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UNIQUENESS RESULT OF ENTROPY SOLUTION TO NONLINEAR NEUMANN PROBLEMS WITH VARIABLE EXPONENT AND L^1-DATA

AHMED JAMEA*
Département de Mathématiques,
Centre Régional des Métiers de l'Education et de Formation, El Jadida, Morocco

Laboratoire de Mathématiques Appliquées à la Physique et Industrie,
Faculté des Sciences, Université Chouaib Doukkali, El Jadida, Morocco

ABDERRAHMANE EL HACHIMI†
Département de Mathématiques, Faculté des Sciences,
Université Mohammed V, Agdal, Rabat, Morocco

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Abstract. Our aim in this paper is to study the uniqueness of solutions to the nonlinear Neumann problem with variable exponent

$$|u|^{p(x)-2}u - \operatorname{div}\big(|\nabla u - \Theta(u)|^{p(x)-2}\,(\nabla u - \Theta(u))\big) + \alpha(u) = f \quad \text{in} \quad \Omega,$$

where Ω is a connected open bounded set in $\mathbb{R}^N$, $p(.)$ is a continuous function defined on $\overline{\Omega}$ with $p(.) \in L^\infty(\Omega)$ and $p(x) > 1$ for all $x \in \overline{\Omega}$. We will be especially interested in the case when data are in L^1.

Keywords: Entropy solution, Neumann-type boundary condition, nonlinear elliptic problem, uniqueness, variable exponent.

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*e-mail address: a.jamea77@gmail.com
†e-mail address: aelhachi@yahoo.fr

1 Introduction

In this paper we study the uniqueness of entropy solutions to the nonlinear elliptic problem with variable exponent

$$\begin{cases} |u|^{p(x)-2}u - \operatorname{div}\big(|\nabla u - \Theta(u)|^{p(x)-2}\,(\nabla u - \Theta(u))\big) + \alpha(u) = f & \text{in } \Omega, \\ \big(|\nabla u - \Theta(u)|^{p(x)-2}\,(\nabla u - \Theta(u))\big).\eta + \gamma(u) = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p(.)$ is a continuous function defined on $\overline{\Omega}$ with $p(x) > 1$ for all $x \in \overline{\Omega}$, Ω is a connected open bounded set in $\mathbb{R}^N$, $N \geq 3$, with a connected Lipschitz boundary $\partial\Omega$, and η is the unit outward normal on $\partial\Omega$. Moreover, $\alpha(0) = \gamma(0) = 0$. We will be especially interested in the case when all the right-hand side data belong to L^1.

The operator $\Delta_{p(x)}u = \operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big)$ is called the $p(x)$-Laplacian; it becomes the p-Laplacian when $p(x) \equiv p$ (*i.e.*, when $p(.)$ is a constant function) and the Laplacian when $p(x) \equiv 2$.

We recall that the notion of an entropy solution was introduced by Ph. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J. L. Vázquez in [6]. This notion was then adapted by many authors to study some nonlinear elliptic and parabolic problems with a constant or variable exponent and Dirichlet or Neumann boundary conditions (see for example [1, 3, 4, 5, 9, 15, 16, 17] and references therein).

In the particular case where $p(.)$ is a constant function the existence and uniqueness of entropy solutions to the nonlinear elliptic problem (1.1) was treated by A. Abassi, A. El Hachimi and A. Jamea in [1]. The existence of entropy solutions to the nonlinear elliptic problem (1.1) with variable exponent was treated by E. Azroul, M. B. Benboubker and S. Ouaro in [5].

In recent years, the interest in the study of various mathematical problems with variable exponent has increased considerably (see for example [5, 11, 13, 17, 18] and references therein). Problems with variable exponent are interesting because of their applications and raise many difficult mathematical questions. Some of models leading to the problems of such type are models of motion of electrorheological fluids, models of stationary thermo-rheological viscous flows of non-Newtonian fluids, as well as models of filtration of an ideal barotropic gas through a porous medium (see, *e.g.*, [7, 13, 14] and references therein).

2 Preliminaries and notations

In this section we give some notations and definitions and state some results which we shall use in this work.

Let Ω be a measurable connected open bounded set in $\mathbb{R}^N$, $N \geq 3$, and let $\operatorname{meas}(\Omega)$ denote its measure. We write

$$C^+(\overline{\Omega}) = \big\{p\colon \overline{\Omega} \to \mathbb{R}^+ : p \text{ is continuous and such that } 1 < p_- < p_+ < \infty\big\},$$

where

$$p_- = \min_{x\in\Omega} p(x) \quad \text{and} \quad p_+ = \max_{x\in\Omega} p(x).$$

For $p(.) \in C^+(\overline{\Omega})$ we define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ by

$$L^{p(\cdot)}(\Omega) = \left\{ u\colon \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u|^{p(x)}\,\mathrm{d}x < \infty \right\},$$

and endow it with the Luxemburg norm

$$\|u\|_{p(.)} = \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \le 1 \right\}.$$

The space $\left(L^{p(\cdot)}(\Omega), \|.\|_{p(.)}\right)$ is a uniformly convex (and thus reflexive) Banach space and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $p'(.)$ is such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \Omega$ (see [10]).

Proposition 1 (Hölder type inequality [10]) *Let* $p(.)$ *and* $p'(.)$ *be two elements of* $C^+(\overline{\Omega})$ *such that* $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ *for all* $x \in \Omega$*. Then for any* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$ *we have*

$$\left|\int_\Omega u.v\,\mathrm{d}x\right| \le \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\|u\|_{p(.)}\|u\|_{p'(.)}.$$

On the space $L^{p(\cdot)}(\Omega)$ we also consider the function $\rho_{p(.)}\colon L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(.)}(u) = \rho_{L^{p(\cdot)}(\Omega)}(u) = \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x.$$

The connection between $\rho_{p(.)}$ and $\|.\|_{p(.)}$ is established by the next result.

Proposition 2 (Fan and Zhao [10])

(a) *Let* u *be an element of* $L^{p(\cdot)}(\Omega)$*. Then we have*

- (i) $\|u\|_{p(.)} < 1$ (*respectively* $>, = 1$) *if and only if* $\rho_{p(.)}(u) < 1$ (*respectively* $>, = 1$);
- (ii) $\|u\|_{p(.)} = \alpha$ *if and only if* $\rho_{p(.)}(u) = \alpha$ (*when* $\alpha \neq 0$);
- (iii) *if* $\|u\|_{p(.)} < 1$*, then* $\|u\|_{p(.)}^{p+} \le \rho_{p(.)}(u) \le \|u\|_{p(.)}^{p-}$;
- (iv) *if* $\|u\|_{p(.)} > 1$*, then* $\|u\|_{p(.)}^{p-} \le \rho_{p(.)}(u) \le \|u\|_{p(.)}^{p+}$.

(b) *For a sequence* $(u_n)_{n\in\mathbb{N}} \subset L^{p(\cdot)}(\Omega)$ *and an element* $u \in L^{p(\cdot)}(\Omega)$ *the following statements are equivalent*:

- (i) $\lim_{n\to\infty} u_n = u$ *in* $L^{p(\cdot)}(\Omega)$;
- (ii) $\lim_{n\to\infty} \rho_{p(.)}(u_n - u) = 0$;
- (iii) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho_{p(.)}(u_n) = \rho_{p(.)}(u)$.

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ consists of all $u \in L^{p(\cdot)}(\Omega)$ such that the absolute value of the gradient is in $L^{p(\cdot)}(\Omega)$. Let the norm $\|.\|_{1,p(.)}$ be given by

$$\|u\|_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

The space $\left(W^{1,p(.)}(\Omega), \|.\|_{1,p(.)}\right)$ is a separable and reflexive Banach space.

For a given constant $k > 0$, we define the cut function $T_k \colon \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) := \begin{cases} s, & \text{if } |s| \leq k, \\ k \operatorname{sign}(s), & \text{if } |s| > k, \end{cases}$$

where

$$\operatorname{sign}(s) := \begin{cases} 1, & \text{if } s > 0, \\ 0, & \text{if } s = 0, \\ -1, & \text{if } s < 0. \end{cases}$$

And for a function $u = u(x)$ defined on Ω, we define the truncated function $T_k u$ as follows: for every $x \in \Omega$ the value of $(T_k u)$ at x is just $T_k(u(x))$.

We define also the space

$$\mathcal{T}^{1,p(.)}(\Omega) = \left\{u \colon \Omega \to \mathbb{R} : u \text{ is measurable and } T_k(u) \in W^{1,p(.)}(\Omega) \text{ for all } k > 0\right\}.$$

By [6] we have the following result.

Proposition 3 *For every function $u \in \mathcal{T}^{1,p(.)}(\Omega)$ there exists a unique measurable function $v \colon \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_k(u) = \chi_{\{|u| \leq k\}} \nabla v \quad \textit{for all} \quad k > 0,$$

where χ_B is the characteristic function of the measurable set $B \subset \mathbb{R}^N$. The function v is denoted by ∇u. Moreover, if $u \in W^{1,p(.)}(\Omega)$, then $v \in \left(L^{p(.)(\Omega)}\right)^N$ and $v = \nabla u$ in the usual sense.

For $u \in W^{1,p(.)}(\Omega)$ by τu or u we denote the trace of u on $\partial\Omega$ in the usual sense.

On the other hand, as in [2], $\mathcal{T}_{tr}^{1,p(.)}(\Omega)$ denotes the set of functions u of $\mathcal{T}^{1,p(.)}(\Omega)$ which satisfy the following conditions:

there exists a sequence $(u_n)_{n \in \mathbb{N}}$ in $W^{1,p(.)}(\Omega)$ and a measurable function v on $\partial\Omega$ such that:

(a) $u_n \to u$ a.e. in Ω;

(b) $\nabla T_k(u_n) \to \nabla T_k(u)$ in $\left(L^1(\Omega)\right)^N$ for every $k > 0$;

(c) $u_n \to v$ a.e. on $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [2]. For $u \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$, the trace of u on $\partial\Omega$ is denoted by $\operatorname{tr}(u)$ or u. The operator $\operatorname{tr}(.)$ has the following properties (see [2]):

(i) if $u \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$, then $\tau T_k(u) = T_k(\operatorname{tr}(u))$ for all $k > 0$;

(ii) if $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$, then for all $u \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$ we have $u - \varphi \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$ and $\operatorname{tr}(u - \varphi) = \operatorname{tr}(u) - \tau\varphi$.

In the case when $u \in W^{1,p(.)}(\Omega)$, $\operatorname{tr}(u)$ coincides with τu.

Obviously, we have

$$W^{1,p(.)}(\Omega) \subset \mathcal{T}_{tr}^{1,p(.)}(\Omega) \subset \mathcal{T}^{1,p(.)}(\Omega).$$

Lemma 1 ([1]) *For $\xi, \eta \in \mathbb{R}^N$ and $1 < p < \infty$, we have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi(\xi - \eta). \tag{2.1}$$

Lemma 2 (Y. Fu [12]) *Let $p(.)$ be an element of $L^\infty(\Omega)$ and let $u \in W^{1,p(.)}(\Omega)$. Then there exists a constant C^0_Ω depending only on Ω such that*

$$\int_\Omega |u|^{p(x)} \leq C^0_\Omega \int_\Omega |\nabla u|^{p(x)}. \tag{2.2}$$

Lemma 3 ([8]) *Let p, p' be two real numbers such that $p > 1, p' > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then we have*

$$\left||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right|^{p'} \leq C\left\{(\xi - \eta)(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\right\}^{\frac{\beta}{2}}\left\{|\xi|^p + |\eta|^p\right\}^{1-\frac{\beta}{2}} \quad \textit{for all } \xi, \eta \in \mathbb{R}^d,$$

where $\beta = 2$ if $1 < p \leq 2$, and $\beta = p'$ if $p \geq 2$.

Remark 1 *Hereinafter, c_i, $i \in \{1, 2, \ldots, 7\}$, is a positive constant.*

3 Assumptions and statement of the uniqueness result

In this section, we introduce the concept of an entropy solution for the problem (1.1) and state a uniqueness result for this type of solution. We impose the following assumptions:

(H_1) α, γ are continuous real functions on $\mathbb{R}$ such that $\alpha(x).x \geq 0$ and $\gamma(x).x \geq 0$ for all $x \in \mathbb{R}$;

(H_2) $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$;

(H_3) Θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^N$ such that $\Theta(0) = 0$ and $|\Theta(x) - \Theta(y)| \leq \lambda_0|x - y|$ for all $x, y \in \mathbb{R}$, where λ_0 is a positive constant such that

$$\lambda_0 < \min\left(\left(\frac{p_-}{2}\right)^{\frac{1}{p_-}}, \left(\frac{p_-}{2}\right)^{\frac{1}{p_+}}, \left(\frac{p_-}{2C^0_\Omega p_+}\right)^{\frac{1}{p_+}}\right).$$

Definition 1 *A measurable function $u\colon \Omega \to \mathbb{R}$ is an entropy solution of the nonlinear elliptic problem* (1.1) *if $u \in \mathcal{T}^{1,p(.)}_{tr}(\Omega)$, $|u|^{p(.)-2}u \in L^1(\Omega)$, $\alpha(u) \in L^1(\Omega)$, $\gamma(u) \in L^1(\partial\Omega)$ and*

$$\begin{aligned}&\int_\Omega |u|^{p(x)-2}uT_k(u - \varphi) + \int_\Omega |\nabla u - \Theta(u)|^{p(x)-2}(\nabla u - \Theta(u))\, DT_k(u - \varphi)\\ &\quad + \int_\Omega \alpha(u)T_k(u - \varphi) + \int_{\partial\Omega} \gamma(u)T_k(u - \varphi) \leq \int_\Omega fT_k(u - \varphi) + \int_{\partial\Omega} gT_k(u - \varphi)\end{aligned} \tag{3.1}$$

for all $k > 0$ and $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.

Lemma 4 *Let hypotheses* (H_1)*,* (H_2) *and* (H_3) *be satisfied. If u is an entropy solution of the problem* (1.1)*, then*

1. $\displaystyle\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<k+h\}}|\nabla u-\Theta(u)|^{p(x)-2}\,|\nabla u-\Theta(u)|\,\nabla u=0;$
2. $\displaystyle\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<k+h\}}|\nabla u|^{p(x)}=0;$
3. $\displaystyle\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<k+h\}}|\nabla u-\Theta(u)|^{p(x)}=0.$

Proof. We divide the proof into three parts.

1. Taking $\varphi=T_h(u)$ in the inequality (3.1), we get

$$\begin{aligned}\int_\Omega &|u|^{p(x)-2}uT_k(u-T_h(u))\\ &+\int_\Omega\Big(|\nabla u-\Theta(u)|^{p(x)-2}\,(\nabla u-\Theta(u))\Big)\,DT_k(u-T_h(u))\\ &\quad+\int_\Omega\alpha(u)T_k(u-T_h(u))+\int_{\partial\Omega}\gamma(u)T_k(u-T_h(u))\\ &\qquad\le\int_\Omega fT_k(u-T_h(u))+\int_{\partial\Omega}gT_k(u-T_h(u)).\end{aligned}\tag{3.2}$$

Firstly, we have

$$\int_\Omega|u|^{p(x)-2}uT_k(u-T_h(u))=\int_{\{|u|>h\}}|u|^{p(x)-2}uT_k(u-h\,\mathrm{sign}(u)),$$

and

$$\mathrm{sign}(u)\chi_{\{|u|>h\}}=\mathrm{sign}(u-h\,\mathrm{sign}(u))\chi_{\{|u|>h\}}=\mathrm{sign}(T_k(u-h\,\mathrm{sign}(u)))\chi_{\{|u|>h\}},$$

where χ_B is the characteristic function of the measurable set $B\subset\mathbb{R}^N$.

Then

$$\int_\Omega|u|^{p(x)-2}uT_k(u-T_h(u))\ge 0.$$

Using the same approach, we show that

$$\int_\Omega\alpha(u)T_k(u-T_h(u))+\int_{\partial\Omega}\gamma(u)T_k(u-T_h(u))\ge 0.$$

Therefore, the inequality (3.2) becomes

$$\begin{aligned}\int_{\{h\le|u|\le h+k\}}&\Big(|\nabla u-\Theta(u)|^{p(x)-2}\,(\nabla u-\Theta(u))\Big)\,DT_k(u-T_h(u))\\ &\le k\left(\int_{\{|u|>h\}}|f|+\int_{\partial\Omega\cap\{|u|>h\}}|g|\right).\end{aligned}$$

By hypothesis (H_2), we have

$$\int_{\{|u|>h\}} |f| + \int_{\partial\Omega\cap\{|u|>h\}} |g| \geq 0.$$

This implies that

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<k+h\}} |\nabla u-\Theta(u)|^{p(x)-2}\,|\nabla u-\Theta(u)|\,\nabla u = 0.$$

2. Using Lemma 1, we get

$$\frac{1}{p(x)}\,|\nabla u-\Theta(u)|^{p(x)} - \frac{1}{p(x)}|\Theta(u)|^{p(x)} \leq |\nabla u-\Theta(u)|^{p(x)-2}\,|\nabla u-\Theta(u)|\,\nabla u. \tag{3.3}$$

On the other hand, it is known that $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ for all $a, b\in\mathbb{R}^+$ and $p\geq 1$. Therefore, the inequality (3.3) becomes

$$\frac{1}{p(x)}\,|\nabla u-|^{p(x)} - \frac{2}{p(x)}|\Theta(u)|^{p(x)} \leq |\nabla u-\Theta(u)|^{p(x)-2}\,|\nabla u-\Theta(u)|\,\nabla u. \tag{3.4}$$

We use hypothesis (H_3) to get

$$\frac{1}{p(x)}\,|\nabla u|^{p(x)} - \frac{2\lambda_0^{p(x)}}{p(x)}|u|^{p(x)} \leq |\nabla u-\Theta(u)|^{p(x)-2}\,|\nabla u-\Theta(u)|\,\nabla u. \tag{3.5}$$

This implies that

$$\frac{1}{p^+}\int_{\Omega_k^h} |\nabla u|^{p(x)} - \frac{2\lambda_0^{p_+}}{p_-}\int_{\Omega_k^h} |u|^{p(x)} \leq \int_{\Omega_k^h} |\nabla u-\Theta(u)|^{p(x)-2}\,|\nabla u-\Theta(u)|\,\nabla u, \tag{3.6}$$

where $\Omega_k^h = \{h\leq |u|\leq h+k\}$. And applying Lemma 2, we obtain

$$\frac{1}{p_+}\int_{\Omega_k^h} |\nabla u|^{p(x)} - \frac{2\lambda_0^{p_+}C_\Omega^0}{p_-}\int_{\Omega_k^h} |\nabla u|^{p(x)} \leq \int_{\Omega_k^h} |\nabla u-\Theta(u)|^{p(x)-2}\,|\nabla u-\Theta(u)|\,\nabla u. \tag{3.7}$$

Then, by hypothesis (H_3), there exists a positive constant c_1 such that

$$\int_{\Omega_k^h} |\nabla u|^{p(x)} \leq c_1\int_{\Omega_k^h} |\nabla u-\Theta(u)|^{p(x)-2}\,|\nabla u-\Theta(u)|\,\nabla u. \tag{3.8}$$

This allows us to deduce that

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<k+h\}} |\nabla u|^{p(x)} = 0.$$

3. By Lemma 1 we have

$$\frac{1}{p(x)}\,|\nabla u-\Theta(u)|^{p(x)} - \frac{1}{p(x)}|\Theta(u)|^{p(x)} \leq |\nabla u-\Theta(u)|^{p(x)-2}\,|\nabla u-\Theta(u)|\,\nabla u. \tag{3.9}$$

Then

$$
\begin{aligned}
\frac{1}{p_+}\int_{\Omega_k^h} |\nabla u-\Theta(u)|^{p(x)} &\le \int_{\Omega_k^h} |\nabla u-\Theta(u)|^{p(x)-2}\,|\nabla u-\Theta(u)|\,\nabla u+\frac{1}{p_+}\int_{\Omega_k^h}|\Theta(u)|^{p(x)}\\
&\le \int_{\Omega_k^h} |\nabla u-\Theta(u)|^{p(x)-2}\,|\nabla u-\Theta(u)|\,\nabla u+\frac{\lambda_0^{p_+}}{p_+}\int_{\Omega_k^h}|u|^{p(x)}\\
&\le \int_{\Omega_k^h} |\nabla u-\Theta(u)|^{p(x)-2}\,|\nabla u-\Theta(u)|\,\nabla u+\frac{\lambda_0^{p_+}C_\Omega^0}{p_+}\int_{\Omega_k^h}|\nabla u|^{p(x)}.
\end{aligned}
$$

Applying the previous results 1 and 2, we obtain

$$
\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<k+h\}} |\nabla u-\Theta(u)|^{p(x)}=0. \qquad \square
$$

Theorem 1 *Let hypotheses* (H_1)–(H_3) *be satisfied. If* $1<p_-<p_+\le 2$*, then the nonlinear elliptic problem* (1.1) *has a unique entropy solution.*

Remark 2 *We recall that the existence of an entropy solution to the nonlinear elliptic problem* (1.1) *was already treated by E. Azroul, M. B. Benboubker and S. Ouaro in* [5].

Proof. Let u and v be two entropy solutions of the elliptic problem (1.1) and let h,k be two positive real numbers such that $k<1$. As a test function for the solution u we take $\varphi=T_k(v)$; similarly, for v as a test function we take $\varphi=T_h(u)$. Dividing the two inequalities by k and passing to the limit with $h\to\infty$, $k\to 0$, we arrive to

$$
\int_\Omega \left||u|^{p(x)-2}u-|v|^{p(x)-2}v\right|+\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}(k;h)\le 0, \tag{3.10}
$$

where

$$
\begin{aligned}
\mathcal{I}(k;h)=&\int_\Omega \Big(|\nabla u-\Theta(u)|^{p(x)-2}\left(\nabla u-\Theta(u)\right)\Big)\nabla T_k(u-T_h(v))\\
&+\int_\Omega \Big(|\nabla v-\Theta(v)|^{p(x)-2}\left(\nabla v-\Theta(v)\right)\Big)\nabla T_k(v-T_h(u)).
\end{aligned}
$$

We consider the following decomposition

$$
\begin{aligned}
\Omega_h^1&=\{|u|\le h;|v|\le h\}, &\quad \Omega_h^2&=\{|u|\le h;|v|>h\},\\
\Omega_h^3&=\{|u|>h;|v|\le h\}, &\quad \Omega_h^4&=\{|u|>h;|v|>h\},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{I}_i(k;h)=&\int_{\Omega_h^i} \Big(|\nabla u-\Theta(u)|^{p(x)-2}\left(\nabla u-\Theta(u)\right)\Big)\nabla T_k(u-T_h(v))\\
&+\int_{\Omega_h^i} \Big(|\nabla v-\Theta(v)|^{p(x)-2}\left(\nabla v-\Theta(v)\right)\Big)\nabla T_k(v-T_h(u))
\end{aligned}
$$

for $i=1,\dots,4$.

Firstly, we have

$$\begin{aligned}\mathcal{I}_1(k;h) &= \int_{\Omega_h^k(1)} \Big(|\nabla u - \Theta(u)|^{p(x)-2} (\nabla u - \Theta(u)) \\ &\qquad\qquad - |\nabla v - \Theta(v)|^{p(x)-2} (\nabla v - \Theta(v)) \Big) \nabla T_k(u-v) \\ &= \mathcal{I}_1^1(k;h) + \mathcal{I}_1^2(k;h),\end{aligned}$$

where

$$\begin{aligned}\Omega_h^k(1) &= \{|u-v| \leq k; |u| \leq h; |v| \leq h\}, \\ \mathcal{I}_1^1(k;h) &= \int_{\Omega_h^k(1)} \Big(|\nabla u - \Theta(u)|^{p(x)-2} (\nabla u - \Theta(u)) \\ &\qquad\qquad - |\nabla v - \Theta(v)|^{p(x)-2} (\nabla v - \Theta(v)) \Big) \Phi_\theta(u;v), \\ \mathcal{I}_1^2(k;h) &= \int_{\Omega_h^k(1)} \Big(|\nabla u - \Theta(u)|^{p(x)-2} (\nabla u - \Theta(u)) \\ &\qquad\qquad - |\nabla v - \Theta(v)|^{p(x)-2} (\nabla v - \Theta(v)) \Big) \Psi_\theta(u;v),\end{aligned}$$

and

$$\Phi_\theta(u;v) = (\nabla u - \Theta(u)) - (\nabla v - \Theta(v)), \qquad \Psi_\theta(u;v) = \Theta(u) - \Theta(v).$$

Let $\varepsilon > 0$. Applying Young's inequality, we find that

$$\begin{aligned}&\mathcal{I}_1^2(k;h) \\ &\quad \leq \frac{\varepsilon}{p'_-} \int_{\Omega_h^k(1)} \left| \Big(|\nabla u - \Theta(u)|^{p(x)-2} (\nabla u - \Theta(u)) \Big) - \Big(|\nabla v - \Theta(v)|^{p(x)-2} (\nabla v - \Theta(v)) \Big) \right|^{p'(x)} \\ &\qquad + \frac{1}{\varepsilon_{\min} p_-} \int_{\Omega_h^k(1)} |\Theta(u) - \Theta(v)|^{p(x)},\end{aligned}$$

where

$$\varepsilon_{\min} = \min\Big(\varepsilon^{\frac{p_+}{p_-}}, \varepsilon^{\frac{p_-}{p_+}} \Big).$$

Now, we apply Lemma 3 and hypothesis (H_3) and we get

$$\left|\mathcal{I}_1^2(k;h)\right| \leq \varepsilon c_2 \mathcal{I}_1^1(k;h) + \frac{c_3}{\varepsilon_{\min}} k^{p_-},$$

which implies that

$$\lim_{k\to 0} \frac{1}{k} \left|\mathcal{I}_1^2(k;h)\right| \leq \varepsilon c_2 \lim_{k \to 0} \frac{1}{k} \mathcal{I}_1^1(k;h). \tag{3.11}$$

If $\lim_{k\to 0} \frac{1}{k}\mathcal{I}_1^1(k,h) = 0$, the above inequality (3.11) becomes

$$\lim_{k\to 0} \frac{1}{k} \mathcal{I}_1^2(k,h) = 0,$$

i.e.

$$\lim_{h\to\infty} \lim_{k\to 0} \frac{1}{k} \mathcal{I}_1(k,h) = 0.$$

If $0 < \lim_{k\to 0} \frac{1}{k}\mathcal{I}_1^1(k,h) < \infty$, we take $\varepsilon = \frac{1}{h \lim_{k\to 0} \frac{1}{k}\mathcal{I}_1^1(k,h)}$ in (3.11) and we deduce that

$$\lim_{h\to\infty} \lim_{k\to 0} \frac{1}{k}\mathcal{I}_1^2(k,h) = 0.$$

It follows that

$$\lim_{h\to\infty} \lim_{k\to 0} \frac{1}{k}\mathcal{I}_1(k,h) \geq 0.$$

If $\lim_{k\to 0} \frac{1}{k}\mathcal{I}_1^1(k,h) = +\infty$, by hypothesis (H$_3$), we have

$$\begin{aligned} \left|\mathcal{I}_1^2(k;h)\right| &\leq k\lambda_0 \int_{\Omega_h^k(1)} \left| |\nabla u - \Theta(u)|^{p(x)-2} (\nabla u - \Theta(u)) - |\nabla v - \Theta(v)|^{p(x)-2} (\nabla v - \Theta(v)) \right| \\ &\leq k\lambda_0 \left(\int_{\Omega_h^k(1)} |\nabla u - \Theta(u)|^{p(x)-1} + |\nabla v - \Theta(v)|^{p(x)-1} \right). \end{aligned}$$

Consequently,

$$\frac{1}{k}\left|\mathcal{I}_1^2(k;h)\right| \leq \lambda_0 \left(\int_{\Omega_h^k(1)} |\nabla u - \Theta(u)|^{p(x)-1} + |\nabla v - \Theta(v)|^{p(x)-1} \right). \tag{3.12}$$

On the other hand, if for the solution u we take $\varphi = 0$ in the inequality (3.1), we arrive for $m > 1$ to

$$\int_{\{|u|\leq m\}} \left(|\nabla u - \Theta(u)|^{p(x)-2} (\nabla u - \Theta(u)) \right) \nabla u \leq mc_3.$$

This implies that

$$\begin{aligned} \int_{\{|u|\leq m\}} |\nabla u - \Theta(u)|^{p(x)} &\leq mc_3 + c_4 \int_{\{|u|\leq m\}} |\Theta(u)|^{p(x)} \\ &\leq mc_3 + c_5 m^{p_+} \\ &\leq c_6 m^{p_+}. \end{aligned}$$

Therefore,

$$\frac{1}{k}\left|\mathcal{I}_1^2(k;h)\right| \leq \lambda_0 c_7 (h+k)^{p_+},$$

i.e.

$$\lim_{k\to 0} \frac{1}{k}\left|\mathcal{I}_1^2(k;h)\right| \leq \lambda_0 c_7 h^{p_+}.$$

Thus, it follows that

$$\lim_{k\to 0} \frac{1}{k}\mathcal{I}_1^1(k,h) + \lim_{k\to 0} \frac{1}{k}\mathcal{I}_1^2(k,h) = +\infty.$$

Then

$$\lim_{h\to\infty} \lim_{k\to 0} \frac{1}{k}\mathcal{I}_1(k,h) = +\infty.$$

Secondly, we have

$$\begin{aligned}
\mathcal{I}_2(k;h) &= \int_{\Omega_h^2} \Big(|\nabla u - \Theta(u)|^{p(x)-2} (\nabla u - \Theta(u)) \Big) \nabla T_k(u - T_h(v)) \\
&\quad + \int_{\Omega_h^2} \Big(|\nabla v - \Theta(v)|^{p(x)-2} (\nabla v - \Theta(v)) \Big) \nabla T_k(v - T_h(u)) \\
&= \int_{\Omega_h^2} \Big(|\nabla u - \Theta(u)|^{p(x)-2} (\nabla u - \Theta(u)) \Big) \nabla T_k(u - h\,\mathrm{sign}(v)) \\
&\quad + \int_{\Omega_h^2} \Big(|\nabla v - \Theta(v)|^{p(x)-2} (\nabla v - \Theta(v)) \Big) \nabla T_k(v - u) \\
&= \int_{\Omega_h^{2,1}} \Big(|\nabla u - \Theta(u)|^{p(x)-2} (\nabla u - \Theta(u)) \Big) \nabla u \\
&\quad + \int_{\Omega_h^{2,2}} \Big(|\nabla v - \Theta(v)|^{p(x)-2} (\nabla v - \Theta(v)) \Big) \nabla v \\
&\quad - \int_{\Omega_h^{2,2}} \Big(|\nabla v - \Theta(v)|^{p(x)-2} (\nabla v - \Theta(v)) \Big) \nabla u,
\end{aligned}$$

where

$$\Omega_h^{2,1} = \{|u| \le h; |v| > h; |u - h\,\mathrm{sign}(u)| \le k\}, \quad \Omega_h^{2,2} = \{|u| \le h; |v| > h; |u - v| \le k\}.$$

On the one hand, we use hypothesis (H_3) to find that

$$\int_{\Omega_h^{2,1}} \Big(|\nabla u - \Theta(u)|^{p(x)-2} (\nabla u - \Theta(u)) \Big) \nabla u + \int_{\Omega_h^{2,2}} \Big(|\nabla v - \Theta(v)|^{p(x)-2} (\nabla v - \Theta(v)) \Big) \nabla v \ge 0.$$

On the other hand, by Lemma 1 we have

$$\begin{aligned}
\frac{1}{k}\mathcal{J}_2(k;h) &:= \frac{1}{k}\int_{\Omega_h^{2,2}} \Big(|\nabla v - \Theta(v)|^{p(x)-2} (\nabla v - \Theta(v)) \Big) \nabla u \\
&= \int_{\Omega_h^{2,2}} \left(\frac{1}{k}\right)^{\frac{1}{p'(x)}} \Big(|\nabla v - \Theta(v)|^{p(x)-2} (\nabla v - \Theta(v)) \Big) \left(\frac{1}{k}\right)^{\frac{1}{p(x)}} \nabla u \\
&\le c_7 \left\| \left(\frac{1}{k}\right)^{\frac{1}{p'(x)}} (\nabla v - \Theta(v))^{p(x)-1} \right\|_{L^{p'(\cdot)}(\Omega_h^{2,2})} \left\| \left(\frac{1}{k}\right)^{\frac{1}{p(x)}} \nabla u \right\|_{L^{p(\cdot)}(\Omega_h^{2,2})}.
\end{aligned}$$

And using Lemma 2, we get

$$\begin{aligned}
&\left\| \left(\frac{1}{k}\right)^{\frac{1}{p'(x)}} (\nabla v - \Theta(v))^{p(x)-1} \right\|_{L^{p'(\cdot)}(\Omega_h^{2,2})} \\
&\qquad \le \max\Bigg[\left(\rho_{L^{p'(\cdot)}(\Omega_h^{2,2})} \left(\left(\frac{1}{k}\right)^{\frac{1}{p'(x)}} \left(\Theta^\theta_{p(x)}(v)\right) \right) \right)^{\frac{1}{p'_+}}, \\
&\qquad\qquad \left(\rho_{L^{p'(\cdot)}(\Omega_h^{2,2})} \left(\left(\frac{1}{k}\right)^{\frac{1}{p'(x)}} \left(\Theta^\theta_{p(x)}(v)\right) \right) \right)^{\frac{1}{p'_-}} \Bigg],
\end{aligned}$$

where $\Theta^{\theta}_{p(x)}(v) = (\nabla v - \Theta(v))^{p(x)-1}$. Moreover,

$$\rho_{L^{p'(\cdot)}(\Omega_h^{2,2})}\left(\left(\frac{1}{k}\right)^{\frac{1}{p'(x)}}\left((\nabla v - \Theta(v))^{p(x)-1}\right)\right) = \int_{\Omega_h^{2,2}} \frac{1}{k}\left|\nabla v - \Theta(v)\right|^{p(x)}$$
$$\leq \frac{1}{k}\int_{\{h\leq|v|\leq h+k\}} \left|\nabla v - \Theta(v)\right|^{p(x)}.$$

This, in view of Lemma 4, implies that

$$\lim_{h\to\infty}\lim_{k\to 0}\left\|\left(\frac{1}{k}\right)^{\frac{1}{p'(x)}}(\nabla v - \Theta(v))^{p(x)-1}\right\|_{L^{p'(\cdot)}(\Omega_h^{2,2})} = 0.$$

And we follow the same method to prove that

$$\lim_{h\to\infty}\lim_{k\to 0}\left\|\left(\frac{1}{k}\right)^{\frac{1}{p(x)}}\nabla u\right\|_{L^{p(\cdot)}(\Omega_h^{2,2})} = 0.$$

Then

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{J}_2(k;h) = 0.$$

Consequently,

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_2(k;h) \geq 0. \tag{3.13}$$

Finally, in the same manner, we show that

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\left(\mathcal{I}_3(k;h) + \mathcal{I}_4(k;h)\right) \geq 0. \tag{3.14}$$

Hence

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}(k;h) \geq 0. \tag{3.15}$$

Therefore, the inequality (3.10) becomes

$$\int_\Omega \left||u|^{p(x)-2}u - |v|^{p(x)-2}v\right| \leq 0. \tag{3.16}$$

This implies that $u = v$ a.e. in Ω. □

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