\frac{a}{2\,\delta'}-\frac{a}{2}-l(\varepsilon)\Big(\frac{c^2}{2\,\delta_1}+\frac{b}{2\,\delta_2}+\frac{1}{2\,\delta_3}\Big)\right\}\|U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&+\left\{\frac{b}{2}-\frac{c^2}{2\,\delta_6}-\frac{c^2\,\varepsilon}{2\,\delta_1}-\frac{b\,\varepsilon}{2\,\delta_2}-\frac{\varepsilon}{2\,\delta_3}-\frac{\bar g}{2\,\delta_4}-\frac{\bar g}{2\,\delta_5}\right\}\|\nabla U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}\\
&+\frac{a(1-\delta')}{2}\|U_{\tau\tau}(x\,,\,\tau)\|^2_{L^2(\Omega)}+\frac{1-\delta_4}{2}(g\Box\nabla U)(\tau)+\frac{c^2}{2}\|\nabla v(x\,,\,0)\|^2_{L^2(\Omega)}\\
&\le \gamma_6\int_0^\tau\|U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t+\frac{(\sup g(t))^2\,T^2}{4}\int_0^\tau\|\nabla U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\
&+\gamma_7\int_0^\tau\|U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t+\gamma_8\int_0^\tau\|\nabla U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\
&+a\int_0^\tau\|U_{tt}(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t+\frac{1+c^2\,\varepsilon+b\,\varepsilon+\varepsilon}{2}\int_0^\tau\|\nabla v(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,, \qquad (4.45)
\end{aligned}
$$

where

$$
\gamma_6 := \gamma_4+\gamma_5+\|\Omega\|\|\partial\Omega\|\Big(\frac{c^2\,T^2}{4}+\frac{b}{2}+\frac{(\sup g(t))^2\,T^4}{16}\Big)\,,
$$

and

$$
\gamma_7 := a+\frac{l(\varepsilon)(c^2+b+1)}{2}+\frac{\left[l(\varepsilon)(c^2+1+g(0))+b(l(\varepsilon)+\|\Omega\|\|\partial\Omega\|)\right]}{2}\,,
$$

and

$$\gamma_8 := \frac{\varepsilon(c^2+b+1)}{2} + \frac{2\,c^2 + c^2\,\varepsilon + b\,\varepsilon + \varepsilon + g(0)\varepsilon}{2} \,.$$

Now to deal with the last term on the **RHS** of (4.45), we define the function $\theta(x\,,\,t)$ by the relation

$$\theta(x\,,\,t) := \int_0^t U(x,s)\,\mathrm{d}s \,.$$

Hence using (4.3) it follows that

$$v(x\,,\,t) = \theta(x\,,\,\tau) - \theta(x\,,\,t)\,,\ \nabla v(x\,,\,0) = \nabla\theta(x\,,\,\tau)\,, \tag{4.46}$$

and

$$\begin{aligned} \|\nabla v(x\,,\,t)\|^2_{L^2(Q_\tau)} &= \|\nabla\theta(x\,,\,\tau) - \nabla\theta(x\,,\,t)\|^2_{L^2(Q_\tau)} \\ &\le 2(\tau\|\nabla\theta(x\,,\,\tau)\|^2_{L^2(\Omega)} + \|\nabla\theta(x\,,\,t)\|^2_{L^2(Q_\tau)})\,. \end{aligned} \tag{4.47}$$

And make use of the following inequalities

$$\chi_1\|U(x\,,\,\tau)\|^2_{L^2(\Omega)} \le \chi_1\|U(x\,,\,t)\|^2_{L^2(Q_\tau)} + \chi_1\|U_t(x\,,\,t)\|^2_{L^2(Q_\tau)}\,, \tag{4.48}$$

$$\chi_2\|U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} \le \chi_2\|U_t(x\,,\,t)\|^2_{L^2(Q_\tau)} + \chi_2\|U_{tt}(x\,,\,t)\|^2_{L^2(Q_\tau)}\,, \tag{4.49}$$

and

$$\chi_3\|\nabla U(x\,,\,\tau)\|^2_{L^2(\Omega)} \le \chi_3\|\nabla U(x\,,\,t)\|^2_{L^2(Q_\tau)} + \chi_3\|\nabla U_t(x\,,\,t)\|^2_{L^2(Q_\tau)}, \tag{4.50}$$

by replacement (4.46)–(4.50) into (4.45), we get

$$\begin{aligned} &\left\{\chi_1 + \frac{\beta}{2} - \frac{a}{2} - \frac{b\,\delta_2\|\Omega\|\|\partial\Omega\|}{2}\right\}\|U(x\,,\,\tau)\|^2_{L^2(\Omega)} \\ &+ \left\{\chi_3 + \frac{c^2-\bar g}{2} - \frac{c^2\,\delta_6}{2} - \frac{\delta_5\,\bar g}{2}\right\}\|\nabla U(x\,,\,\tau)\|^2_{L^2(\Omega)} \\ &+ \left\{\chi_2 + \frac{\beta}{2} - \frac{a}{2\,\delta'} - \frac{a}{2} - l(\varepsilon)(\frac{c^2}{2\,\delta_1} + \frac{b}{2\,\delta_2} + \frac{1}{2\,\delta_3})\right\}\|U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} \\ &+ \left\{\frac{b}{2} - \frac{c^2\,\varepsilon}{2\,\delta_1} - \frac{b\,\varepsilon}{2\,\delta_2} - \frac{\varepsilon}{2\,\delta_3} - \frac{\bar g}{2\,\delta_4} - \frac{\bar g}{2\,\delta_5} - \frac{c^2}{2\,\delta_6}\right\}\|\nabla U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} \\ &+ \frac{a(1-\delta')}{2}\|U_{\tau\tau}(x\,,\,\tau)\|^2_{L^2(\Omega)} + \frac{1-\delta_4}{2}(g\Box\nabla U)(\tau) \\ &+ \left\{\frac{c^2}{2} - \tau(1 + c^2\,\varepsilon + b\,\varepsilon + \varepsilon)\right\}\|\nabla\theta(x\,,\,\tau)\|^2_{L^2(\Omega)} \\ &\le (\gamma_6 + \chi_1)\int_0^\tau \|U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + \left\{\frac{(\sup g(t))^2\,T^2}{4} + \chi_3\right\}\int_0^\tau \|\nabla U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\ &+ (\gamma_7 + \chi_1 + \chi_2)\int_0^\tau \|U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + (\gamma_8 + \chi_3)\int_0^\tau \|\nabla U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\ &+ (a + \chi_2)\int_0^\tau \|U_{tt}(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + (1 + c^2\,\varepsilon + b\,\varepsilon + \varepsilon)\int_0^\tau \|\nabla\theta(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,. \end{aligned} \tag{4.51}$$

Let

$$\begin{cases} \chi_1 := \dfrac{a + b\,\delta_2 \|\Omega\| \|\partial\Omega\|}{2}\,, \\ \chi_2 := \dfrac{a}{2\,\delta'} + \dfrac{a}{2} + l(\varepsilon)(\dfrac{c^2}{2\,\delta_1} + \dfrac{b}{2\,\delta_2} + \dfrac{1}{2\,\delta_3})\,, \\ \chi_3 := \dfrac{c^2\,\delta_6}{2} + \dfrac{\delta_5\,\bar{g}}{2}\,, \\ \delta' := \dfrac{1}{2}\,, \\ \dfrac{3\,\bar{g}}{b} < \delta_4 < \dfrac{1}{2}\,, \end{cases}$$

and choose $\delta_1\,,\ \delta_2\,,\ \delta_3\,,\ \delta_5$ and δ_6 sufficiently large such that

$$\frac{c^2\,\varepsilon}{2\,\delta_1} + \frac{b\,\varepsilon}{2\,\delta_2} + \frac{\varepsilon}{2\,\delta_3} + \frac{\bar{g}}{2\,\delta_5} + \frac{c^2}{2\,\delta_6} < \frac{b}{6}\,.$$

Since τ is arbitrary we assume that $\dfrac{c^2}{2} - \tau\,\varepsilon(c^2 + b) > 0$, thus inequality (4.51) takes the form

$$\begin{aligned} &\frac{\beta}{2}\|U(x\,,\,\tau)\|^2_{L^2(\Omega)} + \frac{c^2 - \bar{g}}{2}\|\nabla U(x\,,\,\tau)\|^2_{L^2(\Omega)} \\ &+ \frac{\beta}{2}\|U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} + \frac{b}{6}\|\nabla U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} + \frac{a}{4}\|U_{\tau\tau}(x\,,\,\tau)\|^2_{L^2(\Omega)} \\ &+ \left\{\frac{c^2}{2} - \tau(1 + c^2\,\varepsilon + b\,\varepsilon + \varepsilon)\right\}\|\nabla\theta(x\,,\,\tau)\|^2_{L^2(\Omega)} \\ &\le (\gamma_6 + \chi_1)\int_0^\tau \|U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + \left\{\frac{(\sup g(t))^2\,T^2}{4} + \chi_3\right\}\int_0^\tau \|\nabla U(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\ &+ (\gamma_7 + \chi_1 + \chi_2)\int_0^\tau \|U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + (\gamma_8 + \chi_3)\int_0^\tau \|\nabla U_t(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\ &+ (a + \chi_2)\int_0^\tau \|U_{tt}(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + (1 + c^2\,\varepsilon + b\,\varepsilon + \varepsilon)\int_0^\tau \|\nabla\theta(x\,,\,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\,. \end{aligned}$$

Thus we obtain

$$\begin{aligned} &\|U(x\,,\,\tau)\|^2_{L^2(\Omega)} + \|\nabla U(x\,,\,\tau)\|^2_{L^2(\Omega)} + \|U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} \\ &+ \|\nabla U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)} + \|U_{\tau\tau}(x\,,\,\tau)\|^2_{L^2(\Omega)} + \|\nabla\theta(x\,,\,\tau)\|^2_{L^2(\Omega)} \\ &\le D'\int_0^\tau (\|U(x\,,\,t)\|^2_{L^2(\Omega)} + \|\nabla U(x\,,\,t)\|^2_{L^2(\Omega)} + \|U_t(x\,,\,t)\|^2_{L^2(\Omega)} \\ &+ \|\nabla U_t(x\,,\,t)\|^2_{L^2(\Omega)} + \|U_{tt}(x\,,\,t)\|^2_{L^2(\Omega)} + \|\nabla\theta(x\,,\,t)\|^2_{L^2(\Omega)})\mathrm{d}t\,, \end{aligned} \tag{4.52}$$

where

$$D' := \frac{\begin{gathered}\max\Big\{\gamma_6 + \chi_1\,,\ \dfrac{(\sup g(t))^2\,T^2}{4} + \chi_3\,,\ \gamma_7 + \chi_1 + \chi_2\,, \\ \gamma_8 + \chi_3\,,\ a + \chi_2\,,\ 1 + c^2\,\varepsilon + b\,\varepsilon + \varepsilon\Big\}\end{gathered}}{\min\left\{\dfrac{\beta}{2}\,,\ \dfrac{c^2 - \bar{g}}{2}\,,\ \dfrac{\beta}{2}\,,\ \dfrac{b}{6}\,,\ \dfrac{a}{4}\,,\ \dfrac{c^2}{2} - \tau(1 + c^2\,\varepsilon + b\,\varepsilon + \varepsilon)\right\}}\,.$$

If we apply **Gronwall's lemma** to (4.52), we get

$$\begin{aligned}&\|U(x\,,\,\tau)\|^2_{L^2(\Omega)}+\|\nabla U(x\,,\,\tau)\|^2_{L^2(\Omega)}+\|U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}\\&+\|\nabla U_\tau(x\,,\,\tau)\|^2_{L^2(\Omega)}+\|U_{\tau\tau}(x\,,\,\tau)\|^2_{L^2(\Omega)}+\|\nabla\theta(x\,,\,\tau)\|^2_{L^2(\Omega)}\\&\le 0\,,\quad \forall\tau\in\left[0\,,\,\frac{c^2}{2(1+c^2\,\varepsilon+b\,\varepsilon+\varepsilon)}\right].\end{aligned}$$

Proceeding in the same way for the intervals $\tau\in\left[\frac{(m-1)c^2}{2(1+c^2\,\varepsilon+b\,\varepsilon+\varepsilon)}\,,\,\frac{m\,c^2}{2(1+c^2\,\varepsilon+b\,\varepsilon+\varepsilon)}\right]$ to cover the whole interval $[0\,,\,T]$, and thus proving that $U(x\,,\,\tau)=0$, for all τ in $[0\,,\,T]$. □

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