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NONLINEAR PARABOLIC EQUATION WITH VARIABLE EXPONENTS AND DIFFUSE MEASURE DATA

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Abstract. We study the existence of solutions of the nonlinear parabolic equation with variable exponents of the type:

$$\frac{\partial b(x\,,\,u)}{\partial t} - \operatorname{div}(a(x\,,\,t\,,\,\nabla u)) = \mu \ \text{ in } \Omega \times (0\,,\,T),$$

where the right side is a diffuse measure, $b(x\,,\,u)$ is an unbounded function of u and $-\operatorname{div}(a(x\,,\,t\,,\,\nabla u))$ is a Leray–Lions type operator with growth $|\nabla u|^{p(\cdot)-1}$ in ∇u.

Keywords: nonlinear parabolic equations, existence, measure, renormalized solutions.

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1 Introduction

This paper is devoted to the study of the following nonlinear problem:

$$\begin{cases} \dfrac{\partial b(x\,,\,u)}{\partial t} - \operatorname{div}(a(x\,,\,t\,,\,\nabla u)) = \mu & \text{in } Q, \\ b(x\,,\,u)|_{t=0} = b(x\,,\,u_0(x)) & \text{in } \Omega, \\ u(x\,,\,t) = 0 & \text{on } \partial\Omega \times (0\,,\,T). \end{cases} \tag{1.1}$$

In Problem (1.1) the framework is the following: Q is the cylinder $\Omega \times (0\,,\,T)$, Ω is a bounded open subset of $\mathbb{R}^N$, $T > 0$ and $N \geq 2$. Moreover $p : \overline{\Omega} \to [1\,,\,+\infty)$ is a continuous function and

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let $p^- = \min_{x\in\overline{\Omega}} p(x)$ and $p^+ = \max_{x\in\overline{\Omega}} p(x)$ with $1 < p(\cdot) < N$. The operator $-\mathrm{div}(a(x\,,\,t\,,\,\nabla u))$ is a Leray-Lions operator defined from the generalized Sobolev space $V \subset L^{p^-}(0\,,\,T; W_0^{1\,,p(\cdot)}(\Omega))$, into its dual $V^* \subset L^{(p^+)'}(0\,,\,T; W^{-1\,,p'(\cdot)}(\Omega))$ (see (2.4) and (2.6) for the definition), the operator is coercive and which grows like $|\nabla u|^{p(x)-1}$ with respect to ∇u. The function $b : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that for every $x \in \Omega$, $b(x\,,\cdot)$ is a strictly increasing C^1-function, the data μ is a bounded Radon measure on Q and the function u_0 is measurable in Ω such that $b(\cdot\,,\,u_0)$ in $L^1(\Omega)$.

Under our assumptions, it is natural to use Lebesgue and Sobolev spaces with variable exponents. The study of differential equations with variable exponents has been a very active field in recent years. Our motivations for studying (1.1) comes from applications to electro-rheological fluids (see [32]) and image processing (see [33]).

In the case where $p(\cdot) = p$ is a constant, $b(x\,,\,u) = u$, and the right hand side is a bounded measure, the existence of a distributional solution was proved in [1], but due to the lack of regularity of solution, the distributional formulation is not strong enough to provide uniqueness. To overcome this difficulty the notion of renormalized solutions firstly introduced by DiPerna and Lions [27] for the study of Boltzmann equation was adapted to parabolic equations (and elliptic equations) with L^1 data (see [2, 3, 22]).

Concerning the datum μ (where $p(\cdot) = p$ is a constant), the existence and uniqueness of renormalized solution of (1.1) has proved in [28] in the case where $b(x\,,\,u) = u$, $u_0 \in L^1(\Omega)$ and for every measure μ which do not charge the sets of zero p-capacity (see Section 2 for the definition of $p(x)$-capacity), the so-called diffuse measures or soft measures, and we will use the symbol $\mu \in \mathcal{M}_0(Q)$ to denote them. The importance of the measures not charging sets of null p-capacity was first observed in the stationary case in [9], and developed in the evolution case in [28].

In the case of problem (1.1) in the classical Sobolev spaces, Redwane [29, 30] proved the existence and uniqueness of a renormalized solution of problem (1.1) where $\mu \in L^1(Q)$.

For $\mu \in \mathcal{M}_0(Q)$, $b(x\,,\,u) = b(u)$ and $u_0 \in L^1(\Omega)$, the existence and uniqueness of renormalized solution was proved in [7] (see also [11]), and with the parabolic term on $b(x\,,\,u)$, the existence of renormalized solution was proved in [31].

In the case of $p(x)$-Laplace, where $b(x\,,\,u) = u$ and $\mu \in L^1(Q)$, Bendahmane, Wittbold and Zimmermann (see [12]) have proved the existence and uniqueness of renormalized solution. We also point out that the existence and uniqueness of renormalized (or entropy) solutions to parabolic equations with variable exponents can be found in [13].

Our goal of this paper is to extend the results of [12, 13, 29, 32], and our main ideas and methods come from [3, 16, 28].

The paper is organized as follows. In Section 2, we collect some important propositions and results of variable exponents, Lebesgue-Sobolev spaces that will be used throughout the paper. In Section 3, we give our basic assumptions and the definition of a renormalized solution of (1.1). In Section 4, we establish the existence of such solution (Theorem 4.1).

2 Some preliminaries and notations

In what follows, we recall some definitions and basic properties of Lebesgue and Sobolev spaces with variable exponents.

2.1 Sobolev spaces with variable exponents

We first state some elementary results for the generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$, and the generalized Lebesgue-Sobolev spaces $W^{1\,,p(\cdot)}(\Omega)$. We refer to [15] for further properties of variable exponent Lebesgue-Sobolev spaces. Let $p:\overline{\Omega}\to[1\,,+\infty)$ be a continuous function, let $p^-=\min_{x\in\overline{\Omega}}p(x)$ and $p^+=\max_{x\in\overline{\Omega}}p(x)$ with $1<p(\cdot)<N$. The Lebesgue space with variable exponents defined by

$$L^{p(\cdot)}(\Omega)=\left\{u:\Omega\to\mathbb{R};\,u\,\text{is measurable with}\int_\Omega|u(x)|^{p(x)}\,\mathrm{d}x<\infty\right\},$$

we define a norm, the so-called Luxemburg norm, on this space by the formula

$$\|u\|_{L^{p(\cdot)}(\Omega)}=\inf\left\{\mu>0;\int_\Omega\left|\frac{u(x)}{\mu}\right|^{p(x)}\,\mathrm{d}x\le 1\right\}.$$

The following inequality will be used later

$$\min\left\{\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}\,,\,\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\right\}\le\int_\Omega|u(x)|^{p(x)}\,\mathrm{d}x\le\max\left\{\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}\,,\,\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\right\}. \quad (2.1)$$

If $p^->1$, then $L^{p(\cdot)}(\Omega)$ is reflexive and the dual space of $L^{p(\cdot)}(\Omega)$ can be identified with $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)}+\frac{1}{p'(\cdot)}=1$. For any $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$ the Hölder type inequality:

$$\Big|\int_\Omega u\,v\,\mathrm{d}x\Big|\le\Big(\frac{1}{p^-}+\frac{1}{(p^+)'}\Big)\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}\,, \quad (2.2)$$

holds true.

Extending a variable exponent $p:\overline{\Omega}\to[1\,,\infty)$ to $\overline{Q}=[0\,,T]\times\overline{\Omega}$ by setting $p(t\,,x)=p(x)$ for all $(t\,,x)\in\overline{Q}$, we may also consider the generalized Lebesgue space

$$L^{p(\cdot)}(Q)=\left\{u:Q\to\mathbb{R};u\,\text{is measurable with}\int_Q|u(t\,,x)|^{p(x)}\,\mathrm{d}(t\,,x)<\infty\right\}$$

endowed with the norm

$$\|u\|_{L^{p(\cdot)}(Q)}=\inf\left\{\mu>0;\int_Q\left|\frac{u(t\,,x)}{\mu}\right|^{p(x)}\,\mathrm{d}(t\,,x)\le 1\right\},$$

which share the same type of properties as $L^{p(\cdot)}(\Omega)$. We define also the variable Sobolev space

$$W^{1\,,p(\cdot)}(\Omega)=\left\{u\in L^{p(\cdot)}(\Omega);|\nabla u|\in L^{p(\cdot)}(\Omega)\right\},$$

which is Banach space equipped with the following norm $\|u\|_{W^{1\,,p(\cdot)}(\Omega)}=\|u\|_{L^{p(\cdot)}(\Omega)}+\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$, and denote $W_0^{1\,,p(\cdot)}(\Omega)=\overline{C_c^\infty(\Omega)}^{W^{1\,,p(\cdot)}(\Omega)}$. Assuming $1<p^-\le p^+<\infty$, the spaces $W^{1\,,p(\cdot)}(\Omega)$ and $W_0^{1\,,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces. We denote the dual space of $W_0^{1\,,p(\cdot)}(\Omega)$ by $W^{-1\,,p'(\cdot)}(\Omega)$.

Remark 2.1 *The variable exponent $p:\overline{\Omega}\to[1\,,\,\infty)$ is said to satisfy the* log*-continuity condition ($p\in\mathcal{C}^{\log}(\Omega)$) if*

$$\forall\, x_1\,,\, x_2\in\overline{\Omega}\,,\quad |x_1-x_2|<1\,,\quad |p(x_1)-p(x_2)|<w(|x_1-x_2|)\,, \tag{2.3}$$

where $w:\ (0\,,\,\infty)\to\mathbb{R}$ is a nondecreasing function with $\limsup_{\alpha\to0^+}w(\alpha)\ln(\frac{1}{\alpha})<+\infty$.

Lemma 2.2 ([14]) *Let $p\in\mathcal{C}^{\log}(\Omega)$ such that $1\le p^-\le p^+<N$, we have*

$$\forall\, u\in W_0^{1\,,\,p(\cdot)}(\Omega)\,,\quad ||u||_{L^{p^*(\cdot)}(\Omega)}\le C||\nabla u||_{L^{p(\cdot)}(\Omega)}$$

with $C=C(N\,,\,C_{\log}(p)\,,\,p^+)$ and $\frac{1}{p^*(\cdot)}=\frac{1}{p(\cdot)}-\frac{1}{N}$ for $p(x)<N$ a.e. $x\in\Omega$. $p^*(\cdot)=\infty$ otherwise.

Let introduce the functional space

$$V=\Big\{v\in L^{p^-}(0\,,\,T;W_0^{1\,,\,p(\cdot)}(\Omega));\ |\nabla v|\in L^{p(\cdot)}(Q)\Big\}\,, \tag{2.4}$$

which, endowed with the norm

$$\|v\|_V=\|\nabla v\|_{L^{p(\cdot)}(Q)}\ ,$$

or, the equivalent norm

$$\|v\|_V=\|v\|_{L^{p^-}(0\,,\,T;W_0^{1\,,\,p(\cdot)}(\Omega))}+\|\nabla v\|_{L^{p(\cdot)}(Q)}\ ,$$

is a separable and reflexive Banach space. We state some further properties of V in the following lemma.

Lemma 2.3 ([12]) Let V be defined as in (2.4) and its dual space be denoted by V^*. Then
(i) we have the following continuous dense embedding:

$$L^{p^+}(0\,,\,T;W_0^{1\,,\,p(\cdot)}(\Omega))\hookrightarrow V\hookrightarrow L^{p^-}(0\,,\,T;W_0^{1\,,\,p(\cdot)}(\Omega))\,. \tag{2.5}$$

In particular, since $\mathcal{D}(Q)$ is dense in $L^{p^+}(0\,,\,T;W_0^{1\,,\,p(\cdot)}(\Omega))$, it is dense in V and for the corresponding dual space. We have

$$L^{(p^-)'}(0\,,\,T;W^{-1\,,\,p'(\cdot)}(\Omega))\hookrightarrow V^*\hookrightarrow L^{(p^+)'}(0\,,\,T;W^{-1\,,\,p'(\cdot)}(\Omega))\,. \tag{2.6}$$

(ii) One can represent the elements of V^* as follows: if $T\in V^*$, then there exists
$F=(f_1\,,\,\cdots\,,\,f_N)\in(L^{p(\cdot)}(Q))^N$ such that $T=\mathrm{div}_x(F)$ and

$$<T\,,\,\xi>_{V^*,V}=\int_0^T\int_\Omega F\cdot\nabla\xi\ \mathrm{d}x\mathrm{d}t$$

for any $\xi\in V$, moreover, we have, $\|T\|_{V^*}=\max\Big\{\|f_i\|_{L^{p(\cdot)}(Q)},\,i=1\,,\cdots\,,\,n\Big\}$.

Remark 2.4 *Remark that $V\cap L^\infty(Q)$, endowed with the norm*

$$\|v\|_{V\cap L^\infty(Q)}:=\max\left\{\|v\|_V\,,\,\|v\|_{L^\infty(Q)}\right\}\quad \forall v\in V\cap L^\infty(Q)\,,$$

is a Banach space. In fact, it is the dual space of the Banach space $V^*+L^1(Q)$, endowed with the norm

$$\|v\|_{V^*+L^1(Q)}=\inf\left\{\|v_1\|_{V^*}+\|v_2\|_{L^1(Q)};\ \text{such that } v=v_1+v_2,\ \text{with } v_1\in V^*\,,\,v_2\in L^1(Q)\right\}.$$

Lemma 2.5 ([16]) We have

$$W = \Big\{u \in V;\ u_t \in V^* + L^1(Q)\Big\} \hookrightarrow \mathcal{C}([0\,,\,T]; L^1(\Omega))\,, \tag{2.7}$$

and

$$W \cap L^\infty(Q) \hookrightarrow \mathcal{C}([0\,,\,T]; L^2(\Omega))\,. \tag{2.8}$$

2.2 Parabolic capacity and measures

In this part, we introduce the notion of capacity, following the approach developed in [28].

Definition 2.6 *Let us define $X = W_0^{1\,,\,p(\cdot)}(\Omega) \cap L^2(\Omega)$, endowed with its natural norm $\|\cdot\|_{W_0^{1\,,\,p(\cdot)}(\Omega)} + \|\cdot\|_{L^2(\Omega)}$ and the space*

$$W_{p(\cdot)}(0\,,\,T) = \Big\{u \in L^{p^-}(0\,,\,T; X);\ \nabla u \in (L^{p(\cdot)}(Q))^N,\ u_t \in L^{(p^-)'}(0\,,\,T; X')\Big\}$$

endowed with its natural norm

$$\|u\|_{W_{p(\cdot)}(0\,,\,T)} = \|u\|_{L^{p^-}(0\,,\,T;X)} + \|\nabla u\|_{(L^{p(\cdot)}(Q))^N} + \|u_t\|_{L^{(p^-)'}(0\,,\,T;X')}\,.$$

Since $W_0^{1\,,\,p(\cdot)}(\Omega)$ and $L^2(\Omega)$ are separable and reflexive Banach spaces, it follows that X is a separable and reflexive Banach space. Consequently, the following result can be proved similarly to that in [17].

Theorem 2.7 [16] The space $W_{p(\cdot)}(0\,,\,T)$ is a separable and reflexive Banach space.

We now give the definition and some properties of capacity.

Definition 2.8 *If $U \subset Q$ is an open set, we define the parabolic $p(\cdot)$-capacity of U as*

$$Cap_{p(\cdot)}(U) = \inf\Big\{\|u\|_{W_{p(\cdot)}(0\,,\,T)};\ u \in W_{p(\cdot)}(0\,,\,T),\ u \geq \chi_U \text{ almost everywhere in } Q\Big\},$$

where as usual we set $\inf\emptyset = +\infty$*. For any Borel set $B \subset Q$ we then define*

$$Cap_{p(\cdot)}(B) = \inf\Big\{Cap_{p(\cdot)}(U);\ U \text{ is an open set of } Q,\ B \subset U\Big\}.$$

Proposition 2.9 [16] If u is cap-quasi continuous and belongs to $W_{p(\cdot)}(0\,,\,T)$, then for all $k > 0$,

$$Cap_{p(\cdot)}(\{|u| > k\}) = \frac{C}{k}\max\Big\{\|u\|_{W_{p(\cdot)}(0\,,\,T)}^{\frac{p^-}{(p')^-}}\,,\ \|u\|_{W_{p(\cdot)}(0\,,\,T)}^{\frac{(p^-)'}{p^-}}\Big\}.$$

Let us denote with $\mathcal{M}_b(Q)$ the set of all Radon measures with bounded variation on Q, and $\mathcal{M}_0(Q)$ will denote the set of all measures with bounded variation over Q that do not charge the sets of zero $p(\cdot)$-capacity.

Definition 2.10 *We define*

$$\mathcal{M}_0(Q) = \Big\{\mu \in \mathcal{M}_b(Q);\ \mu(E) = 0 \text{ for all } E \subset Q, \text{ such that } Cap_{p(\cdot)}(E) = 0\Big\}.$$

In [16] (see also [28]) the authors also proved the following decomposition theorem:

Theorem 2.11 [16] Let $\mu \in \mathcal{M}_0(Q)$ then there exists $(f\,,\,F\,,\,g_1\,,\,g_2)$ such that $f \in L^1(Q),\ F \in (L^{p'(\cdot)}(Q))^N$, $g_1 \in L^{(p^-)'}(0\,,\,T; W^{-1\,,p'(\cdot)}(\Omega)),\ g_2 \in L^{p^-}(0\,,\,T; X)$ such that

$$\int_Q \varphi\,\mathrm{d}\mu = \int_Q f\,\varphi\,\mathrm{d}x\mathrm{d}t + \int_Q F\cdot\nabla u\,\mathrm{d}x\mathrm{d}t + \int_0^T \langle g_1\,,\,\varphi\rangle\,\mathrm{d}t - \int_0^T \langle \varphi_t\,,\,g_2\rangle\,\mathrm{d}t \tag{2.9}$$

$\forall\varphi \in \mathcal{C}_c^\infty([0\,,\,T]\times\Omega)$, where $\langle\cdot\ ,\ \cdot\rangle$ denote the duality between X' and X. Such a triplet $(f\,,\,F\,,\,g_1\,,\,g_2)$ will be called a decomposition of μ.

Remark 2.12 *(i) Notice that the decomposition of $\mu \in \mathcal{M}_0(Q)$ given by the previous theorem is not unique.*

(ii) Observe that, by Lemma 2.3 the embedding of $L^{(p^-)'}(0\,,\,T; W^{-1\,,p'(\cdot)}(\Omega))$ into V^* is continuous and since $g_1 \in L^{(p^-)'}(0\,,\,T; W^{-1\,,p'(\cdot)}(\Omega))$, then there exists $\overline{F} \in (L^{p(\cdot)}(Q))^N$ such that $g_1 = \mathrm{div}_x(\overline{F})$. Applying (2.9) of Theorem 2.7, we deduce that for $\mu \in \mathcal{M}_0(Q)$, there exists $(f\,,\,G\,,\,g)$ (with $G = F + \overline{F}$ and $g_2 = g$) such that $f \in L^1(Q),\ F \in (L^{p'(\cdot)}(Q))^N$ and $g \in L^{p^-}(0\,,\,T; X)$ such that

$$\int_Q \varphi\,\mathrm{d}\mu = \int_Q f\,\varphi\,\mathrm{d}x\mathrm{d}t + \int_Q G\cdot\nabla u\,\mathrm{d}x\mathrm{d}t - \int_0^T \langle \varphi_t\,,\,g\rangle\,\mathrm{d}t \qquad \forall\varphi \in \mathcal{C}_c^\infty([0\,,\,T]\times\Omega)\,. \tag{2.10}$$

Here are some notations we will use throughout the paper. For any nonnegative real number k we denote by $T_k(r) = \min(k\,,\,\max(r\,,\,-k))$ the truncation function at level k. By $\langle\cdot\,,\,\cdot\rangle$ we mean the duality between suitable spaces in which functions are involved. In particular we will consider both the duality between $W_0^{1\,,p(\cdot)}(\Omega)$ and $W^{-1\,,p'(\cdot)}(\Omega)$ and the duality between $W_0^{1\,,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ and $W^{-1\,,p'(\cdot)}(\Omega) + L^1(\Omega)$.

3 Assumptions and statement of main results

Throughout the paper, we assume that the following assumptions hold true: Ω is a bounded open set on $\mathbb{R}^N$ ($N \geq 1$), $T > 0$ is given and we set $Q = \Omega\times(0\,,\,T)$.

$$b,\ \frac{\partial b}{\partial s} \equiv b_s : \Omega\times\mathbb{R}\to\mathbb{R} \text{ and } \nabla_x b : \Omega\times\mathbb{R}\to\mathbb{R}^N \tag{3.1}$$

are Carathéodory functions such that, for almost every $x \in \Omega$, $b(x\,,\,s)$ is a strictly increasing $\mathcal{C}^1$-function with $b(x\,,\,0) = 0$. For every $s \in \mathbb{R}$, the function $b(x\,,\,s)$ is in $W^{1\,,p(\cdot)}(\Omega)$.

There exist $\lambda,\ \Lambda > 0$ such that

$$\lambda \leq b_s(x\,,\,s) \leq \Lambda\,, \tag{3.2}$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$. There exists a function B in $L^{p(\cdot)}(\Omega)$ such that

$$\Big|\nabla_x b(x\,,\,s)\Big| \le B(x)\,, \tag{3.3}$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$.

$$a : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \quad \text{is a Carathéodory function}\,, \tag{3.4}$$

$$a(x\,,\,t\,,\,\xi)\cdot\xi \ge \alpha|\xi|^{p(\cdot)}\,, \tag{3.5}$$

for almost every $(x\,,\,t) \in Q$, for every $\xi \in \mathbb{R}^N$, where $\alpha > 0$ is a given real number.

$$|a(x\,,\,t\,,\,\xi)| \le \beta(L(x\,,\,t) + |\xi|^{p(x)-1})\,, \tag{3.6}$$

for almost every $(x\,,\,t) \in Q$, for every $\xi \in \mathbb{R}^N$, where $\beta > 0$ is a given real number, L is a non negative function in $L^{p'(\cdot)}(Q)$.

$$[a(x\,,\,t\,,\,\xi) - a(x\,,\,t\,,\,\xi')]\cdot[\xi - \xi'] > 0\,, \tag{3.7}$$

for any $(\xi\,,\,\xi') \in \mathbb{R}^{2N}$ and for almost every $(x\,,\,t) \in Q$.

$$\mu \in \mathcal{M}_0(Q)\,, \tag{3.8}$$

$$u_0 \text{ is a measurable function in } \Omega \text{ such that } b(x\,,\,u_0) \in L^1(\Omega)\,. \tag{3.9}$$

The definition of a renormalized solution for Problem (1.1) is given below.

Definition 3.1 *Let $\mu \in \mathcal{M}_0(Q)$, $(f\,,\,G\,,\,g)$ be a decomposition of μ. A measurable function u defined on Q (let $v := b(x\,,\,u) - g$) is a renormalized solution of Problem (1.1) if*

$$T_k(v) \in L^{p^-}(0\,,\,T; W_0^{1\,,\,p(\cdot)}(\Omega))\,,\ \nabla T_k(v) \in (L^{p(\cdot)}(Q))^N\,,\ \forall k \ge 0 \text{ and } \ v \in L^\infty(0\,,\,T; L^1(\Omega))\,, \tag{3.10}$$

$$\int_{\{(t\,,x)\in Q\ ;\ n\le|v|\le n+1\}} a(x\,,\,t\,,\,\nabla u)\cdot\nabla u\,\mathrm{d}x\mathrm{d}t \ \longrightarrow 0 \ \text{ as } n \to +\infty\,, \tag{3.11}$$

and if, for every function S in $W^{2\,,\,\infty}(\mathbb{R})$, which is piecewise $\mathcal{C}^1$ and such that S' has a compact support, we have

$$\begin{aligned}&\frac{\partial S(v)}{\partial t} - \mathrm{div}\Big(S'(v)a(x\,,\,t\,,\,\nabla u)\Big) + S''(v)a(x\,,\,t\,,\,\nabla u)\cdot\nabla v\\ &= f\,S'(v) - \mathrm{div}\Big(G\,S'(v)\Big) + S''(v)\,G\cdot\nabla v \quad \text{in } \mathcal{D}'(Q)\,,\end{aligned} \tag{3.12}$$

$$S(v)(t=0) = S(b(x\,,\,u_0)) \ \text{ in } L^1(\Omega)\,. \tag{3.13}$$

Remark 3.2 *Note that, all terms in (3.12) are well defined, indeed, let $k > 0$ such that* $\mathrm{supp}(S') \subset [-k\,,\,k]$ *we have:*

$$\int_Q |\nabla S(v)|^{p(x)}\,\mathrm{d}x\mathrm{d}t \le \max\Big(\|S'\|^{p^-}_{L^\infty(\mathbb{R})}\,,\,\|S'\|^{p^+}_{L^\infty(\mathbb{R})}\Big)\int_Q |\nabla T_k(v)|^{p(x)}\,\mathrm{d}x\mathrm{d}t\,.$$

Then $S(v) \in L^{p^-}(0\,,T;W_0^{1\,,p(\cdot)}(\Omega))$ *and* $\dfrac{\partial S(v)}{\partial t} \in \mathcal{D}'(Q)$. *Since* $\nabla T_k(v) = \nabla(b(x\,,u) - g)\chi_{\{|v|\le k\}} = \Big(b_s(x\,,u)\nabla u + \nabla_x b(x\,,u) - \nabla g\Big)\chi_{\{|v|\le k\}}$, *the term* $S'(v)a(x\,,t\,,\nabla u)$ *identifies with:*

$$S'(T_k(v))a\Big(x\,,t\,,(b_s(x\,,u))^{-1}(\nabla T_k(v) + (\nabla g - \nabla_x b(x\,,u))\chi_{\{|v|\le k\}})\Big) \text{ a.e. in } Q\,.$$

Assumptions (3.2) and (3.6) imply that

$$\begin{aligned}
&\int_Q |S'(v)a(x\,,t\,,\nabla u)|^{p'(x)}\,\mathrm{d}x\mathrm{d}t\\
&\le \int_Q \beta^{p(x)}\,\|S'\|^{p(x)}_{L^\infty(\mathbb{R})}\Big|L(x\,,t) + \Big|\frac{1}{\lambda}|\nabla T_k(v) + \nabla g - \nabla_x b(x\,,u)|\Big|^{p(x)-1}\Big|^{p'(x)}\,\mathrm{d}x\mathrm{d}t\\
&\le \max\Big((\beta\|S'\|_{L^\infty(\mathbb{R})})^{(p^-)'}\,,\,(\beta\|S'\|_{L^\infty(\mathbb{R})})^{(p^+)'}\Big)\Big[2^{(p^-)'-1}\int_Q |L(x\,,t)|^{p'(x)}\,\mathrm{d}x\mathrm{d}t \qquad (3.14)\\
&+ \max\Big\{\Big(\frac{2}{\lambda}\Big)^{(p^+)'}\,,\,\Big(\frac{2}{\lambda}\Big)^{(p^-)'}\Big\}\Big(\int_Q |\nabla T_k(v)|^{p(x)}\,\mathrm{d}x\mathrm{d}t + \int_Q (|\nabla g| + |B(x)|)^{p(x)}\,\mathrm{d}x\mathrm{d}t\Big)\Big].
\end{aligned}$$

Using (3.4), (3.10) and (3.14) it follows that:

$$S'(v)a(x\,,t\,,\nabla u) \in (L^{p'(\cdot)}(Q))^N. \qquad (3.15)$$

The term $S''(v)a(x\,,t\,,\nabla u)\cdot\nabla v$ *identifies with*

$$S''(T_k(v))a\Big(x\,,t\,,(b_s(x\,,u))^{-1}(\nabla T_k(v) + (\nabla g - \nabla_x b(x\,,u))\chi_{\{|v|\le k\}})\Big)\cdot\nabla T_k(v) \text{ a.e. in } Q$$

and in view of (3.4), (3.10), (3.15) and by Hölder inequality we obtain

$$S''(v)a(x\,,t\,,\nabla u)\cdot\nabla v \in L^1(Q)\,.$$

Finally $f\,S'(v)$ *and* $G\cdot\nabla S'(v)$ *belongs to* $L^1(Q)$ *and* $G\,S'(v) \in (L^{p'(\cdot)}(Q))^N$.

We also have $S(v) \in V$ *and* $\dfrac{\partial S(v)}{\partial t} \in V^* + L^1(Q)$, *which implies by Lemma 2.5 that* $S(v) \in \mathcal{C}^0([0\,,T]\,,L^1(\Omega))$ *and (3.13) makes sense.*

Note that the formulation of renormalized solution does not depend on the decomposition of μ, the proof of this fact relies on the following result.

Lemma 3.3 *Let* $\mu \in \mathcal{M}_0(Q)$, *and let* $(f\,,G\,,g)$ *and* $(\overline{f}\,,\overline{G}\,,\overline{g})$ *be two different decompositions of* μ *according to Theorem 2.11 (see also Remark 2.12). Then we have* $(g-\overline{g})_t = \overline{f} - f - \operatorname{div}\Big(\overline{G} - G\Big)$ *in distributional sense,* $g - \overline{g} \in \mathcal{C}([0\,,T];L^1(\Omega))$ *and* $(g-\overline{g})(0) = 0$.

Proof. See [28] and [16]. □

Proposition 3.4 *Let* u *be a renormalized solution of (1.1). Then* u *satisfies (3.10)–(3.13) for every decomposition* $(f\,,G\,,g)$ *of* μ.

Proof. The proof of Proposition 3.4 follows the same lines as the proof of the corresponding result in the case of a constant exponent p, Proposition 4.2 of [7], and therefore is omitted here. □

4 Existence result

This section is devoted to establish the following existence theorem.

Theorem 4.1 *Under assumptions (3.1)–(3.9) there exists at least a renormalized solution u of Problem (1.1).*

Proof. The proof is divided into 3 steps. In Step 1, we introduce an approximate problem, establish a few a priori estimates, the limit u of the approximate solutions u^ε is introduced and $v := b(x\,,\,u) - g$ is shown to belong to $L^\infty(0\,,\,T; L^1(\Omega))$ and to satisfy (3.10). In Step 2, we define a time regularization of the field $T_k(u)$ and we establish Lemma 4.2, which allows us to control the parabolic contribution that aries in the monotonicity method when passing to the limit and we prove an energy estimate Lemma 4.3. At last, Step 3 is devoted to prove that u satisfies (3.11), (3.12), and (3.13) of the Definition 3.1.

$\star$ **Step 1**. For $\varepsilon > 0$ fixed, let us introduce the following regularization of the data

$$u_0^\varepsilon \in \mathcal{C}_c^\infty(\Omega) : b(x\,,\,u_0^\varepsilon) \to b(x\,,\,u_0) \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0\,. \tag{4.1}$$

$$\mu^\varepsilon \in \mathcal{C}_c^\infty(Q)\ : \|\mu^\varepsilon\|_{L^1(Q)} \leq C \text{ and } \mu^\varepsilon = f^\varepsilon - \operatorname{div}(G^\varepsilon) + \frac{\partial g^\varepsilon}{\partial t} \tag{4.2}$$

and such that

$$f^\varepsilon \in \mathcal{C}_c^\infty(Q) : f^\varepsilon \to f \text{ in } L^1(Q) \text{ as } \varepsilon \to 0\,, \tag{4.3}$$

$$G^\varepsilon \in (\mathcal{C}_c^\infty(Q))^N : G^\varepsilon \to G \text{ in } (L^{p'(\cdot)}(Q))^N \text{ as } \varepsilon \to 0\,, \tag{4.4}$$

$$g^\varepsilon \in \mathcal{C}_c^\infty(Q) : g^\varepsilon \to g \text{ in } L^{p^-}(0\,,\,T; W_0^{1\,,\,p(\cdot)}(\Omega) \cap L^2(\Omega)) \text{ as } \varepsilon \to 0\,. \tag{4.5}$$

Let us now consider the following regularized problem:

$$\frac{\partial b(x\,,\,u^\varepsilon)}{\partial t} - \operatorname{div}(a(x\,,\,t\,,\,\nabla u^\varepsilon)) = f^\varepsilon - \operatorname{div}(G^\varepsilon) + \frac{\partial g^\varepsilon}{\partial t} \equiv \mu^\varepsilon \text{ in } Q\,, \tag{4.6}$$

$$u^\varepsilon = 0 \text{ in } (0\,,\,T) \times \partial\Omega\,, \tag{4.7}$$

$$b(x\,,\,u^\varepsilon)(t=0)\ =\ b(x\,,\,u_0^\varepsilon) \text{ in } \Omega\,, \tag{4.8}$$

where $v^\varepsilon := b(x\,,\,u^\varepsilon) - g^\varepsilon$. As a consequence, proving existence of a weak solution $u^\varepsilon \in V$ of (4.6)–(4.8) is an easy task (see *e.g.* [20]).

Using $T_k(v^\varepsilon) \in V \cap L^\infty(Q)$ as a test function in (4.6). Employing the integration by parts formula for the evolution term, leads to

$$\begin{aligned} &\int_\Omega \overline{T_k}(v^\varepsilon)\,\mathrm{d}x + \int_0^t \int_\Omega a(x\,,\,s\,,\,\nabla u^\varepsilon) \cdot \nabla T_k(v^\varepsilon)\ \mathrm{d}x\mathrm{d}s \\ &= \int_0^t \int_\Omega f^\varepsilon T_k(v^\varepsilon)\ \mathrm{d}x\mathrm{d}s + \int_0^t \int_\Omega G^\varepsilon \cdot \nabla T_k(v^\varepsilon)\ \mathrm{d}x\mathrm{d}s + \int_\Omega \overline{T_k}(b(x\,,\,u_0^\varepsilon))\,\mathrm{d}x \end{aligned} \tag{4.9}$$

for almost every $t \in (0\,,\,T)$ and where $\overline{T_k}(r) = \displaystyle\int_0^r T_k(s)\ \mathrm{d}s$.

Using assumptions (2.4)–(2.5), the definition of $\overline{T_k}$ and by means of Young's inequality in (4.9), we obtain

$$\begin{aligned}
&\int_\Omega \overline{T_k}(b(x\,,\,u^\varepsilon)-g^\varepsilon)\,\mathrm{d}x+\alpha\int_{E_k} b_s(x\,,\,u^\varepsilon)|\nabla u^\varepsilon|^{p(x)}\,\mathrm{d}x\mathrm{d}s\\
&\le k\|f^\varepsilon\|_{L^1(Q)}+\max\Big(\Big(\frac{2\,\alpha}{\lambda}\Big)^{(p^-)'-1}\,,\,\Big(\frac{2\,\alpha}{\lambda}\Big)^{(p^+)'-1}\Big)\int_Q |G^\varepsilon|^{p'(x)}\,\mathrm{d}x\mathrm{d}t\\
&+\frac{\alpha}{2\,p^-}\int_{E_k} b_s(x\,,\,u^\varepsilon)|\nabla u^\varepsilon|^{p(x)}\,\mathrm{d}x\mathrm{d}s\\
&+\beta\Big(\frac{1}{p^-}+\frac{1}{(p^+)'}\Big)\Big(\|B(x)\|_{L^{p(\cdot)}(Q)}+\|\nabla g^\varepsilon\|_{L^{p(\cdot)}(Q)}\Big)\|L\|_{L^{p'(\cdot)}(Q)}\\
&+\frac{\alpha}{2}\max\Big(\Big(\frac{2\,\beta}{\alpha}\Big)^{p^-}\,,\,\Big(\frac{2\,\beta}{\alpha}\Big)^{p^+}\Big)\int_Q\Big(|B(x)|+|\nabla g^\varepsilon|\Big)^{p(x)}\,\mathrm{d}x\mathrm{d}t\\
&+\frac{\alpha}{2\,p^-}\int_{E_k} b_s(x\,,\,u^\varepsilon)|\nabla u^\varepsilon|^{p(x)}\,\mathrm{d}x\mathrm{d}s+k\|b_\varepsilon(x\,,\,u_0^\varepsilon)\|_{L^1(\Omega)}\,,
\end{aligned}\tag{4.10}$$

where $E_k=\{(x\,,\,s):|v^\varepsilon|\le k\}$. As a consequence, we find

$$\begin{aligned}
&\int_\Omega \overline{T_k}(b(x\,,\,u^\varepsilon)-g_2^\varepsilon)\,\mathrm{d}x+\alpha\Big(\frac{p^--1}{p^-}\Big)\int_{E_k} b_s(x\,,\,u^\varepsilon)|\nabla u^\varepsilon|^{p(x)}\,\mathrm{d}x\mathrm{d}s\\
&\le k\|f^\varepsilon\|_{L^1(Q)}+\max\Big(\Big(\frac{2\,\alpha}{\lambda}\Big)^{(p^-)'-1}\,,\,\Big(\frac{2\,\alpha}{\lambda}\Big)^{(p^+)'-1}\Big)\int_Q |G^\varepsilon|^{p'(x)}\,\mathrm{d}x\mathrm{d}t\\
&+\beta\Big(\frac{1}{p^-}+\frac{1}{(p^+)'}\Big)\Big(\|B(x)\|_{L^{p(\cdot)}(Q)}+\|\nabla g^\varepsilon\|_{L^{p(\cdot)}(Q)}\Big)\|L\|_{L^{p'(\cdot)}(Q)}\\
&+\frac{\alpha}{2}\max\Big(\Big(\frac{2\,\beta}{\alpha}\Big)^{p^-}\,,\,\Big(\frac{2\,\beta}{\alpha}\Big)^{p^+}\Big)\int_Q\Big(|B(x)|+|\nabla g^\varepsilon|\Big)^{p(x)}\,\mathrm{d}x\mathrm{d}t+k\|b(x\,,\,u_0^\varepsilon)\|_{L^1(\Omega)}\,.
\end{aligned}\tag{4.11}$$

Moreover, by definition of $v^\varepsilon=b(x\,,\,u^\varepsilon)-g$ and by the Young's inequality, we deduce

$$\begin{aligned}
&\int_Q |\nabla T_k(v^\varepsilon)|^{p(x)}\,\mathrm{d}x\mathrm{d}t=\int_{E_k}|\nabla v^\varepsilon|^{p(x)}\,\mathrm{d}x\mathrm{d}t\\
&\le 2^{p^+-1}\max\Big(\Lambda^{p^--1}\,,\,\Lambda^{p^+-1}\Big)\int_{E_k} b_s(x\,,\,u^\varepsilon)|\nabla u^\varepsilon|^{p(x)}\,\mathrm{d}x\mathrm{d}s\\
&+2^{p^+-1}\int_\Omega\Big(|B(x)|+|\nabla g^\varepsilon|\Big)^{p(x)}\,\mathrm{d}x\mathrm{d}t\,.
\end{aligned}\tag{4.12}$$

The properties of $G^\varepsilon,\ g^\varepsilon,\ f^\varepsilon$ and $\|b(x\,,\,u_0^\varepsilon)\|_{L^1(\Omega)}$, we deduce from (4.11) and (4.12) that

$$|\nabla T_k(v^\varepsilon)| \text{ is bounded in } (L^{p(\cdot)}(Q))^N,\tag{4.13}$$

and

$$T_k(v^\varepsilon) \text{ is bounded in } L^{p^-}(0\,,\,T;W_0^{1\,,\,p(\cdot)}(\Omega))\,,\tag{4.14}$$

independently of ε and for any $k\ge 0$.

For any $S\in W^{1\,,\,\infty}(\mathbb{R})$ such that S' has a compact support (supp$(S')\subset[-k\,,\,k]$), we have from (4.13), (4.14) and by Stampacchia's Theorem that

$$S(v^\varepsilon) \text{ is bounded in } V \text{ independently of } \varepsilon\,.\tag{4.15}$$

We multiply the equation (4.6) by $S'(v^\varepsilon)$ then we deduce

$$\frac{\partial S(v^\varepsilon)}{\partial t} = \mathrm{div}\Big(S'(v^\varepsilon)a(x\,,\,t\,,\,\nabla u^\varepsilon)\Big) - S''(v^\varepsilon)a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla v^\varepsilon \tag{4.16}$$
$$+ f^\varepsilon S'(v^\varepsilon) - \mathrm{div}\Big(G^\varepsilon S'(v^\varepsilon)\Big) + S''(v^\varepsilon)G^\varepsilon\cdot\nabla v^\varepsilon \text{ in } \mathcal{D}'(Q)\,.$$

Remark that

$$\Big|S'(v^\varepsilon)a(x\,,\,t\,,\,\nabla u^\varepsilon)\Big| \le \beta\|S'\|_{L^\infty(\mathbb{R})}\Big(L(x\,,\,t) + c\Big|\nabla T_k(v^\varepsilon) + \nabla g^\varepsilon - \nabla_x b_\varepsilon(x\,,\,u^\varepsilon)\Big|^{p(x)-1}\Big), \tag{4.17}$$

where $c = \max\Big(\Big(\frac{1}{\lambda}\Big)^{p^+-1}\,,\,\Big(\frac{1}{\lambda}\Big)^{p^--1}\Big)$. As a consequence each terms in the right hand side of (4.16) is bounded either in V^* or in $L^1(Q)$, and we then deduce that

$$\frac{\partial S(v^\varepsilon)}{\partial t} \text{ is bounded in } V^* + L^1(Q)\,, \tag{4.18}$$

independently of ε. Estimates (4.15) and (4.18) imply that, for a subsequence still indexed by ε,

$$v^\varepsilon \;\to\; v := b(x\,,\,u) - g \;\text{ a.e. in } Q\,, \tag{4.19}$$

$$u^\varepsilon \to\; u \text{ a.e. in } Q\,, \tag{4.20}$$

$$T_k(v^\varepsilon) \rightharpoonup T_k(v) \text{ weakly in } L^{p^-}(0\,,\,T; W_0^{1\,,\,p(\cdot)}(\Omega))\,, \tag{4.21}$$

$$a(x\,,\,t\,,\,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}} \rightharpoonup \sigma_k \text{ weakly in } (L^{p(\cdot)'}(Q))^N\,. \tag{4.22}$$

By using (4.1), (4.3), (4.20) and by Lebesgue's convergence theorem we obtain

$$\nabla_x b(x\,,\,u^\varepsilon) \to \nabla_x b(x\,,\,u) \text{ strongly in } (L^p(Q))^N\,, \tag{4.23}$$

as ε tends to zero for any $k > 0$ and where for any $k > 0$, σ_k belongs to $(L^{p'(\cdot)}(Q))^N$.

Now we establish that $b(x\,,\,u) - g$ belongs to $L^\infty(0\,,\,T; L^1(\Omega))$. Indeed using (4.11) and $\overline{T_k}(s) \ge |s| - 1$ leads to

$$\int_\Omega |v^\varepsilon(t)|\,\mathrm{d}x \le k\|f^\varepsilon\|_{L^1(Q)} + \max\Big(\Big(\frac{2\,\alpha}{\lambda}\Big)^{(p^-)'-1}\,,\,\Big(\frac{2\,\alpha}{\lambda}\Big)^{(p^+)'-1}\Big)\int_Q |G^\varepsilon|^{p'(x)}\,\mathrm{d}x\mathrm{d}t$$
$$+ \beta\Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)\Big(\|B(x)\|_{L^{p(\cdot)}(Q)} + \|\nabla g^\varepsilon\|_{L^{p(\cdot)}(Q)}\Big)\|L\|_{L^{p'(\cdot)}(Q)} \tag{4.24}$$
$$+ \frac{\alpha}{2}\max\Big(\Big(\frac{2\,\beta}{\alpha}\Big)^{p^-}\,,\,\Big(\frac{2\,\beta}{\alpha}\Big)^{p^+}\Big)\int_Q\Big(|B(x)| + |\nabla g^\varepsilon|\Big)^{p(x)}\mathrm{d}x\mathrm{d}t + k\|b(x\,,\,u_0^\varepsilon)\|_{L^1(\Omega)} + \mathrm{meas}(\Omega)\,,$$

a.e. t in $(0\,,\,T)$. Using (4.1)–(4.5) and (4.20), we deduce that $v := b(x\,,\,u) - g$ belongs to $L^\infty(0\,,\,T; L^1(\Omega))$.

Now we look of an energy estimate of the approximating solutions. For any integer $n \ge 1$, consider the Lipschitz continuous function θ_n defined through: $\theta_n(r) = T_{n+1}(r) - T_n(r)$. Remark that $\|\theta_n\|_{L^\infty(\mathbb{R})} \le 1$ for any $n \ge 1$ and that $\theta_n(r) \to 0$ as $n \to \infty$, $\forall r \in \mathbb{R}$. Using $\theta_n(v^\varepsilon)$ as a test function in (4.6) leads to

$$\int_\Omega \overline{\theta_n}(v^\varepsilon)\,\mathrm{d}x + \int_Q a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla\theta_n(v^\varepsilon)\,\mathrm{d}x\mathrm{d}t \tag{4.25}$$
$$= \int_Q f^\varepsilon\theta_n(v^\varepsilon)\,\mathrm{d}x\mathrm{d}t + \int_Q G^\varepsilon\cdot\nabla\theta_n(v^\varepsilon)\,\mathrm{d}x\mathrm{d}t + \int_\Omega \overline{\theta_n}(b(x\,,\,u_0^\varepsilon))\,\mathrm{d}x\,,$$

where $\overline{\theta_n}(r) = \displaystyle\int_0^r \theta_n(s)\,\mathrm{d}s \geq 0$ for all $r \in \mathbb{R}$. Hence

$$\begin{aligned}
&\int_{\{n\leq|v^\varepsilon|\leq n+1\}} b_s(x\,,\,u^\varepsilon)a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla u^\varepsilon\,\mathrm{d}x\mathrm{d}t\\
&\leq \int_{\{n\leq|v^\varepsilon|\leq n+1\}} a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\Big(\nabla g^\varepsilon - \nabla_x b(x\,,\,u^\varepsilon)\Big)\,\mathrm{d}x\mathrm{d}t + \int_Q f^\varepsilon\theta_n(v^\varepsilon)\,\mathrm{d}x\mathrm{d}t \qquad (4.26)\\
&+ \int_{\{n\leq|v^\varepsilon|\leq n+1\}} G^\varepsilon\cdot\Big(b_s(x\,,\,u^\varepsilon)\nabla u^\varepsilon + \nabla_x b(x\,,\,u^\varepsilon) - \nabla g^\varepsilon\Big)\,\mathrm{d}x\mathrm{d}t + \int_\Omega \overline{\theta_n}(b(x\,,\,u_0^\varepsilon))\,\mathrm{d}x\,.
\end{aligned}$$

By using (4.1)-(4.5) and Young's inequality we obtain

$$\begin{aligned}
&\lambda\int_{\{n\leq|v^\varepsilon|\leq n+1\}} a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla u^\varepsilon\,\mathrm{d}x\mathrm{d}t \leq \frac{\beta}{(p^+)'}\int_{\{|v^\varepsilon|\geq n\}} |L(x\,,\,t)|^{p'(x)}\,\mathrm{d}x\mathrm{d}t\\
&+ \frac{\beta}{p^-}\int_{\{|v^\varepsilon|\geq n\}}\Big(|B(x)| + |\nabla g^\varepsilon|\Big)^{p(x)}\,\mathrm{d}x\mathrm{d}t + \frac{\alpha\,\lambda}{2(p^+)'}\int_{\{n\leq|v^\varepsilon|\leq n+1\}} |\nabla u^\varepsilon|^{p(x)}\,\mathrm{d}x\mathrm{d}t\\
&+ \frac{1}{p^-}\max\Big(\Big(\frac{2}{\alpha\,\lambda}\Big)^{p^--1}\,,\,\Big(\frac{2}{\alpha\,\lambda}\Big)^{p^+-1}\Big)\int_{\{|v^\varepsilon|\geq n\}} |\nabla g^\varepsilon|^{p(x)}\,\mathrm{d}x\mathrm{d}t \qquad (4.27)\\
&+ \int_{\{|v^\varepsilon|\geq n\}} |f^\varepsilon|\,\mathrm{d}x\mathrm{d}t + \frac{\alpha\,\lambda}{2\,(p^+)'}\int_{\{n\leq|v^\varepsilon|\leq n+1\}} |\nabla u^\varepsilon|^{p(x)}\,\mathrm{d}x\mathrm{d}t\\
&+ \frac{\lambda}{(p^+)'}\max\Big(\Big(\frac{2\,(p^+)'}{\alpha\,p^-}\Big)^{(p^-)'-1}\,,\,\Big(\frac{2\,(p^+)'}{\alpha\,p^-}\Big)^{(p^+)'-1}\Big)\int_{\{|v^\varepsilon|\geq n\}} |G^\varepsilon|^{p(x)}\,\mathrm{d}x\mathrm{d}t\\
&+ \int_{\{|b(x\,,\,u_0^\varepsilon)|\geq n\}} |b(x\,,\,u_0^\varepsilon)|\,\mathrm{d}x\,.
\end{aligned}$$

Hence

$$\begin{aligned}
&\lambda\Big(\frac{(p^+)'-1}{(p^+)'}\Big)\int_{\{n\leq|v^\varepsilon|\leq n+1\}} a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla u^\varepsilon\,\mathrm{d}x\mathrm{d}t\\
&\leq \frac{\beta}{(p^+)'}\int_{\{|v^\varepsilon|\geq n\}} |L(x\,,\,t)|^{p'(x)}\,\mathrm{d}x\mathrm{d}t + \frac{\beta}{p^-}\int_{\{|v^\varepsilon|\geq n\}}\Big(|B(x)| + |\nabla g^\varepsilon|\Big)^{p(x)}\,\mathrm{d}x\mathrm{d}t\\
&+ \frac{1}{p^-}\max\Big(\Big(\frac{2}{\alpha\,\lambda}\Big)^{p^--1}\,,\,\Big(\frac{2}{\alpha\,\lambda}\Big)^{p^+-1}\Big)\int_{\{|v^\varepsilon|\geq n\}} |\nabla g^\varepsilon|^{p(x)}\,\mathrm{d}x\mathrm{d}t \qquad (4.28)\\
&+ \int_{\{|v^\varepsilon|\geq n\}} |f^\varepsilon|\,\mathrm{d}x\mathrm{d}t + \frac{\lambda}{(p^+)'}\max\Big(\Big(\frac{2(p^+)'}{\alpha\,p^-}\Big)^{(p^-)'-1}\,,\,\Big(\frac{2(p^+)'}{\alpha\,p^-}\Big)^{(p^+)'-1}\Big)\int_{\{|v^\varepsilon|\geq n\}} |G^\varepsilon|^{p(x)}\,\mathrm{d}x\mathrm{d}t\\
&+ \int_{\{|b(x\,,\,u_0^\varepsilon)|\geq n\}} |b(x\,,\,u_0^\varepsilon)|\,\mathrm{d}x\,.
\end{aligned}$$

To pass to the limit-sup as ε tends to 0 in (4.28), let us remark that (4.24) imply that v^ε is bounded in $L^\infty(0\,,\,T; L^1(\Omega))$, and we deduce

$$\lim_{n\to\infty}\Big(\sup_\varepsilon \operatorname{meas}\{|v^\varepsilon|\geq n\}\Big) = 0\,. \qquad (4.29)$$

Using the equi-integrability of the sequences $|f^\varepsilon|,\ |b(x\,,\,u_0^\varepsilon)|,\ |\nabla g^\varepsilon|^{p(x)}$ and $|G^\varepsilon|^{p'(x)}$ in $L^1(Q)$ we deduce

$$\lim_{n\to\infty}\Big(\sup_\varepsilon\int_{\{n\leq|v^\varepsilon|\leq n+1\}} a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla u^\varepsilon\,\mathrm{d}x\mathrm{d}t\Big) = 0\,. \qquad (4.30)$$

$\star$ **Step 2**. This step is devoted to introduce for $k \geq 0$ fixed, a time regularization of the function $T_k(v)$. This kind of regularization has been first introduced by Landes [19], and employed by several authors to solve nonlinear time dependent problems with L^1 or measure data (see *e.g.* [4, 22]).

This specific time regularization of $T_k(v)$ (for fixed $k \geq 0$) is defined as follows. Let $(v_0^\zeta)_\mu$ in $W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ such that $\|v_0^\zeta\|_{L^\infty(\Omega)} \leq k$, for all $\zeta > 0$, and $v_0^\zeta \to T_k(v_0)$ a.e. in Ω with $\frac{1}{\zeta}\|v_0^\zeta\|_{L^p(\Omega)} \to 0$ as $\zeta \to +\infty$. Let us consider the unique solution $T_k(v)_\zeta \in V \cap L^\infty(Q)$ of the monotone problem:

$$\frac{\partial T_k(v)_\zeta}{\partial t} + \zeta(T_k(v)_\zeta - T_k(v)) = 0 \text{ in } \mathcal{D}'(Q)\,, \tag{4.31}$$

$$T_k(v)_\zeta(t=0) = v_0^\zeta \text{ in } \Omega\,. \tag{4.32}$$

Following [19] (see also [18]) we can easily prove

$$T_k(v)_\zeta \to T_k(v) \text{ strongly in } V \text{ a.e. in } Q \text{ as } \zeta \to +\infty \tag{4.33}$$

with $\|T_k(v)_\zeta\|_{L^\infty(\Omega)} \leq k$ for any ζ, and $\dfrac{\partial T_k(v)_\zeta}{\partial t} \in V \cap L^\infty(Q)$.

The main estimate is the following:

Lemma 4.2 *Fix $k \geq 0$. Let $S \in W^{1,\infty}(\mathbb{R})$ be an increasing function such that $S(r) = r$ for $|r| \leq k$ and* suppS' *is compact. Then $v^\varepsilon = b_\varepsilon(x\,,\,u^\varepsilon) - g^\varepsilon$, we have*

$$\underline{\lim_{\zeta\to\infty}}\lim_{\varepsilon\to 0}\int_0^T \left\langle \frac{\partial S(v^\varepsilon)}{\partial t}\,,\ \Big(T_k(v^\varepsilon) - T_k(v)_\zeta\Big)\right\rangle \mathrm{d}t \geq 0\,.$$

Proof. The proof is very similar to that in [4, 12, 30] with constant exponents. □

In the following lemma we identify the weak limit σ_k and we prove the weak L^1 convergence of the truncated energy $a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla T_k(v^\varepsilon)$ as ε tends to zero.

Lemma 4.3 *The subsequence of u^ε in Step 1 satisfies for any $k \geq 0$.*

$$\overline{\lim_{\varepsilon\to 0}}\int_Q a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla T_k(v^\varepsilon)\,\mathrm{d}x\mathrm{d}t \leq \int_Q \sigma_k\cdot\nabla T_k(v)\,\mathrm{d}x\mathrm{d}t\,, \tag{4.34}$$

$$\lim_{\varepsilon\to 0}\int_Q b_s(x,u^\varepsilon)\Big[a(x,t,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}}) - a(x,t,\nabla u\chi_{\{|v|\leq k\}})\Big]\cdot\Big[\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}} - \nabla u\chi_{\{|v|\leq k\}}\Big]\,\mathrm{d}x\mathrm{d}t = 0. \tag{4.35}$$

Moreover, for fixed $k \geq 0$, we have

$$\sigma_k = a(x\,,\,t\,,\,\nabla u)\chi_{\{|v|\leq k\}} \text{ a.e. in } Q\,, \tag{4.36}$$

and

$$a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla T_k(v^\varepsilon) \rightharpoonup a(x\,,\,t\,,\,\nabla u)\cdot\nabla T_k(v) \text{ weakly in } L^1(Q) \tag{4.37}$$

as ε tends to 0.

Proof. We first prove that (4.34) holds true. For fixed $k \geq 0$ let $W_\zeta^\varepsilon = (T_k(v^\varepsilon) - T_k(v)_\zeta)$. Let us introduce a sequence of increasing $\mathcal{C}^\infty(\mathbb{R})$-functions S_n such that

$$S_n(r) = r \text{ for } |r| \leq n\,,\ \operatorname{supp}(S_n') \subset [-(n+1)\,,\, n+1]\,,\ \|S_n''\|_{L^\infty(\mathbb{R})} \leq 1\,,\ \text{for any } n \geq 1\,.$$

We choose $S_n'(v^\varepsilon)W_\zeta^\varepsilon$ as test function in (4.6), we obtain

$$\begin{aligned}
&\int_0^T \langle \frac{\partial S_n(v^\varepsilon)}{\partial t}\,,\, W_\zeta^\varepsilon\rangle\ \mathrm{d}t + \int_Q S_n'(v^\varepsilon)a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla W_\zeta^\varepsilon\,\mathrm{d}x\mathrm{d}t\\
&+ \int_Q S_n''(v^\varepsilon)W_\zeta^\varepsilon a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla v^\varepsilon\,\mathrm{d}x\mathrm{d}t \qquad (4.38)\\
&= \int_Q f^\varepsilon S_n'(v^\varepsilon)W_\zeta^\varepsilon\,\mathrm{d}x\mathrm{d}t + \int_Q G^\varepsilon S_n'(v^\varepsilon)\cdot\nabla W_\zeta^\varepsilon\,\mathrm{d}x\mathrm{d}t + \int_Q S_n''(v^\varepsilon)W_\zeta^\varepsilon G^\varepsilon\cdot\nabla v^\varepsilon\,\mathrm{d}x\mathrm{d}t\,.
\end{aligned}$$

In the following we pass to the limit in (4.38) as ε tends to 0, then μ tends to ∞ then n tends to ∞, the real number $k \geq 0$ being kept fixed. In order to perform this task we prove bellow the following results for fixed $k \geq 0$:

$$\underline{\lim_{\zeta\to\infty}}\lim_{\varepsilon\to 0}\int_0^T \Big\langle \frac{\partial S_n(v^\varepsilon)}{\partial t}\,,\, W_\zeta^\varepsilon\Big\rangle\,\mathrm{d}t \geq 0\,, \qquad (4.39)$$

for any $n \geq k$

$$\lim_{n\to\infty}\lim_{\zeta\to\infty}\lim_{\varepsilon\to 0}\int_Q S_n''(v^\varepsilon)a(x\,,\,t\,,\,\nabla u^\varepsilon)W_\zeta^\varepsilon\cdot\nabla v^\varepsilon\,\mathrm{d}x\mathrm{d}t = 0\,, \qquad (4.40)$$

$$\lim_{\zeta\to\infty}\lim_{\varepsilon\to 0}\int_Q f^\varepsilon S_n'(v^\varepsilon)W_\zeta^\varepsilon\,\mathrm{d}x\mathrm{d}t = 0\,, \qquad (4.41)$$

$$\lim_{\zeta\to\infty}\lim_{\varepsilon\to 0}\int_Q G^\varepsilon S_n'(v^\varepsilon)\cdot\nabla W_\zeta^\varepsilon\,\mathrm{d}x\mathrm{d}t = 0\,, \qquad (4.42)$$

$$\lim_{\zeta\to\infty}\lim_{\varepsilon\to 0}\int_Q S_n''(v^\varepsilon)W_\zeta^\varepsilon G^\varepsilon\cdot\nabla v^\varepsilon\,\mathrm{d}x\mathrm{d}t = 0\,. \qquad (4.43)$$

Proof of (4.39). In view of the definition of W_ζ^ε and using Lemma 4.2 for fixed $n \geq k$, then (4.39) holds true.

Proof of (4.40). For any $n \geq 1$, and any $\zeta > 0$, we have $\operatorname{supp}(S_n'') \subset [-(n+1)\,,\,-n]\cup[n\,,\,n+1]$, $\|W_\zeta^\varepsilon\|_{L^\infty(Q)} \leq 2\,k$ and $\|S_n''\|_{L^\infty(\mathbb{R})} \leq 1$. As a consequence

$$\begin{aligned}
&\Big|\int_Q S_n''(v^\varepsilon)a(x\,,\,t\,,\,\nabla u^\varepsilon)W_\zeta^\varepsilon\cdot\nabla v^\varepsilon\,\mathrm{d}x\mathrm{d}t\Big|\\
&\leq 2\,k\int_{\{n\leq|v^\varepsilon|\leq n+1\}}|a(x\,,\,t\,,\,\nabla u^\varepsilon)|\Big(|\nabla_x b(x\,,\,u^\varepsilon)| + |\nabla g^\varepsilon|\Big)\,\mathrm{d}x\mathrm{d}t \qquad (4.44)\\
&+ 2\,k\,\lambda\int_{\{n\leq|v^\varepsilon|\leq n+1\}} a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla u^\varepsilon\,\mathrm{d}x\mathrm{d}t\,.
\end{aligned}$$

By using (4.26), (4.27) and Young's inequality, it is easy to see that

$$\begin{aligned}
&\Big|\int_Q S_n''(v^\varepsilon)a(x\,,\,t\,,\,\nabla u^\varepsilon)W_\zeta^\varepsilon\cdot\nabla v^\varepsilon\,\mathrm{d}x\mathrm{d}t\Big| \qquad (4.45)\\
&\leq c_1\int_{\{n\leq|v^\varepsilon|\leq n+1\}} a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla u^\varepsilon\,\mathrm{d}x\mathrm{d}t + c_2\int_{\{|v^\varepsilon|\geq n\}}\Big(|B|^{p(x)} + |L|^{p'(x)} + |\nabla g^\varepsilon|^{p(x)}\Big)\,\mathrm{d}x\mathrm{d}t\,,
\end{aligned}$$

for any $n \geq 1$, where c_1 and c_2 are constants independent of n. Using assumptions (4.29)–(4.30) and the equi-integrability of the sequences $|\nabla g^\varepsilon|^{p(x)}$ in $L^1(Q)$, permits to pass to the limit in (4.45) as n tends to ∞ and to establish (4.40).

Proof of (4.41). For fixed $n \geq 1$, and in view (4.3) and (4.19), Lebesgue's convergence theorem implies that for any $\zeta > 0$:

$$\lim_{\varepsilon \to 0} \int_Q f^\varepsilon S_n'(v^\varepsilon) W_\zeta^\varepsilon \,\mathrm{d}x\mathrm{d}t = \int_Q f S_n'(v) W_\zeta \,\mathrm{d}x\mathrm{d}t \,.$$

Using (4.33) permits to pass to the limit as ζ tends to ∞ in the above equality to obtain (4.41).

Proof of (4.42). Using (4.4) and (4.21) leads to $S_n'(v^\varepsilon)G^\varepsilon$ tends to $S_n'(v)G$ strongly in $(L^{p'(x)}(Q))^N$ as ε tends to 0. For fixed $\zeta > 0$, we have W_ζ^ε tends to $T_k(v) - T_k(v)_\zeta$ weakly in $L^{p^-}(0\,, T; W_0^{1\,,p(\cdot)}(\Omega))$, and a.e. in Q as ε tends to 0, we deduce that

$$\lim_{\varepsilon \to 0} \int_Q G^\varepsilon S_n'(v^\varepsilon) \cdot \nabla W_\zeta^\varepsilon \,\mathrm{d}x\mathrm{d}t = \int_Q G S_n'(v) \cdot \nabla (T_k(v) - T_k(v)_\zeta) \,\mathrm{d}x\mathrm{d}t \tag{4.46}$$

for any $\zeta > 0$. Appealing now to (4.33) and passing to the limit as $\zeta \to \infty$ in (4.45) allows to conclude that (4.42) holds true.

Proof of (4.43). From (4.5) and (4.21), it follows that

$$\lim_{\zeta \to \infty} \lim_{\varepsilon \to 0} \int_Q \nabla S_n'(v^\varepsilon) W_\zeta^\varepsilon \cdot G^\varepsilon \,\mathrm{d}x\mathrm{d}t = \lim_{\zeta \to \infty} \int_Q \nabla S_n'(v) W_\zeta \cdot G \,\mathrm{d}x\mathrm{d}t = 0$$

for any $n \geq 1$.

Now we are in a position to pass to the limit-sup when ε tends to zero, then to the limit-sup when ζ tends to ∞ and then to the limit as n tends to ∞ in (4.38) we obtain for any $k \geq 0$:

$$\lim_{n \to \infty} \overline{\lim_{\zeta \to \infty}} \, \overline{\lim_{\varepsilon \to 0}} \int_Q S_n'(v^\varepsilon) a(x\,, t\,, \nabla u^\varepsilon) \cdot \nabla (T_k(v^\varepsilon) - T_k(v)_\zeta) \,\mathrm{d}x\mathrm{d}t \leq 0 \,.$$

Since $S_n'(v^\varepsilon) a(x\,, t\,, \nabla u^\varepsilon) \cdot \nabla T_k(v^\varepsilon) = a(x\,, t\,, \nabla u^\varepsilon) \cdot \nabla T_k(v^\varepsilon)$ for $k \leq n$, the above inequality implies for $k \leq n$

$$\overline{\lim_{\varepsilon \to 0}} \int_Q a(x\,, t\,, \nabla u^\varepsilon) \cdot \nabla T_k(v^\varepsilon) \,\mathrm{d}x\mathrm{d}t \leq \lim_{n \to \infty} \overline{\lim_{\zeta \to \infty}} \, \overline{\lim_{\varepsilon \to 0}} \int_Q S_n'(v^\varepsilon) a(x\,, t\,, \nabla u^\varepsilon) \cdot \nabla T_k(v)_\zeta \,\mathrm{d}x\mathrm{d}t \,. \tag{4.47}$$

Due to (4.19) and (4.22), we have $S_n'(v^\varepsilon) a(x\,, t\,, \nabla u^\varepsilon)$ converges to $S_n'(v)\sigma_{n+1}$ weakly in $(L^{p'(\cdot)}(Q))^N$ as ε tends to zero. The strong convergence of $T_k(v)_\zeta$ to $T_k(v)$ in V as ζ tends to ∞, then allows to conclude that

$$\begin{aligned} &\lim_{\zeta \to \infty} \lim_{\varepsilon \to 0} \int_Q S_n'(v^\varepsilon) a(x\,, t\,, \nabla u^\varepsilon) \cdot \nabla T_k(v)_\zeta \,\mathrm{d}x\mathrm{d}t \\ &= \int_Q S_n'(v) \sigma_{n+1} \cdot \nabla T_k(v) \,\mathrm{d}x\mathrm{d}t = \int_Q \sigma_{n+1} \cdot \nabla T_k(v) \,\mathrm{d}x\mathrm{d}t \,. \end{aligned} \tag{4.48}$$

Now for $k \leq n$, we have $S_n'(v^\varepsilon) a(x\,, t\,, \nabla u^\varepsilon) \chi_{\{|v^\varepsilon| \leq k\}} = a(x\,, t\,, \nabla u^\varepsilon) \chi_{\{|v^\varepsilon| \leq k\}}$ a.e. in Q. Letting ε tend to 0, we obtain $\sigma_{n+1} \chi_{\{|v| \leq k\}} = \sigma_k \chi_{\{|v| \leq k\}}$ a.e. in $Q \backslash \{|v| = k\}$ for $k \leq n$, then we have for all $k \le n$, $\sigma_{n+1}\cdot \nabla T_k(v) = \sigma_k \cdot \nabla T_k(v)$ a.e. in Q. Recalling (4.47) and (4.48) allows to conclude that (4.34) holds true.

Proof of (4.35). Let $k \ge 0$ be fixed, we use (3.2) and the monotone character (3.7) of $a(x\,,\,t\,,\,\xi)$ with respect to ξ, we obtain

$$
\begin{aligned}
0 &\le \int_Q b_s(x,u^\varepsilon)\Big(a(x,t,\nabla u^\varepsilon \chi_{\{|v^\varepsilon|\le k\}}) - a(x,t,\nabla u \chi_{\{|v|\le k\}})\Big)\cdot\Big(\nabla u^\varepsilon \chi_{\{|v^\varepsilon|\le k\}} - \nabla u \chi_{\{|v|\le k\}}\Big)\,\mathrm{d}x\mathrm{d}t\\
&= \int_Q b_s(x,u^\varepsilon)a(x,t,\nabla u^\varepsilon)\cdot\nabla u^\varepsilon \chi_{\{|v^\varepsilon|\le k\}}\,\mathrm{d}x\mathrm{d}t - \int_Q b_s(x,u^\varepsilon)a(x,t,\nabla u^\varepsilon)\cdot \nabla u \chi_{\{|v^\varepsilon|\le k\}}\chi_{\{|v|\le k\}}\,\mathrm{d}x\mathrm{d}t\\
&- \int_Q b_s(x\,,\,u^\varepsilon)a(x\,,\,t\,,\,\nabla u)\chi_{\{|v|\le k\}}\cdot\Big(\nabla u^\varepsilon \chi_{\{|v^\varepsilon|\le k\}} - \nabla u\chi_{\{|v|\le k\}}\Big)\,\mathrm{d}x\mathrm{d}t\,. \qquad (4.49)
\end{aligned}
$$

We pass to the limit-sup as ε tends to 0 in (4.49). Let us remark that we have $v^\varepsilon = b(x\,,\,u^\varepsilon) - g^\varepsilon$ and $b_s(x\,,\,u^\varepsilon)\nabla u^\varepsilon \chi_{\{|v^\varepsilon|\le k\}} = \nabla T_k(v^\varepsilon) - (\nabla_x b(x\,,\,u^\varepsilon) - \nabla g^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}$ a.e. in Q, and we have also $\chi_{\{|v^\varepsilon|\le k\}}$ almost everywhere converges to $\chi_{\{|v|\le k\}}$ for almost every k (see [8]), using (4.34), we obtain

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_Q b_s(x\,,\,u^\varepsilon)a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla u^\varepsilon \chi_{\{|v^\varepsilon|\le k\}}\,\mathrm{d}x\mathrm{d}t\\
&= \lim_{\varepsilon\to 0}\int_Q a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla T_k(v^\varepsilon)\,\mathrm{d}x\mathrm{d}t - \lim_{\varepsilon\to 0}\int_Q a(x\,,\,t\,,\,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}\cdot\Big(\nabla_x b(x\,,\,u^\varepsilon) - \nabla g^\varepsilon\Big)\,\mathrm{d}x\mathrm{d}t\\
&\le \int_Q \sigma_k\cdot\nabla T_k(v)\,\mathrm{d}x\mathrm{d}t - \int_Q \sigma_k\cdot\Big(\nabla_x b(x\,,\,u) - \nabla g\Big)\,\mathrm{d}x\mathrm{d}t\,. \qquad (4.50)
\end{aligned}
$$

Since $\nabla u\chi_{\{|v|\le k\}} = (b_s(x\,,\,u))^{-1}\Big(\nabla T_k(v) - (\nabla_x b(x\,,\,u) - \nabla g)\chi_{\{|v|\le k\}}\Big) \in (L^{p(\cdot)}(Q))^N$, using (3.2), (4.20) and (4.22) it follows that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_Q b_s(x\,,\,u^\varepsilon)a(x\,,\,t\,,\,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}\cdot\nabla u\chi_{\{|v|\le k\}}\,\mathrm{d}x\mathrm{d}t\\
&= \int_Q b_s(x\,,\,u)\sigma_k\cdot\nabla u\chi_{\{|v|\le k\}}\,\mathrm{d}x\mathrm{d}t = \int_Q \sigma_k\cdot\Big(\nabla T_k(v) - \nabla_x b(x\,,\,u) + \nabla g\Big)\,\mathrm{d}x\mathrm{d}t\,. \qquad (4.51)
\end{aligned}
$$

In view of (3.6), we have

$$
\begin{aligned}
&\int_Q \Big|a(x\,,\,t\,,\,\nabla u)\chi_{\{|v|\le k\}}\Big|^{p'(x)}\,\mathrm{d}x\mathrm{d}t \le 2^{(p^-)'-1}\max\Big(\beta^{(p^+)'}\,,\,\beta^{(p^-)'}\Big)\int_Q |L(x\,,\,t)|^{p'(x)}\,\mathrm{d}x\mathrm{d}t \qquad (4.52)\\
&+2^{(p^-)'-1}\max\Big(\beta^{(p^+)'}\,,\,\beta^{(p^-)'}\Big)\max\Big(\Big(\frac{1}{\lambda}\Big)^{p^+}\,,\,\Big(\frac{1}{\lambda}\Big)^{p^-}\Big)\int_Q (|\nabla T_k(v)|+|B|+|\nabla g)|)^{p(x)}\,\mathrm{d}x\mathrm{d}t\,.
\end{aligned}
$$

Using (4.25) and (4.27) we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_Q a(x\,,\,t\,,\,\nabla u)\chi_{\{|v|\le k\}}\cdot\Big[\nabla T_k(v^\varepsilon) - (\nabla_x b(x\,,\,u^\varepsilon) - \nabla g^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}} \qquad (4.53)\\
&-b_s(x\,,\,u^\varepsilon)\Big(b_s(x\,,\,u)\Big)^{-1}\Big(\nabla T_k(v) - (\nabla_x b(x\,,\,u) - \nabla g)\chi_{\{|v|\le k\}}\Big)\Big]\,\mathrm{d}x\mathrm{d}t = 0\,.
\end{aligned}
$$

Taking the limit-sup as ε tends to 0 in (4.49) and using (4.50), (4.51) and (4.53) shows that (4.35) holds true.

Proof of (4.36)–(4.37). (4.35) and the usual Minty argument imply (4.36)–(4.37). □

$\star$ **Step 3**. In this step we prove that u satisfies (3.11), (3.12) and (3.13), to this end, we have for any fixed $n \geq 1$:

$$\begin{aligned}
&\int_{\{n\leq |v^\varepsilon|\leq n+1\}} b_s(x\,,\,u^\varepsilon)a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla u^\varepsilon\,\mathrm{d}x\mathrm{d}t\\
&=\int_Q a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla T_{n+1}(v^\varepsilon)\,\mathrm{d}x\mathrm{d}t-\int_Q a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla T_n(v^\varepsilon)\,\mathrm{d}x\mathrm{d}t\\
&+\int_Q a(x\,,\,t\,,\,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\leq n+1\}}\cdot\Big(\nabla g^\varepsilon-\nabla_x b(x\,,\,u^\varepsilon)\Big)\,\mathrm{d}x\mathrm{d}t\\
&-\int_Q a(x\,,\,t\,,\,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\leq n\}}\cdot\Big(\nabla g^\varepsilon-\nabla_x b(x\,,\,u^\varepsilon)\Big)\,\mathrm{d}x\mathrm{d}t\,.
\end{aligned}\tag{4.54}$$

According to (4.5), (4.22), (4.23), (4.36), and (4.37) we can pass to the limit as ε tends to 0 in (4.54) for fixed $n \geq 0$ and obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\{n\leq |v^\varepsilon|\leq n+1\}} b_s(x\,,\,u^\varepsilon)a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla u^\varepsilon\,\mathrm{d}x\mathrm{d}t\\
&=\int_Q a(x\,,\,t\,,\,\nabla u)\cdot\nabla T_{n+1}(v)\,\mathrm{d}x\mathrm{d}t-\int_Q a(x\,,\,t\,,\,\nabla u)\cdot\nabla T_n(v)\,\mathrm{d}x\mathrm{d}t\\
&+\int_Q a(x\,,\,t\,,\,\nabla u)\chi_{\{|v|\leq n+1\}}\cdot\Big(\nabla g-\nabla_x b(x\,,\,u)\Big)\,\mathrm{d}x\mathrm{d}t\\
&-\int_Q a(x\,,\,t\,,\,\nabla u)\chi_{\{|v|\leq n\}}\cdot\Big(\nabla g-\nabla_x b(x\,,\,u)\Big)\,\mathrm{d}x\mathrm{d}t\\
&=\int_{\{n\leq |v|\leq n+1\}} b_s(x\,,\,u)a(x\,,\,t\,,\,\nabla u)\cdot\nabla u\,\mathrm{d}x\mathrm{d}t\,.
\end{aligned}\tag{4.55}$$

Taking the limit as n tends to ∞ in (4.55) and using the estimate (4.30) and in view of (3.2), imply that u satisfies (3.11).

Let S be $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support, let k be a positive real number such that $\operatorname{supp}(S') \subset [-k\,,\,k]$. It is easy to obtain by multiplication of the approximate equation (4.6) by $S'(v^\varepsilon)$ that

$$\begin{aligned}
&\frac{\partial S(v^\varepsilon)}{\partial t}-\operatorname{div}\Big(S'(v^\varepsilon)a(x\,,\,t\,,\,\nabla u^\varepsilon)\Big)+S''(v^\varepsilon)a(x\,,\,t\,,\,\nabla u)\cdot\nabla v^\varepsilon\\
&=f^\varepsilon S'(v^\varepsilon)-\operatorname{div}\Big(G^\varepsilon S'(v^\varepsilon)\Big)+G^\varepsilon S''(v^\varepsilon)\cdot\nabla v^\varepsilon\ \text{ in } \mathcal{D}'(Q)\,.
\end{aligned}\tag{4.56}$$

In what follows we pass as ε tends to 0 in each terms of (4.56). Since S is bounded, and $S(v^\varepsilon)$ converges to $S(v)$ a.e. in Q and in $L^\infty(Q)$ *-weak, $\frac{\partial S(v^\varepsilon)}{\partial t}$ converges to $\frac{\partial S(v)}{\partial t}$ in $\mathcal{D}'(Q)$ as ε tends to 0. Since $\operatorname{supp}(S') \subset [-k\,,\,k]$, we have $S'(v^\varepsilon)a(x\,,\,t\,,\,\nabla u^\varepsilon) = S'(v^\varepsilon)a(x\,,\,t\,,\,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\leq k\}}$ a.e. in Q, the pointwise convergence of u^ε to u as ε tends to 0, the bounded character of S and (4.37) of Lemma 4.3 imply that $S'(v^\varepsilon)a(x\,,\,t\,,\,\nabla u^\varepsilon)$ converges to $S'(v)a(x\,,\,t\,,\,\nabla u)$ weakly in $(L^{p'(\cdot)}(Q))^N$ as ε tends to 0. The pointwise convergence of v^ε to v, the bounded character of S'' and (4.37) of Lemma 4.3 allow to conclude that $S''(v^\varepsilon)a(x\,,\,t\,,\,\nabla u^\varepsilon)\cdot\nabla T_k(v^\varepsilon)$ converges to $S''(v)a(x\,,\,t\,,\,\nabla u)\cdot\nabla T_k(v)$ weakly in $L^1(Q)$ as ε tends to 0. It is easy to see that $f^\varepsilon S'(v^\varepsilon)$ converges to $fS'(v)$ strongly in $L^1(Q)$, the term $G^\varepsilon S'(v^\varepsilon)$ converges to $GS'(v)$ strongly in $(L^{p'(\cdot)}(Q))^N$ and $G^\varepsilon S''(v^\varepsilon)\cdot\nabla v^\varepsilon$ converges to $GS''(v)\cdot\nabla v$ weakly in $L^1(Q)$. As consequence of the above convergence result, we are in a position to pass to the limit as ε tends to 0 in equation (4.56) and to conclude that u satisfies (3.12).

It remains to show that $S(v)$ satisfies the initial condition (3.13). To this end, firstly remark that $S(v^\varepsilon)$ is bounded in $V \cap L^\infty(Q)$, secondly, (4.56) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial S(v^\varepsilon)}{\partial t}$ is bounded in $V^* + L^1(Q)$. As a consequence, an Aubin's type lemma (see *e.g.* [22, 34], the proof of this Corollary follows the same lines as the corresponding result in the case of a constant exponent p) implies that $S(v^\varepsilon)$ lies in a compact set of $\mathcal{C}([0\,,\,T]; L^1(\Omega))$. It follows that, on one hand, $S(v^\varepsilon)(t=0)$ converges to $S(v)(t=0)$ strongly in $L^1(\Omega)$, on the other hand, the smoothness of S implies that $S(v^\varepsilon)(t=0)$ converges to $S(b(x\,,\,u))(t=0)$ strongly in $L^q(\Omega)$ for all $q<\infty$. Due to (4.1) and (4.5), we conclude that $S(v^\varepsilon)(t=0) = S(b_\varepsilon(x\,,\,u_0^\varepsilon))$ converges to $S(b(x\,,\,u))(t=0)$ strongly in $L^q(\Omega)$. Then we conclude that $S(v)(t=0) = S(b(x\,,\,u_0))$ in Ω. As a conclusion of Step 1, Step 2 and Step 3, the proof of Theorem 4.1 is complete. □

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