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INFINITELY MANY SOLUTIONS FOR AN ELLIPTIC NEUMANN PROBLEM IN WEIGHTED VARIABLE EXPONENT SOBOLEV SPACES

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Abstract. In this paper, we consider the Neumann elliptic problems of the form

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(w_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2}\frac{\partial u}{\partial x_i}\right) + w_0(x)|u|^{p(x)-2}u = f(x,u) + g(x,u) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} w_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2}\frac{\partial u}{\partial x_i}\gamma_i = 0 & \text{on } \partial\Omega. \end{cases}$$

We prove the existence of infinitely many weak solutions in the weighted variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega, w)$, which generalizes the corresponding result from [8].

Keywords: Neumann problem, variational principle, weighted variable exponent Lebesgue–Sobolev space.

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1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ with boundary of class C^1, and let γ_i be the components of the outer normal unit vector.

Ricceri [15], Anello and Cordaro [5] studied the existence of solutions for the following problem

$$\begin{cases} -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) + \lambda(x)|u|^{p-2}u = \alpha(x)f(u) + \beta(x)g(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda(x)$ is a positive function such that $\lambda(\cdot) \in L^\infty(\Omega)$ with $\lambda^- = \operatorname{ess\,inf}_{x\in\Omega} \lambda(x) > 0$ and $p > N$. The existence of solutions of problem (1.1) was proved by applying some critical point theorem recently obtained by B. Ricceri as a consequence of a more general variational principle (see [14]).

In [8], X. Fan, C. Ji treated the problem

$$\begin{cases} -\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) + \lambda(x)|u|^{p(x)-2}u = f(x,u) + g(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and they proved the existence of infinitely many solutions in the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$.

Even though the problem (1.2) has been studied by some other authors in anisotropic variable exponent Sobolev spaces and weighted Sobolev space (see [1, 2, 3]), the hypotheses we use in this paper are totally different from those ones and so are our results.

In this paper, we are interested in the following degenerate $p(x)$-Laplacian equation with Neumann boundary value condition:

$$\begin{cases} -\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(w_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2}\frac{\partial u}{\partial x_i}\right) + w_0(x)|u|^{p(x)-2}u = f(x,u) + g(x,u) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^N w_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2}\frac{\partial u}{\partial x_i}\gamma_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

We are interested in the existence of infinitely many weak solutions to such a problem. Precisely, we deal with the existence of weak solutions for the problem (1.3) in the Sobolev spaces $W^{1,p(\cdot)}(\Omega, w)$, where $p \in L^\infty(\Omega)$ satisfies the condition

$$1 < p^- := \operatorname*{ess\,inf}_{\Omega} p(x) \le p^+ := \operatorname*{ess\,sup}_{\Omega} p(x) < \infty, \tag{1.4}$$

and $w = \{w_i(x),\ 0 \le i \le N\}$ is a vector of weight functions on Ω, *i.e.*, each $w_i(x)$ is measurable, strictly positive a.e. on Ω and satisfies some integrability conditions (see Section 2). We refer the reader to [9, 10, 11] where the authors were concerned with Dirichlet problems.

In this paper we introduce the following theorem, which will be essential to establish the existence of weak solutions for our main problem.

Theorem 1 (see [8, Theorem 2.2]) *Let X be a reflexive real Banach space, and let $\Phi, \Psi \colon X \longrightarrow \mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functionals. Assume also that Ψ is (strongly) continuous and satisfies $\lim_{\|u\|\to+\infty} \Psi(u) = +\infty$. For each $\rho > \inf_X \Psi$, put*

$$\varphi(\rho) = \inf_{u\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(u) - \inf_{v\in\overline{(\Psi^{-1}(]-\infty,\rho[))}} \Phi(v)}{\rho - \Psi(u)}, \tag{1.5}$$

where $\overline{(\Psi^{-1}(]-\infty,\rho[))}$ is the closure of $\Psi^{-1}(]-\infty,\rho[)$ in the weak topology. Then, the following conclusions hold.

(a) *If there exist $\rho_0 > \inf_X \Psi$ and $u_0 \in X$ such that*

$$\Psi(u_0) < \rho_0 \tag{1.6}$$

and

$$\Phi(u_0) - \inf_{v\in\overline{(\Psi^{-1}(]-\infty,\rho_0[))}} \Phi(v) < \rho_0 - \Psi(u_0), \tag{1.7}$$

then the restriction of $\Psi + \Phi$ to $\Psi^{-1}(]-\infty,\rho_0[)$ has a global minimum.

(b) *If there exists a sequence $\{r_n\} \subset (\inf_X \Psi, +\infty)$ with $r_n \to +\infty$ and a sequence $\{u_n\} \subset X$ such that for each n we have*

$$\Psi(u_n) < r_n \tag{1.8}$$

and

$$\Phi(u_n) - \inf_{v\in\overline{(\Psi^{-1}(]-\infty,r_n[))}} \Phi(v) < r_n - \Psi(u_n), \tag{1.9}$$

and, in addition,

$$\liminf_{\|u\|\to+\infty} (\Psi(u) + \Phi(u)) = -\infty, \tag{1.10}$$

then there exists a sequence $\{v_n\}$ of local minima of $\Psi + \Phi$ such that $\Psi(v_n) \to +\infty$ as $n \to \infty$.

(c) *If there exists a sequence $\{r_n\} \subset (\inf_X \Psi, +\infty)$ with $r_n \to \inf_X \Psi$ and a sequence $\{u_n\} \subset X$ such that for each n the conditions* (1.8) *and* (1.9) *are satisfied, and, in addition,*

$$\text{every global minimizer of } \Psi \text{ is not a local minimizer of } \Phi + \Psi, \tag{1.11}$$

then there exists a sequence $\{v_n\}$ of pairwise distinct local minimizers of $\Phi + \Psi$ such that $\lim_{n\to\infty} \Psi(v_n) = \inf_X \Psi$, and $\{v_n\}$ weakly converges to a global minimizer of Ψ.

This paper is organized as follows. In Section 2, we present some necessary preliminary results concerning weighted variable exponent Sobolev spaces. In particular, we prove a compact embedding theorem of $W^{1,p(\cdot)}(\Omega, w)$ into $C^0(\overline{\Omega})$, which plays an important role in this paper. In Section 3, we introduce some assumptions for which our problem has solutions and we present we present the main results of the paper, Theorems 3 and 4). Finally, we give the proof of the main results in Section 4.

2 Preliminary

In this section, we state some elementary properties of the (weighted) variable exponent Lebesgue–Sobolev spaces which will be used in the next sections. The basic properties of the variable exponent Lebesgue–Sobolev spaces, that is, when $w(x) \equiv 1$ can be found in [6].

Let γ be a weighted function in Ω and let $w = \{w_i(x), 0 \le i \le N\}$ be a vector of weight functions, *i.e.*, every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω, satisfying the following condition

(V1) $w_i \in L^1_{loc}(\Omega)$ and $w_i^{\frac{-1}{p(x)-1}} \in L^1_{loc}(\Omega)$ for all $i = 0, \dots, N$.

We define the (weighted) variable exponent Lebesgue–Sobolev spaces as

$$L^{p(\cdot)}(\Omega, \gamma) = \big\{u(x) : u\gamma^{\frac{1}{p(x)}} \in L^{p(\cdot)}(\Omega)\big\},$$

and endow them with the norm defined by

$$\|u\|_{L^{p(\cdot)}(\Omega,\gamma)} = \|u\|_{p(\cdot),\Omega,\gamma} = \inf\bigg\{\sigma > 0 : \int_\Omega \gamma(x)\bigg|\frac{u(x)}{\sigma}\bigg|^{p(x)} \mathrm{d}x \le 1\bigg\}.$$

We denote by $W^{1,p(\cdot)}(\Omega, w)$ the space of all real-valued functions $u \in L^{p(\cdot)}(\Omega, w_0)$ such that the derivatives in the sense of distributions satisfy

$$\frac{\partial u}{\partial x_i} \in L^{p(\cdot)}(\Omega, w_i) \text{ for all } i = 1, \dots, N,$$

considered with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega,w)} = \|u\|_{L^{p(\cdot)}(\Omega,w_0)} + \sum_{i=1}^N \bigg\|\frac{\partial u}{\partial x_i}\bigg\|_{L^{p(\cdot)}(\Omega,w_i)}.$$

It is easy to see that the norm

$$|||u|||_{1,p(\cdot),\Omega,w} = \inf\bigg\{\mu > 0 : \int_\Omega \bigg(w_0(x)\bigg|\frac{u(x)}{\mu}\bigg|^{p(x)} + \sum_{i=1}^N w_i(x)\bigg|\frac{\frac{\partial u(x)}{\partial x_i}}{\mu}\bigg|^{p(x)}\bigg) \mathrm{d}x \le 1\bigg\} \tag{2.1}$$

is a norm on $W^{1,p(\cdot)}(\Omega, w)$ equivalent to $\|\cdot\|_{W^{1,p(\cdot)}(\Omega,w)}$. The theory of such spaces was developed in [9, 10, 11, 12, 13]. When $p(x)$ is a constant function, some results were proved in [4, 7]. If $w_0(x) = w_1(x) = \dots = w_N(x) = 1$, we write $W^{1,p(\cdot)}(\Omega)$ instead of $W^{1,p(\cdot)}(\Omega, w)$ and $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ instead of $\|u\|_{W^{1,p(\cdot)}(\Omega,w)}$.

Proposition 1 ([11]) *Assume that* **(V1)** *holds. Then,* $W^{1,p(\cdot)}(\Omega, w)$ *is a reflexive Banach space.*

Lemma 1 ([11]) *Let $\rho(u) = \int_\Omega \gamma(x)|u|^{p(x)}\,\mathrm{d}x$ for $u \in L^{p(\cdot)}(\Omega,\gamma)$. We have*

(i) $\|u\|_{L^{p(\cdot)}(\Omega,\gamma)} < 1\ (=1, >1)$ *if and only if* $\rho(u) < 1\ (=1, >1)$,

(ii) *if* $\|u\|_{L^{p(\cdot)}(\Omega,\gamma)} \le 1$, *then* $\|u\|^{p^+}_{L^{p(\cdot)}(\Omega,\gamma)} \le \rho(u) \le \|u\|^{p^-}_{L^{p(\cdot)}(\Omega,\gamma)}$,

(iii) *if* $\|u\|_{L^{p(\cdot)}(\Omega,\gamma)} \ge 1$, *then* $\|u\|^{p^-}_{L^{p(\cdot)}(\Omega,\gamma)} \le \rho(u) \le \|u\|^{p^+}_{L^{p(\cdot)}(\Omega,\gamma)}$.

We suppose that the function weight $w = \{w_i(x), 0 \le i \le N\}$ satisfies

(V2) $w_i^{-\nu(x)} \in L^1(\Omega)$ with $\nu(x) \in \left]\frac{N}{p(x)-N},\infty\right[\cap \left[\frac{1}{p(x)-1},\infty\right[$ for all $i = 0,\dots,N$ and $\nu^- > \frac{N}{p^- - N}$.

The following compact imbedding result is crucial.

Theorem 2 *Let* **(V1)** *and* **(V2)** *be satisfied. Then,* $W^{1,p(\cdot)}(\Omega,w) \hookrightarrow\hookrightarrow C^0(\overline{\Omega})$.

Proof. Let $u \in W^{1,p(\cdot)}(\Omega,w)$. We denote $\frac{\partial u}{\partial x_0} = u$, $p_1(x) = \frac{\nu(x)p(x)}{\nu(x)+1} < p(x)$. By the Hölder inequality in [9, Proposition 2.1] with parameters $q(x) = \frac{p(x)}{p_1(x)} = \frac{\nu(x)+1}{\nu(x)}$ and its conjugate $q'(x) = \nu(x)+1$ for all $i = 0,\dots,N$ we have

$$\begin{aligned}
\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_1(x)} \mathrm{d}x &= \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{\frac{\nu(x)p(x)}{\nu(x)+1}} \mathrm{d}x \\
&= \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{\frac{p(x)\nu(x)}{\nu(x)+1}} w_i^{\frac{\nu(x)}{\nu(x)+1}} w_i^{\frac{-\nu(x)}{\nu(x)+1}} \mathrm{d}x \\
&\le 2 \left\| w_i^{\frac{\nu(x)}{\nu(x)+1}} \left|\frac{\partial u}{\partial x_i}\right|^{\frac{p(x)\nu(x)}{\nu(x)+1}} \right\|_{L^{\frac{\nu(x)+1}{\nu(x)}}(\Omega)} \left\| w_i^{-\frac{\nu(x)}{\nu(x)+1}} \right\|_{L^{\nu(x)+1}(\Omega)}.
\end{aligned}$$

Assumption **(V2)** and Lemma 1 imply that

$$\left\| w_i^{-\frac{\nu(x)}{\nu(x)+1}} \right\|_{L^{\nu(x)+1}(\Omega)} \le \left(\int_\Omega w_i^{-\nu(x)}(x)\,\mathrm{d}x + 1 \right)^{\frac{1}{\nu^-+1}} \le C.$$

Thus, we get

$$\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_1(x)} \mathrm{d}x \le C \left\| w_i^{\frac{\nu(x)}{\nu(x)+1}} \left|\frac{\partial u}{\partial x_i}\right|^{\frac{p(x)\nu(x)}{\nu(x)+1}} \right\|_{L^{\frac{\nu(x)+1}{\nu(x)}}(\Omega)}. \tag{2.2}$$

Without loss of generality, we may assume that $\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_1(x)} \mathrm{d}x > 1$. (If not, it is easy to see from Lemma 1 that $u \in W^{1,p_1(\cdot)}(\Omega)$.) If $\int_\Omega w_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p(x)} \mathrm{d}x < 1$, then from (2.2) and Lemma 1 we have

$$\begin{aligned}\Big\|\frac{\partial u}{\partial x_i}\Big\|_{L^{p_1(\cdot)}(\Omega)}^{\frac{p^-\nu^-}{\nu^-+1}} &\le \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_1(x)}\,\mathrm{d}x\\ &\le C\Big(\int_\Omega w_i(x)\Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)}\,\mathrm{d}x\Big)^{\frac{\nu^-}{\nu^-+1}}\\ &\le C\Big\|\frac{\partial u}{\partial x_i}\Big\|_{L^{p(\cdot)}(\Omega,w_i)}^{\frac{p^-\nu^-}{\nu^-+1}},\end{aligned}$$

i.e.,

$$\Big\|\frac{\partial u}{\partial x_i}\Big\|_{L^{p_1(\cdot)}(\Omega)} \le C\Big\|\frac{\partial u}{\partial x_i}\Big\|_{L^{p(\cdot)}(\Omega,w_i)}. \tag{2.3}$$

On the other hand, if $\int_\Omega w_i(x)\big|\frac{\partial u}{\partial x_i}\big|^{p(x)}\,\mathrm{d}x > 1$, then from (2.2) and Lemma 1 we obtain

$$\begin{aligned}\Big\|\frac{\partial u}{\partial x_i}\Big\|_{L^{p_1(\cdot)}(\Omega)}^{\frac{p^-\nu^-}{\nu^-+1}} &\le \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_1(x)}\,\mathrm{d}x\\ &\le C\Big(\int_\Omega w_i(x)\Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)}\,\mathrm{d}x\Big)^{\frac{\nu^+}{\nu^++1}}\\ &\le C\Big\|\frac{\partial u}{\partial x_i}\Big\|_{L^{p(\cdot)}(\Omega,w_i)}^{\frac{p^+\nu^+}{\nu^++1}},\end{aligned}$$

i.e.,

$$\Big\|\frac{\partial u}{\partial x_i}\Big\|_{L^{p_1(\cdot)}(\Omega)} \le C\Big\|\frac{\partial u}{\partial x_i}\Big\|_{L^{p(\cdot)}(\Omega,w_i)}^{\beta}, \tag{2.4}$$

where $\beta = \frac{p^+\nu^+}{\nu^++1}\cdot\frac{1+\nu^-}{p^-\nu^-}$. From the inequalities (2.3) and (2.4), we obtain $\frac{\partial u}{\partial x_i} \in L^{p(\cdot)}(\Omega, w_i)$ for all $i = 0,\dots,N$. Therefore, we conclude that $W^{1,p(\cdot)}(\Omega, w) \hookrightarrow W^{1,p_1(\cdot)}(\Omega)$. By **(V2)** we have $\nu^- > \frac{N}{p^- - N}$. Then, $p_1^- > N$. Since $W^{1,p_1(\cdot)}(\Omega)$ is continuously embedded in $W^{1,p_1^-}(\Omega)$, and $W^{1,p_1^-}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$, we deduce the result using the classic injections. This finishes the proof. □

Set

$$C_0 = \sup_{u\in W^{1,p(\cdot)}(\Omega,w)\setminus\{0\}} \frac{\|u\|_{L^\infty(\Omega)}}{|||u|||_{1,p(\cdot),\Omega,w}}; \tag{2.5}$$

then, C_0 is a positive constant.

3 Assumptions and statement of main results

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with the boundary $\partial\Omega$ of class C^1, and let γ be the outward unit normal to $\partial\Omega$. Assume that $f, g\colon \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ are Carathéodory functions satisfying the following condition

$$\sup_{|t|\le r} |f(x,t)| \in L^1(\Omega) \text{ and } \sup_{|t|\le r} |g(x,t)| \in L^1(\Omega) \text{ for each } r > 0. \tag{3.1}$$

We set

$$F(x,t)=\int_0^t f(x,s)\,\mathrm{d}s \quad\text{and}\quad G(x,t)=\int_0^t g(x,s)\,\mathrm{d}s. \tag{3.2}$$

We define, for any $u\in W^{1,p(\cdot)}(\Omega,w)$, the functionals

$$J(u)=\int_\Omega \frac{1}{p(x)}\left(\sum_{i=1}^N w_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p(x)}+w_0(x)|u|^{p(x)}\right)\mathrm{d}x, \tag{3.3}$$

$$\Psi(u)=J(u)-\int_\Omega G(x,u)\,\mathrm{d}x \quad\text{and}\quad \Phi(u)=-\int_\Omega F(x,u)\,\mathrm{d}x. \tag{3.4}$$

Definition 1 *A measurable function $u\in W^{1,p(\cdot)}(\Omega,w)$ is called a weak solution of the Neumann elliptic problem* (1.3) *if for every $v\in W^{1,p(\cdot)}(\Omega,w)$ we have*

$$\int_\Omega\sum_{i=1}^N w_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x+\int_\Omega w_0(x)|u|^{p(x)-2}uv\,\mathrm{d}x$$
$$=\int_\Omega f(x,u)v\,\mathrm{d}x+\int_\Omega g(x,u)v\,\mathrm{d}x.$$

Definition 2 *A function $F(x,t)$ satisfies the condition (S) if for each compact subset E of $\mathbb{R}$, there exists $\xi\in E$ such that*

$$F(x,\xi)=\sup_{t\in E}F(x,t)\ \textit{for a.e. } x\in\Omega. \tag{3.5}$$

We assume that G satisfies one of the following two conditions.

(G1) There exist $M>0$, $\epsilon\in(0,1)$ and $\beta,\theta\in L^1(\Omega)$ with $\|\beta\|_{L^1(\Omega)}\neq 0$ such that

$$\text{for any } |t|\geq M,\quad G(x,t)\leq\frac{(1-\epsilon)\beta(x)}{p^+C_0^{p^-}\|\beta\|_{L^1(\Omega)}}|t|^{p^-}+\theta(x) \text{ a.e. in } \Omega,$$

(G2) There exist $M>0$, $\epsilon\in(0,1)$ and $\theta'\in L^1(\Omega)$ such that

$$\text{for any } |t|\geq M,\quad G(x,t)\leq\frac{(1-\epsilon)w_0(x)}{p(x)}|t|^{p(x)}+\theta'(x) \text{ a.e. in } \Omega.$$

From now on, we always assume that

(V3) $w_0\in L^1(\Omega)$.

We take u_0 and u_n in Theorem 1 as the constant value functions ξ_0 and ξ_n, and we assume that

$$\liminf_{|\xi|\to+\infty}\int_\Omega\left(\frac{w_0(x)}{p(x)}|\xi|^{p(x)}-G(x,\xi)-F(x,\xi)\right)\mathrm{d}x=-\infty. \tag{3.6}$$

Also, we will need the following condition

$$\int_\Omega\frac{w_0(x)}{p(x)}|\xi|^{p(x)}\,\mathrm{d}x-\int_\Omega G(x,\xi)\,\mathrm{d}x\leq d_1|\xi|^{p^+}+d_2 \quad\text{for every } \xi\in\mathbb{R}, \tag{3.7}$$

where d_1 and d_2 are two positive constants. It is easy to see that (3.7) holds true under the condition **(G1)** or **(G2)**.

Our main results are as follows.

Theorem 3 *Let the conditions* **(V1)–(V3)**, (3.6)–(3.7) *be satisfied, and let* **(G1)** *or* **(G2)** *hold. Moreover, let F satisfy the condition (S). Suppose that $\{a_n\}$ and $\{b_n\}$ are two positive sequences such that*

$$\lim_{n\to\infty} b_n = +\infty \ \text{ and } \ \lim_{n\to\infty} \frac{a_n^{p^+}}{b_n^{p^-}} = 0. \tag{3.8}$$

If there exists a positive function $h \in L^1(\Omega)$ with $\|h\|_{L^1(\Omega)} \neq 0$ such that for each n we have

$$F(x, a_n) + \frac{h(x)}{\|h\|_{L^1(\Omega)}} \left(d_0 \left(\frac{b_n}{C_0} \right)^{p^-} - d_1 a_n^{p^+} - d_2 \right) \geq \sup_{t\in[a_n,b_n]} F(x,t) \ \text{ a.e. in } \Omega, \tag{3.9}$$

$$F(x, -a_n) + \frac{h(x)}{\|h\|_{L^1(\Omega)}} \left(d_0 \left(\frac{b_n}{C_0} \right)^{p^-} - d_1 a_n^{p^+} - d_2 \right) \geq \sup_{t\in[-b_n,-a_n]} F(x,t) \ \text{ a.e. in } \Omega, \tag{3.10}$$

and the inequalities (3.9) *and* (3.10) *are strict on a subset of Ω with positive measure, then there exists a sequence $\{v_n\}$ of local minima of $\Psi + \Phi$ such that $\lim_{n\to\infty} \Psi(v_n) = +\infty$. Consequently, the problem* (1.3) *admits an unbounded sequence of weak solutions.*

Theorem 4 *Assume that* **(V1)–(V3)** *hold. Suppose that*

$$G(x,t) \leq 0 \ \text{ for } \ t \in \mathbb{R} \text{ and a.e. } x \in \Omega, \tag{3.11}$$

and that there exist two positive constants M and ϵ such that

$$-G(x,t) \leq M|t|^{p^-} \text{ for } t \leq \epsilon \text{ and a.e. } x \in \Omega. \tag{3.12}$$

Moreover, let the functional F satisfy the condition (S) and

$$\limsup_{|\xi|\to 0} \frac{\int_\Omega F(x,\xi)\,\mathrm{d}x + \int_\Omega G(x,\xi)\,\mathrm{d}x}{|\xi|^{p^-}} > \int_\Omega \frac{w_0(x)}{p(x)}\,\mathrm{d}x. \tag{3.13}$$

Suppose that $\{a_n\}$ and $\{b_n\}$ are two positive sequences such that

$$\lim_{n\to\infty} b_n = 0 \text{ and } \lim_{n\to\infty} \frac{a_n^{p^-}}{b_n^{p^+}} = 0, \tag{3.14}$$

and that there exists a positive function $h \in L^1(\Omega)$ with $\|h\|_{L^1(\Omega)} \neq 0$ such that for each n we have

$$F(x, a_n) + \frac{h(x)}{\|h\|_{L^1(\Omega)}} \left(\frac{1}{p^+} \left(\frac{b_n}{C_0} \right)^{p^+} - d_3 a_n^{p^-} \right) \geq \sup_{t\in[a_n,b_n]} F(x,t) \ \text{ a.e. in } \Omega, \tag{3.15}$$

$$F(x, -a_n) + \frac{h(x)}{\|h\|_{L^1(\Omega)}} \left(\frac{1}{p^+} \left(\frac{b_n}{C_0} \right)^{p^+} - d_3 a_n^{p^-} \right) \geq \sup_{t\in[-b_n,-a_n]} F(x,t) \ \text{ a.e. in } \Omega, \tag{3.16}$$

and the inequalities (3.15) *and* (3.16) *are strict on a subset of Ω with positive measure, where $d_3 = \int_\Omega \frac{w_0(x)}{p(x)}\,\mathrm{d}x + M|\Omega|$. Then, there exists a sequence $\{v_n\}$ of pairwise distinct local minima of $\Psi + \Phi$ such that $v_n \to 0$ in $W^{1,p(\cdot)}(\Omega, w)$. Consequently, the problem* (1.3) *admits a sequence of non-zero weak solutions which strongly converges to* 0 *in $W^{1,p(\cdot)}(\Omega, w)$.*

4 Proofs of the main results

In this section, we are ready to prove the main results. We will begin with the proof of Theorem 3.

Proof of Theorem 3. Since the proof is quite long and challenging, for simplicity's sake, we will divide it into several steps.

Step 1: We start with establishing some technical lemmas.

Lemma 2 *Assume that* **(V1)**, **(V2)** *and* (3.1) *are satisfied. Then,* $\Psi, \Phi \in C^1(W^{1,p(\cdot)}(\Omega, w), \mathbb{R})$ *and their Gâteaux derivatives are given by*

$$\langle \Psi'(u), v\rangle = \int_\Omega \sum_{i=1}^N w_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2}\left|\frac{\partial u}{\partial x_i}\right|\frac{\partial v}{\partial x_i}\,\mathrm{d}x + \int_\Omega w_0(x)|u|^{p(x)-2}uv\,\mathrm{d}x - \int_\Omega g(x,u)v\,\mathrm{d}x$$

and

$$\langle \Phi'(u), v\rangle = -\int_\Omega f(x,u)v\,\mathrm{d}x$$

for any $u, v \in W^{1,p(\cdot)}(\Omega, w)$.

Proof. For $i = 0, \dots, N$ and any $u \in W^{1,p(\cdot)}(\Omega, w)$ define $H, J_i \colon W^{1,p(\cdot)}(\Omega, w) \longrightarrow \mathbb{R}$ by

$$J_i(u) = \int_\Omega \frac{w_i(x)}{p(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p(x)}\mathrm{d}x, \quad \text{where} \quad \frac{\partial u}{\partial x_0} = u, \tag{4.1}$$

$$H(u) = \int_\Omega G(x,u)\,\mathrm{d}x. \tag{4.2}$$

Claim 1: $J_i \in C^1(W^{1,p(\cdot)}(\Omega, w), \mathbb{R})$ and for any $u, v \in W^{1,p(\cdot)}(\Omega, w)$,

$$\langle J_i'(u), v\rangle = \int_\Omega w_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x \quad \text{for all} \quad i = 0, \dots, N.$$

For a fixed $x \in \Omega$ let us consider $\phi \colon \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\phi(\zeta) = \frac{w_i(x)}{p(x)}|\zeta|^{p(x)}$. Obviously, $\phi \in C^1(\mathbb{R}, \mathbb{R})$ and $\frac{\partial \phi(\zeta)}{\partial \zeta} = w_i(x)|\zeta|^{p(x)-2}\zeta$. Thus, for all $\zeta, \vartheta \in \mathbb{R}$, we have

$$\lim_{t\to 0}\frac{\phi(\zeta + t\vartheta) - \phi(\vartheta)}{t} = w_i(x)|\zeta|^{p(x)-2}\zeta \cdot \vartheta.$$

As a consequence, for u, $v \in W^{1,p(\cdot)}(\Omega, w)$, we obtain

$$\lim_{t\to 0}\frac{\frac{w_i(x)}{p(x)}\left|\frac{\partial u}{\partial x_i} + t\frac{\partial v}{\partial x_i}\right|^{p(x)} - \frac{w_i(x)}{p(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p(x)}}{t} = w_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2}\frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i}. \tag{4.3}$$

By the mean value theorem, there exists $\theta \in \mathbb{R}$ with $0 < |\theta| < |t|$ such that for each $t \in \mathbb{R}$ with $0 < |t| < 1$ we have

$$\begin{aligned}
&\left|\frac{\frac{w_i(x)}{p(x)}\left|\frac{\partial u}{\partial x_i} + t\frac{\partial v}{\partial x_i}\right|^{p(x)} - \frac{w_i(x)}{p(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p(x)}}{t}\right| \\
&\qquad = \left|w_i(x)\left|\frac{\partial u}{\partial x_i} + \theta\frac{\partial v}{\partial x_i}\right|^{p(x)-2}\left(\frac{\partial u}{\partial x_i} + \theta\frac{\partial v}{\partial x_i}\right)\cdot\frac{\partial v}{\partial x_i}\right| \qquad (4.4)\\
&\qquad \le w_i(x)\left(\left|\frac{\partial u}{\partial x_i}\right| + \left|\frac{\partial v}{\partial x_i}\right|\right)^{p(x)-1}\left|\frac{\partial v}{\partial x_i}\right|.
\end{aligned}$$

Noting that $\left\|w_i(x)^{\frac{1}{p(x)}}\left|\frac{\partial v}{\partial x_i}\right|\right\|_{L^{p(\cdot)}(\Omega)} = \|v\|_{p(\cdot),\Omega,w_i}$, from the definition, Hölder's inequality and Lemma 1, we obtain

$$\begin{aligned}
&\int_\Omega w_i(x)\left(\left|\frac{\partial u}{\partial x_i}\right| + \left|\frac{\partial v}{\partial x_i}\right|\right)^{p(x)-1}\left|\frac{\partial v}{\partial x_i}\right| \mathrm{d}x \\
&\le 2\left\|w_i^{\frac{1}{p'(x)}}\left(\left|\frac{\partial u}{\partial x_i}\right| + \left|\frac{\partial v}{\partial x_i}\right|\right)^{p(x)-1}\right\|_{L^{p(\cdot)'}(\Omega)}\left\|w_i(x)^{\frac{1}{p(x)}}\left|\frac{\partial v}{\partial x_i}\right|\right\|_{L^{p(\cdot)}(\Omega)} \\
&\le 2\left[1 + \left(\int_\Omega w_i(x)\left(\left|\frac{\partial u}{\partial x_i}\right| + \left|\frac{\partial v}{\partial x_i}\right|\right)^{p(x)} \mathrm{d}x\right)^{\frac{1}{(p')^-}}\right]\|v\|_{p(\cdot),\Omega,w_i} \\
&\le 2\left[1 + 2^{\frac{p^+-1}{(p')^-}}\left(\int_\Omega w_i(x)\left(\left|\frac{\partial u}{\partial x_i}\right|^{p(x)} + \left|\frac{\partial v}{\partial x_i}\right|^{p(x)}\right) \mathrm{d}x\right)^{\frac{1}{(p')^-}}\right]\|v\|_{p(\cdot),(\Omega),w_i}.
\end{aligned}$$

Hence, $w_i\left(\left|\frac{\partial u}{\partial x_i}\right| + \left|\frac{\partial v}{\partial x_i}\right|\right)^{p(x)-1}\left|\frac{\partial v}{\partial x_i}\right| \in L^1(\Omega)$, because $u,\, v \in W^{1,p(\cdot)}(\Omega, w)$. Combining this with (4.3) and (4.4) and applying the dominated convergence theorem, we get

$$\lim_{t\to 0}\int_\Omega \frac{\frac{w_i(x)}{p(x)}\left|\frac{\partial u}{\partial x_i} + t\frac{\partial v}{\partial x_i}\right|^{p(x)} - \frac{w_i(x)}{p(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p(x)}}{t} \mathrm{d}x = \int_\Omega w_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2}\frac{\partial u}{\partial x_i}.\frac{\partial v}{\partial x_i} \mathrm{d}x.$$

It means that J_i is Gâteaux differentiable and for $u,\, v \in W^{1,p(\cdot)}(\Omega, w)$ we have

$$\langle J_i'(u), v\rangle = \int_\Omega w_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} \mathrm{d}x.$$

Next, we prove that $J_i'\colon W^{1,p(\cdot)}(\Omega, w) \longrightarrow W^{1,p(\cdot)}(\Omega, w)^*$ is continuous. To this aim we take a sequence $\{u_n\}$ in $W^{1,p(\cdot)}(\Omega, w)$ such that $u_n \longrightarrow u$ in $W^{1,p(\cdot)}(\Omega, w)$ as $n \longrightarrow \infty$. By [9, Proposition 2.3] we have $\lim_{n\longrightarrow\infty}\int_\Omega w_i(x)\left|\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right|^{p(x)} \mathrm{d}x = 0$. Thus, up to a subsequence, we deduce that

$$\frac{\partial u_n}{\partial x_i} \longrightarrow \frac{\partial u}{\partial x_i} \text{ a.e. in } \Omega \text{ as } n \longrightarrow \infty, \tag{4.5}$$

$$w_i(x)\left|\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right|^{p(x)} \le h(x) \text{ for a.e. } x \in \Omega \text{ for some } h \in L^1(\Omega). \tag{4.6}$$

Since

$$w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^{p(x)} \le w_i(x)\left(\left|\frac{\partial u}{\partial x_i}\right| + \left|\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right|\right)^{p(x)}$$

$$\leq 2^{p(x)-1} w_i(x)\bigg(\Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)} + \Big|\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\Big|^{p(x)}\bigg),$$

it follows from (4.6) that

$$w_i(x)\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)} \leq 2^{p^+-1}\bigg(w_i(x)\Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)} + h(x)\bigg). \tag{4.7}$$

Using Hölder's inequality we have that for any $v \in W^{1,p(\cdot)}(\Omega, w)$ with $|||v|||_{1,p(\cdot),\Omega,w} \leq 1$,

$$\begin{aligned}
&\big|\langle J_i'(u_n) - J_i'(u), v\rangle\big| \\
&= \bigg|\int_\Omega w_i(x)\bigg(\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u_n}{\partial x_i} - \Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u}{\partial x_i}\bigg)\cdot\frac{\partial v}{\partial x_i}\,\mathrm{d}x\bigg| \\
&\leq 2\bigg\|w_i^{\frac{1}{p'(x)}}\bigg|\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u_n}{\partial x_i} - \Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u}{\partial x_i}\bigg|\bigg\|_{L^{p'(\cdot)}(\Omega)}\bigg\|w_i^{\frac{1}{p(x)}}\Big|\frac{\partial v}{\partial x_i}\Big|\bigg\|_{L^{p(\cdot)}(\Omega)} \\
&\leq 2\bigg\|w_i^{\frac{1}{p'(x)}}\bigg|\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)-2}\nabla u_n - \Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u}{\partial x_i}\bigg|\bigg\|_{L^{p'(\cdot)}(\Omega)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\big\|J_i'(u_n) - J_i'(u)\big\|_{W^{1,p(\cdot)}(\Omega,w)^*} \\
&\quad\leq 2\bigg\|w_i^{\frac{1}{p'(x)}}\bigg|\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u_n}{\partial x_i} - \Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u}{\partial x_i}\bigg|\bigg\|_{L^{p'(\cdot)}(\Omega)}.
\end{aligned} \tag{4.8}$$

First, observe that

$$\begin{aligned}
&\int_\Omega\bigg|w_i^{\frac{1}{p'(x)}}\bigg|\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u_n}{\partial x_i} - \Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u}{\partial x_i}\bigg|\bigg|^{p'(x)}\,\mathrm{d}x \\
&\quad= \int_\Omega w_i(x)\bigg|\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u_n}{\partial x_i} - \Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u}{\partial x_i}\bigg|^{p'(x)}\,\mathrm{d}x.
\end{aligned}$$

It follows from (4.5) that

$$w_i(x)\bigg|\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u_n}{\partial x_i} - \Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u}{\partial x_i}\bigg|^{p'(x)} \longrightarrow 0 \text{ for a.e. } x \in \Omega,$$

and from (4.7) that

$$\begin{aligned}
&w_i(x)\bigg|\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u_n}{\partial x_i} - \Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)-2}\frac{\partial u}{\partial x_i}\bigg|^{p'(x)} \\
&\quad\leq 2^{p'(x)-1} w_i(x)\bigg(\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)} + \Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)}\bigg) \\
&\quad\leq 2^{(p')^+-1} w_i(x)\bigg(\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)} + \Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)}\bigg) \\
&\quad\leq 2^{(p')^++p^+-1}\bigg(w_i(x)\Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)} + h(x)\bigg).
\end{aligned}$$

Noting that $2^{(p')^+ + p^+ - 1}\big(w_i|\frac{\partial u}{\partial x_i}|^{p(x)} + h\big) \in L^1(\Omega)$ and applying the dominated convergence theorem, we obtain

$$\int_\Omega \left| w_i^{\frac{1}{p'(x)}} \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p(x)-2} \frac{\partial u_n}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2} \frac{\partial u}{\partial x_i} \right) \right|^{p'(x)} \mathrm{d}x \to 0 \ \text{ as } \ n \to \infty.$$

Therefore, [9, Proposition 2.3] implies that

$$\left\| w_i^{\frac{1}{p'(x)}} \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p(x)-2} \frac{\partial u_n}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2} \frac{\partial u}{\partial x_i} \right) \right\|_{L^{p'(x)}(\Omega)} \to 0 \ \text{ as } \ n \to \infty.$$

Combining this with (4.8), gives

$$\|J_i'(u_n) - J_i'(u)\|_{W^{1,p(x)}(\Omega,w)^*} \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty.$$

This completes the proof that $J_i' \colon W^{1,p(\cdot)}(\Omega, w) \longrightarrow W^{1,p(\cdot)}(\Omega, w)^*$ is continuous, and therefore $J_i \in C^1(W^{1,p(\cdot)}(\Omega, w), \mathbb{R})$. Since $J(u) = \sum_{i=0}^N J_i(u)$, $J \in C^1(W^{1,p(\cdot)}(\Omega, w), \mathbb{R})$.

Claim 2: $H \in C^1(W^{1,p(\cdot)}(\Omega, w), \mathbb{R})$ and for any $u, v \in W^{1,p(\cdot)}(\Omega, w)$ we have

$$\langle H'(u), v\rangle = \int_\Omega g(x,u) v \,\mathrm{d}x.$$

Once again, by the mean value theorem, for $u, v \in W^{1,p(\cdot)}(\Omega, w)$ and $t \in \mathbb{R} \setminus \{0\}$, we have

$$\frac{G(x, u(x) + tv(x)) - G(x, u(x))}{t} = v(x) g(x, u(x) + \theta v(x))$$

for some $\theta \in \mathbb{R}$ with $0 < |\theta| < |t|$. Hence,

$$\frac{G(x, u(x) + tv(x)) - G(x, u(x))}{t} \longrightarrow v(x) g(x, u(x)) \text{ as } t \longrightarrow 0 \text{ for a.e. } x \in \Omega. \tag{4.9}$$

It follows from Theorem 2 that for $|t| < 1$ there exists $\ell = \|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} > 0$ such that

$$\begin{aligned} \left| \frac{G(x, u(x) + tv(x)) - G(x, u(x))}{t} \right| &= |v(x)||g(x, u(x) + \theta v(x))| \\ &\leq |v(x)| \sup_{|s| \leq \ell} |g(x,s)|. \end{aligned} \tag{4.10}$$

From Hölder's inequality and (3.1) we obtain

$$\int_\Omega v(x) \sup_{|s|\leq \ell} |g(x,s)| \,\mathrm{d}x \leq 2\|v\|_{L^\infty(\Omega)} \left\| \sup_{|s|\leq\ell} |g(x,s)| \right\|_{L^1(\Omega)}.$$

Therefore, the dominated convergence theorem together with (4.9) and (4.10) implies that

$$\lim_{t\to 0} \int_\Omega \frac{G(x, u(x) + tv(x)) - G(x, u(x))}{t} \,\mathrm{d}x = \int_\Omega g(x, u(x)) v(x) \,\mathrm{d}x,$$

i.e., H is Gâteaux differentiable and

$$\langle H'(u), v\rangle = \int_\Omega g(x, u(x)) v(x) \,\mathrm{d}x.$$

We now show that H' is continuous on $W^{1,p(\cdot)}(\Omega,w)$. Let $\{u_n\}$ be a sequence in $W^{1,p(\cdot)}(\Omega,w)$ such that $u_n \longrightarrow u$ in $W^{1,p(\cdot)}(\Omega,w)$ as $n \longrightarrow \infty$. Then, Theorem 2 implies that $u_n \longrightarrow u$ in $C^0(\overline{\Omega})$ as $n \longrightarrow \infty$. So, up to a subsequence, we deduce that

$$u_n \longrightarrow u \text{ a.e. in } \Omega \text{ as } n \longrightarrow \infty, \tag{4.11}$$

$$\text{for every } n \in \mathbb{N} \text{ there exists } k > 0 \text{ with } \|u_n\|_{L^\infty(\Omega)} \leq k. \tag{4.12}$$

We obtain that for any $v \in W^{1,p(\cdot)}(\Omega,w)$ with $|||u|||_{1,p(\cdot),\Omega,w} \leq 1$,

$$|\langle H'(u_n) - H'(u), v\rangle| \leq \int_\Omega |g(x,u_n(x)) - g(x,u(x))||v(x)|\,\mathrm{d}x \tag{4.13}$$

Therefore,

$$\begin{aligned} |g(x,u_n(x)) - g(x,u(x))| &\leq 2\big[|g(x,u_n)| + |g(x,u)|\big] \\ &\leq 2\Big[\sup_{|s|\leq k} |g(x,s)| + \sup_{|s|\leq k} |g(x,s)|\Big] \\ &\leq 4 \sup_{|s|\leq k} |g(x,s)|. \end{aligned}$$

By (4.13) we have

$$|\langle H'(u_n) - H'(u), v\rangle| \leq 4\int_\Omega \sup_{|s|\leq k} |g(x,s)||v(x)|\,\mathrm{d}x.$$

Note that $\sup_{|s|\leq k} |g(x,s)| \in L^1(\Omega)$. Applying the dominated convergence theorem with (4.11), we obtain

$$\lim_{n\to\infty} \int_\Omega |g(x,u_n(x)) - g(x,u(x))||v(x)|\,\mathrm{d}x = 0.$$

Hence, from (4.13) it follows that

$$\lim_{n\to\infty} \|H'(u_n) - H'(u)\|_{W^{1,p(x)}(\Omega,w)^*} = 0.$$

This completes the proof that $H' \colon W^{1,p(\cdot)}(\Omega,w) \longrightarrow W^{1,p(\cdot)}(\Omega,w)^*$ is continuous, and therefore $H \in C^1(W^{1,p(\cdot)}(\Omega,w),\mathbb{R})$.

In the same way (as in the case of the mapping H) we can show that $\Phi \in C^1(W^{1,p(\cdot)}(\Omega,w),\mathbb{R})$, and since $\Psi = J - H$, the proof is complete. □

Lemma 3 *Assume that* **(V1)**, **(V2)** *and* (3.1) *hold. Then,* Ψ, Φ *are sequentially weakly lower semicontinuous.*

Proof. Suppose that J_i, where $i = 0,\dots,N$, and H are as in (4.1)–(4.2).

Claim 1: Let $\{u_n\}$ be a sequence weakly convergent to u in $W^{1,p(\cdot)}(\Omega,w)$. Since J_i is convex, for any n we have

$$J_i(u) \leq J_i(u_n) + \langle J_i'(u), u - u_n\rangle.$$

Passing to the limit in the above inequality with $n \to \infty$, we see that J_i is sequentially weakly lower semicontinuous.

Claim 2: H is sequentially weakly continuous. Suppose that G is as in (3.2). Let $\{u_n\}$ be a sequence in $W^{1,p(\cdot)}(\Omega, w)$ such that $u_n \longrightarrow u$ (weakly) in $W^{1,p(\cdot)}(\Omega, w)$. By Theorem 2 we have $u_n \longrightarrow u$ in $C^0(\overline{\Omega})$. Hence, up to a subsequence, we have

$$u_n(x) \longrightarrow u(x) \text{ a.e. in } \Omega,$$
$$\text{for every } n \in \mathbb{N} \text{ there exists } k > 0 \text{ with } \|u_n\|_{L^\infty(\Omega)} \leq k.$$

Therefore, $G(x, u_n(x)) \longrightarrow G(x, u(x))$ a.e. in Ω and $|G(x, u_n(x))| \leq k \sup_{|s|\leq k} |g(x, s)|$. Note that $\sup_{|s|\leq k} |g(x, s)| \in L^1(\Omega)$ by (3.1). Thus, the dominated convergence theorem implies that $\lim_{n\to\infty} H(u_n) = H(u)$. So, the functional H is sequentially weakly continuous on $W^{1,p(\cdot)}(\Omega, w)$, and hence H is sequentially weakly lower semicontinuous. In the same way as for H, we prove that $\Psi = J - H$ is sequentially weakly lower semicontinuous, which completes the proof. □

Step 2: Now, we prove the coercivity of Ψ.

Proposition 2 *Assume that $G(x, t)$ satisfies* **(G1)** *or* **(G2)**. *Then, the functional Ψ is coercive,* i.e., *$\Psi(u) \longrightarrow +\infty$ as $|||u|||_{1,p(\cdot),\Omega,w} \longrightarrow \infty$ for $u \in W^{1,p(\cdot)}(\Omega, w)$.*

Proof. Let us assume that the condition **(G1)** is satisfied. Then,

$$G(x, t) \leq \frac{(1-\epsilon)\beta(x)}{p^+ C_0^{p^-} \|\beta\|_{L^1(\Omega)}} |t|^{p^-} + \theta_1(x) \text{ a.e. in } \Omega \text{ for any } |t| \geq M.$$

When $|||u|||_{1,p(\cdot),\Omega,w} \geq 1$, using (2.5) and [13, Proposition 2.3], we obtain

$$\begin{aligned}
\Psi(u) &= J(u) - \int_\Omega G(x, u)\,\mathrm{d}x \\
&= \int_\Omega \frac{1}{p(x)} \left(\sum_{i=1}^N w_i(x) \left| \frac{\partial u}{\partial x_i} \right|^{p(x)} + w_0(x)|u|^{p(x)} \right) \mathrm{d}x - \int_\Omega G(x, u)\,\mathrm{d}x \\
&\geq \frac{1}{p^+} |||u|||_{1,p(\cdot),\Omega,w}^{p^-} - \frac{(1-\epsilon)}{p^+ C_0^{p^-} \|\beta\|_{L^1(\Omega)}} \int_\Omega \beta |u|^{p^-}\,\mathrm{d}x - \int_\Omega \theta_1(x)\,\mathrm{d}x \\
&\geq \frac{1}{p^+} |||u|||_{1,p(\cdot),\Omega,w}^{p^-} - \frac{(1-\epsilon)}{p^+ C_0^{p^-}} \|u\|_{L^\infty(\Omega)}^{p^-} - c_1 \\
&\geq \frac{1}{p^+} |||u|||_{1,p(\cdot),\Omega,w}^{p^-} - \frac{(1-\epsilon)}{p^+} |||u|||_{1,p(\cdot),\Omega,w}^{p^-} - c_1 \\
&\geq \frac{\epsilon}{p^+} |||u|||_{1,p(\cdot),\Omega,w}^{p^-} - c_1.
\end{aligned} \tag{4.14}$$

Under the condition **(G2)** we have

$$G(x, t) \leq \frac{(1-\epsilon)w_0(x)}{p(x)} |t|^{p(x)} + \theta_1'(x) \text{ a.e. in } \Omega \text{ for any } |t| \geq M.$$

When $\|\|u\|\|_{1,p(\cdot),\Omega,w} \geq 1$, we obtain

$$\begin{aligned}
\Psi(u) &= J(u) - \int_\Omega G(x,u)\,\mathrm{d}x \\
&= \int_\Omega \frac{1}{p(x)}\left(w_0(x)|u|^{p(x)} + \sum_{i=1}^N w_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p(x)}\right)\mathrm{d}x - \int_\Omega G(x,u)\,\mathrm{d}x \\
&= \int_\Omega \frac{1}{p(x)}\sum_{i=1}^N w_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p(x)}\mathrm{d}x + \int_\Omega \frac{1}{p(x)} w_0(x)|u|^{p(x)}\,\mathrm{d}x \\
&\quad - \int_\Omega \left(\frac{(1-\epsilon)w_0(x)}{p(x)}|u|^{p(x)} + \theta_1'(x)\right)\mathrm{d}x \qquad (4.15)\\
&\geq \int_\Omega \frac{1}{p(x)}\sum_{i=1}^N w_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p(x)}\mathrm{d}x + \int_\Omega \frac{\epsilon w_0(x)}{p(x)}|u|^{p(x)}\,\mathrm{d}x - c_2 \\
&\geq \frac{\epsilon}{p^+}\left(\int_\Omega \sum_{i=1}^N w_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p(x)}\mathrm{d}x + \int_\Omega w_0(x)|u|^{p(x)}\,\mathrm{d}x\right) - c_2 \\
&\geq \frac{\epsilon}{p^+}\|\|u\|\|_{1,p(\cdot),\Omega,w}^{p^-} - c_2.
\end{aligned}$$

Thanks to (4.14)–(4.15) we conclude that Ψ is coercive. Moreover, there exist two positive constants d_0 and σ_0 such that

$$\Psi(u) \geq d_0\|\|u\|\|_{1,p(\cdot),\Omega,w}^{p^-} \text{ for } \|\|u\|\|_{1,p(\cdot),\Omega,w} \geq \sigma_0. \qquad (4.16)$$

This ends the proof. □

Step 3: Now, let us move to proving some *a priori* estimates. For $r > \inf_{W^{1,p(\cdot)}(\Omega,w)} \Psi$ we define

$$K(r) = \inf\left\{\sigma > 0 : \Psi^{-1}(]-\infty, r[) \subset \overline{B_{W^{1,p(\cdot)}(\Omega,w)}(0,\sigma)}\right\}, \qquad (4.17)$$

where

$$B_{W^{1,p(\cdot)}(\Omega,w)}(0,\sigma) = \left\{u \in W^{1,p(\cdot)}(\Omega,w) : \|\|u\|\|_{1,p(\cdot),\Omega,w} < \sigma\right\},$$

and $\overline{B_{W^{1,p(\cdot)}(\Omega,w)}(0,\sigma)}$ denotes the closure of $B_{W^{1,p(\cdot)}(\Omega,w)}(0,\sigma)$ in $W^{1,p(\cdot)}(\Omega,w)$ with respect to the norm topology.

We know that Ψ is coercive. So, $0 < K(r) < +\infty$ for each $r > \inf_{W^{1,p(\cdot)}(\Omega,w)} \Psi$. In view of (4.16), we deduce that

$$\text{if } \Psi(u) < d_0\|\|u\|\|_{1,p(\cdot),\Omega,w}^{p^-}, \text{ then } \|\|u\|\|_{1,p(\cdot),\Omega,w} < \sigma_0.$$

Thanks to (4.17) we have $\Psi^{-1}(]-\infty,r[) \subset \overline{B_{W^{1,p(\cdot)}(\Omega,w)}(0,K(r))}$, and so $\overline{(\Psi^{-1}(]-\infty,r[))} \subset \overline{B_{W^{1,p(\cdot)}(\Omega,w)}(0,K(r))}$. Using (2.5), we get $\|u\|_{L^\infty(\Omega)} \leq C_0\|\|u\|\|_{1,p(\cdot),\Omega,w}$. Then,

$$\overline{B_{W^{1,p(\cdot)}(\Omega,w)}(0,K(r))} \subset \left\{u \in C(\overline{\Omega}) : \|u\|_{L^\infty(\Omega)} \leq C_0K(r)\right\}.$$

It follows that

$$\inf_{v \in \overline{(\Psi^{-1}(]-\infty,r[))}} \Phi(v) \geq \inf_{\|\|v\|\|_{1,p(\cdot),\Omega,w} \leq K(r)} \Phi(v) \geq \inf_{\|v\|_{L^\infty(\Omega)} \leq C_0K(r)} \Phi(v). \qquad (4.18)$$

By taking u_0 and u_n as constant value functions ξ_0 and ξ_n in Theorem 1 and using (4.18), we conclude the following Theorem 5, which relies on Theorem 1.

Theorem 5 *Let the conditions* **(V1)** *and* **(V2)** *be satisfied. Suppose that* Ψ *and* Φ *are as in* (3.4)*,* Φ *is coercive, and* $K(r)$ *is as in* (4.17)*.*

(a) *If there exist* $\rho_0 > \inf_{W^{1,p(\cdot)}(\Omega,w)} \Psi$ *and* $\xi_0 \in \mathbb{R}$ *such that*

$$\int_\Omega \frac{w_0(x)}{p(x)}|\xi_0|^{p(x)}\,\mathrm{d}x - \int_\Omega G(x,\xi_0)\,\mathrm{d}x := e_0 < \rho_0 \tag{4.19}$$

and

$$\int_\Omega F(x,\xi_0)\,\mathrm{d}x + (\xi_0 - e_0) > \sup_{v\in C(\overline{\Omega}),\, \|v\|_{L^\infty(\Omega)} \le C_0 K(\rho_0)} \int_\Omega F(x,v(x))\,\mathrm{d}x, \tag{4.20}$$

then the restriction of $\Psi+\Phi$ *to* $\Psi^{-1}(]-\infty,\rho_0[)$ *has a global minimum.*

(b) *If there exist a sequence* $\{r_n\} \subset \left(\inf_{W^{1,p(\cdot)}(\Omega,w)} \Psi, +\infty\right)$ *with* $\lim_{n\to\infty} r_n \to +\infty$ *and a sequence* $\{\xi_n\} \subset \mathbb{R}$ *such that for each* n *we have*

$$\int_\Omega \frac{w_0(x)}{p(x)}|\xi_n|^{p(x)}\,\mathrm{d}x - \int_\Omega G(x,\xi_n)\,\mathrm{d}x := e_n < r_n \tag{4.21}$$

and

$$\int_\Omega F(x,\xi_n)\,\mathrm{d}x + (r_n - e_n) > \sup_{v\in C(\overline{\Omega}),\, \|v\|_{L^\infty(\Omega)} \le C_0 K(r_n)} \int_\Omega F(x,v(x))\,\mathrm{d}x, \tag{4.22}$$

and, in addition, (3.6) *holds, then there exists a sequence* $\{v_n\}$ *of local minima of* $\Psi+\Phi$ *such that* $\lim_{n\to\infty}\Psi(v_n) \to +\infty$*.*

(c) *If there exists a sequence* $\{r_n\} \subset \left(\inf_{W^{1,p(\cdot)}(\Omega,w)} \Psi, +\infty\right)$ *with* $\lim_{n\to\infty} r_n = \inf_{u\in W^{1,p(\cdot)}(\Omega,w)} \Psi(u)$ *and a sequence* $\{\xi_n\} \subset \mathbb{R}$ *such that for each* n *the conditions* (4.21) *and* (4.22) *are satisfied, and in addition, the condition* (1.11) *is satisfied, then there exists a sequence* $\{v_n\}$ *of pairwise distinct local minima of* $\Psi+\Phi$ *such that* $\lim_{n\to\infty}\Psi(v_n) = \inf_{u\in W^{1,p(\cdot)}(\Omega,w)} \Psi(u)$ (i.e., *the sequence* $\{v_n\}$ *converges weakly to the global minimizer of* Ψ).

Proof. Using (4.19), if there exist $\rho_0 > \inf_{u\in W^{1,p(\cdot)}(\Omega,w)} \Psi(u)$ and $\xi_0 \in \mathbb{R}$ such that

$$\int_\Omega \frac{w_0(x)}{p(x)}|\xi_0|^{p(x)}\ \mathrm{d}x - \int_\Omega G(x,\xi_0)\ \mathrm{d}x := e_0 < \rho_0,$$

then $\Psi(\xi_0) < \rho_0$, and therefore (1.6) holds. Thanks to (4.20) we have

$$\int_\Omega F(x,\xi_0)\ \mathrm{d}x + (\rho_0 - e_0) > \sup_{v\in C(\overline{\Omega}),\, \|v\|_{L^\infty(\Omega)} \le C_0 K(\rho_0)} \int_\Omega F(x,v)\ \mathrm{d}x.$$

Then,

$$\rho_0 - \Psi(\xi_0) > -\int_\Omega F(x,\xi_0)\,dx + \sup_{v\in C(\overline{\Omega}),\, \|v\|_{L^\infty(\Omega)} \le C_0 K(\rho_0)} -\Phi(v).$$

Thanks to (4.18) we get

$$\rho_0 - \Psi(\xi_0) > \Phi(\xi_0) - \inf_{v\in\overline{\Psi^{-1}(]-\infty,\rho_0[)}} \Phi(v).$$

Therefore, the hypotheses (1.6) and (1.7) of Theorem 1 (a) are satisfied. Then, the restriction of $\Psi+\Phi$ to $\Psi^{-1}(]-\infty,\rho_0[)$ has a global minimum. Assuming that the hypotheses of Theorem 1 (b) and Theorem 1 (c) are satisfied, using the same approach we can conclude the proof Theorem 5. $\square$

For the condition (4.20) in Theorem 5 (a), we give the following proposition.

Proposition 3 *Assume that $\rho_0 > \inf_{W^{1,p(\cdot)}(\Omega,w)}\Psi$, $\xi_0\in\mathbb{R}$ and* (4.19) *holds. If there exists a positive function $\alpha\in L^1(\Omega)$ with $\|\alpha\|_{L^1(\Omega)}\neq 0$ such that*

$$F(x,\xi_0)+\frac{\alpha(x)}{\|\alpha\|_{L^1(\Omega)}}(\rho_0-e_0) > \sup_{|t|\leq C_0K(\rho_0)} F(x,t) \text{ for a.e. } x\in\Omega, \tag{4.23}$$

and the inequality (4.23) *is strict on a subset of Ω with positive measure, then* (4.20) *holds.*

Proof. Integrating (4.23) over Ω and noting that

$$\int_\Omega \sup_{|t|\leq C_0K(\rho_0)} F(x,t)\,\mathrm{d}x \geq \sup_{v\in C(\overline{\Omega}),\|v\|_{L^\infty(\Omega)}\leq C_0K(\rho_0)} \int_\Omega F(x,v(x))\,\mathrm{d}x,$$

we obtain (4.20). $\square$

Proposition 4 *Assume that Ψ is coercive and* (4.16) *holds. For $r\geq d_0\,\sigma_0^{p^-}$ we have*

$$K(r)\leq\Big(\frac{r}{d_0}\Big)^{\frac{1}{p^-}}. \tag{4.24}$$

Proof. Let $r\geq d_0\sigma_0^{p^-}$ and $u\in W^{1,p(\cdot)}(\Omega,w)$ be such that $\Psi(u)<r$. When $|||u|||_{1,p(\cdot),\Omega,w}\geq\sigma_0$, by (4.16), one has

$$r>\Psi(u)\geq d_0|||u|||^{p^-}_{1,p(\cdot),\Omega,w},$$

which implies that $|||u|||_{1,p(\cdot),\Omega,w}\leq\big(\frac{r}{d_0}\big)^{\frac{1}{p^-}}$. When $|||u|||_{1,p(\cdot),\Omega,w}<\sigma_0$, it is clear that $|||u|||_{1,p(\cdot),\Omega,w}\leq\big(\frac{r}{d_0}\big)^{\frac{1}{p^-}}$. Using the definition of $K(r)$, we conclude (4.24). $\square$

Step 4: Finally, we provide a proof of the statements (4.21) and (4.22). We set $r_n=d_0\big(\frac{b_n}{C_0}\big)^{p^-}$. Then, $\lim_{n\to\infty}r_n\to+\infty$, and thanks to (4.24) we obtain

$$K(r_n)\leq\frac{b_n}{C_0}\,,\quad\text{whence}\quad C_0\,K(r_n)\leq b_n. \tag{4.25}$$

Since F satisfies the condition (S), for each n there exists $\xi_n\in[-a_n,a_n]$ such that

$$F(x,\xi_n)=\sup_{t\in[-a_n,a_n]}F(x,t)\ \text{ for a.e. } x\in\Omega. \tag{4.26}$$

By (3.7), one has

$$e_n=\int_\Omega\frac{w_0(x)}{p(x)}|\xi_n|^{p(x)}\,\mathrm{d}x-\int_\Omega G(x,\xi_n)\,\mathrm{d}x\leq d_1|\xi_n|^{p^+}+d_2\leq d_1|a_n|^{p^+}+d_2.$$

It follows from (3.8) that for n sufficiently large,

$$d_1|a_n|^{p^+} + d_2 < d_0\Big(\frac{b_n}{C_0}\Big)^{p^-} = r_n,$$

and consequently $e_n < r_n$, that is, (4.21) holds. Without loss of generality, we may assume that for all n (4.21) holds. By combining (3.9)–(3.10) and (4.26), we obtain

$$F(x,\xi_n) + \frac{h(x)}{\|h\|_{L^1(\Omega)}}(r_n - e_n) \geq \sup_{|t|\leq b_n} F(x,t) \ \text{ for a.e. } x \in \Omega, \tag{4.27}$$

and the inequality (4.27) is strict on a subset of Ω with positive measure. Using (4.25) and Proposition 3, we obtain (4.22).

Therefore, all hypotheses of Theorem 5 (b) are satisfied, and the proof of the Theorem 3 is concluded. □

Now, let us move on to the proof of Theorem 4.

Proof of Theorem 4. Let us verify all the hypotheses of Theorem 5 (c). Using (3.11), for $|||u|||_{1,p(\cdot),\Omega,w} \leq 1$ we have

$$\begin{aligned}\Psi(u) &= J(u) - \int_\Omega G(x,u)\,\mathrm{d}x \\ &= \int_\Omega \frac{1}{p(x)}\left(\sum_{i=1}^N w_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p(x)} + w_0(x)|u|^{p(x)}\right)\mathrm{d}x \\ &\geq \frac{1}{p^+}|||u|||^{p^+}_{1,p(\cdot),\Omega,w}.\end{aligned}$$

Then, Ψ is coercive, $\inf_{W^{1,p(\cdot)}(\Omega,w)} \Psi = \Psi(0) = 0$ and 0 is the unique global minimizer of Ψ. Thanks to (3.13) we have

$$\begin{aligned}&\limsup_{|\xi|\to 0}\big\{\Psi(\xi)+\Phi(\xi)\big\} \\ &= \limsup_{|\xi|\to 0}\left\{\int_\Omega \frac{w_0(x)}{p(x)}|\xi|^{p(x)}\ \mathrm{d}x - \int_\Omega G(x,\xi)\ \mathrm{d}x - \int_\Omega F(x,\xi)\ \mathrm{d}x\right\} \\ &\leq \limsup_{|\xi|\to 0}\left\{\int_\Omega \frac{w_0(x)}{p(x)}|\xi|^{p^-}\ \mathrm{d}x - \int_\Omega G(x,\xi)\ \mathrm{d}x - \int_\Omega F(x,\xi)\ \mathrm{d}x\right\} < 0,\end{aligned}$$

that is, 0 is not a local minimizer of $\Psi + \Phi$; so (1.11) is satisfied.

For $r > 0$ sufficiently small, the condition $\Psi(u) < r$ implies that $|||u|||_{1,p(\cdot),\Omega,w} < (p^+r)^{\frac{1}{p^+}}$, which shows that $K(r) \leq (p^+r)^{\frac{1}{p^+}}$. Now put $r_n = \frac{1}{p^+}\big(\frac{b_n}{C_0}\big)^{p_+}$. Then, $C_0K(r_n) \leq b_n$. By (3.12), there exists a sequence $\{\xi_n\} \subset \mathbb{R}$ with $\xi_n \in [-a_n, a_n]$ such that for $|\xi_n|$ sufficiently small,

$$\begin{aligned}e_n &= \int_\Omega \frac{w_0(x)}{p(x)}|\xi_n|^{p(x)}\ \mathrm{d}x - \int_\Omega G(x,\xi_n)\ \mathrm{d}x \\ &\leq \left(\int_\Omega \frac{w_0(x)}{p(x)}\ \mathrm{d}x + M|\Omega|\right)|\xi_n|^{p^-} \\ &= d_3|\xi_n|^{p^-} \\ &= d_3|a_n|^{p^-}.\end{aligned} \tag{4.28}$$

It follows from (3.14) that for n large enough,

$$d_3|a_n|^{p^-} < \frac{1}{p^+}\Big(\frac{b_n}{C_0}\Big)^{p^+} = r_n,$$

and consequently $e_n < r_n$, that is, (4.21) holds. Noting that F satisfies the condition (S), thanks to (3.15)–(3.16) and (4.26), we obtain

$$F(x,\xi_n) + \frac{h(x)}{\|h\|_{L^1(\Omega)}}(r_n - e_n) \geq \sup_{|t|\leq b_n} F(x,t) \ \text{ a.e. in } \ \Omega, \tag{4.29}$$

and the inequality (4.29) is strict on a subset of Ω with positive measure. By Proposition 3 and (4.29), we get (4.22). Therefore, all hypotheses of Theorem 5(c) are satisfied. Consequently, there exists a sequence $\{v_n\}$ of pairwise distinct local minima of $\Psi + \Phi$ such that $\Psi(v_n) \to 0$. Thus, $|||u|||_{1,p(\cdot),\Omega,w} \longrightarrow 0$, which completes our proof. □

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