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RENORMALIZED SOLUTIONS OF STEFAN DEGENERATE ELLIPTIC NONLINEAR PROBLEMS WITH VARIABLE EXPONENT

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Abstract. In this paper, we study a general class of nonlinear elliptic degenerate problems associated with the differential inclusion $\beta(u) - \operatorname{div}(a(x, Du) + F(u)) \ni f$ in Ω, where $f \in L^\infty(\Omega)$. Using truncation techniques and the generalized monotonicity method in the framework of weighted variable exponent Sobolev spaces, we prove the existence of renormalized solutions for general L^∞-data.

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1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$ with Lipschitz boundary if $N \geq 2$. The variable exponent $p\colon \overline{\Omega} \to (1, \infty)$ is a continuous function, and ω is a weight function on Ω, *i.e.*, ω is measurable and a.e. positive on Ω. Let $W_0^{1,p(\cdot)}(\Omega, \omega)$ be the weighted variable exponent Sobolev space associated with the weight ω. Our aim is to show the existence of renormalized solutions to

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the following nonlinear elliptic inclusion

$$(E,f)\begin{cases}\beta(u)-\operatorname{div}(a(x,Du)+F(u))\ni f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

with the right-hand side $f\in L^\infty(\Omega)$. Furthermore, F and β are two functions satisfying the following assumptions:

($\mathbf{A}_0$) $F\colon\mathbb{R}\to\mathbb{R}^N$ is locally Lipschitz continuous and $\beta\colon\mathbb{R}\to 2^{\mathbb{R}}$ is a set-valued, maximal monotone mapping such that $0\in\beta(0)$. Moreover, we assume that

$$\beta^0(l)\in L^1(\Omega) \tag{1.1}$$

for each $l\in\mathbb{R}$, where β^0 denotes the minimal selection of the graph of β. Namely, $\beta_0(l)$ is the minimal in the norm element of $\beta(l)$, that is,

$$\beta_0(l)=\inf\{|r|: r\in\mathbb{R} \text{ and } r\in\beta(l)\}.$$

Moreover, $a\colon\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function satisfying the following assumptions:

($\mathbf{A}_1$) there exists a positive constant λ such that $a(x,\xi)\cdot\xi\geq\lambda\omega(x)|\xi|^{p(x)}$ for all $\xi\in\mathbb{R}^N$ and almost every $x\in\Omega$;

($\mathbf{A}_2$) $|a_i(x,\xi)|\leq\alpha\omega^{1/p(x)}(x)[k(x)+\omega^{1/p'(x)}(x)|\xi|^{p(x)-1}]$ for almost every $x\in\Omega$, all $i=1,\dots,N$, every $\xi\in\mathbb{R}^N$, where $k(x)$ is a non-negative function in $L^{p'(\cdot)}(\Omega)$, $p'(x):=p(x)/(p(x)-1)$, and $\alpha>0$;

($\mathbf{A}_3$) $(a(x,\xi)-a(x,\eta))\cdot(\xi-\eta)\geq 0$ for almost every $x\in\Omega$ and every $\xi,\eta\in\mathbb{R}^N$.

It is well-known that for L^1-data a weak solution may not exist in general or may not be unique. In order to obtain the well-posedness of this type of problems, the notion of a renormalized solution was introduced by R. J. DiPerna and P.-L. Lions [7] in their study of the Boltzmann equation. This notion was then adapted to an elliptic version of (E,f) by L. Boccardo et al. [5] when the right-hand side is in $W^{-1,p'}(\Omega)$, by J.-M. Rakotoson [17] when the right-hand side is in $L^1(\Omega)$, and finally by G. Dal Maso, F. Murat, L. Orsina and A. Prignet [10] in the case when the right-hand side is general measure data. The equivalent notion of an entropy solution was introduced by Bénilan et al. in [4]. For results on the existence of renormalized solutions of elliptic problems of type (E,f) with $a(\cdot,\cdot)$ satisfying a variable growth condition, we refer to [2, 3, 12, 19] and [1]. One of the motivations for studying (E,f) comes from applications to electro-rheological fluids (see [18] for more details) as an important class of non-Newtonian fluids.

For the convenience of the readers, we recall some definitions and basic properties of the weighted variable exponent Lebesgue spaces $L^{p(x)}(\Omega,\omega)$ and the weighted variable exponent Sobolev spaces $W^{1,p(x)}(\Omega,\omega)$. Set

$$C_+(\overline{\Omega})=\big\{p\in C(\overline{\Omega}): \min_{x\in\overline{\Omega}}p(x)>1\big\}.$$

For any $p\in C_+(\overline{\Omega})$, we define

$$p^+=\max_{x\in\overline{\Omega}}p(x),\qquad p^-=\min_{x\in\overline{\Omega}}p(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce the weighted variable exponent Lebesgue space $L^{p(x)}(\Omega, \omega)$ that consists of all measurable real-valued functions u such that $\int_\Omega |u(x)|^{p(x)}\omega(x)\,\mathrm{d}x < \infty$, that is,

$$L^{p(x)}(\Omega, \omega) = \left\{u \colon \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\omega(x)\,\mathrm{d}x < \infty\right\}.$$

Then $L^{p(x)}(\Omega, \omega)$ endowed with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega,\omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \omega(x)\mathrm{d}x \leq 1\right\}$$

becomes a normed space. When $\omega(x) \equiv 1$ we have $L^{p(x)}(\Omega, \omega) \equiv L^{p(x)}(\Omega)$ and we use the notation $|u|_{L^{p(x)}(\Omega)}$ instead of $|u|_{L^{p(x)}(\Omega,\omega)}$. The weighted variable exponent Sobolev space $W^{1,p(x)}(\Omega, \omega)$ is defined by

$$W^{1,p(x)}(\Omega, \omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega, \omega)\},$$

where the norm is given by

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega,\omega)}, \tag{1.2}$$

or, equivalently, by

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} + \omega(x)\left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leq 1\right\}$$

for all $u \in W^{1,p(x)}(\Omega, \omega)$.

It should be emphasised that smooth functions are not dense in $W^{1,p(x)}(\Omega)$ without additional assumptions on the exponent $p(x)$. This feature was observed by Zhikov [21] in connection with the Lavrentiev phenomenon. However, if the exponent $p(x)$ is log-Hölder continuous, *i.e.*, there is a constant C such that

$$|p(x) - p(y)| \leq \frac{C}{-\log|x - y|} \tag{1.3}$$

for every x, y with $|x - y| \leq \frac{1}{2}$, then smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values, $W_0^{1,p(x)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega)}$ (see [13]).

The space $W_0^{1,p(x)}(\Omega, \omega)$ is defined as the completion of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega, \omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega,\omega)}$.

Throughout the paper, we assume that $p \in C_+(\overline{\Omega})$ and ω is a measurable positive and a.e. finite function in Ω.

The plan of the paper is as follows. In Section 2, we give some preliminaries and the definition of weighted variable exponent Sobolev spaces. In Section 3, we introduce the notion of a renormalized solution for problem (E, f) and prove the main result on the existence of a renormalized solution to (E, f) for any L^∞-data f.

2 Preliminaries

In this section, we state some elementary properties of the (weighted) variable exponent Lebesgue–Sobolev spaces which will be used in the next sections. The basic properties of the variable exponent Lebesgue-Sobolev spaces, that is when $\omega(x) \equiv 1$, can be found in [11, 15].

Lemma 1 (Generalised Hölder inequality, see [11, 15])

(i) *For any functions $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\left| \int_\Omega uv \, \mathrm{d}x \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) ||u||_{p(\cdot)} ||v||_{p'(.)} \leq 2||u||_{p(\cdot)} ||v||_{p'(.)}.$$

(ii) *For all $p, q \in C_+(\bar{\Omega})$ such that $p(x) \leq q(x)$ a.e. in Ω, we have that $L^{q(\cdot)} \hookrightarrow L^{p(\cdot)}$ and the embedding is continuous.*

Lemma 2 (see [14]) *Denote $\rho(u) = \int_\Omega \omega(x)|u(x)|^{p(x)} \, \mathrm{d}x$ for all $u \in L^{p(x)}(\Omega, \omega)$. Then*

$$|u|_{L^{p(x)}(\Omega,\omega)} < 1 \ (= 1; > 1) \text{ if and only if } \rho(u) < 1 \ (= 1; > 1); \tag{2.1}$$

$$\text{if } |u|_{L^{p(x)}(\Omega,\omega)} > 1, \text{ then } |u|^{p^-}_{L^{p(x)}(\Omega,\omega)} \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}(\Omega,\omega)}; \tag{2.2}$$

$$\text{if } |u|_{L^{p(x)}(\Omega,\omega)} < 1, \text{ then } |u|^{p^+}_{L^{p(x)}(\Omega,\omega)} \leq \rho(u) \leq |u|^{p^-}_{L^{p(x)}(\Omega,\omega)}. \tag{2.3}$$

Remark 1 *If we set*

$$I(u) = \int_\Omega |u(x)|^{p(x)} + \omega(x)|\nabla u(x)|^{p(x)} \, \mathrm{d}x,$$

then following the same argument, we have

$$\begin{aligned} \min\big\{ \|u\|^{p^-}_{W^{1,p(x)}(\Omega,\omega)}, \|u\|^{p^+}_{W^{1,p(x)}(\Omega,\omega)} \big\} &\leq \\ &\leq I(u) \leq \max\big\{ \|u\|^{p^-}_{W^{1,p(x)}(\Omega,\omega)}, \|u\|^{p^+}_{W^{1,p(x)}(\Omega,\omega)} \big\}. \end{aligned} \tag{2.4}$$

Throughout the paper, we assume that ω is a measurable positive and a.e. finite function in Ω satisfying the conditions:

($\mathbf{H}_1$) $\omega \in L^1_{\text{loc}}(\Omega)$ and $\omega^{-\frac{1}{(p(x)-1)}} \in L^1_{\text{loc}}(\Omega)$;

($\mathbf{H}_2$) $\omega^{-s(x)} \in L^1(\Omega)$ with $s(x) \in \big(\frac{N}{p(x)}, \infty\big) \cap \big[\frac{1}{p(x)-1}, \infty\big)$.

The reasons why we assume ($\mathbf{H}_1$) and ($\mathbf{H}_2$) can be found in [14].

Remark 2 ([14])

(i) *If ω is a positive measurable and finite function, then $L^{p(x)}(\Omega, \omega)$ is a reflexive Banach space.*

(ii) *Moreover, if ($\mathbf{H}_1$) holds, then $W^{1,p(x)}(\Omega, \omega)$ is a reflexive Banach space.*

For $p, s \in C_+(\overline{\Omega})$, denote $p_s(x) = \frac{p(x)s(x)}{s(x)+1} < p(x)$, where $s(x)$ is given in ($\mathbf{H}_2$). Assume that we fix the variable exponent restrictions

$$\begin{cases} p_s^*(x) = \dfrac{p(x)s(x)N}{(s(x)+1)N - p(x)s(x)}, & \text{if } N > p_s(x), \\ p_s^*(x) \text{ arbitrary}, & \text{if } N \le p_s(x), \end{cases}$$

for almost all $x \in \Omega$. These definitions play a key role in our paper.

In the next sections, we shall frequently make use of the following (compact) imbedding theorem for the weighted variable exponent Lebesgue–Sobolev space.

Lemma 3 ([14]) *Let $p, s \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition* (1.3) *and let* ($\mathbf{H}_1$) *and* ($\mathbf{H}_2$) *be satisfied. If $r \in C_+(\overline{\Omega})$ and $1 < r(x) \le p_s^*(x)$, then we obtain the continuous embedding*

$$W^{1,p(x)}(\Omega, \omega) \hookrightarrow L^{r(x)}(\Omega).$$

Moreover, we have the compact embedding

$$W^{1,p(x)}(\Omega, \omega) \hookrightarrow L^{r(x)}(\Omega),$$

provided that $1 < r(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$.

From Lemma 3, immediately we have the Poincaré-type inequality.

Corollary 1 ([14]) *Let $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition* (1.3)*. If* ($\mathbf{H}_1$) *and* ($\mathbf{H}_2$) *hold, then the estimate*

$$\|u\|_{L^{p(x)}(\Omega)} \le C \|\nabla u\|_{L^{p(x)}(\Omega,\omega)}$$

holds for every $u \in C_0^\infty(\Omega)$ with a positive constant C independent of u.

Throughout this paper, let $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (1.3) and $X := W_0^{1,p(x)}(\Omega, \omega)$ be the weighted variable exponent Sobolev space that consists of all real-valued functions u from $W^{1,p(x)}(\Omega, \omega)$ which vanish on the boundary $\partial\Omega$, endowed with the norm

$$\|u\|_X = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} \omega(x)\,\mathrm{d}x \le 1\right\},$$

which is equivalent to the norm (1.2) due to Corollary 1.

The following proposition gives the characterization of the dual space $(W_0^{k,p(x)}(\Omega, \omega))^*$, which is analogous to [15, Theorem 3.16].

Proposition 1 *Let $p \in C_+(\overline{\Omega})$ and let α be a multi-index with $|\alpha| \le k$. Further, let ω be a given family $\{\omega_\alpha(x) : |\alpha| \le k\}$ of weight functions $\omega_\alpha(x)$, $x \in \Omega$, satisfying* ($\boldsymbol{H}_1$)*. Then for every $G \in \big(W_0^{k,p(x)}(\Omega, \omega)\big)^*$ there exists a unique system of functions $\{g_\alpha \in L^{p'(x)}\big(\Omega, \omega_\alpha^{1-p'(x)}\big) : |\alpha| \le k\}$ such that*

$$G(f) = \sum_{|\alpha| \le k} \int_\Omega D^\alpha f(x) g_\alpha(x)\,\mathrm{d}x \qquad \text{for all } f \in W_0^{1,p(x)}(\Omega, \omega).$$

We recall that the dual space of the weighted Sobolev space $W_0^{1,p(x)}(\Omega,\omega)$ is equivalent to $W^{-1,p'(x)}(\Omega,\omega^*)$, where $\omega^* = \omega^{1-p'(x)}$.

The following notations will be used throughout the paper: for $k \geq 0$, the truncation at heigth k is defined by

$$T_k(r) := \begin{cases} -k, & \text{if } r \leq -k, \\ r, & \text{if } |r| < k, \\ k, & \text{if } r \geq k, \end{cases}$$

and let $h_l\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$h_l(r) := \min\Big((l+1-|r|)^+, 1\Big) \qquad \text{for each } r \in \mathbb{R}.$$

Remark 3 *The Lipschitz character of F and the Stokes formula together with the boundary condition ($u = 0$ on $\partial\Omega$) of the problem give $\int_\Omega F(u) DT_k(u)\,\mathrm{d}x = 0$ (see [19]).*

3 Notion of a solution and existence results

Definition 1 *A renormalized solution to (E, f) is a pair of functions (u, b) satisfying the following conditions*:

(R1) $u\colon \Omega \to \mathbb{R}$ *is measurable,* $b \in L^1(\Omega)$, $u(x) \in D(\beta(x))$ *and* $b(x) \in \beta(u(x))$ *for a.e.* $x \in \Omega$;

(R2) *for each* $k > 0$, $T_k(u) \in W_0^{1,p(\cdot)}(\Omega,\omega)$ *and*

$$\int_\Omega b \cdot h(u)\varphi + \int_\Omega (a(x,Du) + F(u)) \cdot D(h(u)\varphi) = \int_\Omega f h(u)\varphi \tag{3.1}$$

holds for all $h \in \mathcal{C}_c^1(\mathbb{R})$ *and all* $\varphi \in W_0^{1,p(\cdot)}(\Omega,\omega) \cap L^\infty(\Omega)$;

(R3) $\int_{\{k<|u|<k+1\}} a(x,Du) \cdot Du \longrightarrow 0$ *as* $k \longrightarrow \infty$.

Remark 4 *For $p \in (1,\infty)$, $\tau_0^{p(\cdot)}(\Omega)$ is defined as the set of measurable functions $u\colon \Omega \to \mathbb{R}$ such that for $k > 0$ the truncated functions $T_k(u) \in W_0^{1,p(\cdot)}(\Omega,\omega)$ and for every $u \in \tau_0^{p(\cdot)}(\Omega)$ there exists a unique measurable function $v\colon \Omega \to \mathbb{R}^\mathbb{N}$ such that $\nabla T_K(u) = v\chi_{\{|u|<k\}}$ for a.e. $x \in \Omega$ (see [4, 20] for more details).*

Theorem 1 *Under assumptions* ($\mathbf{H}_1$)–($\mathbf{H}_2$), ($\mathbf{A}_0$)–($\mathbf{A}_3$) *and $f \in L^\infty(\Omega)$ there exists at least one renormalized solution (u, b) to (E, f).*

The proof of Theorem 1 is divided into several steps.

Step 1: Approximate problem. First we approximate (E, f) for $f \in L^\infty(\Omega)$ by problems for which existence can be proved by standard variational arguments. For $0 < \varepsilon \leq 1$, let $\beta_\varepsilon\colon \mathbb{R} \longrightarrow \mathbb{R}$ be the Yosida approximation (see [6]) of β. We introduce the operators

$$A_{1,\varepsilon}\colon W_0^{1,p(\cdot)}(\Omega,\omega) \longrightarrow W^{-1,p'(\cdot)}(\Omega,\omega^*), \qquad u \longrightarrow \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u)) - \operatorname{div} a(x,Du),$$

$$A_{2,\varepsilon}\colon W_0^{1,p(\cdot)}(\Omega,\omega) \longrightarrow W^{-1,p'(\cdot)}(\Omega,\omega^*), \qquad u \longrightarrow -\operatorname{div} F(T_{\frac{1}{\varepsilon}}(u)).$$

Because of ($\mathbf{A}_2$) and ($\mathbf{A}_3$), $A_{1,\varepsilon}$ is well-defined and monotone (see [16, p. 157]). Since $\beta_\varepsilon \circ T_{\frac{1}{\varepsilon}}$ is bounded and continuous and thanks to the growth condition ($\mathbf{A}_2$) on a, it follows that $A_{1,\varepsilon}$ is hemicontinuous (see [16, p. 157]). From the continuity and boundedness of $F \circ T_{\frac{1}{\varepsilon}}$ it follows that $A_{2,\varepsilon}$ is strongly continuous. Therefore the operator $A_\varepsilon := A_{1,\varepsilon} + A_{2,\varepsilon}$ is pseudomonotone. Using the monotonicity of β_ε, the Gauss–Green theorem for Sobolev functions and the boundary condition on the convection term $\int_\Omega F(T_{\frac{1}{\varepsilon}}(u)) \cdot Du$, we show by arguments similar to those in [14] that A_ε is coercive and bounded. Then it follows from [16, Theorem 2.7] that A_ε is surjective, *i.e.*, for each $0 < \varepsilon \le 1$ and $f \in W^{-1,p'(\cdot)}(\Omega, \omega^*)$ there exists at least one solution $u_\varepsilon \in W_0^{1,p(\cdot)}(\Omega, \omega)$ to the problem

$$(E_\varepsilon, f) \begin{cases} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - \operatorname{div}(a(x, Du_\varepsilon) + F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

such that

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\varphi + \int_\Omega \big(a(x, Du_\varepsilon) + F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\big) \cdot D\varphi = \langle f, \varphi \rangle \tag{3.2}$$

holds for all $\varphi \in W_0^{1,p(\cdot)}(\Omega, \omega)$, where $\langle ., . \rangle$ denotes the duality pairing between $W_0^{1,p(\cdot)}(\Omega, \omega)$ and $W^{-1,p'(\cdot)}(\Omega, \omega^*)$.

Step 2: A priori estimates. We begin with the following

Lemma 4 *For* $0 < \varepsilon \le 1$ *and* $f \in L^\infty(\Omega)$ *let* $u_\varepsilon \in W_0^{1,p(\cdot)}(\Omega, \omega)$ *be a solution of* (E_ε, f)*. Then*

(i) *there exists a constant* $C_1 = C_1(\|f\|_\infty, \lambda, p(\cdot), N) > 0$*, not depending on* ε*, such that*

$$\|Du_\varepsilon\|_{L^{p(\cdot)}(\Omega,\omega)} \le C_1; \tag{3.3}$$

(ii) $\|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\|_\infty \le \|f\|_\infty;$ (3.4)

(iii) *for all* $l, k > 0$*, we have*

$$\int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon \le k \int_{\{|u_\varepsilon|>l\}} |f|. \tag{3.5}$$

Proof. (i) Taking u_ε as a test function in (3.2), we obtain

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))u_\varepsilon \,\mathrm{d}x + \int_\Omega a(x, Du_\varepsilon) \cdot Du_\varepsilon \,\mathrm{d}x + \int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot Du_\varepsilon \,\mathrm{d}x = \int_\Omega f u_\varepsilon \,\mathrm{d}x.$$

As the first term on the left-hand side is non-negative and the integral over the convection term vanishes by ($\mathbf{A}_1$), we have

$$\begin{aligned} \lambda \int_\Omega |Du_\varepsilon|^{p(\cdot)}\omega(x) \,\mathrm{d}x &\le \int_\Omega a(x, Du_\varepsilon) \cdot Du_\varepsilon \,\mathrm{d}x \\ &\le \int_\Omega f u_\varepsilon \,\mathrm{d}x = \int_\Omega f u_\varepsilon \omega^{1/p(x)} \omega^{-1/p(x)} \,\mathrm{d}x \\ &\le C(p(\cdot), N)\|f\|_\infty \|Du_\varepsilon\|_{L^{p(\cdot)}(\Omega,\omega)}, \end{aligned} \tag{3.6}$$

where $C(p(\cdot), N) > 0$ is a constant coming from the Hölder and the Poincaré inequalities. From (2.4) and (3.6) it follows that either

$$\|Du_\varepsilon\|_{L^{p(\cdot)}(\Omega,\omega)} \leq \Big(\frac{1}{\lambda} C(p(\cdot), N)\|f\|_\infty\Big)^{\frac{1}{p^- -1}}$$

or

$$\|Du_\varepsilon\|_{L^{p(\cdot)}(\Omega,\omega)} \leq \Big(\frac{1}{\lambda} C(p(\cdot), N)\|f\|_\infty\Big)^{\frac{1}{p^+ -1}}.$$

Setting

$$C_1 := \max\Big(\Big(\frac{1}{\lambda} C(p(\cdot), N)\|f\|_\infty\Big)^{\frac{1}{p^+ -1}}, \Big(\frac{1}{\lambda} C(p(\cdot), N)\|f\|_\infty\Big)^{\frac{1}{p^- -1}}\Big),$$

we get (i).

(ii) Taking $\frac{1}{\delta}[T_{k+\delta}(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) - T_k(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)))]$ as a test function in (3.2), passing to the limit with $\delta \to 0$ and choosing $k > \|f\|_\infty$, we obtain (ii).

(iii) For $k, l > 0$ fixed, we take $T_k(u_\varepsilon - T_l(u_\varepsilon))$ as a test function in (3.2). Using

$$\int_\Omega a(x, Du_\varepsilon) \cdot DT_k(u_\varepsilon - T_l(u_\varepsilon))\,\mathrm{d}x = \int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon\,\mathrm{d}x$$

and as the first and the second term on the left-hand side are non-negative and the convection term vanishes, we get

$$\int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon\,\mathrm{d}x \leq \int_\Omega f T_k(u_\varepsilon - T_l(u_\varepsilon))\,\mathrm{d}x \leq k \int_{\{|u_\varepsilon|>l\}} |f|\,\mathrm{d}x.$$

□

Remark 5 *For $k > 0$, from Lemma* 4 (iii)*, we deduce that*

$$|\{|u_\varepsilon| \geq l\}| \leq l^{-(p_s^*)^-} C(p(\cdot), p^-, \lambda, C_1), \tag{3.7}$$

$$\int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon \leq k\|f\|_\infty |\{|u_\varepsilon| > l\}| \leq C_2(k) l^{-(p_s^*)^-}. \tag{3.8}$$

Indeed, we have the following continuous embeddings

$$W_0^{1,p(x)}(\Omega, \omega) \hookrightarrow L^{p_s^*(x)}(\Omega) \hookrightarrow L^{(p_s^*)^-}(\Omega),$$

where $(p_s^*(x))^- := \frac{p^- s - N}{(s^- +1)N - p^- s^-}$.

From Remark 1 *it follows that for every* $l > 1$,

$$\|T_l(u)\|_{L^{(p_s^*)^-}(\Omega)} \leq C\|DT_l(u)\|_{L^{p(x)}(\Omega,\omega)} \leq C\Big(\int_\Omega \omega(x)|DT_l(u)|^{p(x)}\,\mathrm{d}x\Big)^\nu,$$

where

$$\nu = \begin{cases} \frac{1}{p^-}, & \textit{if } \|DT_l(u)\|_{L^{p(x)}(\Omega,\omega)} \geq 1, \\ \frac{1}{p^+}, & \textit{if } \|DT_k(u)\|_{L^{p(x)}(\Omega,\omega)} \leq 1. \end{cases}$$

Noting that $\{|u_\varepsilon| \geq l\} = \{|T_l(u_\varepsilon)| \geq l\}$, *we have*

$$\begin{aligned} |\{|u_\varepsilon| \geq l\}| &\leq \left(\frac{\|T_l(u)\|_{L^{(p_s^*)^-}(\Omega)}}{l}\right)^{(p_s^*)^-} \\ &\leq l^{-(p_s^*)^-}\left[C\left(\int_\Omega \omega(x)|DT_l(u)|^{p(x)}\,\mathrm{d}x\right)^{\nu}\right]^{(p_s^*)^-}. \end{aligned} \tag{3.9}$$

Combining (3.3), (3.6) *and* (3.9), *and setting*

$$C(p(\cdot),(p_s^*)^-,\lambda,C_1) = C^{(p_s^*)^-}\left(\frac{C(p(\cdot),N)\|f\|_\infty}{\lambda}C_1\right)^{\nu(p_s^*)^-} > 0,$$

we obtain

$$|\{|u_\varepsilon| \geq l\}| \leq C(p(\cdot),(p_s^*)^-,\lambda,C_1)l^{-(p_s^*)^-}. \tag{3.10}$$

So we have

$$\lim_{l\to+\infty} |\{|u_\varepsilon| \geq l\}| = 0.$$

Hence (3.10) *gives* (3.8) *with* $C_2(k) := C(p(\cdot),(p_s^*)^-,\lambda,C_1)k\|f\|_\infty$.

Step 3: Basic convergence results. First, let us prove the following

Lemma 5 *For* $0 < \varepsilon \leq 1$ *and* $f \in L^\infty(\Omega)$, *let* $u_\varepsilon \in W_0^{1,p(\cdot)}(\Omega,\omega)$ *be a solution of* (E_ε, f). *There exist* $u \in W_0^{1,p(\cdot)}(\Omega,\omega)$ *and* $b \in L^\infty(\Omega)$ *such that for a not relabelled subsequence of* $(u_\varepsilon)_{0<\varepsilon\leq 1}$ *as* $\varepsilon \downarrow 0$:

$$u_\varepsilon \rightharpoonup u \text{ in } L^{p(\cdot)}(\Omega,\omega) \text{ and a.e. in } \Omega, \tag{3.11}$$

$$Du_\varepsilon \rightharpoonup Du \text{ in } (L^{p(\cdot)}(\Omega,\omega))^N, \tag{3.12}$$

$$\text{and } \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \rightharpoonup b \text{ weakly-* in } L^\infty(\Omega). \tag{3.13}$$

Moreover, for any $k > 0$:

$$DT_k(u_\varepsilon) \rightharpoonup DT_k(u) \text{ in } (L^{p(\cdot)}(\Omega,\omega))^N, \tag{3.14}$$

$$a(x,DT_k(u_\varepsilon)) \rightharpoonup a(x,DT_k(u)) \text{ in } (L^{p'(\cdot)}(\Omega,\omega^*))^N. \tag{3.15}$$

Proof. Since (3.11)–(3.14) follow directly from Lemma 4 and Remark 5, it is sufficient to prove (3.15). For this end, by ($\mathbf{A}_2$) and (3.3) it follows that given any subsequence of $(a(x,DT_k(u_\varepsilon))_\varepsilon)$, there exists a subsequence, still denoted by $(a(x,DT_k(u_\varepsilon))_\varepsilon)$, such that $a(x,DT_k(u_\varepsilon)) \rightharpoonup \Phi_k$ in $(L^{p'(\cdot)}(\Omega,\omega^*))^N$.

We will we prove that $\Phi_k = a(x,DT_k(u))$ a.e. on Ω. The proof consists of three assertions.

Assertion (i). For every function $h \in W^{1,\infty}(\mathbb{R})$, $h \geq 0$ with $\mathrm{supp}(h)$ compact, we will prove that

$$\limsup_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))\cdot D[h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))]\,\mathrm{d}x \leq 0. \tag{3.16}$$

Taking $h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))$ as a test function in (3.2), we have

$$\begin{aligned}\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) \\ + \int_\Omega \Big(a(x, Du_\varepsilon) + F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\Big) \cdot D[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))] \\ = \int_\Omega fh(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)).\end{aligned} \tag{3.17}$$

Using $|h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))| \le 2k\|h\|_\infty$, by Lebesgue's dominated convergence theorem, we find that

$$\lim_{\varepsilon\to 0} \int_\Omega fh(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) = 0$$

and

$$\lim_{\varepsilon\to 0} \int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot D[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))] = 0.$$

By using the same arguments as in [2], we can prove that

$$\limsup_{\varepsilon\to 0} \int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot [h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))] \,\mathrm{d}x \ge 0.$$

Passing to the limit in (3.17) and using the above results, yields (3.16).

Assertion (ii). We claim that for every $k > 0$ we have

$$\limsup_{\varepsilon\to 0} \int_\Omega a(x, DT_k(u_\varepsilon)) \cdot [DT_k(u_\varepsilon) - DT_k(u)] \,\mathrm{d}x \le 0 \tag{3.18}$$

(see [1]).

Assertion (iii). In this assertion, we prove by monotonicity arguments that for $k > 0$, $\Phi_k = a(x, DT_k(u))$ for almost every $x \in \Omega$. Let $\varphi \in D(\Omega)$ and $\tilde\alpha \in \mathbb{R}$. Using (3.18), we have

$$\begin{aligned}&\tilde\alpha \lim_{\varepsilon\to 0} \int_\Omega a(x, DT_k(u_\varepsilon)) \cdot D\varphi \,\mathrm{d}x \\ &\quad\ge \limsup_{\varepsilon\to 0} \int_\Omega a(x, DT_k(u_\varepsilon)) \cdot [DT_k(u_\varepsilon) - DT_k(u) + D(\tilde\alpha\varphi)] \,\mathrm{d}x \\ &\quad\ge \limsup_{\varepsilon\to 0} \int_\Omega a(x, D(T_k(u) - \tilde\alpha\varphi)) \cdot [DT_k(u_\varepsilon) - DT_k(u) + D(\tilde\alpha\varphi)] \,\mathrm{d}x \\ &\quad\ge \limsup_{\varepsilon\to 0} \int_\Omega a(x, D(T_k(u) - \tilde\alpha\varphi)) \cdot D(\tilde\alpha\varphi)] \,\mathrm{d}x \\ &\quad\ge \tilde\alpha \int_\Omega a(x, D(T_k(u) - \tilde\alpha\varphi)) \cdot D\varphi \,\mathrm{d}x.\end{aligned}$$

Dividing by $\tilde\alpha > 0$ and by $\tilde\alpha < 0$ and letting $\tilde\alpha \to 0$, we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega a(x, DT_k(u_\varepsilon)) \cdot D\varphi \,\mathrm{d}x = \int_\Omega a(x, DT_k(u)) \cdot D\varphi \,\mathrm{d}x.$$

This means that for every $k > 0$,

$$\int_\Omega \Phi_k \cdot D\varphi \,\mathrm{d}x = \int_\Omega a(x, DT_k(u)) \cdot D\varphi \,\mathrm{d}x,$$

and so

$$\Phi_k = a(x, DT_k(u)) \quad \text{in} \quad D^{'}(\Omega)$$

for all $k > 0$. Hence $\Phi_k = a(x, DT_k(u))$ a.e. in Ω and so $a(x, DT_k(u_\varepsilon)) \rightharpoonup a(x, DT_k(u))$ weakly in $(L^{p'(\cdot)}(\Omega, \omega^*))^N$. □

Remark 6 *As an immediate consequence of* (3.18) *and* ($\mathbf{A}_3$)*, we obtain*

$$\lim_{\varepsilon \to 0} \int_\Omega a(x, DT_k(u_\varepsilon)) - a(x, DT_k(u)) \cdot (DT_k(u_\varepsilon) - DT_k(u)) = 0. \tag{3.19}$$

Lemma 6 *The limit u of the approximate solution u_ε of (E_ε, f) satisfies*

$$\lim_{l \to \infty} \int_{\{l<|u|<l+1\}} a(x, Du) \cdot Du \,\mathrm{d}x = 0. \tag{3.20}$$

Proof. To this end, observe that for any fixed $l \geq 0$, one has

$$\begin{aligned} \int_{\{l<|u_\varepsilon|<l+1\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon \,\mathrm{d}x &= \int_\Omega a(x, Du_\varepsilon) \cdot (DT_{l+1}(u_\varepsilon) - DT_l(u_\varepsilon)) \,\mathrm{d}x \\ = \int_\Omega a(x, DT_{l+1}(u_\varepsilon)) \cdot DT_{l+1}(u_\varepsilon) \,\mathrm{d}x &- \int_\Omega a(x, DT_l(u_\varepsilon)) \cdot DT_l(u_\varepsilon)) \,\mathrm{d}x. \end{aligned}$$

According to (3.19), one can pass to the limit with $\varepsilon \to 0$ for fixed $l \geq 0$ to obtain

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_{\{l<|u_\varepsilon|<l+1\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon \,\mathrm{d}x \\ &\quad = \int_\Omega a(x, DT_{l+1}(u)) \cdot DT_{l+1}(u) \,\mathrm{d}x - \int_\Omega a(x, DT_l(u)) \cdot DT_l(u)) \,\mathrm{d}x \\ &\quad = \int_{\{l<|u|<l+1\}} a(x, Du) \cdot Du \,\mathrm{d}x. \end{aligned} \tag{3.21}$$

Taking the limit in (3.21) as $l \to +\infty$ and using the estimate (3.8), we infer that u satisfies (R3), which means that the proof of Lemma 6 is complete. □

Now, we are able to conclude the proof of Theorem 1.

Step 4: Proof of the existence. Let $h \in \mathcal{C}_c^1(\mathbb{R})$ and $\phi \in W_0^{1,p(\cdot)}(\Omega, \omega) \cap L^\infty(\Omega)$ be arbitrary. Taking $h_l(u_\varepsilon)h(u)\phi$ as a test function in (3.2), we obtain

$$I^1_{\varepsilon,l} + I^2_{\varepsilon,l} + I^3_{\varepsilon,l} = I^4_{\varepsilon,l}, \tag{3.22}$$

where

$$\begin{aligned} I^1_{\varepsilon,l} &= \int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) h_l(u_\varepsilon) h(u)\phi, \\ I^2_{\varepsilon,l} &= \int_\Omega a(x, Du_\varepsilon) \cdot D(h_l(u_\varepsilon)h(u)\phi), \\ I^3_{\varepsilon,l} &= \int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot D(h_l(u_\varepsilon)h(u)\phi), \\ I^4_{\varepsilon,l} &= \int_\Omega f h_l(u_\varepsilon) h(u)\phi. \end{aligned}$$

Step 4 (i). Letting $\varepsilon \downarrow 0$ and using the convergence results (3.11) and (3.13) from Lemma 5, we can immediately calculate the following limits:

$$\lim_{\varepsilon\to 0} I^1_{\varepsilon,l} = \int_\Omega b h_l(u)h(u)\phi, \tag{3.23}$$

$$\lim_{\varepsilon\to 0} I^4_{\varepsilon,l} = \int_\Omega f h_l(u)h(u)\phi. \tag{3.24}$$

We write $I^2_{\varepsilon,l} = I^{2,1}_{\varepsilon,l} + I^{2,2}_{\varepsilon,l}$ where,

$$I^{2,1}_{\varepsilon,l} = \int_\Omega h'_l(u_\varepsilon)a(x, Du_\varepsilon)\cdot Du_\varepsilon h(u)\phi$$

$$I^{2,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)a(x, Du_\varepsilon)\cdot D(h(u)\phi).$$

Using (3.8), we get the estimate

$$|\lim_{\varepsilon\to 0} I^{2,1}_{\varepsilon,l}| \le \|h\|_\infty\|\phi\|_\infty \cdot C_2(1)l^{-(p_s^*)^-}. \tag{3.25}$$

Since modular convergence is equivalent to norm convergence in $L^{p(\cdot)}(\Omega,\omega)$, by Lebesgue's dominated convergence theorem it follows that for any $i \in \{1,\dots,N\}$, we have

$$h_l(u_\varepsilon)\frac{\partial}{\partial x_i}(h(u)\phi) \to h_l(u)\frac{\partial}{\partial x_i}(h(u)\phi) \quad \text{in} \quad L^{p(\cdot)}(\Omega,\omega) \quad \text{as } \varepsilon \downarrow 0.$$

Keeping in mind that $I^{2,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)a(x, DT_{l+1}(u_\varepsilon))\cdot D(h(u)\phi)$, by (3.15), we get

$$\lim_{\varepsilon\to 0} I^{2,2}_{\varepsilon,l} = \int_\Omega h_l(u)a(x, DT_{l+1}(u))\cdot D(h(u)\phi). \tag{3.26}$$

Let us write $I^3_{\varepsilon,l} = I^{3,1}_{\varepsilon,l} + I^{3,2}_{\varepsilon,l}$, where

$$I^{3,1}_{\varepsilon,l} = \int_\Omega h'_l(u_\varepsilon)F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\cdot Du_\varepsilon h(u)\phi,$$

$$I^{3,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\cdot D(h(u)\phi).$$

For any $l \in \mathbb{N}$, there exists $\varepsilon_0(l)$ such that for all $\varepsilon < \varepsilon_0(l)$,

$$I^{3,1}_{\varepsilon,l} = \int_\Omega h'_l(T_{l+1}(u_\varepsilon))F(T_{l+1}(u_\varepsilon))\cdot DT_{l+1}(u_\varepsilon)h(u)\phi. \tag{3.27}$$

Using the Gauss–Green theorem for Sobolev functions in (3.27), we get

$$I^{3,1}_{\varepsilon,l} = -\int_\Omega \int_0^{T_{l+1}(u_\varepsilon)} h'_l(r)F(r)\,\mathrm{d}r \cdot D(h(u)\phi). \tag{3.28}$$

Now, using (3.11) and the Gauss–Green theorem, after letting $\varepsilon \downarrow 0$, we get

$$\lim_{\varepsilon\to 0} I^{3,1}_{\varepsilon,l} = \int_\Omega h'_l(u)F(u)\cdot Du\ h(u)\phi. \tag{3.29}$$

Choosing ε small enough, we can write

$$I_{\varepsilon,l}^{3,2} = \int_\Omega h_l(u_\varepsilon)F(T_{l+1}(u_\varepsilon)) \cdot D(h(u)\phi) \tag{3.30}$$

and conclude that

$$\lim_{\varepsilon\to 0} I_{\varepsilon,l}^{3,2} = \int_\Omega h_l(u)F(u) \cdot D(h(u)\phi). \tag{3.31}$$

Step 4 (ii): Passing to the limit with $l \to \infty$. Combining (3.22) and (3.23)–(3.31), we find that

$$I_l^1 + I_l^2 + I_l^3 + I_l^4 + I_l^5 = I_l^6, \tag{3.32}$$

where

$$\begin{aligned}
I_l^1 &= \int_\Omega b h_l(u)h(u)\phi,\\
I_l^2 &= \int_\Omega h_l(u)a(x, DT_{l+1}(u)) \cdot D(h(u)\phi),\\
|I_l^3| &\le C_2(1) l^{-(p_s^*)^-} \|h\|_\infty \|\phi\|_\infty,\\
I_l^4 &= \int_\Omega h_l(u)F(u) \cdot D(h(u)\phi),\\
I_l^5 &= \int_\Omega h_l'(u)F(u) \cdot Du\, h(u)\phi,\\
I_l^6 &= \int_\Omega f\, h_l(u)h(u)\phi.
\end{aligned}$$

Obviously, we have

$$\lim_{l\to\infty} I_l^3 = 0. \tag{3.33}$$

Choosing $m > 0$ such that $\operatorname{supp}(h) \subset [-m, m]$, we can replace u by $T_m(u)$ in $I_l^1, I_l^2, \ldots, I_l^6$, and $h_l'(u) = h_l'(T_m(u)) = 0$ if $l + 1 > m$ and $h_l(u) = h_l(T_m(u)) = 1$ if $l > m$. Therefore, letting $l \to \infty$ and combining (3.32) with (3.33), yields

$$\int_\Omega b h(u)\phi + \int_\Omega (a(x, Du) + F(u)) \cdot D(h(u)\phi) = \int_\Omega f h(u)\phi, \tag{3.34}$$

for all $h \in C_c^1(\mathbb{R})$ and all $\phi \in W_0^{1,p(\cdot)}(\Omega, \omega) \cap L^\infty(\Omega)$.

Step 4 (iii): Subdifferential argument. It is left to prove that $u(x) \in D(\beta(x))$ and $b(x) \in \beta(u(x))$ for almost all $x \in \Omega$. Since β is a maximal monotone graph, there exists a convex, l.s.c. and proper function $j\colon \mathbb{R} \to [0, \infty]$ such that $\beta(r) = \partial j(r)$ for all $r \in \mathbb{R}$. According to [6], for $0 < \varepsilon \le 1$ the function $j_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ defined by $j_\varepsilon(r) = \int_0^r \beta_\varepsilon(s)\,\mathrm{d}s$ has the following properties (see [19]):

(i) for any $0 < \varepsilon \le 1$, j_ε is convex and differentiable for all $r \in \mathbb{R}$ and such that

$$j_\varepsilon'(r) = \beta_\varepsilon(r) \quad \text{for all } r \in \mathbb{R} \text{ and any } 0 < \varepsilon \le 1;$$

(ii) $j_\varepsilon(r) \uparrow j(r)$ pointwise in $\mathbb{R}$ as $\varepsilon \to 0$.

Using the same argument as in [19], we can prove that for all $r \in \mathbb{R}$ and almost every $x \in \Omega$, $u \in D(\beta)$ and $b \in \beta(u)$ almost everywhere in Ω.

With this last step the proof of Theorem 1 is completed.

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