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PSEUDO ALMOST PERIODIC SOLUTIONS FOR CONTINUOUS ALGEBRAIC DIFFERENCE EQUATIONS

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Abstract. Many phenomena in mathematical physics and in the theory of dynamical populations are described by difference equations. The aim of this work is to present a new approach to study the qualitative properties of solutions for some algebraic difference equations. The technique used in the paper is based on convergence of series associated with the forcing term. We also consider the problem of the existence of almost periodic solutions using the compactly characterization of family associated with the forcing term. For illustration, we provide some applications. Our results generalize the main results from our previous work [5].

Keywords: Almost periodic, asymptotic behaviour, bounded solution, ergodic function, greatest integer function, kernels theorem decomposition, pseudo almost periodic, simple convergence, uniform convergence, weak convergence.

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1 Introduction

Difference equations have many applications in population dynamics. They can be used to describe the evolution of many phenomena over the course of time.

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In recent years the theory of almost periodic functions has been extensively developed in connection with problems of differential equations, stability theory, dynamical systems.

The theory of almost periodic functions was introduced by H. Bohr in 1923. The problem of the existence of almost periodic solutions has been extensively studied in the literature. The theory of almost periodic functions, as a generalization of periodic ones and their Fourier series, has been developed by the effort of many mathematicians (see, for example, [1, 2, 7]), and already extends over a period of over one century. Indeed, the year 1893 is marked by the publication of Poincaré's treatise *Nouvelles Méthodes de la mécanique Celeste* as well as the paper of P. Bohl.

In this work, we consider the problem of the existence of periodic and almost periodic solutions of linear difference systems using elementary methods. The main goal of the second part of this work is to conduct the qualitative and quantitative study of the following special difference equation:

$$x(t+1) - x(t) = f(t). \tag{1.1}$$

The work is motivated by some quantitative and qualitative results for the difference equation considered in Ait Dads *et al.* [4, 5], Lancaster [9], Banasiak [6] and Francinou *et al.* [8].

The paper is organized as follows. Section 2 concerns the existence of almost periodic solutions of the difference equation (1.1). In Section 3, we consider the compactly characterization of almost periodic solutions. The second part of Section 3 deals with the characterization of the almost periodicity of solutions in relation to their Fourier coefficients for the equation (1.1). In Section 4, we consider the problem of qualitative solution when the second member is pseudo almost periodic (for more details on this collection of functions we refer the reader to [3]). The last section considers the qualitative properties of the solution in connection with the forcing term for the more general difference equation (5.1). To illustrate the work, some examples and counterexamples are given.

2 Almost periodic solutions for special difference equations

2.1 Preliminary results

The translation of f by s is the function $R_s f(t) := f(t+s)$ defined for all $t \in \mathbb{R}$. A subset $\mathbb{F}$ of $C(\mathbb{R}, \mathbb{E})$ $(C_b(\mathbb{R}, \mathbb{E}))$ is said to be translation stable if $R_s\mathbb{F} = \{R_s f : f \in \mathbb{F}\} \subset \mathbb{F}$ for all $s \in \mathbb{R}$. For a function $f : \mathbb{R} \to \mathbb{C}^m$, we define $T(f, \varepsilon)$ by

$$T(f, \varepsilon) = \{\tau \in \mathbb{R} : |f(t+\tau) - f(t)| < \varepsilon \text{ for all } t \in \mathbb{R}\}.$$

The following definition is well-known.

Definition 1 *A function $f \in C_b(\mathbb{R}, \mathbb{E})$ is called almost periodic, if for each $\varepsilon > 0$ there exists a relatively dense set $p(\varepsilon)$ in $\mathbb{R}$ such that $\|R_\tau f(t) - f(t)\| < \varepsilon$ for all $t \in \mathbb{R}$ and $\tau \in p(\varepsilon)$. By $AP(\mathbb{R}, \mathbb{E})$ we denote the set of all such functions.*

In the sequel, we consider only complex-valued functions, so we will assume $\mathbb{E} = \mathbb{C}$. For $f \in AP(\mathbb{R}, \mathbb{C})$ and $\lambda \in \mathbb{R}$, by $c_\lambda(f)$ we denote the mean value of the function defined by $t \mapsto f(t)e^{-i\lambda t}$. Moreover, we set $F = \{f \in AP(\mathbb{R}, \mathbb{C}) : c_\lambda(f) = 0 \text{ for all } \lambda \in 2\pi\mathbb{Z}\}$.

2.2 Almost periodic case

Proposition 1 *If $f \in F$, then the sequence $\frac{1}{n+1}\sum_{k=0}^{n} f(t+k)$ is uniformly convergent to 0 on $\mathbb{R}$.*

Proof. Let $\lambda \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. Note that $\frac{1}{n+1}\sum_{k=0}^{n} e^{i\lambda(t+k)}$ converges uniformly to 0 on $\mathbb{R}$ as

$$\left|\frac{1}{n+1}\sum_{k=0}^{n} e^{i\lambda(t+k)}\right| = \left|\frac{e^{i\lambda t}}{n+1}\cdot\frac{1-e^{i(n+1)\lambda}}{1-e^{i\lambda}}\right| \leq \frac{2}{(n+1)|1-e^{i\lambda}|}.$$

By linearity, we deduce that for $P(t) = \sum_{k=1}^{d} c_k e^{i\lambda_k t}$ with $\lambda_k \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, the sequence $\frac{1}{n+1}\sum_{k=0}^{n} P(t+k)$ converges uniformly to 0 on $\mathbb{R}$.

Let $f \in F$ and $\varepsilon > 0$. Then, there exists $P(t) = \sum_{k=1}^{d} c_k e^{i\lambda_k t}$, where $\lambda_k \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, such that $||f - P||_\infty \leq \varepsilon$. So,

$$\begin{aligned}\left|\frac{1}{n+1}\sum_{k=0}^{n} f(t+k)\right| &\leq \frac{1}{n+1}\sum_{k=0}^{n} ||f-P||_\infty + \left|\frac{1}{n+1}\sum_{k=0}^{n} P(t+k)\right| \\ &\leq \varepsilon + \left|\frac{1}{n+1}\sum_{k=0}^{n} P(t+k)\right|.\end{aligned}$$

Thus, $\frac{1}{n+1}\sum_{k=0}^{n} f(t+k)$ converges uniformly to 0 on $\mathbb{R}$. □

Let $f \in AP(\mathbb{R}, \mathbb{C})$. We know that if the equation (1.1) admits a solution $x \in AP(\mathbb{R}, \mathbb{C})$, then $f \in F$ (see [5, Proposition 2.13]). For the reverse, one has the following proposition.

Proposition 2 *Let $f \in F$ and $s_n(t) = \sum_{k=0}^{n} f(t+k)$. Then, the equation* (1.1) *has a solution $x \in F$ if and only if the sequence $\frac{1}{N+1}\sum_{n=0}^{N} s_n(t)$ converges uniformly on $\mathbb{R}$. Under these conditions, the equation* (1.1) *has a unique solution in F which is given by*

$$x(t) = -\lim_{N\to+\infty}\frac{1}{N+1}\sum_{n=0}^{N} s_n(t).$$

Proof. ($\Rightarrow$) Let x be a solution in F for the equation (1.1). Then,

$$x(t+n+1) - x(t) = s_n(t).$$

It follows that

$$\frac{1}{N+1}\sum_{n=0}^{N} s_n(t) = \left[\frac{1}{N+1}\sum_{n=0}^{N} x(t+1+n)\right] - x(t).$$

Since $x \in F$, the sequence $\frac{1}{N+1}\sum_{n=0}^{N} x(t+1+n)$ converges uniformly to 0. Therefore, $\frac{1}{N+1}\sum_{n=0}^{N} s_n(t)$ converges uniformly to $-x$. Thus, x is unique and $x(t) = -\lim_{N\to+\infty}\frac{1}{N+1}\sum_{n=0}^{N} s_n(t)$.

($\Leftarrow$) For $f \in F$, set $\tau f(t) = f(t+1)$. Then, $c_\lambda(\tau f) = e^{i\lambda} c_\lambda(f)$ and $\tau f \in F$. Since F is a vector subspace of $AP(\mathbb{R}, \mathbb{C})$, we infer that $s_n = \sum_{k=0}^{n} \tau^k f \in F$. It follows that $\frac{1}{N+1}\sum_{n=0}^{N} s_n \in F$. Since F is closed in $\|.\|_\infty$, we have $x = -\lim_{N\to+\infty}\frac{1}{N+1}\sum_{n=0}^{N} s_n \in F$.

One has

$$s_n(t+1) = \sum_{k=0}^{n} f(t+1+k) = \sum_{k=1}^{n+1} f(t+k) = s_{n+1}(t) - f(t).$$

Then,

$$\begin{aligned} -\frac{1}{N+1}\sum_{n=0}^{N} s_n(t+1) = f(t) - \frac{1}{N+1}\sum_{n=0}^{N} s_{n+1}(t) = f(t) - \frac{1}{N+1}\sum_{n=1}^{N+1} s_n(t) \\ = f(t) - \frac{N+2}{N+1}\left[\frac{1}{N+2}\sum_{n=0}^{N+1} s_n(t)\right] + \frac{1}{N+1}s_0(t). \end{aligned}$$

When N goes to $+\infty$, $x(t+1) = f(t) + x(t)$, and thus x is a solution of the equation (1.1). $\square$

Notation 1 *By $\mathcal{P}_1$ denote the space of continuous functions from $\mathbb{R}$ to $\mathbb{C}$ which are 1-periodic.*

Proposition 3 *Let $f \in AP(\mathbb{R},\mathbb{C})$. Then, the sequence $\frac{1}{n+1}\sum_{k=0}^{n} f(t+k)$ converges uniformly on $\mathbb{R}$. Furthermore, the limit $L \in \mathcal{P}_1$, $f - L \in F$ and $AP(\mathbb{R},\mathbb{C}) = \mathcal{P}_1 \oplus F$.*

Proof. Let $\lambda \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. Note that $\frac{1}{n+1}\sum_{k=0}^{n} e^{i\lambda(t+k)}$ converges uniformly to 0 on $\mathbb{R}$ as

$$\left|\frac{1}{n+1}\sum_{k=0}^{n} e^{i\lambda(t+k)}\right| = \left|\frac{e^{i\lambda t}}{n+1}\cdot\frac{1-e^{i(n+1)\lambda}}{1-e^{i\lambda}}\right| \leq \frac{2}{(n+1)|1-e^{i\lambda}|}.$$

If $\lambda \in 2\pi\mathbb{Z}$, then we simply have $\frac{1}{n+1}\sum_{k=0}^{n} e^{i\lambda(t+k)} = e^{i\lambda t}$ for every $t \in \mathbb{R}$. By linearity, we deduce that for $P(t) = \sum_{k=1}^{d} c_k e^{i\lambda_k t}$, where $\lambda_k \in \mathbb{R}$, the sequence $\frac{1}{n+1}\sum_{k=0}^{n} P(t+k)$ converges uniformly on $\mathbb{R}$. Let $f \in AP(\mathbb{R},\mathbb{C})$ and $\varepsilon > 0$. Then, there exists $P(t) = \sum_{k=1}^{d} c_k e^{i\lambda_k t}$ with $\lambda_k \in \mathbb{R}$ such that $||f - P||_\infty \leq \varepsilon$. Since $\frac{1}{n+1}\sum_{k=0}^{n} P(t+k)$ converges uniformly on $\mathbb{R}$, it satisfies the Cauchy uniform criterion, namely there is $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ and any $t \in \mathbb{R}$ we have

$$\left|\frac{1}{n+1}\sum_{k=0}^{n} P(t+k) - \frac{1}{m+1}\sum_{k=0}^{m} P(t+k)\right| \leq \varepsilon.$$

Then,

$$\begin{aligned} &\left|\frac{1}{n+1}\sum_{k=0}^{n} f(t+k) - \frac{1}{m+1}\sum_{k=0}^{m} f(t+k)\right| \\ &\qquad \leq 2||f-P||_\infty + \left|\frac{1}{n+1}\sum_{k=0}^{n} P(t+k) - \frac{1}{m+1}\sum_{k=0}^{m} P(t+k)\right| \leq 3\varepsilon. \end{aligned}$$

So, $\frac{1}{n+1}\sum_{k=0}^{n} f(t+k)$ satisfies the Cauchy uniform criterion and, since $(AP(\mathbb{R},\mathbb{C}), ||.||_\infty)$ is complete, this means that $\frac{1}{n+1}\sum_{k=0}^{n} f(t+k)$ converges uniformly on $\mathbb{R}$.

Let

$$L(t) = \lim_{n\to+\infty}\frac{1}{n+1}\sum_{k=0}^{n} f(t+k).$$

Note that L is continuous, as the uniform limit of continuous mappings. Moreover, it is 1-periodic, since

$$\begin{aligned} L(t+1) &= \lim_{n\to+\infty} \frac{1}{n+1}\sum_{k=0}^{n} f(t+1+k) \\ &= \lim_{n\to+\infty} \frac{1}{n+1}\sum_{k=1}^{n+1} f(t+k) \\ &= \lim_{n\to+\infty} \frac{1}{n+1}\sum_{k=0}^{n+1} f(t+k) = L(t). \end{aligned}$$

Let $\lambda \in 2\pi\mathbb{Z}$. Observe that $c_\lambda(\tau f) = e^{i\lambda}c_\lambda(f) = c_\lambda(f)$. Then, $c_\lambda(\tau^k f) = c_\lambda(f)$, and so

$$c_\lambda\left(\frac{1}{n+1}\sum_{k=0}^{n}\tau^k f\right) = c_\lambda(f).$$

By the continuity of $f \mapsto c_\lambda(f)$ with respect to the norm $\|.\|_\infty$ we deduce that when n goes to $+\infty$ we have $c_\lambda(L) = c_\lambda(f)$ for all $\lambda \in 2\pi\mathbb{Z}$. And we get $f - L \in F$. So, one has $AP(\mathbb{R},\mathbb{C}) = \mathcal{P}_1 + F$. Let $f \in F \cap \mathcal{P}_1$; in particular, $f \in \mathcal{P}_1$. Then, $c_\lambda(f) = 0$ for every $\lambda \in \mathbb{R}\setminus 2\pi\mathbb{Z}$. But $f \in F$, and so $c_\lambda(f) = 0$ for every $\lambda \in 2\pi\mathbb{Z}$. Thus, $c_\lambda(f) = 0$ for any $\lambda \in \mathbb{R}$, from which it follows that $f = 0$. Consequently, $AP(\mathbb{R},\mathbb{C}) = \mathcal{P}_1 \oplus F$. □

Proposition 4 *Let $f \in F$ and $s_n(t) = \sum_{k=0}^{n} f(t+k)$. Then, the equation* (1.1) *has a solution $x \in AP(\mathbb{R},\mathbb{C})$ if and only if $\frac{1}{N+1}\sum_{n=0}^{N} s_n(t)$ converges uniformly on $\mathbb{R}$. Under these conditions the equation* (1.1) *has a unique solution in F which is given by*:

$$x(t) = -\lim_{N\to+\infty}\frac{1}{N+1}\sum_{n=0}^{N} s_n(t).$$

Proof. Let $\tau\colon AP(\mathbb{R},\mathbb{C}) \to AP(\mathbb{R},\mathbb{C})$, $f \mapsto \tau f$. From Proposition 2 we have

$$(\tau - \mathrm{id})(F) = \left\{ f \in F : \frac{1}{N+1}\sum_{n=0}^{N} s_n \text{ converges uniformly on } \mathbb{R} \right\},$$

where $s_n = \sum_{k=0}^{n}\tau^k f$. On the other hand, one has $\ker(\tau - \mathrm{id}) = \mathcal{P}_1$, and from Proposition 3 one has $AP(\mathbb{R},\mathbb{C}) = \mathcal{P}_1 \oplus F$. Then, $(\tau - \mathrm{id})(AP(\mathbb{R},\mathbb{C})) = (\tau - \mathrm{id})(F)$. It follows that (1.1) has a solution in $AP(\mathbb{R},\mathbb{C})$ if and only if $\frac{1}{N+1}\sum_{n=0}^{N} s_n$ converges uniformly on $\mathbb{R}$, and from Proposition 2 we deduce that

$$x(t) = -\lim_{N\to+\infty}\frac{1}{N+1}\sum_{n=0}^{N} s_n(t)$$

is the unique solution in F. □

Remark 1 *Looking at the proof of Proposition 2, we see that if $f \in AP(\mathbb{R},\mathbb{C})$ and $\frac{1}{N+1}\sum_{n=0}^{N} s_n$ converges uniformly on $\mathbb{R}$, then*

$$x = -\lim_{N\to+\infty}\frac{1}{N+1}\sum_{n=0}^{N} s_n$$

is a solution of the equation (1.1) *in $AP(\mathbb{R},\mathbb{C})$, it follows that $f \in F$.*

Conclusion 1

$$\begin{aligned}&(\tau - \mathrm{id})(AP(\mathbb{R},\mathbb{C}))\\ &\quad = \left\{ f \in AP(\mathbb{R},\mathbb{C}) : \frac{1}{N+1}\sum_{n=0}^{N} s_n \textit{ converges uniformly on } \mathbb{R} \right\}\\ &\quad = \left\{ f \in AP(\mathbb{R},\mathbb{C}) : \frac{1}{n+1}\sum_{k=0}^{n} (n+1-k)\tau^k f \textit{ converges uniformly on } \mathbb{R} \right\}.\end{aligned}$$

3 Characterization of almost periodic solutions

Now, we give the compactly characterization of the existence of almost periodic solutions.

Lemma 1 *Let $g\colon \mathbb{R} \to \mathbb{C}$ be continuous. Then, g is uniformly continuous on $\mathbb{R}$ if and only if the family $t \mapsto g(t+n)$, $n \in \mathbb{Z}$, is uniformly equicontinuous on $[0,1]$.*

Proof. ($\Rightarrow$) Let $\varepsilon > 0$. Then, there exists $\alpha > 0$ such that for all $t, s \in \mathbb{R}$ satisfying the condition $|t-s| \le \alpha$ we have $|g(t) - g(s)| \le \varepsilon$.

Now, let $t, s \in [0,1]$ be such that $|t-s| \le \alpha$. Then, for any $n \in \mathbb{Z}$ we have $|t+n-(s+n)| = |t-s| \le \alpha$. Hence, $|g(t+n) - g(s+n)| \le \varepsilon$, which implies that the family $t \mapsto g(t+n)$, $n \in \mathbb{Z}$, is uniformly equicontinuous on $[0,1]$.

($\Leftarrow$) Let $\varepsilon > 0$. Then, there exists $\alpha \in (0,1)$ so that for all $u, v \in [0,1]$ satisfying the condition $|u-v| \le \alpha$ we have $\sup_{n\in\mathbb{Z}} |g(u+n) - g(v+n)| \le \varepsilon$.

Now, let $t, s \in \mathbb{R}$ be such that $|t-s| \le \alpha$. By the symmetry of the distance, we can assume that $t \le s$. Let $n = E(t)$. Then, $t \in [n, n+1]$, and consequently $s \in [n, t+\alpha] \subset [n, t+1] \subset [n, n+2]$. Let us consider two cases.

Case 1: $s \in [n, n+1]$. Then, there exist $u, v \in [0,1]$ such that $t = n+u$, $s = n+v$. Since $|u-v| = |t-s| \le \alpha$, we have $|g(u+n) - g(v+n)| \le \varepsilon$, meaning that $|g(t) - g(s)| \le \varepsilon$.

Case 2: $s \in [n+1, n+2]$. Then, t and $n+1$ are in $[n, n+1]$ and $|n+1-t| \le |t-s| \le \alpha$. Applying the reasoning from Case 1 we have $|g(t) - g(n+1)| \le \varepsilon$. Similarly, $n+1$ and s are in $[n+1, n+2]$ and $|n+1-s| \le |t-s| \le \alpha$, and so by the above argument $|g(n+1) - g(s)| \le \varepsilon$. This gives $|g(t) - g(s)| \le 2\varepsilon$. □

In the sequel, for $f \in AP(\mathbb{R},\mathbb{C})$ and $t \in \mathbb{R}$ we set $F_n(t) = \sum_{k=0}^{n} f(t+k)$.

Lemma 2 *If the family $(F_n)_{n\ge 0}$ is uniformly equicontinuous on $[0,1]$, then it is uniformly equicontinuous on $\mathbb{R}$; moreover, so is the family*

$$t \mapsto \sum_{k=1}^{n} f(t-k), \quad n \ge 1.$$

Proof. For $p \in \mathbb{N}^*$ we have $F_n(r+p) = \sum_{k=p}^{n+p} f(r+k)$, and so

$$F_n(r+p) = F_{n+p}(r) - F_{p-1}(r). \tag{3.1}$$

Let $\varepsilon > 0$. Then, there exists $\alpha \in (0,1)$ such that for all $s, t \in [0,1]$ satisfying the condition $|t-s| \leq \alpha$ we have $\sup_{n\in\mathbb{N}} |F_n(t) - F_n(s)| \leq \varepsilon$. Let $s, t \in \mathbb{R}^+$ be such that $|t-s| \leq \alpha$. By the symmetry of the distance, we can assume that $t \leq s$. Let $p = E(t)$. Then, $t \in [p, p+1]$, and therefore $s \in [p, t+\alpha] \subset [p, t+1] \subset [p, p+2]$. Now, let us consider two cases.

Case 1: $s \in [p, p+1]$. Then, there exist $u, v \in [0,1]$ such that $t = p+u$, $s = p+v$. Since $|u-v| = |t-s| \leq \alpha$, for every $n \in \mathbb{N}$ we have $|F_n(u) - F_n(v)| \leq \varepsilon$. Then, by (3.1) one has $|F_n(t) - F_n(s)| = |F_n(p+u) - F_n(p+v)| \leq 2\varepsilon$.

Case 2: $s \in [p+1, p+2]$. We have then that t and $p+1$ belong to $[p, p+1]$ and $|p+1-t| \leq |t-s| \leq \alpha$. Using the first case, one has $|F_n(t) - F_n(p+1)| \leq 2\varepsilon$. Similarly, $p+1$ and s are in the interval $[p+1, p+2]$ and $|p+1-s| \leq |t-s| \leq \alpha$, and so using the first case we have $|F_n(p+1) - F_n(s)| \leq 2\varepsilon$. Thus, $|F_n(t) - F_n(s)| \leq 4\varepsilon$.

We conclude that in both cases for $s, t \in \mathbb{R}^+$, if $|t-s| \leq \alpha$, then $|F_n(t) - F_n(s)| \leq 4\varepsilon$ for every $n \in \mathbb{N}$. So, for $|h| \leq \alpha$, one has $\sup_{t\in[\alpha,\infty)} |F_n(t) - F_n(t+h)| \leq 4\varepsilon$. But the map g_n which to each t assigns the value $|F_n(t) - F_n(t+h)|$ is almost periodic, and so $g_n(\mathbb{R}) \subset \overline{g_n([\alpha,\infty))} \subset [0, 4\varepsilon]$. Hence, for every $t \in \mathbb{R}$, every $|h| \leq \alpha$ and every $n \in \mathbb{N}$ we have $|F_n(t) - F_n(t+h)| \leq 4\varepsilon$. Further, for $n \geq 1$ one gets

$$\left|\sum_{k=1}^{n} f(t-k) - \sum_{k=1}^{n} f(t+h-k)\right| = |F_{n-1}(t-n) - F_{n-1}(t-n+h)| \leq 4\varepsilon.$$

Therefore, the family $t \mapsto \sum_{k=1}^{n} f(t-k)$, $n \geq 1$, is uniformly equicontinuous on $\mathbb{R}$. □

Proposition 5 *Let $f \in AP(\mathbb{R}, \mathbb{C})$. Then, the equation* (1.1) *has a solution x in $AP(\mathbb{R}, \mathbb{C})$ if and only if the set $\{F_n : n \in \mathbb{N}\}$ is a relatively compact subset of $C([0,1])$ endowed with the norm of the uniform convergence.*

Proof. ($\Rightarrow$) Let $x \in AP(\mathbb{R}, \mathbb{C})$ be a solution of the equation (1.1). In particular, x is bounded, and therefore by [5, Proposition 2.5] there exists $c > 0$ such that $|F_n(t)| \leq c$ for all $t \in \mathbb{R}$ and all $n \in \mathbb{N}$. On the other hand, one has $F_n(t) = x(t+n+1) - x(t)$. And as $x \in AP(\mathbb{R}, \mathbb{C})$, x is uniformly continuous on $\mathbb{R}$. So, from Lemma 1 the family $t \mapsto x(t+n)$, $n \in \mathbb{Z}$, is uniformly equicontinuous on $[0,1]$. Therefore, the family $(F_n)_{n\geq 0}$ is equicontinuous on $[0,1]$. Thus, by the Ascoli–Arzéla theorem, we infer that $\{F_n : n \in \mathbb{N}\}$ is a relatively compact subset of $C([0,1])$ for the sup norm.

($\Leftarrow$) Assume that $\{F_n : n \in \mathbb{N}\}$ is a relatively compact subset of $C([0,1])$. In particular, it is bounded, that is, there exists $c > 0$ such that $||F_n||_\infty \leq c$ for all $n \in \mathbb{N}$. So, for every $t \in [0,1]$ and every $n \in \mathbb{N}$ we have $|\sum_{k=0}^{n} f(t+k)| \leq c$.

Let us prove that for every $t \in \mathbb{R}$ and every $n \in \mathbb{N}$ we have $|\sum_{k=0}^{n} f(t+k)| \leq 2c$. Let $t \in \mathbb{R}^+$ and $p = E(t) \in \mathbb{N}$. Then, there exists $u \in [0,1)$ such that $t = p+u$. So,

$$\sum_{k=0}^{n} f(t+k) = \sum_{k=p}^{n+p} f(u+k) = \sum_{k=0}^{n+p} f(u+k) - \sum_{k=0}^{p-1} f(u+k),$$

and thus $|\sum_{k=0}^{n} f(t+k)| \leq 2c$ for all $t \in \mathbb{R}^+$. The map g_n which to each t assigns $|\sum_{k=0}^{n} f(t+k)|$ is almost periodic, so $g_n(\mathbb{R}) \subset \overline{g_n(\mathbb{R}^+)} \subset [0, 2c]$. This implies that for every $t \in \mathbb{R}$ and every $n \in \mathbb{N}$ we have $|\sum_{k=0}^{n} f(t+k)| \leq 2c$. Therefore, the equation (1.1) has a solution x which is bounded on $\mathbb{R}$. From [5, Proposition 2.7] it suffices to prove now that x is uniformly continuous on $\mathbb{R}$. By Lemma 1 it is enough to prove that the family $t \mapsto x(t+n)$, $n \in \mathbb{Z}$, is uniformly equicontinuous on $[0,1]$, which amounts to prove that the families $t \mapsto x(t+n)$, $n \in \mathbb{N}$, and $t \mapsto x(t-n)$, $n \geq 1$, are uniformly equicontinuous on $[0,1]$.

One has $x(t+n+1) = F_n(t) + x(t)$, but as $[0,1]$ is compact, x is uniformly continuous on $[0,1]$. Moreover, $(F_n)_{n\geq 0}$ is uniformly equicontinuous on $[0,1]$. Hence, so is the family $t \mapsto x(t+n)$, $n \in \mathbb{N}$. On the other hand, for $n \geq 1$ we have $x(t) - x(t-n) = \sum_{k=1}^{n} f(t-k)$, and so it suffices to apply Lemma 2. □

3.1 Characterization of AP solutions by Fourier coefficients

3.1.1 A generalization of Bessel's inequality and application

Let g belong to $\mathcal{B} = C_b(\mathbb{R}, \mathbb{C})$, that is, the space of bounded and continuous functions from $\mathbb{R}$ to $\mathbb{C}$, and let $\lambda \in \mathbb{R}$. When the function $r \mapsto \frac{1}{2r}\int_{-r}^{r} e^{-i\lambda x} g(x)\,\mathrm{d}x$ has a finite limit as $r \to +\infty$, we denote it by $c_\lambda(g)$, *i.e.*, $c_\lambda(g) = \lim_{r\to+\infty} \frac{1}{2r}\int_{-r}^{r} e^{-i\lambda x} g(x)\,\mathrm{d}x$. We have the following result.

Proposition 6 *Let $g \in \mathcal{B}$ and let $I = \{\lambda \in \mathbb{R} : c_\lambda(g)$ exists$\}$. If $I \neq \emptyset$, then the family $(c_\lambda(g))_{\lambda\in I}$ is square summable and*

$$\sum_{\lambda\in I} |c_\lambda(g)|^2 \leq \limsup_{r\to+\infty} \frac{1}{2r}\int_{-r}^{r} |g(t)|^2\,\mathrm{d}t.$$

Proof. Note the fact that $g \in \mathcal{B}$ implies that $\frac{1}{2r}\int_{-r}^{r} |g(t)|^2\,\mathrm{d}t \leq ||g||_\infty^2$. Therefore, $\limsup_{r\to+\infty} \frac{1}{2r}\int_{-r}^{r} |g(t)|^2\,\mathrm{d}t$ exists and is finite. Let $e_\lambda \colon \mathbb{R} \to \mathbb{C}$ be the function defined by $e_\lambda(t) = e^{i\lambda t}$. Let J be a finite subset of I and let $P = \sum_{\lambda\in J} c_\lambda(g) e_\lambda$. Then,

$$|P|^2 = \sum_{\lambda,\mu\in J} c_\lambda(g) e_\lambda \overline{c_\mu(g)}\,\overline{e_\mu} = \sum_{\lambda,\mu\in J} c_\lambda(g)\overline{c_\mu(g)} e_{\lambda-\mu}.$$

Hence, $c_0(|P|^2) = \sum_{\lambda\in J} |c_\lambda(g)|^2$. On the other hand, one has $\overline{P}g = \sum_{\lambda\in J} \overline{c_\lambda(g)}\,\overline{e_\lambda} g$, and so $c_0(\overline{P}g) = \sum_{\lambda\in J} |c_\lambda(g)|^2$. Since $|g-P|^2 = |g|^2 + |P|^2 - 2\operatorname{Re}(\overline{P}g) \geq 0$, we deduce that $2\operatorname{Re}(\overline{P}g) - |P|^2 \leq |g|^2$, and thus

$$2\operatorname{Re}\left(\frac{1}{2r}\int_{-r}^{r} \overline{P}g(t)\,\mathrm{d}t\right) - \frac{1}{2r}\int_{-r}^{r} |P(t)|^2\,\mathrm{d}t \leq \frac{1}{2r}\int_{-r}^{r} |g(t)|^2\,\mathrm{d}t.$$

Let $L = \limsup_{r\to+\infty} \frac{1}{2r}\int_{-r}^{r} |g(t)|^2\,\mathrm{d}t$. Then, $2\operatorname{Re}(c_0(\overline{P}g)) - c_0(|P|^2) \leq L$, namely

$$2\sum_{\lambda\in J} |c_\lambda(g)|^2 - \sum_{\lambda\in J} |c_\lambda(g)|^2 \leq L,$$

from which it follows that $\sum_{\lambda\in J} |c_\lambda(g)|^2 \leq L$. Consequently, $(c_\lambda(g))_{\lambda\in I}$ is square summable and $\sum_{\lambda\in I} |c_\lambda(g)|^2 \leq L$. □

3.1.2 Application

As an application of the above inequality, we have the following proposition.

Proposition 7 *Let $f \in AP(\mathbb{R}, \mathbb{C})$. If the equation* (1.1) *admits a solution in $\mathcal{B}$, then the family $\left(\frac{c_\lambda(f)}{e^{i\lambda}-1}\right)_{\lambda \in \mathbb{R}\setminus 2\pi\mathbb{Z}}$ is square summable.*

Proof. Let x a solution in $\mathcal{B}$ of the equation (1.1) and let $\lambda \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. We get

$$\frac{1}{2r}\int_{-r}^{r} e^{-i\lambda x} x(t+1)\,\mathrm{d}t - \frac{1}{2r}\int_{-r}^{r} e^{-i\lambda t} x(t)\,\mathrm{d}t = \frac{1}{2r}\int_{-r}^{r} e^{-i\lambda t} f(t) dt$$

and

$$\begin{aligned}\int_{-r}^{r} e^{-i\lambda t} x(t+1)\,\mathrm{d}t &= \int_{1-r}^{1+r} e^{-i\lambda(t-1)} x(t)\,\mathrm{d}t \\ &= e^{i\lambda}\left(\int_{-r}^{r} e^{-i\lambda t} x(t)\,\mathrm{d}t + \int_{1-r}^{-r} e^{-i\lambda t} x(t)\,\mathrm{d}t + \int_{r}^{1+r} e^{-i\lambda t} x(t)\,\mathrm{d}t\right).\end{aligned}$$

Then,

$$(e^{i\lambda}-1)\frac{1}{2r}\int_{-r}^{r} e^{-i\lambda t} x(t)\,\mathrm{d}t + \varepsilon_r = \frac{1}{2r}\int_{-r}^{r} e^{-i\lambda t} f(t)\,\mathrm{d}t$$

with

$$\varepsilon_r = \frac{1}{2r} e^{i\lambda}\left[\int_{1-r}^{-r} e^{-i\lambda t} x(t)\,\mathrm{d}t + \int_{r}^{1+r} e^{-i\lambda t} x(t)\,\mathrm{d}t\right].$$

Since $|\varepsilon_r| \leq \frac{1}{r}||x||_\infty$, ε_r tends to 0 when r tends to $+\infty$. Thus, $c_\lambda(x)$ exists and $c_\lambda(x) = \frac{c_\lambda(f)}{e^{i\lambda}-1}$. Now, it remains to apply Proposition 6. □

Example 1 *The equation*

$$x(t+1) - x(t) = \sum_{n=2}^{\infty} \frac{e^{2i\pi n! et}}{n(\ln n)^2}$$

does not have a solution in $\mathcal{B}$. In fact, we have

$$|e^{i2\pi e n!} - 1| = 2|\sin(\pi e n!)| = 2\left|\sin\left(\pi \sum_{k=n+1}^{\infty} \frac{n!}{k!}\right)\right|,$$

because

$$\pi \sum_{k=0}^{n} \frac{n!}{k!} \in \pi\mathbb{Z} \quad \text{and} \quad \sum_{k=n+1}^{\infty} \frac{1}{k!} \underset{n\to+\infty}{\sim} \frac{1}{(n+1)!}.$$

Then,

$$|e^{i2\pi e n!} - 1| \underset{n\to+\infty}{\sim} \frac{2\pi}{n}.$$

This, in turn, implies that

$$\left|\frac{c_{2\pi e n!}(f)}{e^{i2\pi e n!} - 1}\right|^2 \underset{n\to+\infty}{\sim} \frac{1}{4\pi^2(\ln n)^4},$$

and the family $\left(\frac{1}{\pi^2(\ln n)^4}\right)_{n\geq 2}$ is not summable. Consequently, the equation has no solution in $\mathcal{B}$.

An example of an equation (1.1) which has a solution in $\mathcal{B}$, but not in $AP(\mathbb{R},\mathbb{C})$ is shown in the following proposition.

Proposition 8 *Let $x\colon \mathbb{R}\to\mathbb{C}$ be the function defined in the following way: for any $k\in\mathbb{Z}$ and $\ell\in\mathbb{N}$ we set $x(k+\frac{1}{\ell})=\sin(\frac{2k\pi}{\ell})$; moreover, on each interval $[k+\frac{1}{\ell+1},k+\frac{1}{\ell}]$ we define x to be affine. We also set $f(t)=x(t+1)-x(t)$. Then, $f\in AP(\mathbb{R},\mathbb{C})$ and $x\in\mathcal{B}\setminus AP(\mathbb{R},\mathbb{C})$.*

Proof. Let $\ell\in\mathbb{N}^*$ and set $s_{k,\ell}=\sin(\frac{2k\pi}{\ell})$. For $t\in[k+\frac{1}{\ell+1},k+\frac{1}{\ell}]$ we have

$$x(t)=\ell(\ell+1)(k+\tfrac{1}{\ell}-t)s_{k+1,\ell}+\ell(\ell+1)(t-k-\tfrac{1}{\ell+1})s_{k,\ell}.$$

It follows that

$$|x(t)|\le\max(|s_{k+1,\ell}|,|s_{k,\ell}|)\le\frac{2(|k|+1)\pi}{\ell}.$$

Then, for

$$t\in(k,k+\tfrac{1}{\ell}]=\bigcup_{j\ge\ell}[k+\tfrac{1}{j+1},k+\tfrac{1}{j}]$$

there exists $j\ge\ell$ such that

$$|x(t)|\le\frac{2(|k|+1)\pi}{j}\le\frac{2(|k|+1)\pi}{\ell}.$$

Hence, for every $t\in(k,k+\frac{1}{\ell}]$ we have

$$|x(t)|\le\frac{2(|k|+1)\pi}{\ell}.$$

Therefore, $\lim_{t\to k^+}x(t)=0$. Moreover, for every $k\in\mathbb{Z}$ one has $x(k+1)=\sin(2k\pi)=0$, which proves that x is continuous on $\mathbb{R}$. Notice that for $t\in[k+\frac{1}{\ell+1},k+\frac{1}{\ell}]$ one has $|x(t)|\le 1$, from which it follows that x is bounded on $\mathbb{R}$. One also has $x(k+\frac{1}{4k})-x(k)=1$. So, x is not uniformly continuous on $\mathbb{R}$. This implies that $x\notin AP(\mathbb{R},\mathbb{C})$.

Since $x(t+1)-x(t)=f(t)$, f is affine on $[k+\frac{1}{\ell+1},k+\frac{1}{\ell}]$ and $f(k+\frac{1}{\ell})=s_{k+1,\ell}-s_{k,\ell}$. For $n\ge 2$, we define f_n on $\mathbb{R}$ as follows: for every $k\in\mathbb{Z}$, f_n coincides with f on $[k+\frac{1}{n},k+1]$, f_n is affine on $[k,k+\frac{1}{n}]$ and $f_n(k)=0$. So, we have $f_n(k+1)=f(k+1)=0$. Then, f_n is also continuous on $\mathbb{R}$.

Let us prove that f_n is periodic. In fact, let $t\in\mathbb{R}$. Then, there exists $k\in\mathbb{Z}$ such that $t\in(k,k+1]$. If $t\in[k,k+\frac{1}{n}]$, then

$$f_n(t)=n[s_{k+1,n}-s_{k,n}](t-k),$$

and as

$$t+n!\in[k+n!,k+n!+\tfrac{1}{n}],$$

we have

$$f_n(t+n!)=n\left[\sin\left(\frac{2(k+n!+1)\pi}{n}\right)-\sin\left(\frac{2(k+n!)\pi}{n}\right)\right](t+n!-(k+n!))=f_n(t).$$

If $t\in[k+\frac{1}{n},k+1]$, then there exists $\ell\le n-1$ such that $t\in[k+\frac{1}{\ell+1},k+\frac{1}{\ell}]$, and so

$$f_n(t)=f(t)=\ell(\ell+1)[(k-t+\tfrac{1}{\ell})f(k+\tfrac{1}{\ell+1})+(t-k-\tfrac{1}{\ell+1})f(k+\tfrac{1}{\ell})].$$

Since $\ell \leq n-1$, $n!$ is a common multiple of ℓ and $\ell+1$. By the periodicity of the sine function, we deduce that

$$f(k+n!+\tfrac{1}{\ell}) = f(k+\tfrac{1}{\ell})$$

and

$$f(k+n!+\tfrac{1}{\ell+1}) = f(k+\tfrac{1}{\ell+1}).$$

Then,

$$\begin{aligned} f_n(t+n!) &= \ell(\ell+1)[(k+n!-(t+n!)+\tfrac{1}{\ell})f(k+n!+\tfrac{1}{\ell+1}) \\ &\quad + (t+n!-(k+n!)-\tfrac{1}{\ell+1})f(k+n!+\tfrac{1}{\ell})] \\ &= \ell(\ell+1)[(k-t+\tfrac{1}{\ell})f(k+\tfrac{1}{\ell+1}) + (t-k-\tfrac{1}{\ell+1})f(k+\tfrac{1}{\ell})] = f_n(t). \end{aligned}$$

So, f_n is periodic.

To end the proof, it suffices to prove that f_n converges uniformly to f. In fact, if $t \in [k, k+\frac{1}{n}]$, then

$$f_n(t) = n[s_{k+1,n} - s_{k,n}](t-k).$$

Hence,

$$|f_n(t)| \leq |s_{k+1,n} - s_{k,n}| \leq \frac{2\pi}{n},$$

and we have

$$|f(t)| \leq \sup_{\ell \geq n} |f(k+\tfrac{1}{\ell})| \leq \sup_{\ell \geq n} |s_{k+1,\ell} - s_{k,\ell}| \leq \sup_{\ell \geq n} \frac{2\pi}{\ell} \leq \frac{2\pi}{n}.$$

Then,

$$|f_n(t) - f(t)| \leq \frac{4\pi}{n}$$

As f_n coincides with f on $[k+\frac{1}{n}, k+1]$, then for every $t \in \mathbb{R}$ we have

$$|f_n(t) - f(t)| \leq \frac{4\pi}{n},$$

from which it follows that f_n converges uniformly to f on $\mathbb{R}$. $\square$

4 Ergodic case

Assume that $f \in PAP_0(\mathbb{R}, \mathbb{C})$. We look for the necessary and sufficient conditions on f ensuring that the equation (1.1) admits a solution in $PAP_0(\mathbb{R}, \mathbb{C})$.

Definition 2 *Let $f_n \in C(\mathbb{R}, \mathbb{C})$. We say that the sequence $(f_n)_{n\geq 0}$ is weakly convergent if there exists $f \in C(\mathbb{R}, \mathbb{C})$ such that $\lim_{n\to+\infty} \int_a^b |f_n(t) - f(t)|\,\mathrm{d}t = 0$ for each $[a,b] \subset \mathbb{R}$. The function f is called the weak limit of $(f_n)_{n\geq 0}$.*

Remark 2 *The weak limit f of the sequence $(f_n)_{n\geq 0}$ is unique. Indeed, let f and g be two weak limits of $(f_n)_{n\geq 0}$ and let $a \in \mathbb{R}$. Then, for every $x \geq a$ we have*

$$\int_a^x |f(t) - g(t)|\,\mathrm{d}t \leq \int_a^x |f_n(t) - f(t)|\,\mathrm{d}t + \int_a^x |g(t) - f_n(t)|\,\mathrm{d}t \to 0$$

as $n \to +\infty$, and so $\int_a^x |f(t) - g(t)|\,\mathrm{d}t = 0$. By derivation, we deduce that $f = g$.

Remark 3 *If f_n converges weakly to f, then for each $[a,b] \subset \mathbb{R}$ we have*

$$\lim_{n\to+\infty} \int_a^b f_n(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t.$$

The linearity of weak limits follows from the following estimate

$$\int_a^b |(\lambda f_n + \mu g_n)(t) - (\lambda f + \mu g)(t)|\,\mathrm{d}t \le |\lambda| \int_a^b |f_n(t) - f(t)|\,\mathrm{d}t + |\mu| \int_a^b |g_n(t) - g(t)|\,\mathrm{d}t.$$

If λ_n is a complex sequence which converges to λ and f_n converges weakly to f, then $\lambda_n f_n$ converges weakly to λf. In fact,

$$\int_a^b |\lambda_n f_n(t) - \lambda f(t)|\,\mathrm{d}t \le |\lambda_n| \int_a^b |f_n(t) - f(t)|\,\mathrm{d}t + |\lambda_n - \lambda| \int_a^b |f(t)|dt \to 0 \text{ as } n \to +\infty.$$

Note also that f_n converges weakly to f if and only if $\int_a^{a+1} |f_n(t) - f(t)|\,\mathrm{d}t \to 0$ as $n \to +\infty$ for every $a \in \mathbb{R}$.

Proposition 9

(i) *Let S belong to $PAP_0(\mathbb{R}, \mathbb{C})$. Then, the sequence $\frac{1}{n}\sum_{k=1}^{n} S(x+k)$ converges weakly to 0.*

(ii) *Let f_n, f be continuous on $\mathbb{R}$. Assume that f_n converges simply to f and that for each $[a,b] \subset \mathbb{R}$ there exists $c > 0$ such that for every $n \in \mathbb{N}$ and every $x \in [a,b]$ we have $|f_n(x)| \le c$. Then, f_n converges weakly to f.*

Proof. Part (i) follows from the estimate

$$\lim_{n\to+\infty} \int_a^{a+1} \left| \frac{1}{n} \sum_{k=1}^{n} S(s+k) \right| \mathrm{d}s \le \lim_{n\to+\infty} \frac{1}{n} \int_{a+1}^{a+1+n} |S(s)|\,\mathrm{d}s = 0.$$

Now, we will prove part (ii). For every $t \in [a,b]$ and every $n \in \mathbb{N}$ we have $|f_n(t)| \le c$. Then, for n near $+\infty$, one has $|f(t)| \le c$. Hence, $|f_n(t) - f(t)| \le 2c$. So, from the Lebesgue dominated convergence theorem, we get $\int_a^b |f_n(t) - f(t)|\,\mathrm{d}t \to 0$ as $n \to +\infty$. □

Proposition 10 *Let $f \in PAP_0(\mathbb{R}, \mathbb{C})$ and let us put $f_n(t) = -\sum_{k=0}^{n} f(t+k)$. Then, the equation* (1.1) *has a solution in $PAP_0(\mathbb{R}, \mathbb{C})$ if and only if there exists $L \in PAP_0(\mathbb{R}, \mathbb{C})$ such that the sequence $\frac{1}{N}\sum_{n=1}^{N} f_n(t)$ converges weakly to L. Under these conditions L is the unique solution of* (1.1) *in $PAP_0(\mathbb{R}, \mathbb{C})$.*

Proof. ($\Rightarrow$) Let x be a solution in $PAP_0(\mathbb{R}, \mathbb{C})$. Then,

$$x(t+n+1) - x(t) = \sum_{k=0}^{n} f(t+k) = -f_n(t),$$

from which it follows that

$$\frac{1}{N}\sum_{n=1}^{N} f_n(t) = x(t) - \frac{1}{N}\sum_{n=1}^{N} x(t+1+n).$$

Thanks to Proposition 9 the sequence $\frac{1}{N}\sum_{n=1}^{N} x(t+1+n)$ converges weakly to 0. Hence, $\frac{1}{N}\sum_{n=1}^{N} f_n$ converges weakly to $x \in PAP_0(\mathbb{R},\mathbb{C})$, and so we have the uniqueness of the solution of (1.1) in $PAP_0(\mathbb{R},\mathbb{C})$.

($\Leftarrow$) Let us put $v_N = \frac{1}{N}\sum_{n=1}^{N} f_n$ and suppose that it converges weakly to $L \in PAP_0(\mathbb{R},\mathbb{C})$. One has

$$f_n(t+1) = -\sum_{k=0}^{n} f(t+1+k) = -\sum_{k=1}^{n+1} f(t+k) = f(t) + f_{n+1}(t),$$

and so

$$\begin{aligned} v_N(t+1) &= \frac{1}{N}\sum_{n=1}^{N}(f(t)+f_{n+1}(t)) = f(t) + \frac{1}{N}\sum_{n=2}^{N+1} f_n(t) \\ &= f(t) + \frac{N+1}{N}\cdot\frac{1}{N+1}\left(\sum_{n=1}^{N} f_n(t) - f_1(t)\right) \\ &= f(t) + \frac{N+1}{N} v_{N+1}(t) - \frac{f_1(t)}{N}. \end{aligned}$$

By Remark 3 and by taking the weak limit, one has

$$L(t+1) = f(t) + L(t).$$

Thus, L is a solution of (1.1) in $PAP_0(\mathbb{R},\mathbb{C})$. □

Corollary 1 *Let $f \in PAP_0(\mathbb{R},\mathbb{C})$ and let us set $f_n(t) = -\sum_{k=0}^{n} f(t+k)$. Assume that f satisfies one of the following conditions*:

(C$_1$) *$\frac{1}{N}\sum_{n=1}^{N} f_n$ converges uniformly on $\mathbb{R}$, or*

(C$_2$) *$\frac{1}{N}\sum_{n=1}^{N} f_n$ converges simply on $\mathbb{R}$ to $L \in PAP_0(\mathbb{R},\mathbb{C})$ and for each $[a,b] \subset \mathbb{R}$ there exists $c > 0$ such that for every $n \in \mathbb{N}$ and every $t \in [a,b]$ we have $|f_n(t)| \le c$.*

Then, the equation (1.1) *has a unique solution in $PAP_0(\mathbb{R},\mathbb{C})$.*

Proof. Put $v_N = \frac{1}{N}\sum_{n=1}^{N} f_n$. Let us assume (C$_1$). The function f is bounded on $\mathbb{R}$, and so v_N is also bounded on $\mathbb{R}$ for every $N \ge 1$. Since $(v_N)_{N\ge 1}$ converges uniformly on $\mathbb{R}$ and $v_N \in PAP_0(\mathbb{R},\mathbb{C})$, its limit L belongs to $PAP_0(\mathbb{R},\mathbb{C})$ and there exists $c > 0$ such that for every $t \in \mathbb{R}$ and every $N \ge 1$ we have $|v_N(t)| \le c$. Hence, from Proposition 9 the sequence v_N converges weakly to L.

Now, let us assume that f satisfies (C$_2$). Then, for every $t \in [a,b]$ and every $N \ge 1$ we have

$$|v_N(t)| \le \frac{1}{N}\sum_{n=1}^{N}|f_n(t)| \le \frac{1}{N}\sum_{n=1}^{N} c = c.$$

Hence, from Proposition 9 one deduces that v_N converges weakly to $L \in PAP_0(\mathbb{R},\mathbb{C})$.

So, in both cases to end the proof, it suffices to apply Proposition 10. □

4.1 Pseudo almost periodic case

Corollary 2 *For $f \in PAP(\mathbb{R}, \mathbb{C})$ the equation* (1.1) *has a solution $x \in PAP(\mathbb{R}, \mathbb{C})$ if and only if the two equations*:

$$x(t+1) - x(t) = f^{ap}(t) \tag{4.1}$$

and

$$x(t+1) - x(t) = f^{e}(t) \tag{4.2}$$

have solutions in $AP(\mathbb{R}, \mathbb{C})$ and $PAP_0(\mathbb{R}, \mathbb{C})$, respectively. In this case the solutions of (1.1) *are of the form $x = x^{ap} + x^{e}$, where x^{ap} is a solution of* (4.1) *and x^{e} is a solution of* (4.2).

Proof. The proof is a direct consequence of the fact that $PAP(\mathbb{R}, \mathbb{C}) = AP(\mathbb{R}, \mathbb{C}) \oplus PAP_0(\mathbb{R}, \mathbb{C})$ and the fact that the spaces $AP(\mathbb{R}, \mathbb{C})$ and $PAP_0(\mathbb{R}, \mathbb{C})$ are stable under the translation operator τ defined by the formula $\tau f(t) = f(t+1)$. □

5 More general case of equation

In this section, we study the existence of solutions of a more general equation than (1.1); namely, we are interested in equations of the following form

$$x(t+a) - bx(t) = f(t), \tag{5.1}$$

where $(a, b) \in \mathbb{R}^* \times \mathbb{C}^*$ and $f \in \mathcal{B}$.

Proposition 11 *Assume that $|b| = 1$ and $f \in AP(\mathbb{R}, \mathbb{C})$. Moreover, let $v_n(t) = \sum_{k=0}^{n} \frac{f(t+ka)}{b^{k+1}}$ and*

$$F_{a,b} = \left\{ x \in AP(\mathbb{R}, \mathbb{C}) : c_{\frac{2k\pi + \arg(b)}{a}}(x) = 0 \text{ for every } k \in \mathbb{Z} \right\}.$$

Then, the equation (5.1) *has a solution in $AP(\mathbb{R}, \mathbb{C})$ if and only if the Césaro mean of the sequence v_n converges uniformly on $\mathbb{R}$. In this case the equation* (5.1) *has a unique solution in $F_{a,b}$ given by*

$$x(t) = -\lim_{N \to +\infty} \frac{1}{N+1} \sum_{n=0}^{N} v_n(t).$$

Proof. By the change of variables $y(t) = x(at)$ and $\varphi(t) = f(at)$, the equation (5.1) can be rewritten in the form $y(t+1) - by(t) = \varphi(t)$. There exists $\theta \in \mathbb{R}$ such that $b = e^{i\theta}$. Then, we have $y(t+1) - e^{i\theta} y(t) = \varphi(t)$. So,

$$\frac{y(t+1)}{e^{(t+1)i\theta}} - \frac{y(t)}{e^{it\theta}} = \frac{\varphi(t)}{e^{(t+1)i\theta}},$$

which is an equation of the form $z(t+1) - z(t) = g(t)$, where $z(t) = \frac{y(t)}{e^{it\theta}}$ and $g(t) = \frac{\varphi(t)}{e^{(t+1)i\theta}}$. Then, thanks to Proposition 4, this equation has a solution in $AP(\mathbb{R}, \mathbb{C})$ if and only if the Césaro mean of the sequence $s_n(t) = \sum_{k=0}^{n} g(t+k)$ converges uniformly on $\mathbb{R}$, and in this case we have a unique solution in F given by $z(t) = -\lim_{N \to +\infty} \frac{1}{N+1} \sum_{n=0}^{N} s_n(t)$. Consequently, $y(t) = e^{it\theta} z(t)$. After what $x(t) = y(\frac{t}{a}) = e^{it\frac{\theta}{a}} z(\frac{t}{a})$ is also almost periodic as a product of almost periodic functions. We have

$$x(t) = e^{it\frac{\theta}{a}} z(\tfrac{t}{a}) = -e^{it\frac{\theta}{a}} \lim_{N\to+\infty} \frac{1}{N+1} \sum_{n=0}^{N} s_n(\tfrac{t}{a})$$

with

$$s_n(\tfrac{t}{a}) = \sum_{k=0}^{n} g(\tfrac{t}{a}+k) = \sum_{k=0}^{n} \frac{\varphi(\frac{t}{a}+k)}{e^{(\frac{t}{a}+k+1)i\theta}} = \sum_{k=0}^{n} \frac{f(t+ka)}{e^{(\frac{t}{a}+k+1)i\theta}}.$$

Then,

$$x(t) = -\lim_{N\to+\infty} \frac{1}{N+1} \sum_{n=0}^{N} v_n(t),$$

where

$$v_n(t) = \sum_{k=0}^{n} \frac{f(t+ka)}{e^{(k+1)i\theta}} = \sum_{k=0}^{n} \frac{f(t+ka)}{b^{k+1}}.$$

On the other hand,

$$\begin{aligned} c_\lambda(x) &= \lim_{r\to+\infty} \frac{1}{2r} \int_{-r}^{r} e^{-i\lambda t} x(t)\,\mathrm{d}t \\ &= \lim_{r\to+\infty} \frac{1}{2r} \int_{-r}^{r} e^{-i\lambda t} e^{it\frac{\theta}{a}} z(\tfrac{t}{a})\,\mathrm{d}t \\ &= \lim_{r\to+\infty} \frac{1}{2r} \int_{-r}^{r} e^{-i(\lambda-\frac{\theta}{a})t} z(\tfrac{t}{a})\,\mathrm{d}t \\ &= \lim_{r\to+\infty} \frac{1}{2(r/a)} \int_{-r/a}^{r/a} e^{-i(a\lambda-\theta)t} z(t)\,\mathrm{d}t \\ &= c_{a\lambda-\theta}(z). \end{aligned}$$

Hence, $c_\lambda(z) = c_{\frac{\lambda+\theta}{a}}(x)$; in particular, $c_{2k\pi}(z) = c_{\frac{2k\pi+\theta}{a}}(x)$. So, $z \in F$ if and only if $c_{\frac{2k\pi+\theta}{a}}(x) = 0$ for every $k \in \mathbb{Z}$. Consequently, one has $x \in F_{a,b}$. □

Proposition 12 *If* $|b| \neq 1$*, then the equation* (5.1) *has an unique solution in* $\mathcal{B}$ *defined by*

$$x(t) = \begin{cases} -\displaystyle\sum_{k=0}^{\infty} \frac{1}{b^{k+1}} f(t+ka), & \text{if } |b| > 1, \\ \displaystyle\sum_{k=0}^{\infty} b^k f(t-ka-a), & \text{if } |b| < 1. \end{cases}$$

Moreover, if $f \in PAP(\mathbb{R},\mathbb{C})$*, then so is* x*.*

Proof. One has $x(at+t) - bx(at) = f(at)$. If we put $y(t) = x(at)$ and $\varphi(t) = f(at)$, the equation becomes $y(t+1) - by(t) = \varphi(t)$. Let us consider two cases.

Case 1: $|b| > 1$. We start with showing the uniqueness of solutions. One has

$$\frac{1}{b^{n+1}} y(t+n+1) - \frac{1}{b^n} y(t+n) = \frac{1}{b^{n+1}} \varphi(t+n).$$

Then,

$$\frac{1}{b^n}y(t+n)-y(t)=\sum_{k=0}^{n-1}\frac{1}{b^{k+1}}\varphi(t+k).$$

And as y is bounded, then when n goes to $+\infty$, we obtain $y(t)=-\sum_{k=0}^{\infty}\frac{1}{b^{k+1}}\varphi(t+k)$. This proves the uniqueness of the bounded solution.

Now, let us move to the existence part. Let us put $y(t)=-\sum_{k=0}^{\infty}\frac{1}{b^{k+1}}\varphi(t+k)$. Since φ is bounded and $\left|\frac{1}{b}\right|<1$, the series $\sum_{k=0}^{\infty}\frac{1}{b^{k+1}}\varphi(t+k)$ converges normally, and thus uniformly on $\mathbb{R}$. So, y is defined and continuous on $\mathbb{R}$. Moreover, one has

$$\begin{aligned} y(t+1) &= -\sum_{k=0}^{\infty}\frac{1}{b^{k+1}}\varphi(t+1+k) \\ &= -\sum_{k=1}^{\infty}\frac{1}{b^{k}}\varphi(t+k) \\ &= \varphi(t)-\sum_{k=0}^{\infty}\frac{1}{b^{k}}\varphi(t+k) \\ &= \varphi(t)+by(t), \end{aligned}$$

from where $y(t+1)-by(t)=\varphi(t)$. The boundedness of y follows from the following estimate

$$|y(t)|\leq\sum_{k=0}^{\infty}\frac{1}{|b|^{k+1}}||\varphi||_\infty=\frac{|b|||\varphi||_\infty}{1-\frac{1}{|b|}}.$$

If $f\in PAP(\mathbb{R},\mathbb{C})$, then $\varphi\in PAP(\mathbb{R},\mathbb{C})$ and since the series defining y converges uniformly, y is a uniform limit in $PAP(\mathbb{R},\mathbb{C})$, which implies that $y\in PAP(\mathbb{R},\mathbb{C})$.

Since $y(t)=x(at)$ and $\varphi(t)=f(at)$, we have

$$x(t)=y(\tfrac{t}{a})=-\sum_{k=0}^{\infty}\frac{1}{b^{k+1}}\varphi(\tfrac{t}{a}+k),$$

meaning that

$$x(t)=-\sum_{k=0}^{\infty}\frac{1}{b^{k+1}}f(t+ka).$$

Case 2: $|b|<1$. One has $y(t+1)-by(t)=\varphi(t)$. Then, $y(t)-\frac{1}{b}y(t+1)=\frac{-1}{b}\varphi(t)$. Putting $u(t)=y(-t)$ and $\psi(t)=\frac{-1}{b}\varphi(-t-1)$, yields

$$u(t+1)-\frac{1}{b}u(t)=y(-t-1)-\frac{1}{b}y(-t)=\frac{-1}{b}\varphi(-t-1)=\psi(t)$$

with $|\frac{1}{b}|>1$. Then, by Case 1, we have a unique bounded solution which is pseudo almost periodic if f is pseudo almost periodic and which is given by

$$u(t)=-\sum_{k=0}^{\infty}b^{k+1}\psi(t+k)=\sum_{k=0}^{\infty}b^{k}\varphi(-t-k-1),$$

meaning that

$$y(t) = \sum_{k=0}^{\infty} b^k \varphi(t-k-1).$$

Then,

$$x(t) = y(\tfrac{t}{a}) = \sum_{k=0}^{\infty} b^k \varphi(\tfrac{t}{a}-k-1),$$

from where

$$x(t) = \sum_{k=0}^{\infty} b^k f(t-ka-a).$$

This ends the proof. □

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