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ENTROPY SOLUTIONS FOR NONLINEAR NONHOMOGENEOUS NEUMANN PROBLEMS INVOLVING THE GENERALIZED $p(x)$-LAPLACE OPERATOR AND MEASURE DATA

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Abstract. Our aim in this paper is to study the existence of entropy solutions for the class of nonlinear $p(x)$-Laplace problems with Neumann nonhomogeneous boundary conditions and diffuse Radon measure data which does not charge the sets of zero $p\,(.)$-capacity.

Keywords: Generalized Sobolev space, Neumann boundary conditions, Entropy solution, Radon measure, $p\,(.)$-capacity.

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1 Introduction

In this paper we study the notion of entropy solutions for the inhomogeneous and nonlinear Neumann boundary value problem

$$P(\mu)\ \begin{cases} -\operatorname{div}\left(\Phi\left(\nabla u-\Theta\left(u\right)\right)\right)+\left|u\right|^{p(x)-2}u+\alpha\left(u\right)=\mu & \text{in } \Omega,\\ \Phi\left(\nabla u-\Theta\left(u\right)\right).\eta+\gamma\left(u\right)=g & \text{on } \partial\Omega, \end{cases}$$

with

$$\Phi\left(\xi\right)=\left|\xi\right|^{p(x)-2}\xi,\quad \forall\xi\in\mathbb{R}^N,$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a smooth bounded open domain with Lipschitz boundary $\partial\Omega$, η is the outer unit normal vector on $\partial\Omega$, α, γ, Θ are real functions defined on $\mathbb{R}$ or $\mathbb{R}^N$, $g\in L^1\left(\partial\Omega\right)$ and μ is a diffuse measure such that $\mu=\mu\lfloor\Omega$.

The existence of entropy solutions for the problem $P(\mu)$ where μ belongs to $L^1(\Omega)$ is also studied in [2]. In [2], under some assumptions the authors proved that the operator associated to the approximated problem is of type (M). So, by some a priori estimates, they obtained the convergence of the approximate sequence to an entropy solution of the initial problem. In this work, we extend the approach developed in [2] to the case of diffuse Radon measure.

We define $\mathcal{M}_b(X)$ as the space of bounded Radon measures in X, equipped with the standard norm $\|.\|_{\mathcal{M}_b(X)}$.

In the context of variable exponent, the $p(.)$-capacity of any subset $B\subset X$ is defined by

$$Cap_{p(.)}\left(B,X\right)=\inf_{u\in S_{p(.)}(B)}\left\{\int_X\left(|u|^{p(x)}+|\nabla u|^{p(x)}\right)\,\mathrm{d}x\right\},$$

with

$$S_{p(.)}\left(B\right)=\left\{u\in W_0^{1,p(.)}\left(X\right):u\geq 1\text{ in an open set containing }B\text{ and }u\geq 0\text{ in }X\right\}.$$

If $S_{p(.)}\left(B\right)=\emptyset$, we set $Cap_{p(.)}\left(B,X\right)=+\infty$.

For $\mu\in\mathcal{M}_b\left(X\right)$, we say that μ is diffuse with respect to the capacity $W^{1,p(.)}\left(X\right)$ ($p\left(.\right)$-capacity for short) if $\mu\left(B\right)=0$ for every set B such that $Cap_{p(.)}\left(B,X\right)=0$.

The set of bounded Radon diffuse measures in variable exponent setting is denoted by $\mathcal{M}_b^{p(.)}\left(X\right)$.

Elliptic problem with measure data was studied by many authors (cf. [1, 3, 7, 8, 13, 14, 15]). Let us recall that within the context of variable exponent, the Dirichlet problem was investigated in [3, 13, 14]. In [14], the authors proved that every measure $\mu\in\mathcal{M}_b^{p(.)}\left(\Omega\right)$ admits a decomposition in $L^1\left(\Omega\right)+W^{-1,p'(.)}\left(\Omega\right)$ and used it to prove the existence and uniqueness of entropy solutions. In the case of Neumann boundary conditions we work in general in $W^{1,p(.)}\left(\Omega\right)$ (see [15]), so we cannot use directly the argument of decomposition of measure, since the second part of the measure is in $W^{-1,p'(.)}\left(\Omega\right)$ (the dual of $W_0^{1,p(.)}\left(\Omega\right)$). To overcome this difficulty, in [15] the authors assumed that Ω is an extension domain (see [9]) which permited them to work with a space like $W_0^{1,p(.)}\left(\Omega\right)$ and return after to the space $W^{1,p(.)}\left(\Omega\right)$. With a view to use the same ideas we suppose that Ω is a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^1. Then, it has an extension domain (cf. [9]), so for any fixed open bounded subset U_Ω of $\mathbb{R}^N$ such that $\overline{\Omega} \subset U_\Omega$, there exists a bounded linear operator

$$E\colon W^{1,p(.)}(\Omega) \to W_0^{1,p(.)}(U_\Omega),$$

for which

(i) $E(u) = u$ a.e. in Ω for each $u \in W^{1,p(.)}(\Omega)$,

(ii) $\|E(u)\|_{W_0^{1,p(.)}(U_\Omega)} \leq C \|u\|_{W^{1,p(.)}(\Omega)}$, where C is a constant depending only on Ω.

We introduce the set

$$\mathfrak{M}_b^{p(.)}(\Omega) := \left\{\mu \in \mathcal{M}_b^{p(.)}(U_\Omega) : \mu \text{ is concentrated on } \Omega\right\}.$$

This definition is independent of the open set U_Ω. Note that for $u \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ and $\mu \in \mathfrak{M}_b^{p(.)}(\Omega)$ we have

$$\langle \mu, E(u)\rangle = \int_\Omega u \, \mathrm{d}\mu.$$

On the other hand, as μ is diffuse, there exist $f \in L^1(U_\Omega)$ and $F \in \left(L^{p(.)}(U_\Omega)\right)^N$ such that $\mu = f - \operatorname{div}(F)$ in $\mathcal{D}'(U_\Omega)$. Therefore, we can also write

$$\langle \mu, E(u)\rangle = \int_{U_\Omega} f E(u) \, \mathrm{d}x + \int_{U_\Omega} F.\nabla E(u) \, \mathrm{d}x.$$

In this paper, we assume that

$$\begin{cases} p(.) : \overline{\Omega} \to \mathbb{R} \text{ is a continuous function such that} \\ 1 < p_- \leq p_+ < +\infty, \end{cases} \tag{1.1}$$

where $p_- := \operatorname{ess\,inf}_{x\in\Omega} p(x)$ and $p_+ := \operatorname{ess\,sup}_{x\in\Omega} p(x)$.

Let us briefly summarize the rest of the paper: in Section 2, we introduce some basic properties of the space $W^{1,p(.)}(\Omega)$ and some useful lemmas. Section 3 is devoted to the proof of the existence of entropy solutions for the problem $P(\mu)$.

2 Preliminaries

We denote the Lebesgue space with variable exponent $L^{p(.)}(\Omega)$ (see [10]) as the set of all measurable function $u\colon \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} \, \mathrm{d}x$$

is finite.

If the exponent is bounded, *i.e.*, if $p_+ < +\infty$, then the expression

$$\|u\|_{p(.)} := \inf\left\{\lambda > 0 : \rho_{p(.)}(u/\lambda) \leq 1\right\}$$

defines a norm in $L^{p(.)}(\Omega)$, called the Luxembourg norm.

The space $\left(L^{p(.)}(\Omega), \|.\|_{p(.)}\right)$ is a separable Banach space. Moreover, if $1 < p_- \leq p_+ < +\infty$, then $L^{p(.)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Finally, we have the Hölder type inequality

$$\left|\int_\Omega uv \,\mathrm{d}x\right| \leq \left(\frac{1}{p_-} + \frac{1}{p_+}\right) \|u\|_{p(.)} \|v\|_{p'(.)} \tag{2.1}$$

for all $u \in L^{p(.)}(\Omega)$ and $v \in L^{p'(.)}(\Omega)$.

Let

$$W^{1,p(.)}(\Omega) := \left\{u \in L^{p(.)}(\Omega) : |\nabla u| \in L^{p(.)}(\Omega)\right\},$$

which is a Banach space equipped with the following norm

$$\|u\|_{1,p(.)} := \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

The space $\left(W^{1,p(.)}(\Omega), \|.\|_{1,p(.)}\right)$ is a separable and reflexive Banach space.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(.)}(\Omega)$. We have the following result.

Proposition 2.1 (see [11, 21]) *If* $u_p, u \in L^{p(.)}(\Omega)$ *and* $p_+ < \infty$*, the following properties hold true*:

(i) $\|u\|_{p(.)} > 1 \;\Rightarrow\; \|u\|_{p(.)}^{p_-} < \rho_{p(.)}(u) < \|u\|_{p(.)}^{p_+}$;

(ii) $\|u\|_{p(.)} < 1 \;\Rightarrow\; \|u\|_{p(.)}^{p_+} < \rho_{p(.)}(u) < \|u\|_{p(.)}^{p_-}$;

(iii) $\|u\|_{p(.)} < 1$ (*respectively* $= 1; > 1$) $\Leftrightarrow\; \rho_{p(.)}(u) < 1$ (*respectively* $= 1; > 1$);

(iv) $\|u_n\|_{p(.)} \to 0$ (*respectively* $\to +\infty$) $\Leftrightarrow\; \rho_{p(.)}(u_n) < 1$ (*respectively* $\to +\infty$);

(v) $\rho_{p(.)}\left(u/\|u\|_{p(.)}\right) = 1$.

For a measurable function $u\colon \Omega \to \mathbb{R}$ we introduce the following notation:

$$\rho_{1,p(.)}(u) = \int_\Omega |u|^{p(x)} \,\mathrm{d}x + \int_\Omega |\nabla u|^{p(x)} \,\mathrm{d}x.$$

Proposition 2.2 (see [19, 20]) *If* $u \in W^{1,p(.)}(\Omega)$*, the following properties hold true*:

(i) $\|u\|_{1,p(.)} > 1 \;\Rightarrow \|u\|_{1,p(.)}^{p_-} < \rho_{1,p(.)}(u) < \|u\|_{1,p(.)}^{p_+}$;

(ii) $\|u\|_{1,p(.)} < 1 \;\Rightarrow\; \|u\|_{1,p(.)}^{p_+} < \rho_{p(.)}(u) < \|u\|_{1,p(.)}^{p_-}$;

(iii) $\|u\|_{1,p(.)} < 1$ (*respectively* $= 1; > 1$) $\Leftrightarrow\; \rho_{1,p(.)}(u) < 1$ (*respectively* $= 1; > 1$).

Put

$$p^{\partial}(x) := (p(x))^{\partial} \begin{cases} \dfrac{(N-1)\,p(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N. \end{cases}$$

Proposition 2.3 (see [20]) *Let $p \in C\left(\bar{\Omega}\right)$ and $p_- > 1$. If $q \in C(\partial\Omega)$ satisfies the condition*

$$1 < q(x) < p^{\partial}(x) \quad \forall x \in \partial\Omega,$$

then there is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\partial\Omega)$. In particular, there is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\partial\Omega)$.

Let us introduce the following notation: given two bounded measurable functions $p(x), q(x) : \Omega \to \mathbb{R}$, we write

$$q(x) \ll p(x) \qquad \text{if} \qquad \operatorname*{ess\,inf}_{x\in\Omega} (p(x) - q(x)) > 0.$$

For the next section, we need the following lemmas

Lemma 2.4 ([2]) *Let $\xi, \eta \in \mathbb{R}^N$ an let $1 < p < \infty$. We have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\,\xi.(\xi - \eta).$$

Lemma 2.5 (Lebesgue generalized convergence theorem) *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions and let f be a measurable function such that $f_n \to f$ a.e. in Ω. Let $(g_n)_{n\in\mathbb{N}} \subset L^1(\Omega)$ be such that for all $n \in \mathbb{N}$, $|f_n| \leq g_n$ a.e. in Ω and $g_n \to g$ in $L^1(\Omega)$. Then*

$$\int_{\Omega} f_n \,\mathrm{d}x \to \int_{\Omega} f \,\mathrm{d}x.$$

We recall the following two technical lemmas (see [12, 18]), the second is a well-known result in measure theory.

Lemma 2.6 *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions in Ω. If v_n converges in measure to v and is uniformly bounded in $L^{p(.)}(\Omega)$ for some $1 \ll p(.) \in L^{\infty}(\Omega)$, then v_n strongly converges to v in $L^1(\Omega)$.*

Lemma 2.7 *Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) < \infty$. Consider a measurable function $\gamma\colon X \to (0,\infty)$ such that*

$$\mu\big(\{x \in X : \gamma(x) = 0\}\big) = 0.$$

Then, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mu(A) < \epsilon \text{ for all } A \in \mathcal{M} \text{ with } \int_A \gamma \,\mathrm{d}\mu < \delta.$$

We end this part by giving the result concerning the decomposition of measure.

Theorem 2.8 (see [16]) *Let $p(.) : \overline{X_1} \subset X \to [1,\infty]$ with $1 < p_- < p_+ < +\infty$ be a continuous function and let $\mu \in \mathcal{M}_b(X)$. Then $\mu \in \mathcal{M}_b^{p(.)}(X)$ if and only if $\mu \in L^1(X) + W^{-1,p'(.)}(X)$.*

3 Existence result

This section is devoted to the proof of the existence of an entropy solution for the problem $P(\mu)$. Our method of proof combines the ideas from [2] and [15]. First of all, we make the following assumptions:

(H_1) α and γ are continuous functions defined on $\mathbb{R}$ such that there exist two positive real numbers M_1, M_2 with $|\alpha(x)| \leq M_1$, $|\gamma(x)| \leq M_2$, $\alpha(x).x \geq 0$, $\gamma(x).x \geq 0$ for all $x \in \mathbb{R}$ and $\alpha(0) = \gamma(0) = 0$.

(H_2) $\mu \in \mathfrak{M}_b^{p(.)}(\Omega)$ and $g \in L^1(\partial\Omega)$.

(H_3) $\Theta\colon \mathbb{R} \to \mathbb{R}^N$ is a continuous function such that $\Theta(0) = 0$ and $|\Theta(x) - \Theta(y)| \leq C|x - y|$ for all $x, y \in \mathbb{R}$, where C is a positive constant such that

$$C < \min\left(\left(\frac{p_-}{2}\right)^{\frac{1}{p_-}}, \left(\frac{p_-}{2}\right)^{\frac{1}{p_+}}\right).$$

Now, we recall some notations and results.

For any $k > 0$, we define the truncation function T_k by $T_k(s) := \max\{-k, \min\{k, s\}\}$. For all $u \in W^{1,p(.)}(\Omega)$ we denote by $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense.

In the sequel, we will identify at the boundary, u and $\tau(u)$.

Set

$$\mathcal{T}^{1,p(.)}(\Omega) = \left\{u\colon \Omega \to \mathbb{R} : u \text{ measurable and such that } T_k(u) \in W^{1,p(.)}(\Omega) \text{ for any } k > 0\right\}.$$

Proposition 3.1 (see [6]) *Let $u \in \mathcal{T}^{1,p(.)}(\Omega)$. Then there exists a unique measurable function $v\colon \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$, for all $k > 0$. The function v is denoted by ∇u. Moreover, if $u \in W^{1,p(.)}(\Omega)$ then $v \in \left(L^{p(.)}(\Omega)\right)^N$ and $v = \nabla u$ in the usual sense.*

We denote by $\mathcal{T}_{\text{tr}}^{1,p(.)}(\Omega)$ (cf. [4, 5, 16, 17]) the set of functions $u \in \mathcal{T}^{1,p(.)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,p(.)}(\Omega)$ satisfying the following conditions:

(C_1) $u_n \to u$ a.e. in Ω;

(C_2) $\nabla T_k(u_n) \to \nabla T_k(u)$ in $\left(L^1(\Omega)\right)^N$ for any $k > 0$;

(C_3) there exists a measurable function v on $\partial\Omega$ such that $u_n \to v$ a.e. on $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [4, 5]. In the sequel, the trace of $u \in \mathcal{T}_{\text{tr}}^{1,p(.)}(\Omega)$ on $\partial\Omega$ will be denoted by $\text{tr}(u)$. If $u \in W^{1,p(.)}(\Omega)$, $\text{tr}(u)$ coincides with $\tau(u)$ in the usual sense. Moreover $u \in \mathcal{T}_{\text{tr}}^{1,p(.)}(\Omega)$ and for every $k > 0$, $\tau(T_k(u)) = T_k(\text{tr}(u))$ and if $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ then $(u - \varphi) \in \mathcal{T}_{\text{tr}}^{1,p(.)}(\Omega)$ and $\text{tr}(u - \varphi) = \text{tr}(u) - \text{tr}(\varphi)$.

Now, we announce our notion of entropy solution for the problem $P(\mu)$.

Definition 3.2 *A measurable function* $u\colon \Omega \to \mathbb{R}$ *is called entropy solution of the elliptic problem* $P(\mu)$ *if* $u \in \mathcal{T}_{\rm tr}^{1,p(.)}(\Omega)$, $|u|^{p(.)-2}u \in L^1(\Omega)$, $\alpha(u) \in L^1(\Omega)$, $\gamma(u) \in L^1(\partial\Omega)$ *and*

$$\begin{aligned}\int_\Omega \Phi(\nabla u - \Theta(u))\nabla T_k(u-\varphi)\,\mathrm{d}x + \int_\Omega |u|^{p(x)-2}uT_k(u-\varphi)\,\mathrm{d}x \\ + \int_\Omega \alpha(u)T_k(u-\varphi)\,\mathrm{d}x + \int_{\partial\Omega}\gamma(u)T_k(u-\varphi)\,\mathrm{d}\sigma \\ \leq \int_\Omega T_k(u-\varphi)\,\mathrm{d}\mu + \int_{\partial\Omega} gT_k(u-\varphi)\,d\sigma,\end{aligned} \tag{3.1}$$

for any $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ *and every* $k > 0$.

Remark 3.3 *Since* $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$, *then* $u - \varphi \in \mathcal{T}_{\rm tr}^{1,p(.)}(\Omega)$, *hence* $T_k(u-\varphi) \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ *and consequently the first, the second, the third and the fifth integral in* (3.1) *are finite. Moreover, in the fourth and sixth integral, we can use the fact that the trace of* $g \in W^{1,p(.)}(\Omega)$ *on* $\partial\Omega$ *is well-defined in* $L^{p(.)}(\partial\Omega)$.

The main result of this paper is the following theorem.

Theorem 3.4 *Let assumptions* (H$_1$), (H$_2$) *and* (H$_3$) *hold true. Then there exists at least one entropy solution of the problem* $P(\mu)$.

Proof. The proof is divided into two steps.

Step 1. The approximate problem. Since $\mu \in \mathcal{M}_b^{p(.)}(U_\Omega)$, recall that $\mu = f - \operatorname{div}(F)$ in $\mathcal{D}'(U_\Omega)$ with $f \in L^1(U_\Omega)$ and $F \in \left(L^{p'(.)}(U_\Omega)\right)^N$, where U_Ω is the open bounded subset of $\mathbb{R}^N$ which extend Ω via the operator E. We regularize μ as follow: $\forall \epsilon > 0$, $\forall x \in U_\Omega$ we define

$$f_\epsilon(x) = T_{\frac{1}{\epsilon}}(f(x))\chi_\Omega(x).$$

We consider $F_R = \chi_\Omega F$ and $\mu_\epsilon = f_\epsilon - \operatorname{div}(F_R)$.

For any $\epsilon > 0$, one has $\mu_\epsilon \in \mathfrak{M}_b^{p(.)}(\Omega) \cap L^\infty(\Omega)$ and $\mu_\epsilon \rightharpoonup \mu$ in $\mathcal{M}_b^{p(.)}(U_\Omega)$. Furthermore, for any $k > 0$ and any $\xi \in \mathcal{T}^{1,p(.)}(\Omega)$

$$\left|\int_\Omega T_k(\xi)\,\mathrm{d}\mu_\epsilon\right| \leq kC(\mu,\Omega).$$

Now, we consider the approximated problem

$$P(\mu_\epsilon)\begin{cases} -\operatorname{div}(\Phi(\nabla u_\epsilon - \Theta(u_\epsilon))) + |u_\epsilon|^{p(x)-2}u_\epsilon + T_{\frac{1}{\epsilon}}(\alpha(u_\epsilon)) = \mu_\epsilon & \text{in } \Omega \\ \Phi(\nabla u_\epsilon - (\Theta(u_\epsilon))).\eta + T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon)) = T_{\frac{1}{\epsilon}}(g) & \text{on } \partial\Omega. \end{cases}$$

Let us prove the following result.

Lemma 3.5 *There exists at least one weak solution u_ϵ for the problem $P(\mu_\epsilon)$ in the sense that $u_\epsilon \in W^{1,p(.)}(\Omega)$ and for all $v \in W^{1,p(.)}(\Omega)$,*

$$\int_\Omega \Phi\left(\nabla u_\epsilon - \Theta\left(u_\epsilon\right)\right)\nabla v \,\mathrm{d}x + \int_\Omega |u_\epsilon|^{p(x)-2} u_\epsilon v \,\mathrm{d}x + \int_\Omega T_{\frac{1}{\epsilon}}\left(\alpha\left(u_\epsilon\right)\right) v \,\mathrm{d}x$$
$$+ \int_{\partial\Omega} T_{\frac{1}{\epsilon}}\left(\gamma\left(u_\epsilon\right)\right) v \,\mathrm{d}\sigma = \int_\Omega v \,\mathrm{d}\mu_\epsilon + \int_{\partial\Omega} T_{\frac{1}{\epsilon}}(g) v \,\mathrm{d}\sigma. \tag{3.2}$$

Proof. We define the reflexive space

$$E = W^{1,p(.)}(\Omega) \times L^{p(.)}(\partial\Omega).$$

Let X_0 be the subspace of E defined by

$$X_0 = \{(u,v) \in E : v = \tau(u)\}.$$

In the sequel, we will identify an element $(u,v) \in X_0$ with its representative $u \in W^{1,p(.)}(\Omega)$.

We define the operator A_ϵ by

$$\langle A_\epsilon u, v\rangle = \langle Au, v\rangle + \int_\Omega T_{\frac{1}{\epsilon}}(\alpha(u)) v \,\mathrm{d}x + \int_{\partial\Omega} T_{\frac{1}{\epsilon}}(\gamma(u)) v \,\mathrm{d}\sigma,$$

where

$$\langle Au, v\rangle = \int_\Omega \Phi(\nabla u - \Theta(u))\nabla v \,\mathrm{d}x + \int_\Omega |u|^{p(x)-2} uv \,\mathrm{d}x, \quad \forall\, u, v \in X_0.$$

According to [2], the operator A_ϵ is of type (M), coercive, bounded and hemi-continuous. Thus, for $F_\epsilon \in E' \subset X_0'$ defined by

$$\langle F_\epsilon, v\rangle = \int_\Omega v \,\mathrm{d}\mu_\epsilon + \int_{\partial\Omega} T_{\frac{1}{\epsilon}}(g) v \,\mathrm{d}\sigma,$$

we deduce the existence of a function $u_\epsilon \in X_0$ such that

$$\langle A_\epsilon u_\epsilon, v\rangle = \langle F_\epsilon, v\rangle \ \ \forall v \in X_0,$$

i.e.,

$$\int_\Omega \Phi\left(\nabla u_\epsilon - \Theta\left(u_\epsilon\right)\right)\nabla v \,\mathrm{d}x + \int_\Omega |u_\epsilon|^{p(x)-2} u_\epsilon v \,\mathrm{d}x + \int_\Omega T_{\frac{1}{\epsilon}}\left(\alpha\left(u_\epsilon\right)\right) v \,\mathrm{d}x$$
$$+ \int_{\partial\Omega} T_{\frac{1}{\epsilon}}\left(\gamma\left(u_\epsilon\right)\right) v \,\mathrm{d}\sigma = \int_\Omega v \,\mathrm{d}\mu_\epsilon + \int_{\partial\Omega} T_{\frac{1}{\epsilon}}(g) v \,\mathrm{d}\sigma. \tag{3.3}$$

□

Step 2. A priori estimates.

Assertion 1. $\nabla\left(T_k\left(u_\epsilon\right)\right)_{\epsilon>0}$ is bounded in $\left(L^{p_-}(\Omega)\right)^N$.

Proof. We take $v = T_k(u_\epsilon)$ as a test function in (3.3) to get

$$\begin{aligned}
&\int_\Omega \Phi(\nabla u_\epsilon - \Theta(u_\epsilon)) \nabla T_k(u_\epsilon)\, \mathrm{d}x + \int_\Omega |u_\epsilon|^{p(x)-2} u_\epsilon T_k(u_\epsilon)\, \mathrm{d}x \\
&\quad + \int_\Omega T_{\frac{1}{\epsilon}}(\alpha(u_\epsilon)) T_k(u_\epsilon)\, \mathrm{d}x + \int_{\partial\Omega} T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon)) T_k(u_\epsilon)\, \mathrm{d}\sigma \\
&\quad = \int_\Omega T_k(u_\epsilon)\, \mathrm{d}\mu_\epsilon + \int_{\partial\Omega} T_{\frac{1}{\epsilon}}(g) T_k(u_\epsilon)\, \mathrm{d}\sigma.
\end{aligned} \tag{3.4}$$

The third and fourth terms on the left-hand side of the above equality are non-negative, and so

$$\begin{aligned}
&\int_\Omega \Phi(\nabla u_\epsilon - \Theta(u_\epsilon)) \nabla T_k(u_\epsilon)\, \mathrm{d}x + \int_\Omega |u_\epsilon|^{p(x)-2} u_\epsilon T_k(u_\epsilon)\, \mathrm{d}x \\
&\qquad \le k \|g\|_{L^1(\partial\Omega)} + \int_\Omega T_k(u_\epsilon)\, \mathrm{d}\mu_\epsilon.
\end{aligned} \tag{3.5}$$

As

$$\begin{aligned}
\int_\Omega |u_\epsilon|^{p(x)-2} u_\epsilon T_k(u_\epsilon)\, \mathrm{d}x &= \int_{[|u_\epsilon|\le k]} |T_k(u_\epsilon)|^{p(x)}\, \mathrm{d}x + \int_{[|u_\epsilon|>k]} |u_\epsilon|^{p(x)-2} u_\epsilon T_k(u_\epsilon)\, \mathrm{d}x \\
&\ge \int_{[|u_\epsilon|\le k]} |T_k(u_\epsilon)|^{p(x)}\, \mathrm{d}x + \int_{[|u_\epsilon|>k]} k^{p(x)}\, \mathrm{d}x \\
&\ge \int_{[|u_\epsilon|\le k]} |T_k(u_\epsilon)|^{p(x)}\, \mathrm{d}x + \int_{[|u_\epsilon|>k]} |T_k(u_\epsilon)|^{p(x)}\, \mathrm{d}x \\
&\ge \int_\Omega |T_k(u_\epsilon)|^{p(x)}\, \mathrm{d}x,
\end{aligned} \tag{3.6}$$

we have

$$\begin{aligned}
\int_\Omega T_k(u_\epsilon)\, \mathrm{d}\mu_\epsilon &= \int_\Omega E(T_k(u_\epsilon))\, \mathrm{d}\mu_\epsilon \\
&= \langle \mu_\epsilon, E(T_k(u_\epsilon)) \rangle \\
&= \int_{U_\Omega} f_\epsilon E(T_k(u_\epsilon))\, \mathrm{d}x + \int_{U_\Omega} F_R.\nabla E(T_k(u_\epsilon))\, \mathrm{d}x \\
&= \int_\Omega T_{\frac{1}{\epsilon}}(f) E(T_k(u_\epsilon))\, \mathrm{d}x + \int_{U_\Omega} F.\nabla E(\chi_\Omega T_k(u_\epsilon))\, \mathrm{d}x.
\end{aligned} \tag{3.7}$$

Firstly, we have

$$\left| \int_\Omega T_{\frac{1}{\epsilon}}(f) E(T_k(u_\epsilon))\, \mathrm{d}x \right| \le k \|f\|_{L^1(\Omega)}. \tag{3.8}$$

Secondly, we have

$$\begin{aligned}
\left| \int_{U_\Omega} F.\nabla E(\chi_\Omega T_k(u_\epsilon))\, \mathrm{d}x \right| &= \left| \int_{[x\in\Omega, |u_\epsilon|<k]} F.\nabla T_k(u_\epsilon)\, \mathrm{d}x \right| \\
&\le \int_\Omega |F| \, |\nabla T_k(u_\epsilon)|\, \mathrm{d}x.
\end{aligned} \tag{3.9}$$

By Young's inequality, we have

$$\begin{aligned}\int_\Omega |F|\,|\nabla T_k(u_\epsilon)|\,\mathrm{d}x &= \int_\Omega \frac{|\nabla T_k(u_\epsilon)|}{(p_+\times 2^{p_+})^{\frac{1}{p(x)}}}(p_+\times 2^{p_+})^{\frac{1}{p(x)}}|F|\,\mathrm{d}x\\ &\le \int_\Omega \frac{1}{p(x)}\frac{1}{(p_+\times 2^{p_+})}|\nabla T_k(u_\epsilon)|^{p(x)}\,\mathrm{d}x\\ &\qquad + \int_\Omega \frac{1}{p'(x)}(p_+\times 2^{p_+})^{\frac{p'(x)}{p(x)}}|F|^{p'(x)}\,\mathrm{d}x\\ &\le \frac{1}{p_-}\frac{1}{(p_+\times 2^{p_+})}\int_\Omega |\nabla T_k(u_\epsilon)|^{p(x)}\,\mathrm{d}x\\ &\qquad + \frac{1}{p'_-}(p_+\times 2^{p_+})^{\frac{p'_+}{p_-}}\int_\Omega |F|^{p'(x)}\,\mathrm{d}x.\end{aligned}\tag{3.10}$$

Combining (3.8) and (3.10), the equality (3.7) becomes

$$\begin{aligned}\int_\Omega T_k(u_\epsilon)\,\mathrm{d}\mu_\epsilon \le k\,\|f\|_{L^1(\Omega)} &+ \frac{1}{p_-}\frac{1}{(p_+\times 2^{p_+})}\int_\Omega |\nabla T_k(u_\epsilon)|^{p(x)}\,\mathrm{d}x\\ &+ \frac{1}{p'_-}(p_+\times 2^{p_+})^{\frac{p'_+}{p_-}}\int_\Omega |F|^{p'(x)}\,\mathrm{d}x.\end{aligned}\tag{3.11}$$

Then, we deduce from (3.5) that

$$\begin{aligned}\langle Au_\epsilon, T_k(u_\epsilon)\rangle &\le k\left(\|g\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)}\right)\\ &+ \frac{1}{p_-}\frac{1}{p_+\times 2^{p_+}}\int_\Omega |\nabla T_k(u_\epsilon)|^{p(x)}\,\mathrm{d}x + \frac{1}{p'_-}(p_+\times 2^{p_+})^{\frac{p'_+}{p_-}}\int_\Omega |F|^{p'(x)}\,\mathrm{d}x.\end{aligned}\tag{3.12}$$

On the other hand, using Lemma 2.4 and (3.6), we obtain

$$\begin{aligned}\langle Au_\epsilon, T_k(u_\epsilon)\rangle &= \int_\Omega \Phi(\nabla u_\epsilon - \Theta(u_\epsilon))\nabla T_k(u_\epsilon)\,\mathrm{d}x + \int_\Omega |u_\epsilon|^{p(x)-2}u_\epsilon T_k(u_\epsilon)\,\mathrm{d}x\\ &= \int_\Omega |\nabla T_k(u_\epsilon) - \Theta(u_\epsilon)|^{p(x)-2}(\nabla T_k(u_\epsilon) - \Theta(u_\epsilon)).\nabla T_k(u_\epsilon)\,\mathrm{d}x\\ &\qquad + \int_\Omega |u_\epsilon|^{p(x)-2}u_\epsilon T_k(u_\epsilon)\,\mathrm{d}x\\ &\ge \int_\Omega \frac{1}{p(x)}|\nabla T_k(u_\epsilon) - \Theta(u_\epsilon)|^{p(x)}\,\mathrm{d}x - \int_\Omega \frac{1}{p(x)}|\Theta(u_\epsilon)|^{p(x)}\,\mathrm{d}x\\ &\qquad + \int_\Omega |T_k(u_\epsilon)|^{p(x)}\,\mathrm{d}x.\end{aligned}\tag{3.13}$$

Since

$$(a+b)^p \le 2^{p-1}(|a|^p + |b|^p) \text{ for all } a,b\in\mathbb{R},$$

we have

$$\begin{aligned}\frac{1}{2^{p_+-1}}|\nabla T_k(u_\epsilon)|^{p(x)} &= \frac{1}{2^{p_+-1}}|\nabla T_k(u_\epsilon) - \Theta(u_\epsilon) + \Theta(u_\epsilon)|^{p(x)}\\ &\le |\nabla T_k(u_\epsilon) - \Theta(u_\epsilon)|^{p(x)} + |\Theta(u_\epsilon)|^{p(x)}.\end{aligned}\tag{3.14}$$

Then

$$\frac{1}{2^{p_+-1}}\left|\nabla T_k\left(u_\epsilon\right)\right|^{p(x)}-\left|\Theta\left(u_\epsilon\right)\right|^{p(x)} \leq\left|\nabla T_k\left(u_\epsilon\right)-\Theta\left(u_\epsilon\right)\right|^{p(x)}.$$

Consequently, for k large enough we have

$$\begin{aligned}
\left\langle A u_\epsilon, T_k\left(u_\epsilon\right)\right\rangle \geq & \int_{\Omega} \frac{1}{p(x)}\left[\frac{1}{2^{p_+-1}}\left|\nabla T_k\left(u_\epsilon\right)\right|^{p(x)}-\left|\Theta\left(u_\epsilon\right)\right|^{p(x)}\right] \mathrm{d} x \\
& -\int_{\Omega} \frac{1}{p(x)}\left|\Theta\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x+\int_{\Omega}\left|T_k\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x \\
\geq & \int_{\Omega} \frac{1}{p(x)} \frac{1}{2^{p_+-1}}\left|\nabla T_k\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x-\int_{\Omega} \frac{2}{p(x)}\left|\Theta\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x \\
& +\int_{\Omega}\left|T_k\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x \\
\geq & \int_{\Omega} \frac{1}{p(x)} \frac{1}{2^{p_+-1}}\left|\nabla T_k\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x-\int_{\Omega} \frac{2}{p(x)} C^{p(x)}\left|u_\epsilon\right|^{p(x)} \mathrm{d} x \\
& +\int_{\Omega}\left|T_k\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x \\
\geq & \int_{\Omega} \frac{1}{p(x)} \frac{1}{2^{p_+-1}}\left|\nabla T_k\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x-\int_{\Omega} \frac{2}{p_-} C^{p(x)}\left|u_\epsilon\right|^{p(x)} \mathrm{d} x \\
& +\int_{\Omega}\left|T_k\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x \\
\geq & \int_{\Omega} \frac{1}{p_+} \frac{1}{2^{p_+-1}}\left|\nabla T_k\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x \\
& +\int_{\Omega}\left(1-\frac{2}{p_-} C^{p(x)}\right)\left|T_k\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x .
\end{aligned} \tag{3.15}$$

So, the choice of C in (H_3) gives the existence of a positive constant C_0 such that

$$\left\langle A u_\epsilon, T_k\left(u_\epsilon\right)\right\rangle \geq \frac{1}{p_+} \frac{1}{2^{p_+-1}} \int_{\Omega}\left|\nabla T_k\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x+C_0 \int_{\Omega}\left|T_k\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x . \tag{3.16}$$

Hence, by using the inequalities (3.12) and (3.16) and the fact that $p_->1$, we deduce that

$$\begin{aligned}
\frac{1}{p_+} \frac{1}{2^{p_+-1}} \int_{\Omega}\left|\nabla T_k\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x+C_0 \int_{\Omega}\left|T_k\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x \leq k\left(\|g\|_{L^1(\partial \Omega)}+\|f\|_{L^1(\Omega)}\right) \\
+\frac{1}{p_+\times 2^{p_+}} \int_{\Omega}\left|\nabla T_k\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x+\frac{1}{p_-^{\prime}}\left(p_+\times 2^{p_+}\right)^{\frac{p_+^{\prime}}{p_-}} \int_{\Omega}|F|^{p^{\prime}(x)} \mathrm{d} x,
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{1}{p_+} \frac{1}{2^{p_+}} \int_{\Omega}\left|\nabla T_k\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x+C_0 \int_{\Omega}\left|T_k\left(u_\epsilon\right)\right|^{p(x)} \mathrm{d} x \leq k\left(\|g\|_{L^1(\partial \Omega)}+\|f\|_{L^1(\Omega)}\right) \\
+\frac{1}{p_-^{\prime}}\left(p_+\times 2^{p_+}\right)^{\frac{p_+^{\prime}}{p_-}} \int_{\Omega}|F|^{p^{\prime}(x)} \mathrm{d} x .
\end{aligned}$$

So

$$\begin{aligned}\min\left\{\frac{1}{p_+}\frac{1}{2^{p_+}},C_0\right\}\rho_{1,p(.)}\left(T_k\left(u_\epsilon\right)\right)\leq k\left(\|f\|_{L^1(\Omega)}\|g\|_{L^1(\partial\Omega)}\right)\\+\frac{1}{p'_-}\left(p_+\times 2^{p_+}\right)^{\frac{p'_+}{p_-}}\int_\Omega|F|^{p'(x)}\,\mathrm{d}x.\end{aligned}\tag{3.17}$$

Consequently

$$\rho_{1,p(.)}\left(T_k\left(u_\epsilon\right)\right)\leq kC_1+C_2,\tag{3.18}$$

where $C_1=\text{const}\left(f,g,p_-,p_+\right)$ and $C_2=\text{const}\left(F,p_-,p_+\right)$. Therefore

$$\|T_k\left(u_\epsilon\right)\|_{1,p(.)}\leq 1+\left(kC_1+C_2\right)^{\frac{1}{p_-}}.\tag{3.19}$$

We deduce that for any $k>0$, the sequence $\left(T_k\left(u_\epsilon\right)\right)_{\epsilon>0}$ is uniformly bounded in $W^{1,p(.)}\left(\Omega\right)$ and so in $W^{1,p_-}\left(\Omega\right)$. Then, up to a subsequence, we can assume that for any $k>0$,

$$T_k\left(u_\epsilon\right)\rightharpoonup v_k \text{ in } W^{1,p_-}\left(\Omega\right)$$

and by compact embedding, we have

$$T_k\left(u_\epsilon\right)\to v_k \text{ in } L^{p_-}\left(\Omega\right) \text{ and a.e. in } \Omega.$$

□

Assertion 2. $\left(u_\epsilon\right)_{\epsilon>0}$ converges in measure to some function u.

Proof. Let $k>1$ be large enough. Using $T_k\left(u_\epsilon\right)$ as a test function in (3.3), we get

$$\rho_{1,p(.)}\left(T_k\left(u_\epsilon\right)\right)\leq kC_1+C_2,$$

which yields

$$\int_{\{|u_\epsilon|>k\}}k^{p_-}\,\mathrm{d}x\leq\int_{\{|u_\epsilon|>k\}}k^{p(x)}\,\mathrm{d}x\leq k\left(C_1+C_2\right).$$

It follows that

$$\text{meas}\left\{|u_\epsilon|>k\right\}\leq k^{1-p_-}\left(C_1+C_2\right).$$

Therefore

$$\text{meas}\left\{|u_\epsilon|>k\right\}\to 0 \text{ as } k\to+\infty \text{ since } 1-p_-<0.\tag{3.20}$$

Moreover, for every fixed $t>0$ and every positive $k>0$, we know that

$$\left\{|u_{\epsilon_1}-u_{\epsilon_2}|>t\right\}\subset\left\{|u_{\epsilon_1}|>k\right\}\cup\left\{|u_{\epsilon_2}|>k\right\}\cup\left\{|T_k\left(u_{\epsilon_1}\right)-T_k\left(u_{\epsilon_2}\right)|>t\right\},$$

and hence

$$\begin{aligned}\text{meas}\big(\left\{|u_{\epsilon_1}-u_{\epsilon_2}|>t\right\}\big)\leq\text{meas}\big(\left\{|u_{\epsilon_1}|>k\right\}\big)+\text{meas}\big(\left\{|u_{\epsilon_2}|>k\right\}\big)\\+\text{meas}\big(\left\{|T_k\left(u_{\epsilon_1}\right)-T_k\left(u_{\epsilon_2}\right)|>t\right\}\big).\end{aligned}\tag{3.21}$$

Let $\delta>0$. Using (3.20), we choose $k=k\left(\delta\right)$ such that

$$\text{meas}\big(\left\{|u_{\epsilon_1}|>k\right\}\big)\leq\frac{\delta}{3}\qquad\text{and}\qquad\text{meas}\big(\left\{|u_{\epsilon_2}|>k\right\}\big)\leq\frac{\delta}{3}.\tag{3.22}$$

Since $T_k(u_\epsilon)$ converges strongly to v_k in $L^{p_-}(\Omega)$ and a.e. in Ω, then it is a Cauchy sequence in $L^{p_-}(\Omega)$. Thus, for all $\epsilon_1, \epsilon_2 \geq n_0(t,\delta)$ we have

$$\operatorname{meas}\big(\{|T_k(u_{\epsilon_1}) - T_k(u_{\epsilon_2})| > t\}\big) \leq \frac{1}{t^{p_-}} \int_\Omega |T_k(u_{\epsilon_1}) - T_k(u_{\epsilon_2})|^{p_-} \,\mathrm{d}x \leq \frac{\delta}{3}. \tag{3.23}$$

Finally, from (3.21), (3.22) and (3.23) we obtain

$$\operatorname{meas}\big(\{|u_{\epsilon_1} - u_{\epsilon_2}| > t\}\big) \leq \delta \text{ for all } \epsilon_1, \epsilon_2 \geq n_0(t,\delta), \tag{3.24}$$

which proves that the sequence $(u_\epsilon)_{\epsilon>0}$ is a Cauchy sequence in measure, and so it converges almost everywhere to some measurable function u. □

As for $k > 0$, T_k is continuous, then $T_k(u_\epsilon) \to T_k(u)$ a.e. in Ω and $v_k = T_k(u)$ a.e. in Ω. Therefore

$$\begin{aligned} &T_k(u_\epsilon) \rightharpoonup T_k(u) \text{ in } W^{1,p_-}(\Omega), \\ &T_k(u_\epsilon) \to T_k(u) \text{ in } L^{p_-}(\Omega) \text{ and a.e. in } \Omega. \end{aligned} \tag{3.25}$$

Assertion 3. $(\nabla u_\epsilon)_{\epsilon>0}$ converges in measure to the weak gradient of u.

Proof. Indeed, let δ, t, k, ν be positive real numbers (it is assumed that $\nu < 1$) and let $\epsilon > 0$. We have

$$\begin{aligned}\{|\nabla u_\epsilon - \nabla u| > t\} \subset \{|u_\epsilon| > k\} \cup \{|u| > k\} \cup \{|\nabla T_k(u_\epsilon)| > k\} \cup \{|\nabla T_k(u)| > k\} \cup \\ \cup \{|u_\epsilon - u| > \nu\} \cup G,\end{aligned}$$

where

$$G = \big\{|\nabla u_\epsilon - \nabla u| > t, |u| \leq k, |u_\epsilon| \leq k, |\nabla T_k(u_\epsilon)| \leq k, |\nabla T_k(u)| \leq k, |u_\epsilon - u| \leq \nu\big\}.$$

The same method used in the proof of Assertion 2 allows us to obtain for k sufficiently large,

$$\operatorname{meas}\big(\{|u_\epsilon| > k\} \cup \{|u| > k\} \cup \{|\nabla T_k(u_n)| > k\} \cup \{|\nabla T_k(u)| > k\}\big) \leq \frac{\delta}{3}. \tag{3.26}$$

On the other hand, the application

$$\mathcal{A}\colon\ (s, \xi_1, \xi_2) \mapsto (\Phi(\xi_1 - \Theta(s)) - \Phi(\xi_2 - \Theta(s))) \,.\, (\xi_1 - \xi_2)$$

is continuous, the set

$$\mathcal{K} := \big\{(s, \xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N : |s| \leq k, |\xi_1| \leq k, |\xi_2| \leq k, |\xi_1 - \xi_2| > t\big\}$$

is compact and

$$(\Phi(\xi_1 - \Theta(s)) - \Phi(\xi_2 - \Theta(s))) \,.\, (\xi_1 - \xi_2) > 0, \ \forall \xi_1 \neq \xi_2.$$

Then, the application $\mathcal{A}$ has its minimum on $\mathcal{K}$; we denote it by β. Therefore, we have $\beta > 0$ and

$$\begin{aligned}\int_G \beta \, \mathrm{d}x &\leq \int_G \left[\Phi\left(\nabla u_\epsilon - \Theta\left(u_\epsilon\right)\right) - \Phi\left(\nabla u - \Theta\left(u_\epsilon\right)\right)\right]\left[\nabla u_\epsilon - \nabla u\right] \mathrm{d}x \\ &\leq \int_G \left[\Phi\left(\nabla u_\epsilon - \Theta\left(u_\epsilon\right)\right) - \Phi\left(\nabla T_k\left(u\right) - \Theta\left(T_{k+\nu}\left(u_\epsilon\right)\right)\right)\right] \nabla T_\nu\left(T_{k+\nu}\left(u_\epsilon\right) - T_k\left(u\right)\right) \mathrm{d}x \\ &\leq \int_G \Phi\left(\nabla u_\epsilon - \Theta\left(u_\epsilon\right)\right) \nabla T_\nu\left(T_{k+\nu}\left(u_\epsilon\right) - T_k\left(u\right)\right) \mathrm{d}x \\ &\quad - \int_G \Phi\left(\nabla T_k\left(u\right) - \Theta\left(T_{k+\nu}\left(u_\epsilon\right)\right)\right) \nabla T_\nu\left(T_{k+\nu}\left(u_\epsilon\right) - T_k\left(u\right)\right) \mathrm{d}x.\end{aligned}$$

We take $v = T_\nu\left(T_{k+\nu}\left(u_n\right) - T_k\left(u\right)\right)$ in (3.3) to obtain

$$\begin{aligned}&\int_\Omega \Phi\left(\nabla u_\epsilon - \Theta\left(u_\epsilon\right)\right) \nabla T_\nu\left(T_{k+\nu}\left(u_\epsilon\right) - T_k\left(u\right)\right) \mathrm{d}x + \int_\Omega \left|u_\epsilon\right|^{p(x)-2} u_\epsilon T_\nu\left(T_{k+\nu}\left(u_\epsilon\right) - T_k\left(u\right)\right) \mathrm{d}x \\ &\qquad \leq \nu\left(\left\|T_{\frac{1}{\epsilon}}\left(\alpha\left(u_\epsilon\right)\right)\right\|_{L^1(\Omega)} + \left\|T_{\frac{1}{\epsilon}}\left(\gamma\left(u_\epsilon\right)\right)\right\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}\right) \\ &\qquad\qquad + \int_{U_\Omega} F.\nabla E\left(\chi_\Omega T_\nu\left(T_{k+\nu}\left(u_\epsilon\right) - T_k\left(u\right)\right)\right) \mathrm{d}x.\end{aligned}$$

Then

$$\begin{aligned}&\int_\Omega \Phi\left(\nabla u_\epsilon - \Theta\left(u_\epsilon\right)\right) \nabla T_\nu\left(T_{k+\nu}\left(u_\epsilon\right) - T_k\left(u\right)\right) \mathrm{d}x \\ &\qquad \leq \nu\left(\left\|T_{\frac{1}{\epsilon}}\left(\alpha\left(u_\epsilon\right)\right)\right\|_{L^1(\Omega)} + \left\|T_{\frac{1}{\epsilon}}\left(\gamma\left(u_\epsilon\right)\right)\right\|_{L^1(\partial\Omega)} + \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}\right) \\ &\qquad\qquad + \int_{U_\Omega} F.\nabla E\left(\chi_\Omega T_\nu\left(T_{k+\nu}\left(u_\epsilon\right) - T_k\left(u\right)\right)\right) \mathrm{d}x \\ &\qquad\qquad\qquad - \int_\Omega \left|u_\epsilon\right|^{p(x)-2} u_\epsilon T_\nu\left(T_{k+\nu}\left(u_\epsilon\right) - T_k\left(u\right)\right) \mathrm{d}x.\end{aligned}$$

Taking $v = \frac{1}{k} T_k\left(u_\epsilon\right)$ in (3.3), we get

$$\begin{aligned}&\int_\Omega T_{\frac{1}{\epsilon}}\left(\alpha\left(u_\epsilon\right)\right) \frac{1}{k} T_k\left(u_\epsilon\right) \mathrm{d}x + \int_{\partial\Omega} T_{\frac{1}{\epsilon}}\left(\gamma\left(u_\epsilon\right)\right) \frac{1}{k} T_k\left(u_\epsilon\right) \mathrm{d}\sigma \\ &\qquad \leq \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)} + \int_{U_\Omega} F.\nabla E\left(\chi_\Omega \frac{1}{k} T_k\left(u_\epsilon\right)\right) \mathrm{d}x.\end{aligned} \tag{3.27}$$

Since

$$\lim_{k \to 0} \frac{1}{k} T_k\left(u_\epsilon\right) = \operatorname{sign}_0\left(u_\epsilon\right),$$

using the Lebesgue dominated convergence theorem, as $k \to 0$, we deduce that

$$\begin{aligned}&\int_\Omega T_{\frac{1}{\epsilon}}\left(\alpha\left(u_\epsilon\right)\right) \frac{1}{k} T_k\left(u_\epsilon\right) \mathrm{d}x + \int_{\partial\Omega} T_{\frac{1}{\epsilon}}\left(\gamma\left(u_\epsilon\right)\right) \frac{1}{k} T_k\left(u_\epsilon\right) \mathrm{d}\sigma \to \\ &\qquad \to \left\|T_{\frac{1}{\epsilon}}\left(\alpha\left(u_\epsilon\right)\right)\right\|_{L^1(\Omega)} + \left\|T_{\frac{1}{\epsilon}}\left(\gamma\left(u_\epsilon\right)\right)\right\|_{L^1(\partial\Omega)}.\end{aligned} \tag{3.28}$$

The sequence $\left(E\left(\chi_\Omega \frac{1}{k} T_k\left(u_\epsilon\right)\right)\right)_{\epsilon>0}$ is bounded in $W_0^{1,p(.)}\left(U_\Omega\right)$. Indeed, $\left(\chi_\Omega \frac{1}{k} T_k\left(u_\epsilon\right)\right)_{\epsilon>0}$ is bounded in $W^{1,p(.)}(\Omega)$ and we use the inequality

$$\|E(v)\|_{W_0^{1,p(.)}(U_\Omega)} \leq C\|v\|_{W^{1,p(.)}(\Omega)} \quad \forall v \in W^{1,p(.)}(\Omega).$$

We also have

$$E\left(\chi_\Omega \frac{1}{k} T_k\left(u_\epsilon\right)\right)=\chi_\Omega \frac{1}{k} T_k\left(u_\epsilon\right) \text{ a.e. in } U_\Omega$$

and

$$\chi_\Omega \frac{1}{k} T_k\left(u_\epsilon\right) \rightarrow \chi_\Omega \operatorname{sign}_0\left(u_\epsilon\right) \text{ a.e. in } U_\Omega \text{ as } k \rightarrow 0.$$

Hence

$$E\left(\chi_\Omega \frac{1}{k} T_k\left(u_\epsilon\right)\right) \rightarrow E\left(\chi_\Omega \operatorname{sign}_0\left(u_\epsilon\right)\right) \text{ a.e. in } U_\Omega \text{ as } k \rightarrow 0.$$

Then

$$\nabla E\left(\chi_\Omega \frac{1}{k} T_k\left(u_\epsilon\right)\right) \rightharpoonup 0 \text{ in }\left(L^{p(.)}\left(U_\Omega\right)\right)^N.$$

Finally, we get

$$\lim_{k \rightarrow 0} \int_{U_\Omega} F . \nabla E\left(\chi_\Omega \frac{1}{k} T_k\left(u_\epsilon\right)\right) \mathrm{d} x=0. \tag{3.29}$$

Therefore, by passing to the limit as $k \rightarrow 0$ in (3.27) and using (3.28) and (3.29), we get

$$\left\|T_{\frac{1}{\epsilon}}\left(\alpha\left(u_\epsilon\right)\right)\right\|_{L^1(\Omega)}+\left\|T_{\frac{1}{\epsilon}}\left(\gamma\left(u_\epsilon\right)\right)\right\|_{L^1(\partial \Omega)} \leq\|f\|_{L^1(\Omega)}+\|g\|_{L^1(\partial \Omega)}.$$

It follows that

$$\begin{aligned}\int_\Omega \Phi\left(\nabla u_\epsilon-\Theta\left(u_\epsilon\right)\right) \nabla T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k(u)\right) \mathrm{d} x \leq \nu C_3 & \\ +\int_\Omega\left|u_\epsilon\right|^{p(x)-1}\left|T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k(u)\right)\right| \mathrm{d} x & \\ +\int_{U_\Omega} F . \nabla E\left(\chi_\Omega T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k(u)\right)\right) \mathrm{d} x .&\end{aligned} \tag{3.30}$$

Now, let us show that

$$\lim_{\nu \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{U_\Omega} F . \nabla E\left(\chi_\Omega T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k(u)\right)\right) \mathrm{d} x=0. \tag{3.31}$$

The sequence $\left(E\left(\chi_\Omega T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k(u)\right)\right)\right)_{\epsilon>0}$ is bounded in $W_0^{1,p(.)}\left(U_\Omega\right)$. We have

$$E\left(\chi_\Omega T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k(u)\right)\right)=\chi_\Omega T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k(u)\right) \text{ a.e. in } U_\Omega$$

and

$$\chi_\Omega T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k(u)\right) \rightarrow \chi_\Omega T_\nu\left(T_{k+\nu}(u)-T_k(u)\right) \text{ a.e. in } U_\Omega \text{ as } \epsilon \rightarrow 0.$$

Hence

$$E\left(\chi_\Omega T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k(u)\right)\right) \rightarrow E\left(\chi_\Omega T_\nu\left(T_{k+\nu}(u)-T_k(u)\right)\right) \text{ a.e. in } U_\Omega \text{ as } \epsilon \rightarrow 0.$$

Then

$$\nabla E\left(\chi_\Omega T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k\left(u\right)\right)\right) \rightharpoonup \nabla E\left(\chi_\Omega T_\nu\left(T_{k+\nu}\left(u\right)-T_k\left(u\right)\right)\right) \text{ in } \left(L^{p(\cdot)}\left(U_\Omega\right)\right)^N.$$

Since $F \in \left(L^{p'(\cdot)}\left(U_\Omega\right)\right)^N$, we deduce that

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{U_\Omega} & F.\nabla E\left(\chi_\Omega T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k\left(u\right)\right)\right) \mathrm{d}x \\ &= \int_{U_\Omega} F.\nabla E\left(\chi_\Omega T_\nu\left(T_{k+\nu}\left(u\right)-T_k\left(u\right)\right)\right) \mathrm{d}x.\end{aligned} \tag{3.32}$$

We have

$$\nabla E\left(\chi_\Omega T_\nu\left(T_{k+\nu}\left(u\right)-T_k\left(u\right)\right)\right) \to 0 \text{ a.e. in } U_\Omega \text{ as } \nu \to 0$$

and as $\nu < 1$, we have

$$F.E\left(\chi_\Omega T_\nu\left(T_{k+\nu}\left(u\right)-T_k\left(u\right)\right)\right) \leq |F|\left|E\left(\chi_\Omega T_1\left(T_{k+1}\left(u\right)-T_k\left(u\right)\right)\right)\right|.$$

Using Hölder inequality, we get

$$|F|\left|E\left(\chi_\Omega T_1\left(T_{k+1}\left(u\right)-T_k\left(u\right)\right)\right)\right| \in L^1\left(U_\Omega\right).$$

Thus, by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{\epsilon\to 0}\int_{U_\Omega} F.\nabla E\left(\chi_\Omega T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k\left(u\right)\right)\right) \mathrm{d}x = 0.$$

Consequently, letting $\nu \to 0$ in (3.32) yields

$$\lim_{\nu\to 0}\lim_{\epsilon\to 0}\int_{U_\Omega} F.\nabla E\left(\chi_\Omega T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k\left(u\right)\right)\right) \mathrm{d}x = 0.$$

Since

$$\begin{aligned}\int_\Omega \left|u_\epsilon\right|^{p(x)-1}\left|T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k\left(u\right)\right)\right| \mathrm{d}x &\leq \nu\int_\Omega \left|u_\epsilon\right|^{p(x)-1} \mathrm{d}x \\ &\leq \nu\left(\rho_{p'(.)}\left(\left|u_\epsilon\right|^{p(.)-1}\right)+\rho_{p(.)}\left(1\right)\right) \\ &\leq \nu\left(\text{meas}\left(\Omega\right)+\rho_{p(.)}\left(u_\epsilon\right)\right),\end{aligned} \tag{3.33}$$

so, letting $\nu \to 0$ in (3.33) and using the fact that Ω is bounded and $\rho_{p(.)}\left(u_\epsilon\right)$ is finite, we deduce that

$$\lim_{\nu\to 0}\int_\Omega \left|u_\epsilon\right|^{p(x)-1}\left|T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k\left(u\right)\right)\right| \mathrm{d}x = 0. \tag{3.34}$$

According to the Assertion 1, the sequence $\left(T_{k+\nu}\left(u_\epsilon\right)\right)_{\epsilon>0}$ is uniformly bounded in $W^{1,p(.)}\left(\Omega\right)$. Then, we have

$$T_{k+\nu}\left(u_\epsilon\right) \rightharpoonup T_{k+\nu}\left(u\right) \text{ in } W^{1,p(.)}\left(\Omega\right) \text{ and a.e. in } \Omega$$

and

$$T_{k+\nu}\left(u_\epsilon\right) \to T_{k+\nu}\left(u\right) \text{ in } L^{p(\cdot)}\left(\Omega\right) \text{ and a.e. in } \Omega.$$

Hence, by using (H_3), we obtain

$$\Theta\left(T_{k+\nu}\left(u_\epsilon\right)\right) \to \Theta\left(T_{k+\nu}\left(u\right)\right) \text{ in } L^{p(\cdot)}\left(\Omega\right) \text{ and a.e. in } \Omega. \tag{3.35}$$

Thus, the sequence $\Phi\left(\nabla T_k\left(u\right)-\Theta\left(T_{k+\nu}\left(u_\epsilon\right)\right)\right)$ converges to $\Phi\left(\nabla T_k\left(u\right)-\Theta\left(T_{k+\nu}\left(u\right)\right)\right)$ a.e. in Ω and $\left|\Phi\left(\nabla T_k\left(u\right)-\Theta\left(T_{k+\nu}\left(u_\epsilon\right)\right)\right)\right|^{p'(.)}$ is equi-integrable. Therefore

$$\Phi\left(\nabla T_k\left(u\right)-\Theta\left(T_{k+\nu}\left(u_\epsilon\right)\right)\right)\to\Phi\left(\nabla T_k\left(u\right)-\Theta\left(T_{k+\nu}\left(u\right)\right)\right)\text{ in }\left(L^{p'(.)}\left(\Omega\right)\right)^N \tag{3.36}$$

and since $T_{k+\nu}\left(u_\epsilon\right)$ converges weakly to $T_{k+\nu}\left(u\right)$ in $W^{1,p(.)}\left(\Omega\right)$, it follows that

$$\nabla T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k\left(u\right)\right)\rightharpoonup\nabla T_\nu\left(T_{k+\nu}\left(u\right)-T_k\left(u\right)\right)\text{ in }\left(L^{p(.)}\left(\Omega\right)\right)^N. \tag{3.37}$$

Consequently

$$\begin{aligned}\lim_{\epsilon\to0}\int_\Omega&\Phi\left(\nabla T_k\left(u\right)-\Theta\left(T_{k+\nu}\left(u_\epsilon\right)\right)\right)\nabla T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k\left(u\right)\right)\,\mathrm{d}x\\&=\int_\Omega\Phi\left(\nabla T_k\left(u\right)-\Theta\left(T_{k+\nu}\left(u\right)\right)\right)\nabla T_\nu\left(T_{k+\nu}\left(u\right)-T_k\left(u\right)\right)\,\mathrm{d}x.\end{aligned} \tag{3.38}$$

Since

$$\lim_{\nu\to0}\nabla T_\nu\left(T_{k+\nu}\left(u\right)-T_k\left(u\right)\right)=0$$

and as $\nu<1$, thanks to (H_3), we get

$$\begin{aligned}\Phi&\left(\nabla T_k\left(u\right)-\Theta\left(T_{k+\nu}\left(u\right)\right)\right)\nabla T_\nu\left(T_{k+\nu}\left(u\right)-T_k\left(u\right)\right)\\&\le C_4\left(\left|T_{k+1}\left(u\right)\right|^{p(x)-1}+\left|\nabla T_k\left(u\right)\right|^{p(x)-1}\right)\left|\nabla T_1\left(T_{k+1}\left(u\right)-T_k\left(u\right)\right)\right|.\end{aligned}$$

Although

$$\left(\left|T_{k+1}\left(u\right)\right|^{p(x)-1}+\left|\nabla T_k\left(u\right)\right|^{p(x)-1}\right)\left|\nabla T_1\left(T_{k+1}\left(u\right)-T_k\left(u\right)\right)\right|\in L^1\left(\Omega\right),$$

thanks to the Lebesgue dominated convergence theorem, we obtain

$$\lim_{\nu\to0}\int_\Omega\Phi\left(\nabla T_k\left(u\right)-\Theta\left(T_{k+\nu}\left(u\right)\right)\right)\nabla T_\nu\left(T_{k+\nu}\left(u\right)-T_k\left(u\right)\right)\,\mathrm{d}x=0.$$

Let $\rho>0$ and $\nu<\frac{\rho}{4C_3}$ be fixed such that

$$\left|\int_\Omega\Phi\left(\nabla T_k\left(u\right)-\Theta\left(T_{k+\nu}\left(u\right)\right)\right)\nabla T_\nu\left(T_{k+\nu}\left(u\right)-T_k\left(u\right)\right)\,\mathrm{d}x\right|\le\frac{\rho}{4}. \tag{3.39}$$

Then, there exists $\epsilon_1>0$ such that for all $\epsilon<\epsilon_1$,

$$\begin{aligned}\Bigg|\int_\Omega&\Phi\left(\nabla T_k\left(u\right)-\Theta\left(T_{k+\nu}\left(u_\epsilon\right)\right)\right)\nabla T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k\left(u\right)\right)\,\mathrm{d}x\\&-\int_\Omega\Phi\left(\nabla T_k\left(u\right)-\Theta\left(T_{k+\nu}\left(u\right)\right)\right)\nabla T_\nu\left(T_{k+\nu}\left(u\right)-T_k\left(u\right)\right)\,\mathrm{d}x\Bigg|\le\frac{\rho}{4}.\end{aligned} \tag{3.40}$$

Combining the two inequalities (3.39) and (3.40), we obtain

$$\left|\int_\Omega\Phi\left(\nabla T_k\left(u\right)-\Theta\left(T_{k+\nu}\left(u_\epsilon\right)\right)\right)\nabla T_\nu\left(T_{k+\nu}\left(u_\epsilon\right)-T_k\left(u\right)\right)\,\mathrm{d}x\right|\le\frac{\rho}{2},\quad\forall\epsilon<\epsilon_1. \tag{3.41}$$

Also, there exists $\epsilon_2 > 0$ such that for all $\epsilon < \epsilon_2$,

$$\begin{aligned}\nu C_3 + \int_\Omega |u_\epsilon|^{p(x)-1} \left|T_\nu\left(T_{k+\nu}\left(u_\epsilon\right) - T_k\left(u\right)\right)\right| \, \mathrm{d}x \\ + \int_{U_\Omega} F.\nabla E\left(\chi_\Omega T_\nu\left(T_{k+\nu}\left(u_\epsilon\right) - T_k\left(u\right)\right)\right) \, \mathrm{d}x \leq \frac{\rho}{2}.\end{aligned} \tag{3.42}$$

Adding the two inequalities (3.41) and (3.42), we get

$$\int_G \beta \, \mathrm{d}x \leq \rho.$$

Thus, by applying Lemma 2.7, we obtain

$$\mathrm{meas}\left(G\right) \leq \frac{\delta}{3}. \tag{3.43}$$

Moreover, by using the Assertion 2, we deduce the existence of $\epsilon_3 > 0$ such that

$$\mathrm{meas}\big(\{|u_\epsilon - u| > \nu\}\big) \leq \frac{\delta}{3}, \quad \forall \epsilon \leq \epsilon_3. \tag{3.44}$$

Therefore, for $\epsilon_0 = \min\left(\epsilon_1, \epsilon_2, \epsilon_3\right)$, it follows that

$$\mathrm{meas}\big(\{|\nabla u_\epsilon - \nabla u| > t\}\big) \leq \delta, \quad \forall \epsilon \leq \epsilon_0. \tag{3.45}$$

So, ∇u_ϵ converges in measure to ∇u. □

Assertion 4. $\left(u_\epsilon\right)_{\epsilon>0}$ converges a.e. on $\partial\Omega$ to some function v.

Proof. We know that the trace operator is compact from $W^{1,1}\left(\Omega\right)$ into $L^1\left(\partial\Omega\right)$; then there exists a constant C_5 such that

$$\left\|T_k\left(u_\epsilon\right) - T_k\left(u\right)\right\|_{L^1(\partial\Omega)} \leq C_5 \left\|T_k\left(u_\epsilon\right) - T_k\left(u\right)\right\|_{W^{1,1}(\Omega)}.$$

Then

$$T_k\left(u_\epsilon\right) \to T_k\left(u\right) \text{ in } L^1\left(\partial\Omega\right) \text{ and a.e. on } \partial\Omega.$$

Therefore, there exists $A \subset \partial\Omega$ such that $T_k\left(u_\epsilon\right)$ converges to $T_k\left(u\right)$ on $\partial\Omega \setminus A$ with $\sigma\left(A\right) = 0$, where σ is the area measure on $\partial\Omega$.

For every $k > 0$, let $A_k = \{x \in \partial\Omega : |T_k\left(u\left(x\right)\right)| < k\}$ and $B = \partial\Omega \setminus \bigcup_{k>0} A_k$. We have

$$\begin{aligned}\sigma\left(B\right) &= \frac{1}{k}\int_B |T_k\left(u\right)| \, \mathrm{d}\sigma \\ &\leq \frac{1}{k}\left\|T_k\left(u\right)\right\|_{L^1(\partial\Omega)} \\ &\leq \frac{C_6}{k}\left\|T_k\left(u\right)\right\|_{W^{1,1}(\Omega)} \\ &\leq \frac{C_7}{k}\left\|T_k\left(u\right)\right\|_{1,p(.)}.\end{aligned} \tag{3.46}$$

We know that for all $k>1$, $\rho_{1,p(.)}\left(T_k\left(u_\epsilon\right)\right)\leq kM$, where M is a positive constant which does not depend on ϵ. Then

$$\int_\Omega \left|T_k\left(u_\epsilon\right)\right|^{p(x)}\,\mathrm{d}x+\int_\Omega \left|\nabla T_k\left(u_\epsilon\right)\right|^{p(x)}\,\mathrm{d}x\leq kM. \tag{3.47}$$

We now use Fatou's lemma in (3.47) to get

$$\int_\Omega \left|T_k\left(u\right)\right|^{p(x)}\,\mathrm{d}x+\int_\Omega \left|\nabla T_k\left(u\right)\right|^{p(x)}\,\mathrm{d}x\leq kM,$$

which is equivalent to

$$\rho_{1,p(.)}\left(T_k\left(u\right)\right)\leq kM. \tag{3.48}$$

According to (3.48), we deduce that

$$\left\|T_k\left(u\right)\right\|_{W^{1,p(.)}(\Omega)}\leq C_7\left(k^{\frac{1}{p_-}}+k^{\frac{1}{p_+}}\right).$$

Therefore, by letting $k\to\infty$ in (3.46), we get that $\sigma\left(B\right)=0$.

Let us now define on $\partial\Omega$, the function v by

$$v\left(x\right)=T_k\left(u\left(x\right)\right)\quad \text{if}\quad x\in A_k.$$

We take $x\in\partial\Omega\setminus\left(A\cup B\right)$; then, there exists $k>0$ such that $x\in A_k$ and we have

$$u_\epsilon\left(x\right)-v\left(x\right)=\left(u_\epsilon\left(x\right)-T_k\left(u_\epsilon\left(x\right)\right)\right)+\left(T_k\left(u_\epsilon\left(x\right)\right)-T_k\left(u\left(x\right)\right)\right).$$

Since $x\in A_k$, we have $\left|T_k\left(u_\epsilon\left(x\right)\right)\right|<k$, from which we deduce that $\left|u_\epsilon\left(x\right)\right|<k$. Therefore

$$u_\epsilon\left(x\right)-v\left(x\right)=\left(T_k\left(u_\epsilon\left(x\right)\right)-T_k\left(u\left(x\right)\right)\right)\to 0 \text{ as } \epsilon\to 0.$$

This means that u_ϵ converges to v a.e. on $\partial\Omega$. □

Assertion 5. u is an entropy solution solution of the problem $P\left(\mu\right)$.

Proof. Since the sequence $\left(\nabla T_k\left(u_\epsilon\right)\right)_{\epsilon>0}$ converges in measure to $\nabla T_k\left(u\right)$, by (3.19) and Lemma 2.6, we get

$$\nabla T_k\left(u_\epsilon\right)\to\nabla T_k\left(u\right)\ \text{in}\ \left(L^1\left(\Omega\right)\right)^N\quad \forall k>0. \tag{3.49}$$

Consequently, the Assertions 2 and 4 as well as (3.49) give $u\in\mathcal{T}^{1,p(.)}_{\mathrm{tr}}\left(\Omega\right)$.

Let $\varphi\in W^{1,p(.)}\left(\Omega\right)\cap L^\infty\left(\Omega\right)$; we take $v=T_k\left(u_\epsilon-\varphi\right)$ as a test function in (3.3) to get

$$\begin{aligned}\int_\Omega \Phi\left(\nabla u_\epsilon-\Theta\left(u_\epsilon\right)\right)\nabla T_k\left(u_\epsilon-\varphi\right)\,\mathrm{d}x&+\int_\Omega\left|u_\epsilon\right|^{p(x)-2}u_\epsilon T_k\left(u_\epsilon-\varphi\right)\,\mathrm{d}x\\ +\int_\Omega T_{\frac{1}{\epsilon}}\left(\alpha\left(u_\epsilon\right)\right)T_k\left(u_\epsilon-\varphi\right)\,\mathrm{d}x&+\int_{\partial\Omega}T_{\frac{1}{\epsilon}}\left(\gamma\left(u_\epsilon\right)\right)T_k\left(u_\epsilon-\varphi\right)\,\mathrm{d}\sigma\\ =\int_{\partial\Omega}T_{\frac{1}{\epsilon}}\left(g\right)T_k\left(u_\epsilon-\varphi\right)\,\mathrm{d}\sigma&+\int_\Omega T_k\left(u_\epsilon-\varphi\right)\,\mathrm{d}\mu_\epsilon.\end{aligned} \tag{3.50}$$

Let $\bar{k} = k + \|\varphi\|_{\infty}$. We have

$$\begin{aligned}\int_\Omega \Phi\left(\nabla u_\epsilon - \Theta\left(u_\epsilon\right)\right)\nabla T_k\left(u_\epsilon - \varphi\right)\,\mathrm{d}x &= \int_\Omega \Phi\left(\nabla T_{\bar{k}}\left(u_\epsilon\right) - \Theta\left(T_{\bar{k}}\left(u_\epsilon\right)\right)\right)\nabla T_k\left(T_{\bar{k}}\left(u_\epsilon\right) - \varphi\right)\,\mathrm{d}x \\ &= \int_\Omega \Phi\left(\nabla T_{\bar{k}}\left(u_\epsilon\right) - \Theta\left(u_\epsilon\right)\right)\nabla T_{\bar{k}}\left(u_\epsilon\right)\chi_{\Omega\left(\epsilon,\bar{k}\right)}\,\mathrm{d}x \\ &\quad - \int_\Omega \Phi\left(\nabla T_{\bar{k}}\left(u_\epsilon\right) - \Theta\left(u_\epsilon\right)\right)\nabla\varphi\chi_{\Omega\left(\epsilon,\bar{k}\right)}\,\mathrm{d}x,\end{aligned}$$

where $\Omega\left(\epsilon,\bar{k}\right) = \left\{\left|T_{\bar{k}}\left(u_\epsilon\right) - \varphi\right| \leq k\right\}$ and χ_B is the characteristic function of the set $B\subset\mathbb{R}^N$.

The inequality (3.50) can be written as

$$\begin{aligned}&\int_\Omega \left(\Phi\left(\nabla T_{\bar{k}}\left(u_\epsilon\right) - \Theta\left(T_{\bar{k}}\left(u_\epsilon\right)\right)\right)\nabla T_{\bar{k}}\left(u_\epsilon\right) + \frac{1}{p\left(x\right)}\left|\Theta\left(T_{\bar{k}}\left(u_\epsilon\right)\right)\right|^{p(x)}\right)\chi_{\Omega\left(\epsilon,\bar{k}\right)}\,\mathrm{d}x \\ &\quad - \int_\Omega \Phi\left(\nabla T_{\bar{k}}\left(u_\epsilon\right) - \Theta\left(T_{\bar{k}}\left(u_\epsilon\right)\right)\right)\nabla\varphi\chi_{\Omega\left(\epsilon,\bar{k}\right)} + \int_\Omega \left|u_\epsilon\right|^{p(x)-2}u_\epsilon T_k\left(u_\epsilon - \varphi\right)\,\mathrm{d}x \\ &\quad + \int_\Omega T_{\frac{1}{\epsilon}}\left(\alpha\left(u_\epsilon\right)\right)T_k\left(u_\epsilon - \varphi\right)\,\mathrm{d}x + \int_{\partial\Omega} T_{\frac{1}{\epsilon}}\left(\gamma\left(u_\epsilon\right)\right)T_k\left(u_\epsilon - \varphi\right)\,\mathrm{d}\sigma \\ &\quad = \int_\Omega T_k\left(u_\epsilon - \varphi\right)\,\mathrm{d}\mu_\epsilon + \int_{\partial\Omega} T_{\frac{1}{\epsilon}}\left(g\right)T_k\left(u_\epsilon - \varphi\right)\,\mathrm{d}\sigma \\ &\quad + \int_\Omega \frac{1}{p\left(x\right)}\left|\Theta\left(T_{\bar{k}}\left(u_\epsilon\right)\right)\right|^{p(x)}\chi_{\Omega\left(\epsilon,\bar{k}\right)}\,\mathrm{d}x.\end{aligned} \tag{3.51}$$

Since

$$\nabla T_{\bar{k}}\left(u_\epsilon\right) \rightharpoonup \nabla T_{\bar{k}}\left(u\right) \text{ in } \left(L^{p(.)}\left(\Omega\right)\right)^N$$

and

$$\Theta\left(T_{\bar{k}}\left(u_\epsilon\right)\right) \to \Theta\left(T_{\bar{k}}\left(u\right)\right) \text{ in } \left(L^{p(.)}\left(\Omega\right)\right)^N, \tag{3.52}$$

then

$$\nabla T_{\bar{k}}\left(u_\epsilon\right) - \Theta\left(T_{\bar{k}}\left(u_\epsilon\right)\right) \rightharpoonup \nabla T_{\bar{k}}\left(u\right) - \Theta\left(T_{\bar{k}}\left(u\right)\right) \text{ in } \left(L^{p(.)}\left(\Omega\right)\right)^N.$$

Thus

$$\Phi\left(\nabla T_{\bar{k}}\left(u_\epsilon\right) - \Theta\left(T_{\bar{k}}\left(u_\epsilon\right)\right)\right) \rightharpoonup \Phi\left(\nabla T_{\bar{k}}\left(u\right) - \Theta\left(T_{\bar{k}}\left(u\right)\right)\right) \text{ in } \left(L^{p'(.)}\left(\Omega\right)\right)^N. \tag{3.53}$$

Furthermore

$$\nabla\varphi\chi_{\Omega\left(\epsilon,\bar{k}\right)} \to \nabla\varphi\chi_{\Omega\left(\bar{k}\right)} \text{ in } \left(L^{p(.)}\left(\Omega\right)\right)^N,$$

with $\Omega\left(\bar{k}\right) = \left\{\left|T_{\bar{k}}\left(u\right) - \varphi\right| \leq k\right\}$. Then

$$\begin{aligned}&\int_\Omega \Phi\left(\nabla T_{\bar{k}}\left(u_\epsilon\right) - \Theta\left(T_{\bar{k}}\left(u_\epsilon\right)\right)\right)\nabla\varphi\chi_{\Omega\left(\epsilon,\bar{k}\right)}\,\mathrm{d}x \to \\ &\qquad \to \int_\Omega \Phi\left(\nabla T_{\bar{k}}\left(u\right) - \Theta\left(T_{\bar{k}}\left(u\right)\right)\right)\nabla\varphi\chi_{\Omega\left(\bar{k}\right)}\,\mathrm{d}x.\end{aligned} \tag{3.54}$$

According to (H_3) and the properties of the truncation function, we get

$$\left|\Theta\left(T_{\bar{k}}\left(u_\epsilon\right)\right)\right|^{p(x)} \leq \left(C\bar{k}\right)^{p(x)}.$$

Using (3.52) and the Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega \frac{1}{p(x)} \left|\Theta\left(T_{\bar{k}}\left(u_\epsilon\right)\right)\right|^{p(x)} \chi_{\Omega\left(\epsilon,\bar{k}\right)} \,\mathrm{d}x \to \int_\Omega \frac{1}{p(x)} \left|\Theta\left(T_{\bar{k}}(u)\right)\right|^{p(x)} \chi_{\Omega\left(\bar{k}\right)} \,\mathrm{d}x.$$

Now, since

$$\left(\Phi\left(\nabla T_{\bar{k}}\left(u_\epsilon\right) - \Theta\left(T_{\bar{k}}\left(u_\epsilon\right)\right)\right)\nabla T_{\bar{k}}\left(u_\epsilon\right) + \frac{1}{p(x)}\left|\Theta\left(T_{\bar{k}}\left(u_\epsilon\right)\right)\right|^{p(x)}\right)\chi_{\Omega\left(\epsilon,\bar{k}\right)} \geq 0 \text{ a.e. in } \Omega,$$

by using Fatou's lemma, we obtain

$$\begin{aligned}
&\int_\Omega \left(\Phi\left(\nabla T_{\bar{k}}(u) - \Theta\left(T_{\bar{k}}(u)\right)\right)\nabla T_{\bar{k}}(u) + \frac{1}{p(x)}\left|\Theta\left(T_{\bar{k}}(u)\right)\right|^{p(x)}\right)\chi_{\Omega\left(\bar{k}\right)} \,\mathrm{d}x \\
&\qquad \leq \liminf_{\epsilon\to 0}\left(\int_\Omega \left(\Phi\left(\nabla T_{\bar{k}}\left(u_\epsilon\right) - \Theta\left(T_{\bar{k}}\left(u_\epsilon\right)\right)\right)\nabla T_{\bar{k}}\left(u_\epsilon\right) + \right.\right. \\
&\qquad\qquad \left.\left. + \frac{1}{p(x)}\left|\Theta\left(T_{\bar{k}}\left(u_\epsilon\right)\right)\right|^{p(x)}\right)\chi_{\Omega\left(\epsilon,\bar{k}\right)} \,\mathrm{d}x\right).
\end{aligned} \tag{3.55}$$

Let us prove that

$$\int_\Omega T_k\left(u_\epsilon - \varphi\right) \,\mathrm{d}\mu_\epsilon = \int_\Omega T_k\left(u - \varphi\right) \,\mathrm{d}\mu. \tag{3.56}$$

We have

$$\begin{aligned}
\int_\Omega T_k\left(u_\epsilon - \varphi\right) \,\mathrm{d}\mu_\epsilon &= \int_\Omega E\left(T_k\left(u_\epsilon - \varphi\right)\right) \,\mathrm{d}\mu_\epsilon \\
&= \left\langle \mu_\epsilon, E\left(T_k\left(u_\epsilon - \varphi\right)\right)\right\rangle \\
&= \int_{U_\Omega} f_\epsilon E\left(T_k\left(u_\epsilon - \varphi\right)\right) \,\mathrm{d}x \\
&\quad + \int_{U_\Omega} F_R.\nabla E\left(T_k\left(u_\epsilon - \varphi\right)\right) \,\mathrm{d}x \\
&= \int_{U_\Omega} T_{\frac{1}{\epsilon}}(f)\,\chi_\Omega E\left(T_k\left(u_\epsilon - \varphi\right)\right) \,\mathrm{d}x \\
&\quad + \int_{U_\Omega} \left(F\chi_\Omega\right).\nabla E\left(T_k\left(u_\epsilon - \varphi\right)\right) \,\mathrm{d}x \\
&= \int_\Omega T_{\frac{1}{\epsilon}}(f)\left(T_k\left(u_\epsilon - \varphi\right)\right) \,\mathrm{d}x \\
&\quad + \int_{U_\Omega} F.\nabla E\left(\chi_\Omega T_k\left(u_\epsilon - \varphi\right)\right) \,\mathrm{d}x.
\end{aligned} \tag{3.57}$$

Thanks to the Lebesgue dominated convergence theorem, we have

$$\lim_{\epsilon\to 0}\int_\Omega T_{\frac{1}{\epsilon}}(f)\,T_k\left(u_\epsilon - \varphi\right) \,\mathrm{d}x = \int_\Omega f T_k\left(u - \varphi\right) \,\mathrm{d}x. \tag{3.58}$$

Since the sequence $\left(\chi_\Omega T_k\left(u_\epsilon - \varphi\right)\right)$ is bounded in $W^{1,p(.)}(\Omega)$, by using the property (ii) of the operator E, we deduce that $E\left(\left(\chi_\Omega T_k\left(u_\epsilon - \varphi\right)\right)\right)$ is bounded in $W_0^{1,p(.)}\left(U_\Omega\right)$. Moreover, we have

$$E\left(\left(\chi_\Omega T_k\left(u_\epsilon - \varphi\right)\right)\right) = \chi_\Omega T_k\left(u_\epsilon - \varphi\right) \text{ a.e. in } U_\Omega$$

and

$$\chi_\Omega T_k\left(u_\epsilon-\varphi\right)\to\chi_\Omega T_k\left(u-\varphi\right)\text{ a.e. in }U_\Omega\text{ as }\epsilon\to 0,$$

which implies that

$$E\left(\chi_\Omega T_k\left(u_\epsilon-\varphi\right)\right)\to E\left(\chi_\Omega T_k\left(u-\varphi\right)\right)\text{ a.e. in }U_\Omega\text{ as }\epsilon\to 0.$$

Consequently, we have

$$\nabla E\left(\chi_\Omega T_k\left(u_\epsilon-\varphi\right)\right)\rightharpoonup\nabla E\left(\chi_\Omega T_k\left(u-\varphi\right)\right)\text{ in }\left(L^{p(\cdot)}\left(U_\Omega\right)\right)^N.$$

Thus, by using the fact that $F\in\left(L^{p'(\cdot)}\left(U_\Omega\right)\right)^N$, we deduce that

$$\lim_{\epsilon\to 0}\int_{U_\Omega}F.\nabla E\left(\chi_\Omega T_k\left(u_\epsilon-\varphi\right)\right)\,\mathrm{d}x=\int_{U_\Omega}F.\nabla E\left(\chi_\Omega T_k\left(u-\varphi\right)\right)\,\mathrm{d}x. \tag{3.59}$$

Hence, by passing to the limit in (3.57) and using (3.58) and (3.59), we obtain

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega T_k\left(u_\epsilon-\varphi\right)\,\mathrm{d}\mu_\epsilon&=\int_\Omega f\left(T_k\left(u-\varphi\right)\right)\,\mathrm{d}x+\int_{U_\Omega}F.\nabla E\left(\chi_\Omega T_k\left(u-\varphi\right)\right)\,\mathrm{d}x.\\&=\int_{U_\Omega}fE\left(\chi_\Omega\left(T_k\left(u-\varphi\right)\right)\right)\,\mathrm{d}x+\int_{U_\Omega}F.\nabla E\left(\chi_\Omega T_k\left(u-\varphi\right)\right)\,\mathrm{d}x.\\&=\left\langle\mu,E\left(T_k\left(u-\varphi\right)\right)\right\rangle\\&=\int_\Omega T_k\left(u-\varphi\right)\,\mathrm{d}\mu.\end{aligned} \tag{3.60}$$

By using again the Lebesgue dominated convergence theorem, we get

$$\lim_{\epsilon\to 0}\int_{\partial\Omega}T_{\frac{1}{\epsilon}}\left(g\right)T_k\left(u_\epsilon-\varphi\right)\,\mathrm{d}\sigma=\int_{\partial\Omega}gT_k\left(u-\varphi\right)\,\mathrm{d}\sigma. \tag{3.61}$$

We can rewrite the third term of (3.51) as

$$\begin{aligned}\int_\Omega\left|u_\epsilon\right|^{p(x)-2}u_\epsilon T_k\left(u_\epsilon-\varphi\right)\,\mathrm{d}x=&\int_\Omega\left(\left|u_\epsilon\right|^{p(x)-2}u_\epsilon-\left|\varphi\right|^{p(x)-2}\varphi\right)T_k\left(u_\epsilon-\varphi\right)\,\mathrm{d}x\\&+\int_\Omega\left|\varphi\right|^{p(x)-2}\varphi T_k\left(u_\epsilon-\varphi\right)\,\mathrm{d}x.\end{aligned}$$

Let us note that $\left(\left|u_\epsilon\right|^{p(x)-2}u_\epsilon-\left|\varphi\right|^{p(x)-2}\varphi\right)T_k\left(u_\epsilon-\varphi\right)$ is non-negative and converges to $\left(\left|u\right|^{p(x)-2}u-\left|\varphi\right|^{p(x)-2}\varphi\right)T_k\left(u-\varphi\right)$ a.e. in Ω. So, thanks to the Fatou's lemma, we have

$$\begin{aligned}&\int_\Omega\left(\left|u\right|^{p(x)-2}u-\left|\varphi\right|^{p(x)-2}\varphi\right)T_k\left(u-\varphi\right)\,\mathrm{d}x\\&\qquad\le\liminf_{\epsilon\to 0}\int_\Omega\left(\left|u_\epsilon\right|^{p(x)-2}u_\epsilon-\left|\varphi\right|^{p(x)-2}\varphi\right)T_k\left(u_\epsilon-\varphi\right)\,\mathrm{d}x.\end{aligned} \tag{3.62}$$

Moreover, $\left(T_k\left(u_\epsilon-\varphi\right)\right)_{\epsilon>0}$ converges weak-$*$ to $T_k\left(u-\varphi\right)$ in $L^\infty\left(\Omega\right)$ and $\left|\varphi\right|^{p(x)-2}\varphi\in L^1\left(\Omega\right)$; it follows that

$$\lim_{\epsilon\to 0}\int_\Omega\left|\varphi\right|^{p(x)-2}\varphi T_k\left(u_\epsilon-\varphi\right)\,\mathrm{d}x=\int_\Omega\left|\varphi\right|^{p(x)-2}\varphi T_k\left(u-\varphi\right)\,\mathrm{d}x. \tag{3.63}$$

Now, we focus our attention on the fourth term of (3.51). We have

$$T_{\frac{1}{\epsilon}}\left(\alpha\left(u_{\epsilon}\right)\right) T_{k}\left(u_{\epsilon}-\varphi\right) \rightarrow \alpha(u) T_{k}(u-\varphi) \text { a.e. in } \Omega \tag{3.64}$$

and

$$\left|T_{\frac{1}{\epsilon}}\left(\alpha\left(u_{\epsilon}\right)\right) T_{k}\left(u_{\epsilon}-\varphi\right)\right| \leq k\left|\alpha\left(u_{\epsilon}\right)\right| \in L^{1}(\Omega) . \tag{3.65}$$

From the assumption (H_1) we know that

$$k\left|\alpha\left(u_{\epsilon}\right)\right| \rightarrow k|\alpha(u)| \text { a.e. in } \Omega \tag{3.66}$$

and

$$k\left|\alpha\left(u_{\epsilon}\right)\right| \leq k M_{1} \in L^{1}(\Omega) . \tag{3.67}$$

Then, thanks to the Lebesgue dominated convergence theorem, we deduce that the sequence $\left(k\left|\alpha\left(u_{\epsilon}\right)\right|\right)_{\epsilon>0}$ converges to $k|\alpha(u)|$ in $L^{1}(\Omega)$. Consequently, using the Lebesgue generalized convergence theorem, we get

$$\lim _{\epsilon \rightarrow 0} \int_{\Omega} T_{\frac{1}{\epsilon}}\left(\alpha\left(u_{\epsilon}\right)\right) T_{k}\left(u_{\epsilon}-\varphi\right) \mathrm{d} x=\int_{\Omega} \alpha(u) T_{k}(u-\varphi) \mathrm{d} x . \tag{3.68}$$

Similarly, we have

$$\lim _{\epsilon \rightarrow 0} \int_{\partial \Omega} T_{\frac{1}{\epsilon}}\left(\gamma\left(u_{\epsilon}\right)\right) T_{k}\left(u_{\epsilon}-\varphi\right) \mathrm{d} \sigma=\int_{\partial \Omega} \gamma(u) T_{k}(u-\varphi) \mathrm{d} \sigma . \tag{3.69}$$

Finally, by passing to the limit as $\epsilon \rightarrow 0$ in the inequality (3.51) and using the results (3.55), (3.60), (3.61), (3.62), (3.63), (3.68) and (3.69), we deduce that u is an entropy solution of the problem $P(\mu)$. □

This concludes the proof of the second step and thus of the Theorem. □

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