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INSENSITIZING CONTROL WITH CONSTRAINTS ON THE CONTROL OF THE SEMILINEAR HEAT EQUATION FOR A MORE GENERAL COST FUNCTIONAL

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Abstract. We consider the problem of insensitizing control when the control is subject to a finite number of linear constraints with a non-standard cost. We prove the existence of such control by solving a problem of null controllability with constraints on the control. The key of our result is an observability inequality adapted to the constraints.

Keywords: Insensitizing control, observability inequality, orthogonality, orthogonal projection, Schauder fixed point, semilinear heat equation.

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1 Introduction

The problem of insensitizing control was originally addressed by J. L. Lions in 1989 (see [9]), and since then many papers have been devoted to this topic. In [5] the authors introduced and

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studied the notion of approximate controllability for a semilinear heat equation. This lead to an approximate controllability problem for a cascade system. In [1] a semilinear case with a globally Lipschitz-continuous nonlinearity is studied. Using the notion of approximate insensitizing control and the boundedness of the control, the authors proved the existence of insensitizing control by using the Schauder fixed point theorem.

The insensitizing control problem for the heat equation with many types of nonlinearities was studied by O. Bodart *et al.* (see [2] and [3]). In [10] the authors studied the insensitizing control problem with constraints on the control for a nonlinear heat equation by means of the Kakutani fixed point theorem and an adapted Carleman's inequality. In a recent paper, E. Zuazua and L. de Teresa [12] analysed the class of initial data for the heat equation that can be insensitized. This completes in some sense the work of [11] where it was shown that when the initial datum is non null, one cannot expect to find an insensitizing control in a usual space. Let us mention that in all the cases cited above, the problem is reduced to a controllability problem for a cascade system of heat equations.

Exact controllability problem for a linear system via Carleman inequalities was studied by G. Lebeau and L. Robbiano in [8] and by A. V. Fursikov and O. Yu. Imanuvilov in [6]. In [1] a non-linear convective heat equation with Dirichlet boundary condition was studied, and a controllability result was proved by means of Kakutatani's fixed point theorem.

In this paper we examine an insensitizing control problem with constraints on the control for the semilinear heat equation with a more general cost function. We reduce our problem to a null controllability of a non-linear cascade system with constraints on the control. This generalizes the work of [10] where the cost function insensitized is the square of the norm of the state function on the observation set.

The rest of this paper is organized as follows. We state the problem and give the main result in Section 2. In Section 3 we study the existence of the approximate insensitizing control for the linear system and finally we prove the main result in the last section.

2 Statement of the problem and the main result

Let Ω be an open and bounded set of class C^2 of $\mathbb{R}^N$, $N \geq 1$. Let f be a globally Lipschitz function of class C^1 defined on $\mathbb{R}$. Let $T > 0$ and let ω and Θ be two non-empty open subsets of Ω. We set $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \partial\Omega$, where $\partial\Omega$ is the boundary of Ω. Moreover, let χ_ω and χ_Θ be the characteristic functions of ω and Θ, respectively.

In this paper we consider the following system:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + f\left(y\right) = \xi + h\chi_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y\left(0,.\right) = y^0 + \tau \hat{y}^0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where ξ and y^0 are given in $L^2\left(Q\right)$ and $L^2\left(\Omega\right)$, respectively. The function h is a control term which belongs to the orthogonal of K_ω, where K_ω is the subspace of $L^2\left(Q\right)$ spanned by $\chi_\omega e_1, \ldots, \chi_\omega e_M$. Here, by $e_1, e_2, \ldots, e_M$, we have denoted functions in $L^2\left(Q\right)$ which admit a trace at T in $L^2\left(\Omega\right)$.

The data of the state equation (2.1) are incomplete in the following sense: τ is unknown, but very small and $\left\|\hat{y}^0\right\|_{L^2(\Omega)} = 1$.

Assume that $f \in C^1(\mathbb{R})$ is globally Lipschitz continuous. Then we have:

Proposition 1 ([4]) *Under the assumptions on f, τ and $\hat{y}^0$, the problem* (2.1) *admits a unique solution in $\mathcal{C}\left([0,T], L^2(\Omega)\right) \cap L^2\left(0,T; H^1(\Omega)\right)$. Moreover, there exists a neighbourhood I of* 0 *such that the function $\tau \mapsto y(\tau)$ is in $\mathcal{C}^1\left(I; L^2(0,T; L^2(\Omega))\right)$.*

The problem of insensitizing controls can be roughly stated as follows. Let Φ be a differentiable functional defined on the set of solutions of (2.1). We say that the control h insensitizes $\Phi(y)$ if

$$\frac{\partial \Phi}{\partial \tau}(y,\tau)\,|_{\tau=0} = 0. \tag{2.2}$$

When (2.2) holds, the functional Φ is locally insensitive to the perturbation $\tau\hat{y}^0$.

In this paper we consider a more general functional, namely, a functional of the form:

$$\Phi(y) = \int_0^T \int_\Theta F(y)\,\mathrm{d}x\mathrm{d}t. \tag{2.3}$$

To the best of our knowledge, only two types of such functionals have been studied in the literature. In [7] the author studied the problem of insensitizing control with $F(y) = \frac{1}{2}|\nabla y|^2$, and in [2, 11] and [10] the case $F(y) = \frac{1}{2}y^2$ was examined. Here, we study the general case with the following assumptions on F: $F \in C^2(\mathbb{R})$ and F' is globally Lipschitz continuous.

The insensitivity condition is then given by

$$\frac{\partial \Phi}{\partial \tau}(y,\tau)\,|_{\tau=0} = \int_0^T \int_\Theta F'(y)\,y'\,\mathrm{d}x\mathrm{d}t = 0,$$

where $y' = \lim_{\tau\to 0} \dfrac{y(\tau)-y(0)}{\tau}$ is the solution of the system:

$$\begin{cases} \dfrac{\partial y'}{\partial t} - \Delta y' + f'(y)\,y' = 0 & \text{in } Q, \\ y' = 0 & \text{on } \Sigma, \\ y'(0,.) = \hat{y}^0 & \text{in } \Omega. \end{cases} \tag{2.4}$$

Proposition 2 *Let us consider the following cascade system of heat equations*:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + f(y) = \xi + h\chi_\omega & \textit{in } Q, \\ y = 0 & \textit{on } \Sigma, \\ y(0,.) = 0 & \textit{in } \Omega, \end{cases} \tag{2.5}$$

$$\begin{cases} -\dfrac{\partial q}{\partial t} - \Delta q + f'(y)\,q = F'(y)\,\chi_\Theta & \textit{in } Q, \\ q = 0 & \textit{on } \Sigma, \\ q(T,.) = 0 & \textit{in } \Omega. \end{cases} \tag{2.6}$$

Then the insensitivity condition (2.2) *is equivalent to $q(0,.) = 0$ a.e. on Ω.*

Proof. Multiplying the first equation of (2.6) by y' and integrating by parts, we obtain:

$$\int_0^T \int_\Theta F'(y)\, y' \,\mathrm{d}x\mathrm{d}t = \int_\Omega \hat{y}^0 q(0)\, \mathrm{d}x \qquad \text{for all } \hat{y}^0 \in L^2(\Omega) \text{ with } \left\|\hat{y}^0\right\|_{L^2(\Omega)} = 1.$$

Then $\frac{\partial \Phi}{\partial \tau}(y,\tau)\,|_{\tau=0} = 0$ is equivalent to

$$\int_\Omega \hat{y}^0 q(0)\, \mathrm{d}x = 0 \qquad \text{for all } \hat{y}^0 \in L^2(\Omega),\ \left\|\hat{y}^0\right\|_{L^2(\Omega)} = 1. \tag{2.7}$$

Relation (2.7) is also equivalent to $q(0,.) = 0$ a.e. on Ω. □

We now state the main result of this paper. For this purpose, we set $K_\omega = \operatorname{span}(e_1\chi_\omega, \ldots, e_M\chi_\omega)$ and we denote by Π the orthogonal projection of $L^2((0,T)\times\omega)$ on K_ω. As mentioned above, we assume that:

- $F \in C^2(\mathbb{R})$, F' is a globally Lipschitz function such that $F'(x) \neq 0$ for all $x \neq 0$ and $F''(0) \neq 0$;
- f is a globally Lipschitz function defined on $\mathbb{R}$;
- the family $\{e_1(T,.)\chi_\omega, \ldots, e_M(T,.)\chi_\omega\}$ is linearly independent.

Theorem 1 *Let $\xi \in L^2(Q)$ and suppose that $\Theta \cap \omega \neq \emptyset$. Under the assumptions on the functions F and f and the family $\{e_1(T,.)\chi_\omega, \ldots, e_M(T,.)\chi_\omega\}$, there exists a control h, in the orthogonal set of K_ω, that insensitizes the functional given by* (2.3).

Since the insensitizing control problem is equivalent to a null controllability problem, first we prove a null controllability result for the cascade linear system and by a fixed point argument we conclude the proof of Theorem 1.

In the sequel, in order to simplify the proof of the main result, we assume without loss of the generality that $f(0) = 0$ and $F'(0) = 0$.

One of the main steps of the proof of Theorem 1 is to show that there exists $C > 0$ such that the following observability inequality holds for any solution z of (3.2) given below:

$$\int_0^T \int_\Omega z^2 \,\mathrm{d}x\mathrm{d}t \leq C \int_0^T \int_\omega |z - \Pi z|^2 \,\mathrm{d}x\mathrm{d}t.$$

3 An approximate insensitizing control result for a linear system

Keeping in mind Proposition 2, the main result of Theorem 1 is then reduced to the following proposition which will be proved later.

Proposition 3 *Assume that the above hypotheses on F, f and the family $\{e_1(T,.)\chi_\omega, \ldots, e_M(T,.)\chi_\omega\}$ are satisfied. Then there exists $h \in K_\omega^\perp$ such that (y,q),* i.e. *the corresponding solution to* (2.5)–(2.6)*, satisfies $q(0,.) = 0$ a.e. in Ω.*

Let H, a and b be three functions in $L^{\infty}(Q)$. We consider the following systems:

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w + aw = 0 & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(0,.) = w^0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

$$\begin{cases} -\dfrac{\partial z}{\partial t} - \Delta z + bz = Hw\chi_{\Theta} & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{3.2}$$

Proposition 4 *We suppose that there exists a ball $B_r \subset \Theta \cap \omega$ such that $|H| > 0$ a.e. in $(0,T) \times B_r$. For any solution (w,z) of the cascade system* (3.1)–(3.2)*, if $z\chi_{\omega} \in K_{\omega}$ and the family $\{e_1(T,.)\chi_{\omega}, \ldots, e_M(T,.)\chi_{\omega}\}$ is linearly independent, then $(w,z) = (0,0)$.*

Proof. If z is a solution of (3.2) and if $z\chi_{\omega} \in K_{\omega}$, then there exists $(\alpha_1, \ldots, \alpha_M) \in \mathbb{R}^M$ such that

$$z\chi_{\omega} = \sum_{k=0}^{M} \alpha_k e_k \chi_{\omega}.$$

Then

$$z\chi_{\omega}(T,.)\chi_{\Omega} = \sum_{k=0}^{M} \alpha_k e_k(T,.)\chi_{\omega} = 0.$$

As the family $\{e_1(T,.)\chi_{\omega}, \ldots, e_M(T,.)\chi_{\omega}\}$ is linearly independent, we infer that $\alpha_k = 0$ for $1 \leq k \leq M$. Consequently, $z\chi_{\omega} = 0$ in $(0,T) \times \omega$.

Since $-\frac{\partial z}{\partial t} - \Delta z + bz = Hw\chi_{\Theta}$ on Q, we see that $w = 0$ on $(0,T) \times B_r$, and this in turn gives via the property of unique continuation that $w = 0$ in Q. Therefore, $z = 0$ in Q. $\square$

Proposition 5 *Assume that there exists a ball $B_r \subset \Theta \cap \omega$ such that $|H| > 0$ a.e. in $(0,T) \times B_r$. If the family $\{e_1(T,.)\chi_{\omega}, \ldots, e_M(T,.)\chi_{\omega}\}$ is linearly independent, then there exists a positive constant $C > 0$ such that for any solution (w,z) of* (3.1)–(3.2) *with initial datum $w^0 \in L^2(\Omega)$ the following inequality holds true*:

$$\int_0^T \int_{\Omega} z^2 \,\mathrm{d}x\mathrm{d}t \leq C \int_0^T \int_{\omega} |z - \Pi z|^2 \,\mathrm{d}x\mathrm{d}t.$$

In the sequel, we shall use the following notation:

- $H^{r,s}(Q) = L^2((0,T);H^r(\Omega)) \cap H^s\left((0,T);L^2(\Omega)\right)$ endowed with the norm $\|.\|_{r,s} = \left(\|.\|^2_{L^2((0,T);H^r(\Omega))} + \|.\|^2_{H^s((0,T);L^2(\Omega))}\right)^{\frac{1}{2}}$;
- $W(0,T) = \left\{y \in L^2((0,T);H^1(\Omega)) : \frac{\partial y}{\partial t} \in L^2((0,T);H^{-1}(\Omega))\right\}$ endowed with the norm $\|.\|_W = \left(\|.\|^2_{L^2((0,T);H^1(\Omega))} + \|.\|^2_{L^2((0,T);H^{-1}(\Omega))}\right)^{\frac{1}{2}}$.

Proof. We argue by contradiction. Let (w_n, z_n) be a sequence of solutions to the system (3.1)–(3.2) with w_0^n instead of w_0 in (3.1) such that

$$\begin{cases} \|w_0^n\|_{L^2(\Omega)} = 1, \\ \dfrac{1}{n} \displaystyle\int_0^T \int_\Omega z_n^2 \,\mathrm{d}x\mathrm{d}t \geq \int_0^T \int_\omega |z_n - \Pi z_n|^2 \,\mathrm{d}x\mathrm{d}t. \end{cases} \tag{3.3}$$

The proof is now broken into three steps.

Step 1. From the classical estimates on the energy of the heat equation we have:

$$\|w_n\|_{2,1} \leq C \|w_0^n\|_{L^2(\Omega)} = C.$$

So, the sequence (w_n) is bounded in $H^{2,1}(Q)$, and therefore it converges weakly (up to extracting a subsequence) to an element w in $H^{2,1}(Q)$. Since the canonical injection of $H^{2,1}(Q)$ in $W(0,T)$ is compact, we infer that $w_n \to w$ strongly in $W(0,T)$. Similarly, z_n converges strongly to z in $W(0,T)$.

Step 2. Using (3.3) we obtain

$$(z_n - \Pi z_n)\chi_\omega \to 0 \text{ in } L^2((0,T) \times \Omega). \tag{3.4}$$

Noticing that

$$\Pi z_n = (-z_n + \Pi z_n) + z_n,$$

we obtain

$$|\Pi z_n|^2 \leq 2\left(|z_n - \Pi z_n|^2 + |z_n|^2\right).$$

Then

$$\int_0^T \int_\omega |\Pi z_n|^2 \,\mathrm{d}x\mathrm{d}t \leq 2\left(\int_0^T \int_\omega z_n^2 \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_\omega |z_n - \Pi z_n|^2 \,\mathrm{d}x\mathrm{d}t\right),$$

and since $(z_n - \Pi z_n)\chi_\omega$ is bounded in $L^2\left((0,T); L^2(\Omega)\right)$, we see that Πz_n is also bounded in $L^2((0,T) \times \Omega)$. So, we can extract a subsequence which converges weakly to an element $g\chi_\omega \in K_\omega$. Since K_ω is a finite dimensional subspace, we have:

$$\Pi z_n \chi_\omega \to g\chi_\omega \text{ in } L^2((0,T) \times \Omega) \text{ strongly.} \tag{3.5}$$

From (3.4) and (3.5) we obtain

$$z_n \chi_\omega \to g\chi_\omega \text{ in } L^2((0,T) \times \Omega) \text{ strongly.}$$

Step 3. Now, as (w_n, z_n) converges strongly to (w, z), classical arguments give that (w, z) is a solution of (3.1)–(3.2). But as $z_n\chi_\omega \to g\chi_\omega = z\chi_\omega \in K_\omega$, Proposition 4 yields that $(w, z) = (0, 0)$ in $(0,T) \times \Omega$.

Since $W(0,T)$ is continuously embedded in $C\left(0,T; L^2(\Omega)\right)$ and using the fact that $H^{2,1}(Q)$ is compactly embedded in $W(0,T)$, it follows that the sequence $(w_n(0,.))_n = (w_0^n)_n$ is compact. Therefore, one can extract a subsequence, still denoted by $(w_0^n)_n$, which converges strongly to $w(0) = 0$. This contradicts the fact that $\|w_0^n\|_{L^2(\Omega)} = 1$ for all $n \in \mathbb{N}^*$. □

Let $\xi \in L^2(Q)$, $\epsilon > 0$ and $H \in L^\infty(Q)$ such that $|H| > 0$ a.e. in $(0,T) \times B_r$ be given. For $w^0 \in L^2(\Omega)$ we define the following functional:

$$J_\epsilon\left(w^0, a, b, H\right) = \frac{1}{2}\int_0^T\int_\omega |z - \Pi z|^2 \, dxdt + \int_0^T\int_\Omega \xi z \, dxdt + \epsilon \left\|w^0\right\|_{L^2(\Omega)}.$$

Proposition 6 *The functional $J_\epsilon(., a, b, H)$ is continuous, strictly convex and*

$$\liminf_{\|w^0\|_{L^2(\Omega)} \to \infty} \frac{J_\epsilon\left(w^0, a, b, H\right)}{\|w^0\|_{L^2(\Omega)}} \geq \epsilon.$$

Therefore, J_ϵ reaches its minimum at a point $\hat{w}^0 \in L^2(\Omega)$.

If $\hat{w}^0 \neq 0$, J_ϵ satisfies the optimality condition:

$$\int_0^T\int_\omega (\hat{z}_\epsilon - \Pi\hat{z}_\epsilon)\, z \, dxdt + \int_0^T\int_\Omega \xi z \, dxdt + \frac{\epsilon}{\|w^0\|_{L^2(\Omega)}}\int_\Omega \hat{w}^0 w^0 dx = 0$$

for every $w^0 \in L^2(\Omega)$ and (w, z) the corresponding solution to (3.1)–(3.2). *Furthermore, there exists $C > 0$ independent of ϵ such that*

$$\|\hat{z}_\epsilon - \Pi\hat{z}_\epsilon\|_{L^2((0,T)\times\omega)} \leq C\,\|\xi\|_{L^2(Q)}. \tag{3.6}$$

Proof. The strict convexity and continuity are obvious. Let us prove the coercitivity.

Let $w_n^0 \in L^2(\Omega)$ be such that $\left\|w_n^0\right\|_{L^2(\Omega)} \to +\infty$ and let (w_n, z_n) be the solution to the following systems

$$\begin{cases} \dfrac{\partial w_n}{\partial t} - \Delta w_n + a(\eta_n)\, w_n = 0 & \text{in } Q, \\ w_n = 0 & \text{on } \Sigma, \\ w_n(0,.) = w_n^0 & \text{in } \Omega, \end{cases} \tag{3.7}$$

$$\begin{cases} -\dfrac{\partial z_n}{\partial t} - \Delta z_n + b(\eta_n)\, z_n = H(\eta_n)\, w_n \chi_\Theta & \text{in } Q, \\ z_n = 0 & \text{on } \Sigma, \\ z_n(T,.) = 0 & \text{in } \Omega, \end{cases} \tag{3.8}$$

where $(b(\eta_n))_n$, $(a(\eta_n))_n$ and $(H(\eta_n))_n$ are sequences in $L^\infty(Q)$ with $(\eta_n)_n$ a sequence in $L^2(Q)$.

Let $\tilde{w}_n^0 = \frac{w_n^0}{\|w_n^0\|_{L^2(\Omega)}}$ and let $(\tilde{w}_n, \tilde{z}_n)$ be the solution of the systems

$$\begin{cases} \dfrac{\partial \tilde{w}_n}{\partial t} - \Delta \tilde{w}_n + a(\eta_n)\, \tilde{w}_n = 0 & \text{in } Q, \\ \tilde{w}_n = 0 & \text{on } \Sigma, \\ \tilde{w}_n(0,.) = \tilde{w}_n^0 & \text{in } \Omega, \end{cases} \tag{3.9}$$

$$\begin{cases} -\dfrac{\partial \tilde{z}_n}{\partial t} - \Delta \tilde{z}_n + b(\eta_n)\, \tilde{z}_n = H(\eta_n)\, \tilde{w}_n \chi_\Theta & \text{in } Q, \\ \tilde{z}_n = 0 & \text{on } \Sigma, \\ \tilde{z}_n(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{3.10}$$

We have

$$\frac{J_\epsilon\left(w_n^0, H\left(\eta_n\right)\right)}{\|w_n^0\|_{L^2(\Omega)}} = \frac{\left\|w_n^0\right\|_{L^2(\Omega)}}{2}\int_0^T\int_\Omega \left|\tilde{z}_n - \Pi\tilde{z}_n\right|^2\, \mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \xi\tilde{z}_n\,\mathrm{d}x\mathrm{d}t + \epsilon.$$

From the conditions on $b\left(\eta_n\right)$, $a\left(\eta_n\right)$ and $H\left(\eta_n\right)$ we have $\left\|a\left(\eta_n\right)\right\|_\infty \leq K$ a.e., $\left\|a\left(\eta_n\right)\right\|_\infty \leq R$ a.e. and $\left\|H\left(\eta_n\right)\right\|_\infty \leq C$ a.e., where C, R and K do not depend on η_n.

We have $\left\|\tilde{w}_n^0\right\|_{L^2(\Omega)} = 1$. So, we can extract a subsequence which converges weakly to $\tilde{w}^0$, $\tilde{w}_n^0 \rightharpoonup \tilde{w}^0$ in $L^2\left(\Omega\right)$ and $\left(\tilde{w}_n, \tilde{z}_n\right) \to \left(w, z\right)$ strongly in $L^2\left(Q\right)$. Letting

$$\left\|w_n^0\right\|_{L^2(\Omega)} \to +\infty,$$

two cases are possible.

Case 1. If $\liminf_{n\to+\infty} \int_0^T\int_\omega \left|\tilde{z}_n - \Pi\tilde{z}_n\right|^2\, \mathrm{d}x\mathrm{d}t > 0$, then we have

$$\liminf_{n\to+\infty}\frac{J_\epsilon\left(w_n^0, H\left(\eta_n\right)\right)}{\|w_n^0\|_{L^2(\Omega)}} = +\infty.$$

Case 2. If $\liminf_{n\to+\infty} \int_0^T\int_\omega \left|\tilde{z}_n - \Pi\tilde{z}_n\right|^2\, \mathrm{d}x\mathrm{d}t = 0$, then

$$\int_0^T\int_\omega \left|\tilde{z} - \Pi\tilde{z}\right|^2\, \mathrm{d}x\mathrm{d}t = 0.$$

Consequently,

$$\tilde{z}\chi_\omega = \Pi\tilde{z}\chi_\omega \in K_\omega.$$

Thus, $\left(\tilde{w}, \tilde{z}\right) = \left(0, 0\right)$. Arguing as above, one gets that

$$\left\|\tilde{w}_n^0\right\|_{L^2(\Omega)} \to 0.$$

This contradicts the fact that

$$\left\|\tilde{w}_n^0\right\|_{L^2(\Omega)} = 1. \qquad \square$$

4 Proof of the main result

Now, we are in position to prove Theorem 1.

Let us define

$$g\left(\eta\right) = \begin{cases} \dfrac{f\left(\eta\right)}{\eta}, & \text{if } \eta \neq 0,\\ f'\left(0\right), & \text{if } \eta = 0, \end{cases} \qquad H\left(\eta\right) = \begin{cases} \dfrac{F'\left(\eta\right)}{\eta}, & \text{if } \eta \neq 0,\\ F''\left(0\right), & \text{if } \eta = 0. \end{cases}$$

The assumptions on F and f guarantee that H and g are both continuous and bounded. Let $\eta \in L^2\left(Q\right)$, $\hat{w}^0$ denote the minimizer of $J_\epsilon\left(w^0, f'(\eta), g(\eta), H\left(\eta\right)\right)$ and $L\left(\hat{z}_\epsilon\right) = \hat{z}_\epsilon - \Pi\hat{z}_\epsilon$, where

$\hat{z}_\epsilon$ is the corresponding solution of (3.2) with $b = f'(\eta)$. When $\hat{w}^0 \neq 0$ the optimality condition allows to prove that (y_ϵ, q_ϵ) is a solution of

$$\begin{cases} \dfrac{\partial y_\epsilon}{\partial t} - \Delta y_\epsilon + g(\eta)\, y_\epsilon = \xi + L(\hat{z}_\epsilon)\,\chi_\omega & \text{in } Q, \\ y_\epsilon = 0 & \text{on } \Sigma, \\ y_\epsilon(0,.) = 0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

$$\begin{cases} -\dfrac{\partial q_\epsilon}{\partial t} - \Delta q_\epsilon + f'(\eta)\, q_\epsilon = H(\eta)\, y_\epsilon \chi_\Theta & \text{in } Q, \\ q_\epsilon = 0 & \text{on } \Sigma, \\ q_\epsilon(T,.) = 0 & \text{in } \Omega, \end{cases} \tag{4.2}$$

satisfying

$$\|q_\epsilon(0,.)\|_{L^2(\Omega)} \leq \epsilon. \tag{4.3}$$

Indeed, let us consider the following systems

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w + g(\eta)\, w = 0 & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(0,.) = w^0 & \text{in } \Omega, \end{cases} \tag{4.4}$$

$$\begin{cases} -\dfrac{\partial z}{\partial t} - \Delta z + f'(\eta)\, z = H(\eta)\, w \chi_\Theta & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{4.5}$$

Multiplying the first equation of (4.1) by z and the first equation of (4.2) by w and integrating over Q, one gets:

$$\int_0^T \int_\Theta H(\eta)\, w y_\epsilon \,\mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \xi z \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_\omega L(\hat{z}_\epsilon)\, z \,\mathrm{d}x\mathrm{d}t \tag{4.6}$$

and

$$\int_0^T \int_\Theta H(\eta)\, w y_\epsilon \,\mathrm{d}x\mathrm{d}t = \int_\Omega q_\epsilon(0,x)\, w^0 \,\mathrm{d}x. \tag{4.7}$$

Combining (4.6) and (4.7) we obtain

$$\int_\Omega q_\epsilon(0,x)\, w^0 \,\mathrm{d}x = \int_0^T \int_\Omega \xi z \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_\omega \hat{z}_\epsilon z \,\mathrm{d}x\mathrm{d}t. \tag{4.8}$$

The optimality condition and equality (4.8) give:

$$\int_\Omega q_\epsilon(0,x)\, w^0 \,\mathrm{d}x = -\frac{\epsilon}{\|w^0\|_{L^2(\Omega)}} \int_\Omega \hat{w}^0 w^0 \,\mathrm{d}x \qquad \text{for all } w^0 \in L^2(\Omega).$$

Therefore,

$$\left| \int_\Omega q_\epsilon (0,x)\, w^0 \,\mathrm{d}x \right| \leq \epsilon \left\| w^0 \right\|_{L^2(\Omega)},$$

$$\| q_\epsilon (0,x) \|_{L^2(\Omega)} = \sup_{w^0 \in L^2(\Omega)} \frac{\left| \int_\Omega q_\epsilon (0,x)\, w^0 \,\mathrm{d}x \right|}{\left\| w^0 \right\|_{L^2(\Omega)}} \leq \frac{\epsilon \left\| w^0 \right\|_{L^2(\Omega)}}{\left\| w^0 \right\|_{L^2(\Omega)}} = \epsilon.$$

Observe that if $\hat{w}^0 = 0$, $\lim_{t \to 0} \frac{J_\epsilon\left(tw^0, H(\eta)\right)}{t} \geq 0$ for every $w^0 \in L^2(\Omega)$. This means that

$$\epsilon \left\| w^0 \right\|_{L^2(\Omega)} \geq - \int_\Omega q_\epsilon (0,x)\, w^0 \,\mathrm{d}x$$

for every $w^0 \in L^2(\Omega)$. In particular, for $w^0 = q_\epsilon (0,x)$ we obtain (4.3) for $h = 0$.

Let us define the function:

$$\begin{gathered} \Lambda_\epsilon \colon L^2(Q) \to L^2(Q) \to L^2(Q), \\ \eta \mapsto \hat{z}_\epsilon (\eta) \mapsto y_\epsilon (\eta). \end{gathered}$$

Lemma 1 *The function Λ_ϵ is bounded, continuous and compact in $L^2(Q)$.*

Proof. The classical estimate of the heat equation, gives

$$\| y_\epsilon \|_{L^2(Q)} \leq \| \xi + L(\hat{z}_\epsilon)\, \chi_\omega \|_{L^2(Q)}.$$

From the estimate (3.6) we obtain

$$\| y_\epsilon \|_{L^2(Q)} \leq \| \xi \|_{L^2(e_M)}.$$

So, Λ_ϵ is bounded.

Let us prove the continuity of Λ_ϵ. Let $\eta_n \to \eta$ strongly in $L^2(Q)$. Let us show that the sequence $y_\epsilon(\eta_n)$ converges to $y_\epsilon(\eta)$ strongly. For all $\eta_n \in L^2(Q)$ we have $\| g(\eta_n) \|_\infty \leq K$, $\| f'(\eta_n) \|_\infty \leq R$ and $\| H(\eta_n) \|_\infty \leq C$, where C, R and K do not depend on n. As well, J_ϵ is uniformly coercive in η_n while $\hat{w}^0_\epsilon(\eta_n)$ remains bounded in $L^2(\Omega)$. One has that $L(\hat{z}_\epsilon(\eta_n))$, $y_\epsilon(\eta_n)$ and $q_\epsilon(\eta_n)$ are bounded in $L^2(Q)$. Indeed, $y_\epsilon(\eta_n)$ and $q_\epsilon(\eta_n)$ are bounded in $H^{2,1}(Q)$. So we can extract a subsequence $(y_\epsilon(\eta_n), q_\epsilon(\eta_n))$ which converges weakly to $(y_\epsilon(\eta), q_\epsilon(\eta))$ in $H^{2,1}(Q)$. Since $H^{2,1}(Q)$ is compact in $L^2(Q)$, we see that $(y_\epsilon(\eta_n), q_\epsilon(\eta_n)) \to (y_\epsilon(\eta), q_\epsilon(\eta))$ strongly and $(y_\epsilon(\eta), q_\epsilon(\eta))$ is the solution of (4.1)–(4.2).

Let us prove the compactness of Λ_ϵ. The functions $g(\eta)$, $H(\eta)$ and $f'(\eta)$ are uniformly bounded. So, $\left\| \hat{w}^0_\epsilon(\eta) \right\|_{L^2(\Omega)}$ is uniformly bounded with respect to η. Then, $w_\epsilon(\eta)$ and $L(\hat{z}_\epsilon(\eta))$ are also bounded in $L^2(Q)$. Therefore, $y_\epsilon(\eta)$ and $\frac{\partial y_\epsilon(\eta)}{\partial t}$ are bounded in $L^2\left((0,T); H^1(\Omega)\right)$ and $L^2\left((0,T); H^{-1}(\Omega)\right)$, respectively. Therefore, $(y_\epsilon(\eta))$ is compact in $L^2\left((0,T); L^2(\Omega)\right)$. $\square$

The Schauder fixed point theorem implies that the operator Λ_ϵ admits a fixed point y_ϵ. Then $g(y_\epsilon)\, y_\epsilon = f(y_\epsilon)$, $H(y_\epsilon)\, y_\epsilon = F'(y_\epsilon)$. The cascade system (4.1)–(4.2) becomes

$$\begin{cases} \dfrac{\partial y_\epsilon}{\partial t} - \Delta y_\epsilon + f(y_\epsilon) = \xi + L(\hat{z}_\epsilon)\, \chi_\omega & \text{in } Q, \\ y_\epsilon = 0 & \text{on } \Sigma, \\ y_\epsilon(0,.) = 0 & \text{in } \Omega, \end{cases} \tag{4.9}$$

$$\begin{cases} -\dfrac{\partial q_\epsilon}{\partial t} - \Delta q_\epsilon + f'(y_\epsilon)\, q_\epsilon = F'(y_\epsilon)\, \chi_\Theta & \text{in } Q, \\ q_\epsilon = 0 & \text{on } \Sigma, \\ q_\epsilon(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{4.10}$$

From this, one has:

- (y_ϵ, q_ϵ) solves (4.1)–(4.2) with $h = L(\hat{z}_\epsilon)$;
- $\|\hat{z}_\epsilon - \Pi \hat{z}_\epsilon\|_{L^2((0,T)\times\omega)} \le C \|\xi\|_{L^2(Q)}$, where C is a constant independent of ϵ;
- $\|q_\epsilon(0)\|_{L^2(\Omega)} \le \epsilon$.

From the boundedness of $L(\hat{z}_\epsilon)$, one deduces that there exists a subsequence, still denoted by $(y_\epsilon, q_\epsilon, L(\hat{z}_\epsilon))$, such that $y_\epsilon \to y$ in $L^2(Q)$, $q_\epsilon \to q$ in $L^2(Q)$ and $L(\hat{z}_\epsilon) \rightharpoonup h$ in $L^2(Q)$ as ϵ goes to zero. Moreover, (y, q) solves (2.5)–(2.6). In addition, as $\|q_\epsilon(0)\|_{L^2(\Omega)} \le \epsilon$ as $\epsilon \to 0$, we get that $q(0,.) = 0$ a.e. in Ω. This ends the proof of Theorem 1.

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