Journal of Nonlinear Evolution Equations and Applications
Volume 2018, Number 2, pp. 11–24 (December 2018)

ISSN 2161-3680
http://www.jneea.com

COUPLED SYSTEMS OF HILFER FRACTIONAL DIFFERENTIAL EQUATIONS WITH MAXIMA

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Received on February 6, 2018, Revised version on April 19, 2018

Accepted on April 20, 2018

Communicated by Martin Bohner

Abstract. In this paper, by applying Mönch's fixed point theorem associated with the technique of measure of weak noncompactness, we present some results concerning the existence of weak solutions for some coupled systems of Hilfer differential equations with maxima.

Keywords: Coupled system, differential equation with maxima, fixed point, Hilfer fractional derivative, Pettis–Riemann–Liouville integral of fractional order, weak solution.

2010 Mathematics Subject Classification: 26A33.

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1 Introduction

Fractional differential equations have been applied in various areas of engineering, mathematics, physics and bio-engineering, and other applied sciences [16, 27]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs of Abbas *et al.* [2, 3], Samko *et al.* [25], Kilbas *et al.* [19] and Zhou [30]. Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hilfer fractional derivative; see [16, 18].

The measure of weak noncompactness was introduced by De Blasi [11]. The strong measure of noncompactness was developed by Banaś and Goebel [6] and used in many papers; see, for example, Abbas *et al.* [1], Akhmerov *et al.* [4], Alvàrez [5], Benchohra *et al.* [9], Guo *et al.* [15], and the references therein. In [9, 22] the authors considered some existence results by applying the techniques of the measure of noncompactness. Recently, several researchers have obtained other results applying the technique of measure of weak noncompactness; see [3, 7, 8] and the references therein.

Differential equations with maxima arise naturally when solving practical problems. For example, many problems in the control theory correspond to the maximal deviation of the regulated quantity. The existence and uniqueness of solutions of differential equations with maxima is considered in [12, 13, 14, 17, 23, 26, 28, 29] (see also the references therein). In this paper, we discuss the existence of weak solutions for the following coupled system of Hilfer fractional differential equations

$$
\begin{aligned}
\big(D_0^{\alpha_1,\beta_1}u\big)(t) &= f_1\Big(t, \max_{0\le\tau\le t}\|u(\tau)\|, \max_{0\le\tau\le t}\|v(\tau)\|\Big),\\
\big(D_0^{\alpha_2,\beta_2}v\big)(t) &= f_2\Big(t, \max_{0\le\tau\le t}\|u(\tau)\|, \max_{0\le\tau\le t}\|v(\tau)\|\Big), \quad t\in I:=[0,T]
\end{aligned}
\tag{1.1}
$$

with the initial conditions

$$
\begin{aligned}
(I_0^{1-\gamma_1}u)(t)|_{t=0} &= \phi_1,\\
(I_0^{1-\gamma_2}v)(t)|_{t=0} &= \phi_2,
\end{aligned}
\tag{1.2}
$$

where $T>0$, $\alpha_i\in(0,1)$, $\beta_i\in[0,1]$, $\gamma_i=\alpha_i+\beta_i-\alpha_i\beta_i$, $\phi_i\in E$, $f_i\colon I\times[0,\infty)\times[0,\infty)\to E$, $i=1,2$, are given functions, E is a real (or complex) Banach space with norm $\|\cdot\|$ and the dual E^* such that E is the dual of a weakly compactly generated Banach space X, $I_0^{1-\gamma_i}$ is the left-sided mixed Riemann–Liouville integral of order $1-\gamma_i$, and $D_0^{\alpha_i,\beta_i}$ is the generalized Riemann–Liouville derivative (Hilfer) operator of order α_i and type β_i, $i=1,2$.

To the best of our knowledge, no papers are devoted to the existence of weak solutions for such class of problems. Thus, the present paper initiates the application of the measure of weak noncompactness to this class of coupled systems.

2 Preliminaries

Let C be the Banach space of all continuous functions v from I into E with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t\in I} \|v(t)\|.$$

As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into E. By $C_\gamma(I)$ and $C^1_\gamma(I)$, we denote the weighted spaces of continuous functions defined by

$$C_\gamma(I) = \{w\colon (0,T] \to E : t^{1-\gamma}w(t) \in C\}$$

with the norm

$$\|w\|_{C_\gamma} := \sup_{t\in I} \|t^{1-\gamma}w(t)\|,$$

and

$$C^1_\gamma(I) = \left\{ w \in C : \frac{\mathrm{d}w}{\mathrm{d}t} \in C_\gamma \right\}$$

with the norm

$$\|w\|_{C^1_\gamma} := \|w\|_\infty + \|w'\|_{C_\gamma}.$$

Also, by $\mathcal{C} := C_{\gamma_1} \times C_{\gamma_2}$ we denote the product weighted space with the norm

$$\|(u,v)\|_{\mathcal{C}} = \|u\|_{C_{\gamma_1}} + \|v\|_{C_{\gamma_2}}.$$

Let $(E,w) = (E, \sigma(E,E^*))$ be the Banach space E with its weak topology.

Definition 1 *A Banach space X is called weakly compactly generated* (*WCG, for short*) *if it contains a weakly compact set whose linear span is dense in X.*

Definition 2 *A function $h\colon E \to E$ is said to be weakly-sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E* (i.e.*, for any (u_n) in E with $u_n \to u$ in (E,w) we have $h(u_n) \to h(u)$ in (E,w)*).

Definition 3 ([24]) *The function $u\colon I \to E$ is said to be Pettis integrable on I if and only if there is an element $u_J \in E$ corresponding to each measurable $J \subset I$ such that $\phi(u_J) = \int_J \phi(u(s))\,\mathrm{d}s$ for all $\phi \in E^*$, where the integral on the right-hand side is assumed to exist in the sense of Lebesgue* (*by definition, $u_J = \int_J u(s)\,\mathrm{d}s$*).

Let $P(I,E)$ be the space of all E-valued Pettis integrable functions defined on I and let $L^1(I,E)$ be the Banach space of all measurable functions $u\colon I \to E$ which are Bochner integrable. Define the class $P_1(I,E)$ by

$$P_1(I,E) = \{u \in P(I,E) : \varphi(u) \in L^1(I,\mathbb{R}) \text{ for every } \varphi \in E^*\}.$$

The space $P_1(I,E)$ is normed by

$$\|u\|_{P_1} = \sup_{\substack{\varphi\in E^*\\ \|\varphi\|_{E^*}\le 1}} \int_0^T |\varphi(u(x))|\,\mathrm{d}\lambda(x),$$

where λ stands for the Lebesgue measure on I.

The following result is due to Pettis (see [24, Theorem 3.4 and Corollary 3.41]).

Proposition 1 ([24]) *If $u \in P_1(I,E)$ and h is a measurable and essentially bounded real-valued function, then $uh \in P_1(I,E)$.*

For all that follows, the symbol "$\int$" denotes the Pettis integral.

Now, we give some results and properties of fractional calculus.

Definition 4 ([2, 19, 25]) *The left-sided mixed Riemann–Liouville integral of order $r > 0$ of a function $w \in L^1(I,E)$ is defined by*

$$(I_0^r w)(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} w(s)\,\mathrm{d}s \quad \textit{for a.e. } t \in I,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t}\,\mathrm{d}t, \quad \xi > 0.$$

Notice that for all $r, r_1, r_2 > 0$ and each $w \in C$, we have $I_0^r w \in C$, and

$$(I_0^{r_1} I_0^{r_2} w)(t) = (I_0^{r_1+r_2} w)(t) \quad \text{for a.e. } t \in I.$$

Definition 5 ([2, 19, 25]) *The Riemann–Liouville fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I,E)$ is defined by*

$$\begin{aligned}(D_0^r w)(t) &= \left(\frac{\mathrm{d}}{\mathrm{d}t} I_0^{1-r} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{-r} w(s)\,\mathrm{d}s \quad \textit{for a.e. } t \in I.\end{aligned}$$

Let $r \in (0,1]$, $\gamma \in [0,1)$ and $w \in C_{1-\gamma}(I)$. Then, the following expression leads to the left inverse operator as follows:

$$(D_0^r I_0^r w)(t) = w(t) \quad \text{for all } t \in (0,T].$$

Moreover, if $I_0^{1-r} w \in C_{1-\gamma}^1(I)$, then the following composition is proved in [25]:

$$(I_0^r D_0^r w)(t) = w(t) - \frac{(I_0^{1-r} w)(0^+)}{\Gamma(r)} t^{r-1} \quad \text{for all } t \in (0,T].$$

Definition 6 ([2, 19, 25]) *The Caputo fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I,E)$ is defined by*

$$\begin{aligned}({}^c D_0^r w)(t) &= \left(I_0^{1-r} \frac{\mathrm{d}}{\mathrm{d}t} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} \frac{\mathrm{d}}{\mathrm{d}s} w(s)\,\mathrm{d}s \quad \textit{for a.e. } t \in I.\end{aligned}$$

In [16], Hilfer studied applications of a generalized fractional operator having the Riemann–Liouville and the Caputo derivatives as special cases (see also [18]).

Definition 7 (Hilfer derivative) *Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $w \in L^1(I,E)$, $I_0^{(1-\alpha)(1-\beta)} w \in AC(I)$. The Hilfer fractional derivative of order α and type β of w is defined as*

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} I_0^{(1-\alpha)(1-\beta)} w \right)(t) \quad \textit{for a.e. } t \in I. \tag{2.1}$$

We will list some properties of the Hilfer derivative. Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, and $w \in L^1(I,E)$.

1. The operator $(D_0^{\alpha,\beta} w)(t)$ can be written as

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} I_0^{1-\gamma} w \right)(t) = \left(I_0^{\beta(1-\alpha)} D_0^{\gamma} w \right)(t) \quad \text{for a.e. } t \in I.$$

   Moreover, the parameter γ satisfies the following estimates: $\gamma \in (0,1]$, $\gamma \geq \alpha$, $\gamma > \beta$, $1-\gamma < 1-\beta(1-\alpha)$.

2. The generalization (2.1) for $\beta = 0$ coincides with the Riemann–Liouville derivative and for $\beta = 1$ with the Caputo derivative, that is,

$$D_0^{\alpha,0} = D_0^{\alpha} \quad \text{and} \quad D_0^{\alpha,1} = {}^{c}D_0^{\alpha}.$$

3. If $D_0^{\beta(1-\alpha)} w$ exists and is in $L^1(I,E)$, then

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = (I_0^{\beta(1-\alpha)} D_0^{\beta(1-\alpha)} w)(t) \quad \text{for a.e. } t \in I.$$

   Furthermore, if $w \in C_\gamma(I)$ and $I_0^{1-\beta(1-\alpha)} w \in C_\gamma^1(I)$, then

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = w(t) \quad \text{for a.e. } t \in I.$$

4. If $D_0^{\gamma} w$ exists and is in $L^1(I,E)$, then

$$(I_0^{\alpha} D_0^{\alpha,\beta} w)(t) = (I_0^{\gamma} D_0^{\gamma} w)(t) = w(t) - \frac{I_0^{1-\gamma}(0^+)}{\Gamma(\gamma)} t^{\gamma-1} \quad \text{for a.e. } t \in I.$$

Corollary 1 *Let $h \in C_\gamma(I)$. Then, the linear problem*

$$\begin{aligned} &(D_0^{\alpha,\beta} u)(t) = h(t), \quad t \in I := [0,T], \\ &(I_0^{1-\gamma} u)(t)|_{t=0} = \phi, \end{aligned}$$

has a unique solution which is given by

$$u(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^{\alpha} h)(t).$$

Remark 1 *Let $g \in P_1(I, E)$. For every $\varphi \in E^*$, we have*

$$\varphi(I_0^\alpha g)(t) = (I_0^\alpha \varphi g)(t) \quad \textit{for a.e. } t \in I.$$

Definition 8 ([11]) *Let E be a Banach space, Ω_E the family of bounded subsets of E and B_1 the unit ball of E. The De Blasi measure of weak noncompactness is the map $\beta\colon \Omega_E \to [0, \infty)$ defined by $\beta(X) = \inf\{\epsilon > 0 :$ there exists a weakly compact subset Ω of E such that $X \subset \epsilon B_1 + \Omega\}$.*

The De Blasi measure of weak noncompactness has the following properties:

(a) if $A \subset B$, then $\beta(A) \leq \beta(B)$,

(b) $\beta(A) = 0$ if and only if A is weakly relatively compact,

(c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$,

(d) $\beta(\overline{A}^\omega) = \beta(A)$, where $\overline{A}^\omega$ denotes the weak closure of A,

(e) $\beta(A + B) \leq \beta(A) + \beta(B)$,

(f) $\beta(\lambda A) = |\lambda|\beta(A)$,

(g) $\beta(\operatorname{conv} A) = \beta(A)$,

(h) $\beta(\bigcup_{|\lambda| \leq h} \lambda A) = h\beta(A)$.

The next result follows directly from the Hahn–Banach theorem.

Proposition 2 *Let E be a normed space and $x_0 \in E$ with $x_0 \neq 0$. Then, there exists $\varphi \in E^*$ with $\|\varphi\|_{E^*} = 1$ and $\varphi(x_0) = \|x_0\|$.*

For a given set V of functions $v\colon I \to E$ let us put

$$V(t) = \{v(t) : v \in V\}, \quad t \in I$$

and

$$V(I) = \{v(t) : v \in V,\ t \in I\}.$$

Lemma 1 ([15]) *Let H be a bounded and equicontinuous subset of C. Then, the function $t \mapsto \beta(H(t))$ is continuous on I, and*

$$\beta_C(H) = \max_{t \in I} \beta(H(t))$$

and

$$\beta\left(\int_I u(s)\,\mathrm{d}s\right) \leq \int_I \beta(H(s))ds,$$

where $H(t) = \{u(t) : u \in H\}$, $t \in I$, and β, β_C are the De Blasi measures of weak noncompactness defined on the bounded sets of E and C, respectively.

For our purpose we will need the following fixed point theorem.

Theorem 1 ([21]) *Let Q be a nonempty, closed, convex and equicontinuous subset of a metrizable locally convex vector space $C(I,E)$ such that $0 \in Q$. Suppose that $T\colon Q \to Q$ is weakly-sequentially continuous. If the implication*

$$\overline{V} = \overline{\operatorname{conv}}(\{0\} \cup T(V)) \quad \Longrightarrow \quad V \textit{ is relatively weakly compact} \tag{2.2}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

3 Existence of weak solutions

Let us start by defining what we mean by a weak solution of the coupled system (1.1)–(1.2).

Definition 9 *By a weak solution of the system* (1.1)–(1.2) *we mean a measurable coupled functions $(u,v) \in \mathcal{C}$ that satisfy the conditions $(I_0^{1-\gamma_1}u)(0^+) = \phi_1$, $(I_0^{1-\gamma_2}v)(0^+) = \phi_2$, and the equations $(D_0^{\alpha_1,\beta_1}u)(t) = f_1(t, \max_{0\le\tau\le t}\|u(\tau)\|, \max_{0\le\tau\le t}\|v(\tau)\|)$ and $(D_0^{\alpha_2,\beta_2}v)(t) = f_2(t, \max_{0\le\tau\le t}\|u(\tau)\|, \max_{0\le\tau\le t}\|v(\tau)\|)$ on I.*

The following hypotheses will be used in the sequel.

(H_1) For a.e. $t \in I$, the functions $(u,v) \mapsto f_i(t,u,v)$, $i = 1,2$, are weakly-sequentially continuous.

(H_2) For each $u,v \in [0,\infty)$, the functions $t \mapsto f_i(t,u,v)$ are Pettis integrable on I.

(H_3) There exist $p_i, q_i \in C(I,[0,\infty))$, $i = 1,2$, such that for all $\varphi \in E^*$, $i = 1,2$, we have

$$|\varphi(f_i(t,u,v))| \le \frac{p_i(t)u + q_i(t)v}{\|\varphi\|_{E^*} + u + v} \quad \text{for a.e. } t \in I \text{ and each } u,v \in [0,\infty).$$

(H_4) For each bounded and measurable set $B \subset E$ and for each $t \in I$, we have

$$\beta(f_i(t,\|B\|,\|B\|)) \le t^{1-r}(p_i(t) + q_i(t))\beta(B), \quad i = 1,2,$$

where $\|B\| = \max_{t\in I}\max_{0\le\tau\le t}\{\|w(\tau)\| :\ w(\tau) \in B\}$.

Set

$$p_i^* = \sup_{t\in I} p_i(t) \quad \text{and} \quad q_i^* = \sup_{t\in I} q_i(t), \quad i = 1,2.$$

Theorem 2 *Assume that the hypotheses* (H_1)–(H_4) *hold. If*

$$L := \frac{(p_1^* + q_1^*)T^{1-\gamma_1+\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{(p_2^* + q_2^*)T^{1-\gamma_2+\alpha_2}}{\Gamma(1+\alpha_2)} < 1, \tag{3.1}$$

then the system (1.1)–(1.2) *has at least one weak solution defined on I.*

Proof. Define the operators $N_1\colon C_{\gamma_1} \to C_{\gamma_1}$ and $N_2\colon C_{\gamma_2} \to C_{\gamma_2}$ by

$$(N_1 u)(t) = \frac{\phi_1}{\Gamma(\gamma_1)} t^{\gamma_1 - 1} + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} f_1\Big(s, \max_{0\le\tau\le s} \|u(\tau)\|, \max_{0\le\tau\le s} \|v(\tau)\|\Big)\, \mathrm{d}s \quad (3.2)$$

and

$$(N_2 v)(t) = \frac{\phi_2}{\Gamma(\gamma_2)} t^{\gamma_2 - 1} + \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2 - 1} f_2\Big(s, \max_{0\le\tau\le s} \|u(\tau)\|, \max_{0\le\tau\le s} \|v(\tau)\|\Big)\, \mathrm{d}s. \quad (3.3)$$

Consider the continuous operator $N\colon \mathcal{C} \to \mathcal{C}$ defined by

$$\big(N(u,v)\big)(t) = \big((N_1 u)(t), (N_2 v)(t)\big). \quad (3.4)$$

First, notice that the hypotheses imply that for each $(u,v) \in C_{\gamma_1} \times C_{\gamma_2}$ the functions $t \mapsto (t-s)^{\alpha-1} f_i(s,u,v)$, for a.e. $t \in I$ are Pettis integrable. Thus, the operator N is well-defined. For all $\varphi \in E^*$, let $R > 0$ be such that $R = R_1 + R_2$ with

$$R_i > \frac{|\varphi(\phi_i)|}{\Gamma(\alpha_i)} + \frac{(p_i^* + q_i^*) T^{1-\gamma_i+\alpha_i}}{\Gamma(1+\alpha_i)}, \quad i = 1,2,$$

and consider the set

$$\begin{aligned} Q = \Big\{ &(u,v) \in C_{\gamma_1} \times C_{\gamma_2} : \|(u,v)\|_{\mathcal{C}} \le R,\ \|t_2^{1-\gamma_1} u(t_2) - t_1^{1-\gamma_1} u(t_1)\| \\ &\le \frac{(p_1^* + q_1^*) T^{1-\gamma_1+\alpha_1}}{\Gamma(1+\alpha_1)} (t_2 - t_1)^{\alpha_1} + \frac{(p_1^* + q_1^*)}{\Gamma(\alpha_1)} \int_0^{t_1} |t_2^{1-\gamma_1} (t_2 - s)^{\alpha_1 - 1} \\ &- t_1^{1-\gamma_1} (t_1 - s)^{\alpha_1 - 1}|\, \mathrm{d}s \text{ and } \|t_2^{1-\gamma_2} v(t_2) - t_1^{1-\gamma_2} v(t_1)\| \\ &\le \frac{(p_2^* + q_2^*) T^{1-\gamma_2+\alpha_2}}{\Gamma(1+\alpha_2)} (t_2 - t_1)^{\alpha_2} + \frac{(p_2^* + q_2^*)}{\Gamma(\alpha_2)} \int_0^{t_1} |t_2^{1-\gamma_2} (t_2 - s)^{\alpha_2 - 1} \\ &- t_1^{1-\gamma_2} (t_1 - s)^{\alpha_2 - 1}|\, \mathrm{d}s \Big\}. \end{aligned}$$

Clearly, the set Q is closed, convex end equicontinuous. We shall show that the operator N satisfies all the assumptions of Theorem 1. The proof will be given in several steps.

Step 1. *N maps Q into itself.* Let $(u,v) \in Q$, $t \in I$ and assume that $(N(u,v))(t) \ne (0,0)$. Then, there exists $\varphi \in E^*$ with $\|\varphi\|_{E^*} = 1$ such that $\|t^{1-\gamma_1}(N_1 u)(t)\| = |\varphi(t^{1-\gamma_1}(N_1 u)(t))|$ and $\|t^{1-\gamma_2}(N_2 v)(t)\| = |\varphi(t^{1-\gamma_2}(N_2 v)(t))|$. Thus,

$$\begin{aligned} &\big\|t^{1-\gamma_1}(N_1 u)(t)\big\| \\ &\quad = \left|\varphi\left(\frac{\phi_1}{\Gamma(\gamma_1)} + \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} f_1\Big(s, \max_{0\le\tau\le s} \|u(\tau)\|, \max_{0\le\tau\le s} \|v(\tau)\|\Big)\, \mathrm{d}s\right)\right|. \end{aligned}$$

Then,

$$\begin{aligned}
&\left\|t^{1-\gamma_1}(N_1u)(t)\right\| \\
&\quad\leq \left|\varphi\left(\frac{\phi_1}{\Gamma(\gamma_1)}\right)\right| + \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\Big|\varphi\Big(f_1\Big(s, \max_{0\leq\tau\leq s}\|u(\tau)\|, \max_{0\leq\tau\leq s}\|v(\tau)\|\Big)\Big)\Big|\,\mathrm{d}s \\
&\quad\leq \frac{|\varphi(\phi_1)|}{\Gamma(\alpha_1)} + \frac{(p_1^*+q_1^*)T^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\,\mathrm{d}s \\
&\quad\leq \frac{|\varphi(\phi_1)|}{\Gamma(\alpha_1)} + \frac{(p_1^*+q_1^*)T^{1-\gamma_1+\alpha_1}}{\Gamma(1+\alpha_1)} \\
&\quad\leq R_1.
\end{aligned}$$

Also, we obtain

$$\begin{aligned}
&\left\|t^{1-\gamma_2}(N_2v)(t)\right\| \\
&\quad\leq \left|\varphi\left(\frac{\phi_2}{\Gamma(\gamma_2)}\right)\right| + \frac{t^{1-\gamma_2}}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}\Big|\varphi\Big(f_2\Big(s, \max_{0\leq\tau\leq s}\|u(\tau)\|, \max_{0\leq\tau\leq s}\|v(\tau)\|\Big)\Big)\Big|\,\mathrm{d}s \\
&\quad\leq \frac{|\varphi(\phi_2)|}{\Gamma(\alpha_2)} + \frac{(p_2^*+q_2^*)T^{1-\gamma_2}}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}\,\mathrm{d}s \\
&\quad\leq \frac{|\varphi(\phi_2)|}{\Gamma(\alpha_2)} + \frac{(p_2^*+q_2^*)T^{1-\gamma_2+\alpha_2}}{\Gamma(1+\alpha_2)} \\
&\quad\leq R_2.
\end{aligned}$$

Hence,

$$\|N(u,v)\|_{\mathcal{C}} \leq R_1 + R_2 = R.$$

Next, let $t_1, t_2 \in I$ be such that $t_1 < t_2$ and let $(u,v) \in Q$ with

$$t_2^{1-\gamma_1}(N_1u)(t_2) - t_1^{1-\gamma_1}(N_1u)(t_1) \neq (0,0).$$

Then, there exists $\varphi \in E^*$ with $\|\varphi\|_{E^*} = 1$ such that

$$\left\|t_2^{1-\gamma_1}(N_1u)(t_2) - t_1^{1-\gamma_1}(N_1u)(t_1)\right\| = \left|\varphi(t_2^{1-\gamma_1}(N_1u)(t_2) - t_1^{1-\gamma_1}(N_1u)(t_1))\right|$$

and

$$\left\|t_2^{1-\gamma_2}(N_2v)(t_2) - t_1^{1-\gamma_2}(N_2v)(t_1)\right\| = \left|\varphi(t_2^{1-\gamma_2}(N_2v)(t_2) - t_1^{1-\gamma_2}(N_2v)(t_1))\right|.$$

Then,

$$\begin{aligned}
&\left\|t_2^{1-\gamma_1}(N_1u)(t_2) - t_1^{1-\gamma_1}(N_1u)(t_1)\right\| = \left|\varphi(t_2^{1-\gamma_1}(N_1u)(t_2) - t_1^{1-\gamma_1}(N_1u)(t_1))\right| \\
&\quad\leq \varphi\Bigg(t_2^{1-\gamma_1}\int_0^{t_2}(t_2-s)^{\alpha_1-1}\frac{f_1\big(s, \max_{0\leq\tau\leq s}\|u(\tau)\|, \max_{0\leq\tau\leq s}\|v(\tau)\|\big)}{\Gamma(\alpha_1)}\,\mathrm{d}s \\
&\qquad -t_1^{1-\gamma_1}\int_0^{t_1}(t_1-s)^{\alpha_1-1}\frac{f_1\big(s, \max_{0\leq\tau\leq s}\|u(\tau)\|, \max_{0\leq\tau\leq s}\|v(\tau)\|\big)}{\Gamma(\alpha_1)}\,\mathrm{d}s\Bigg).
\end{aligned}$$

This gives

$$\begin{aligned}
&\big\|t_2^{1-\gamma_1}(N_1u)(t_2) - t_1^{1-\gamma_1}(N_1u)(t_1)\big\| \\
&\quad\le t_2^{1-\gamma_1}\int_{t_1}^{t_2}(t_2-s)^{\alpha_1-1}\frac{\big|\varphi\big(f_1\big(s,\max_{0\le\tau\le s}\|u(\tau)\|,\max_{0\le\tau\le s}\|v(\tau)\|\big)\big)\big|}{\Gamma(\alpha_1)}\,\mathrm{d}s \\
&\qquad+\int_0^{t_1}\big|t_2^{1-\gamma_1}(t_2-s)^{\alpha_1-1} - t_1^{1-\gamma_1}(t_1-s)^{\alpha_1-1}\big|\times \\
&\qquad\qquad\times\frac{\big|\varphi\big(f_1\big(s,\max_{0\le\tau\le s}\|u(\tau)\|,\max_{0\le\tau\le s}\|v(\tau)\|\big)\big)\big|}{\Gamma(\alpha_1)}\,\mathrm{d}s \\
&\quad\le t_2^{1-\gamma_1}\int_{t_1}^{t_2}(t_2-s)^{\alpha_1-1}\frac{p_1(s)+q_1(s)}{\Gamma(\alpha_1)}\,\mathrm{d}s \\
&\qquad+\int_0^{t_1}\big|t_2^{1-\gamma_1}(t_2-s)^{\alpha_1-1} - t_1^{1-\gamma_1}(t_1-s)^{\alpha_1-1}\big|\frac{p_1(s)+q_1(s)}{\Gamma(\alpha_1)}\,\mathrm{d}s.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\big\|t_2^{1-\gamma_1}(N_1u)(t_2) - t_1^{1-\gamma_1}(N_1u)(t_1)\big\| \\
&\le \frac{(p_1^*+q_1^*)T^{1-\gamma_1+\alpha_1}}{\Gamma(1+\alpha_1)}(t_2-t_1)^{\alpha_1} + \frac{p_1^*+q_1^*}{\Gamma(\alpha_1)}\int_0^{t_1}\big|t_2^{1-\gamma_1}(t_2-s)^{\alpha_1-1} - t_1^{1-\gamma_1}(t_1-s)^{\alpha_1-1}\big|\,\mathrm{d}s.
\end{aligned}$$

Also, we obtain

$$\begin{aligned}
&\big\|t_2^{1-\gamma_2}(N_2v)(t_2) - t_1^{1-\gamma_2}(N_2v)(t_1)\big\| \\
&\le \frac{(p_2^*+q_2^*)T^{1-\gamma_2+\alpha_2}}{\Gamma(1+\alpha_2)}(t_2-t_1)^{\alpha_2} + \frac{p_2^*+q_2^*}{\Gamma(\alpha_2)}\int_0^{t_1}\big|t_2^{1-\gamma_2}(t_2-s)^{\alpha_2-1} - t_1^{1-\gamma_2}(t_1-s)^{\alpha_2-1}\big|\,\mathrm{d}s.
\end{aligned}$$

Hence, $N(Q)\subset Q$.

Step 2. *N is weakly-sequentially continuous.* Let (u_n,v_n) be a sequence in Q and let $(u_n(t))\to u(t)$ and $(v_n(t))\to v(t)$ in (E,ω) for each $t\in I$. Fix $t\in I$. Since for any $i\in\{1,2\}$ the function f_i satisfies the assumption (H_1), we deduce that $f_i(t,\max_{0\le\tau\le t}\|u_n(\tau)\|,\max_{0\le\tau\le t}\|v_n(\tau)\|)$ converges weakly uniformly to $f_i(t,\max_{0\le\tau\le t}\|u(\tau)\|,\max_{0\le\tau\le t}\|v(\tau)\|)$. Hence, the Lebesgue dominated convergence theorem for the Pettis integral implies that $(N(u_n,n_n))(t)$ converges weakly uniformly to $(N(u,v))(t)$ in (E,ω) for each $t\in I$. Thus, $N(u_n,v_n)\to N(u,v)$. Hence, $N\colon Q\to Q$ is weakly-sequentially continuous.

Step 3. *The implication* (2.2) *holds.* Let V be a subset of Q such that $\overline{V}=\overline{\mathrm{conv}}(N(V)\cup\{(0,0)\})$. Obviously,

$$V(t)\subset\overline{\mathrm{conv}}(NV)(t)\cup\{(0,0)\}\quad\text{for every } t\in I.$$

Further, as V is bounded and equicontinuous, by Lemma 3 in [10] the function $t\mapsto\beta(V(t))$ is continuous on I. From (H_3), (H_4), Lemma 1 and the properties of the measure β, for any $t\in I$, we have

$$\begin{aligned}
t^{1-\gamma_1}u(t) &\le \beta\big(t^{1-\gamma_1}(NV)(t)\cup\{(0,0)\}\big) \\
&\le \beta\big(t^{1-\gamma_1}(NV)(t)\big) \\
&\le \frac{T^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_0^t|t-s|^{\alpha_1-1}(p_1(s)+q_1(s))\beta(V(s))\,\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{T^{1-\gamma_1}}{\Gamma(\alpha_1)} \int_0^t |t-s|^{\alpha_1-1} s^{1-\gamma_1}(p_1(s)+q_1(s))u(s)\,\mathrm{d}s \\
&\leq \frac{(p_1^*+q_1^*)T^{1-\gamma_1+\alpha_1}}{\Gamma(1+\alpha_1)} \|u\|_{C_{\gamma_1}}.
\end{aligned}$$

Also, we obtain

$$\begin{aligned}
t^{1-\gamma_2}v(t) &\leq \beta\big(t^{1-\gamma_2}(NV)(t) \cup \{(0,0)\}\big) \\
&\leq \beta\big(t^{1-\gamma_2}(NV)(t)\big) \\
&\leq \frac{T^{1-\gamma_2}}{\Gamma(\alpha_2)} \int_0^t |t-s|^{\alpha_2-1}(p_2(s)+q_2(s))\beta(V(s))\,\mathrm{d}s \\
&\leq \frac{T^{1-\gamma_2}}{\Gamma(\alpha_2)} \int_0^t |t-s|^{\alpha_2-1} s^{1-\gamma_2}(p_2(s)+q_2(s))v(s)\,\mathrm{d}s \\
&\leq \frac{(p_2^*+q_2^*)T^{1-\gamma_2+\alpha_2}}{\Gamma(1+\alpha_2)} \|v\|_{C_{\gamma_2}}.
\end{aligned}$$

Thus, $\|(u,v)\|_{\mathcal{C}} \leq L\|(u,v)\|_{\mathcal{C}}$. From (3.1), we get $\|(u,v)\|_{\mathcal{C}} = 0$, that is, $\beta(V(t)) = 0$ for each $t \in I$. Then, by Theorem 2 in [20], V is weakly relatively compact in $\mathcal{C}$. Applying now Theorem 1, we conclude that N has a fixed point which is a weak solution of the coupled system (1.1)–(1.2). □

4 An example

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots) : \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be a Banach space with the norm

$$\|u\| = \sum_{n=1}^{\infty} |u_n|.$$

As an application of our results we consider the following coupled system of Hilfer fractional differential equations

$$\begin{aligned}
\big(D_0^{\frac12,\frac12} u_n\big)(t) &= f_n\Big(t, \max_{0\leq\tau\leq t} \|u(\tau)\|, \max_{0\leq\tau\leq t} \|v(\tau)\|\Big), \\
\big(D_0^{\frac12,\frac12} v_n\big)(t) &= g_n\Big(t, \max_{0\leq\tau\leq t} \|u(\tau)\|, \max_{0\leq\tau\leq t} \|v(\tau)\|\Big), \quad t \in [0,1]
\end{aligned} \tag{4.1}$$

with the initial conditions

$$\begin{aligned}
(I_0^{\frac14} u)(t)|_{t=0} &= (2^{-1}, 2^{-2}, \ldots, 2^{-n}, \ldots), \\
(I_0^{\frac14} v)(t)|_{t=0} &= (0, 0, \ldots, 0, \ldots),
\end{aligned} \tag{4.2}$$

where

$$f_n(t,u,v) = \frac{ct^2}{1+|u|+|v|} \frac{u_n(t)}{e^{t+4}}, \quad t \in [0,1]$$

and

$$g_n(t,u,v)=\frac{ce^{-4}}{1+|u|+|v|}v_n(t),\quad t\in[0,1]$$

with

$$c:=\frac{e^4}{8}\Gamma\left(\frac{1}{2}\right),\quad u=(u_1,u_2,\ldots,u_n,\ldots),\quad v=(v_1,v_2,\ldots,v_n,\ldots),$$
$$\max_{0\le\tau\le t}\|u(\tau)\|=\Big(\max_{0\le\tau\le t}\|u_1(\tau)\|,\max_{0\le\tau\le t}\|u_2(\tau)\|,\ldots,\max_{0\le\tau\le t}\|u_n(\tau)\|,\ldots\Big)$$

and

$$\max_{0\le\tau\le t}\|v(\tau)\|=\Big(\max_{0\le\tau\le t}\|v_1(\tau)\|,\max_{0\le\tau\le t}\|v_2(\tau)\|,\ldots,\max_{0\le\tau\le t}\|v_n(\tau)\|,\ldots\Big).$$

Set

$$f=(f_1,f_2,\ldots,f_n,\ldots)\quad\text{and}\quad g=(g_1,g_2,\ldots,g_n,\ldots).$$

Clearly, the functions f and g are continuous. Moreover, for each $u,v\in E$ and $t\in[0,1]$, we have

$$\Big\|f\Big(t,\max_{0\le\tau\le t}\|u(\tau)\|,\max_{0\le\tau\le t}\|v(\tau)\|\Big)\Big\|\le ct^2\frac{1}{e^{t+4}}\frac{|u|}{1+|u|+|v|},$$

and

$$\Big\|g\Big(t,\max_{0\le\tau\le t}\|u(\tau)\|,\max_{0\le\tau\le t}\|v(\tau)\|\Big)\Big\|\le ce^{-4}\frac{|v|}{1+|u|+|v|}.$$

Hence, the hypothesis (H_3) is satisfied with $p_1^*=ce^{-4}$, $p_2^*=0$, $q_1^*=0$ and $q_2^*=ce^{-4}$. We shall show that condition (3.1) holds with $T=1$. Indeed,

$$\frac{(p_1^*+q_1^*)T^{1-\gamma_1+\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{(p_2^*+q_2^*)T^{1-\gamma_2+\alpha_2}}{\Gamma(1+\alpha_2)}=\frac{4ce^{-4}}{\Gamma(\frac{1}{2})}=\frac{1}{2}<1.$$

Simple computations show that all conditions of Theorem 2 are satisfied. It follows that the coupled system (4.1)–(4.2) has at least one weak solution defined on $[0,1]$.

Conclusion We have provided some sufficient conditions guaranteeing the existence of weak solutions for some coupled systems of Hilfer differential equations with maxima. The achieved results are obtained using the notion of measure of weak noncompactness. Such notion requires the use of weak conditions on the right-hand side, like the weak sequential continuity.

References

[1] S. Abbas, M. Benchohra, M. Bohner, *Weak solutions for implicit differential equations of Hilfer–Hadamard fractional derivative*, Advances in Dynamical Systems and Applications **12** (2017), no. 1, 1–16.

[2] S. Abbas, M. Benchohra, G. M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer-Verlag, New York, 2012.

[3] S. Abbas, M. Benchohra, G. M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[4] R. R. Akhmerov, M. I. Kamenskii, A. S. Patapov, A. E. Rodkina, B. N. Sadovskii, *Measures of Noncompactness and Condensing Operators*, Birkhauser Verlag, Basel, 1992.

[5] J. C. Alvàrez, *Measure of noncompactness and fixed points of nonexpansive condensing mappings in locally convex spaces*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales de Madrid **79** (1985), 53–66.

[6] J. Banaś, K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.

[7] M. Benchohra, J. Graef, F-Z. Mostefai, *Weak solutions for boundary-value problems with nonlinear fractional differential inclusions*, Nonlinear Dynamics and Systems Theory **11** (2011), no. 3, 227–237.

[8] M. Benchohra, J. Henderson, F-Z. Mostefai, *Weak solutions for hyperbolic partial fractional differential inclusions in Banach spaces*, Computers & Mathematics with Applications **64** (2012), no. 10, 3101–3107.

[9] M. Benchohra, J. Henderson, D. Seba, *Measure of noncompactness and fractional differential equations in Banach spaces*, Communications in Applied Analysis **12** (2008), no. 4, 419–428.

[10] D. Bugajewski, S. Szufla, *Kneser's theorem for weak solutions of the Darboux problem in a Banach space*, Nonlinear Analysis: Theory, Methods & Applications **20** (1993), no. 2, 169–173.

[11] F. S. De Blasi, *On the property of the unit sphere in a Banach space*, Bulletin Mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie, Nouvelle Série **21** (1977), 259–262.

[12] S. B. Dhage, B. C. Dhage, *Dhage iteration method for approximating positive solutions of nonlinear first order ordinary quadratic differential equations with maxima*, Nonlinear Analysis Forum. An International Journal **21** (2016), 87–100.

[13] L. Georgiev, V. G. Angelov, *On the existence and uniqueness of solutions for maximum equations*, Glasnik Matematički **37** (2002), no. 2, 275–281.

[14] P. Gonzáles, M. Pinto, *Convergent solutions of certain nonlinear differential equations with maxima*, Mathematical and Computer Modelling **45** (2007), no. 1–2, 1–10.

[15] D. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers, Dordrecht, 1996.

[16] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[17] R. W. Ibrahim, *Extremal solutions for certain type of fractional differential equations with maxima*, Advances in Difference Equations 2012, **2012**:7, 8 pages.

[18] R. Kamocki, C. Obczńnski, *On fractional Cauchy-type problems containing Hilfer's derivative*, Electronic Journal of Qualitative Theory of Differential Equations, 2016, No. 50, 1-12.

[19] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, 2006.

[20] A. R. Mitchell, Ch. Smith, *Nonlinear equations in abstract spaces*, In: V. Lakshmikantham (ed.), *An existence theorem for weak solutions of differential equations in Banach spaces*, pp. 387–403, Academic Press, New York, 1978.

[21] D. O'Regan, *Fixed point theory for weakly sequentially continuous mapping*, Mathematical and Computer Modelling **27** (1998), no. 5, 1–14.

[22] D. O'Regan, *Weak solutions of ordinary differential equations in Banach spaces*, Applied Mathematics Letters **12** (1999), no. 1, 101–105.

[23] D. Otrocol, *Hybrid differential equations with maxima via Picard operators theory*, Studia Universitatis Babeş-Bolyai Mathematica **61** (2016), no. 4, 421–428.

[24] B. J. Pettis, *On integration in vector spaces*, Transactions of the American Mathematical Society **44** (1938), 277–304.

[25] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Amsterdam, 1987.

[26] E. Stepanov, *On solvability of some boundary value problems for differential equations with "maxima"*, Topological Methods in Nonlinear Analysis **8** (1996), no. 2, 315–326.

[27] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg, Higher Education Press, Beijing, 2010.

[28] P. Thiramanus, J. Tariboon, S. K. Ntouyas, *Integral inequalities with "maxima" and their applications to Hadamard type fractional differential equations*, Journal of Inequalities and Applications **2014**, 2014:398, 15 pages.

[29] Y. Zhang, J. Wang, *Existence and finite-time stability results for impulsive fractional differential equations with maxima*, Journal of Applied Mathematics and Computing **51** (2016), no. 1–2, 67–79.

[30] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.