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SOME PROPERTIES OF THE FUNDAMENTAL SOLUTIONS OF NON-STATIONARY THIRD ORDER COMPOSITE TYPE EQUATION IN MULTIDIMENSIONAL DOMAINS

A. R. KHASHIMOV*
Tashkent Financial Institute, Tashkent, Uzbekistan

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Abstract. In the paper, some basic properties for fundamental solutions of non-stationary third order composite type equation are studied in the neighborhood of some hyperplane of the domain in which fundamental solutions irregular. These properties give possibility to construct the classical solution of the Cauchy problem and the boundary value problems by a method of potentials.

Keywords: composite type, fundamental solutions, non-stationary equation, boundary value problems, solution, multidimensional domain.

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1 Introduction

Fundamental solutions of an equation, i.e. solutions with singularities of a certain type, play the important role in studying partial linear equations. Hence, special interest is spared to study of their properties. Investigations concerned with properties of fundamental solutions for even order equations, for example, for parabolic equations, were carried out sufficiently fully. Review of those investigations can be found in [9].

However, odd order equations have been investigated a little. For example, in work by Block, Del Vecchio, Cattabriga, Roetman, Djuraev and Abdinazarov [1, 3–8, 11] series of properties of

*e-mail address: abdukomil@yandex.ru

fundamental solutions for the equations

$$Lu \equiv \frac{\partial^3 u}{\partial x^3} - \frac{\partial u}{\partial t} = 0, \tag{1.1}$$

$$Lu \equiv \sum_{i=1}^{n} \frac{\partial^3 u}{\partial x^3} - \frac{\partial^2 u}{\partial y^2} = 0, \tag{1.2}$$

$$Lu \equiv \frac{\partial^{2n+1} u}{\partial x^{2n+1}} + (-1)^n \frac{\partial u}{\partial t} = 0 \tag{1.3}$$

were studied.

The obtained results were insufficient for studying third order equations such extensively as parabolic equations. For example, investigations conducted in [3] - [11] were insufficient to construct regular solutions of boundary value problems for equations of type (1.1)–(1.3) in multidimensional domains.

S. Abdinazarov and Z. Sobirov constructed fundamental solutions for third order equations and studied them in multidimensional domains (see [2]). They proved that the functions

$$\begin{aligned} &U(x_1-\xi_1, x_2-\xi_2, \dots, x_n-\xi_n; t-\tau) = \\ &= \frac{1}{(t-\tau)^{n/3}} f\left(\frac{x_1-\xi_1}{(t-\tau)^{1/3}}\right) f\left(\frac{x_2-\xi_2}{(t-\tau)^{1/3}}\right) \cdots f\left(\frac{x_n-\xi_n}{(t-\tau)^{1/3}}\right),\ x_i \neq \xi_i, t > \tau, i = \overline{1,n}, \end{aligned} \tag{1.4}$$

$$\begin{aligned} &V(x_1-\xi_1, x_2-\xi_2, \dots, x_n-\xi_n; t-\tau) = \\ &= \frac{1}{(t-\tau)^{n/3}} \varphi\left(\frac{x_1-\xi_1}{(t-\tau)^{1/3}}\right) \varphi\left(\frac{x_2-\xi_2}{(t-\tau)^{1/3}}\right) \cdots \varphi\left(\frac{x_n-\xi_n}{(t-\tau)^{1/3}}\right),\ x_i > \xi_i, t > \tau, i = \overline{1,n}, \end{aligned} \tag{1.5}$$

$$\begin{aligned} &U_j(x_1-\xi_1, x_2-\xi_2, \dots, x_n-\xi_n; t-\tau) = \\ &= \frac{1}{(t-\tau)^{n/3}} f\left(\frac{x_1-\xi_1}{(t-\tau)^{1/3}}\right) \cdots f\left(\frac{x_{j-1}-\xi_{j-1}}{(t-\tau)^{1/3}}\right) f\left(\frac{x_{j+1}-\xi_{j+1}}{(t-\tau)^{1/3}}\right) \cdots \\ &\qquad \cdots f\left(\frac{x_n-\xi_n}{(t-\tau)^{1/3}}\right) \varphi\left(\frac{x_j-\xi_j}{(t-\tau)^{1/3}}\right),\ x_i \neq \xi_i, x_j > \xi_j, t > \tau, j = \overline{1,n}, i = \overline{1,n}, j \neq i \end{aligned} \tag{1.6}$$

are fundamental solutions for the non-stationary third order composite type equation

$$Lu \equiv \sum_{i=1}^{n} \frac{\partial^3 u}{\partial x_i^3} - \frac{\partial u}{\partial t} = 0, \tag{1.7}$$

were the functions

$$\varphi(z) = \int_0^\infty \left[\exp(-\lambda^3 - \lambda z) + \sin(\lambda^3 - \lambda z)\right] \mathrm{d}\lambda, \ \ 0 < z < \infty,$$

$$f(z) = \int_0^\infty \cos(\lambda^3 - \lambda z)\, \mathrm{d}\lambda, \ \ -\infty < z < \infty, \ \text{ here } \ z = \frac{x-\xi}{(t-\tau)^{1/3}}$$

are Airy functions which satisfy to the equation (see [4])

$$p''(z) + \frac{z}{3} p(z) = 0. \tag{1.8}$$

The following relations are valid for functions $f(z)$, $\varphi(z)$ (see [1]):

$$p^{(n)}(z) \sim c_n^+ \, z^{n/2-1/4} \sin\left(\tfrac{2}{3}\, z^{1/3}\right), \text{ as } z \to \infty, \tag{1.9}$$

$$p^{(n)}(z) \sim c_n^- \, |z|^{n/2-1/4} \exp\left(-\tfrac{2}{3}\, |z|^{3/2}\right), \text{ as } z \to -\infty, \tag{1.10}$$

$$\int\limits_{-\infty}^{\infty} f(z)\,\mathrm{d}z = \pi, \quad \int\limits_{-\infty}^{0} f(z)\,\mathrm{d}z = \frac{\pi}{3}, \quad \int\limits_{0}^{\infty} f(z)\,\mathrm{d}z = \frac{2\pi}{3}, \quad \int\limits_{0}^{\infty} \varphi(z)\,\mathrm{d}z = 0, \tag{1.11}$$

where c_n^+, c_n^- are constants.

In the present paper we establish some important properties for potentials of fundamental solutions of the equation (1.7) in the neighborhood of the hyperplane $(x_1 = x_1^0, x_2, \ldots, x_n, t) \in \mathbb{R}^{n+1}$, $x_1^0 = const$.

It should be noted, analogous study for the case of $i = 2$ was carried out in [10].

2 Basic results

2.1 Properties of fundamental solutions

Let the equation (1.7) be defined in the domain $D = \Omega \times (0, T)$, $\Omega = \mathbb{R}^n$.

Consider the following integrals

$$I_0^l(x,t) = \int\limits_0^t \int\limits_{\Omega'} U_l\left(x_1 - x_1^0, x_2 - \xi_2, \ldots, x_n - \xi_n; t - \tau\right) \alpha(\xi', \tau)\,\mathrm{d}\xi'\,\mathrm{d}\tau, \ \ l = \overline{0, n},$$

$$J_0(x,t) = \int\limits_0^t \int\limits_{\Omega'} V\left(x_1 - x_1^0, x_2 - \xi_2, \ldots, x_n - \xi_n; t - \tau\right) \alpha(\xi', \tau)\,\mathrm{d}\xi'\,\mathrm{d}\tau,$$

$$I_1^l(x,t) = \int\limits_0^t \int\limits_{\Omega'} \frac{\partial}{\partial \xi_1} U_l\left(x_1 - x_1^0, x_2 - \xi_2, \ldots, x_n - \xi_n; t - \tau\right) \alpha(\xi', \tau)\,\mathrm{d}\xi'\,\mathrm{d}\tau, \ \ l = \overline{0, n},$$

$$J_1(x,t) = \int\limits_0^t \int\limits_{\Omega'} \frac{\partial}{\partial \xi_1} V\left(x_1 - x_1^0, x_2 - \xi_2, \ldots, x_n - \xi_n; t - \tau\right) \alpha(\xi', \tau)\,\mathrm{d}\xi'\,\mathrm{d}\tau,$$

where $\Omega' = \{x' : x' \in \mathbb{R}^{n-1}\}$, $x' = (x_2, x_3, \ldots, x_n)$, $\mathrm{d}\xi' = \mathrm{d}\xi_2\,\mathrm{d}\xi_2 \ldots \mathrm{d}\xi_n$.

These integrals are analogs of simple layer potentials and they are continuous in a neighborhood of the hyperplane $(x_1 = x_1^0, x_2, \ldots, x_n, t) \subset D$. Since the scheme of investigation of the integrals is the same as in [6], we shall not turn our attention to investigation of their properties.

It is known, double layer potentials are important in constructing regular solutions of boundary value problems. Therefore we shall study some properties of the following potentials

$$I_2^l(x,t) = \int\limits_0^t \int\limits_{\Omega'} \frac{\partial^2}{\partial \xi_1^2} U_l\left(x_1 - x_1^0, x_2 - \xi_2, \ldots, x_n - \xi_n; t - \tau\right) \alpha\left(\xi', \tau\right)\,\mathrm{d}\xi'\,\mathrm{d}\tau, \ l = \overline{0, n},$$

and

$$J_2(x,t) = \iint\limits_{0\,\Omega'}^{t} \frac{\partial^2}{\partial \xi_1^2} V\left(x_1 - x_2^0, x_2 - \xi_2, \dots, x_n - \xi_n; t-\tau\right) \alpha\left(\xi', \tau\right)\, \mathrm{d}\xi'\, \mathrm{d}\tau,$$

which are analogs of double layer potentials in the neighborhood of the hyperplane $(x_1 = x_1^0, x_2, \dots, x_n, t) \subset D$.

Now we show that expressions $I_2^l(x,t)$ and $J_2(x,t)$ can have a discontinuity of the first kind as $(x_1, x_2, \dots, x_n, t) \to \left(x_1^0 \pm 0, x_2, \dots, x_n, t\right)$ of a different character depending on behavior of fundamental solutions (1.4)–(1.6).

Let K be arbitrary bounded domain from Ω, functions (1.4)–(1.6) be defined in $K \times (0,T)$.

Lemma 1 *Let the function* $\alpha(x',t) \in C\left(\overline{K'} \times [0,T]\right)$. *Then*

$$\lim_{(x_1,x_2,\dots,x_n,t)\to(x_1^0-0,x_2,\dots,x_n,t)} \iint\limits_{0\,\Omega'}^{t} U_{\xi_1\xi_1}(x_1 - x_1^0, x_2 - \xi_2, \dots, x_n - \xi_n; t-\tau)\alpha(\xi',\tau)\,\mathrm{d}\xi'\,\mathrm{d}\tau = \frac{\pi^n}{3}\alpha(x',t),$$

where $K' \subset \Omega'$ *is an arbitrary bounded domain.*

Proof. For definiteness, we suppose $K = \{x : \lambda_i \le x_i \le \lambda_{i+1}\} \subset \Omega$, i.e. it is sufficient to prove the statement of Lemma 1 for domains of the canonical form. For simplicity, here we suppose that the hyperplane $(x_1^0, x_2, \dots, x_n, t) \subset K$ is the boundary of K, then in our case properties of the potentials $I_2^0(x,t)$ as $(x_1, x_2, \dots, x_n, t) \to (\lambda_2 - 0, x_2, \dots, x_n, t)$ will be studied.

Using the relation (1.8), we transform the expression of $I_2^0(x,t)$ in the domain $K \times [0,T]$, then we obtain

$$I_2^0(x,t) = -\int\limits_0^t \frac{x_1 - \lambda_2}{3(t-\tau)^{4/3}} f\left(\frac{x_1-\lambda_2}{(t-\tau)^{1/3}}\right) \mathrm{d}\tau \int\limits_{K'} \frac{1}{(t-\tau)^{n-1/3}} f^*\left(\frac{x'-\xi'}{(t-\tau)^{1/3}}\right) \alpha(\xi',\tau)\,\mathrm{d}\xi',$$

where $K' = \{x_j : \lambda_j \le x_j \le \lambda_{j+1}\} \subset \Omega'$, $j = \overline{2,n}$,

$$f^*\left(\frac{x'-\xi'}{(t-\tau)^{1/3}}\right) = f\left(\frac{x_2-\xi_2}{(t-\tau)^{1/3}}\right) f\left(\frac{x_3-\xi_3}{(t-\tau)^{1/3}}\right) \cdots f\left(\frac{x_n-\xi_n}{(t-\tau)^{1/3}}\right).$$

Transforming this integral, i.e. adding and subtracting the function $\alpha(x',t)$, we obtain

$$\begin{aligned} I_2^0(x,t) = &-\int\limits_0^t \frac{x_1-\lambda_2}{3(t-\tau)^{1/3}} f\left(\frac{x_1-\lambda_2}{(t-\tau)^{1/3}}\right) \mathrm{d}\tau \times \\ &\times \int\limits_{K'} \frac{1}{(t-\tau)^{1/3}} f^*\left(\frac{x'-\xi'}{(t-\tau)^{1/3}}\right) (\alpha(\xi',\tau) - \alpha(x',t))\,\mathrm{d}\xi' - \\ &- \alpha(x',t) \int\limits_0^t \frac{x_1-\lambda_2}{3(t-\tau)^{1/3}} f\left(\frac{x_1-\lambda_2}{(t-\tau)^{1/3}}\right) \mathrm{d}\tau \int\limits_{K'} \frac{1}{(t-\tau)^{1/3}} f^*\left(\frac{x'-\xi'}{(t-\tau)^{1/3}}\right) \mathrm{d}\xi' \\ &= I_{21}^0(x,t) + \alpha(x',t) I_{22}^0(x,t). \end{aligned} \tag{2.1}$$

Now we consider each integral in (2.1) separately.

If we change variables in the second integral $I_{22}(x,t)$ of (2.1): $z=\frac{x_1-\lambda_2}{(t-\tau)^{1/3}}$ and $\theta'=\frac{x'-\xi'}{(t-\tau)^{1/3}}$, then we obtain

$$I_{22}^{0}(x,t)=\int\limits_{-\infty}^{\frac{x_1-\lambda_2}{t^{1/3}}} f(z)\,\mathrm{d}z\int\limits_{\widetilde{K'}} f^{*}(\theta')d\theta'. \tag{2.2}$$

Passing to the limit as $(x_1,x_2,\dots,x_n,t)\to(\lambda_2-0,x_2,\dots,x_n,t)$ and taking (1.11) into account, we have

$$\lim_{(x_1,x_2,\dots,x_n,t)\to(\lambda_2-0,x_2,\dots,x_n,t)} I_{22}^{0}(x,t)=\frac{\pi^n}{3}.$$

Now we change variables in the integral $I_{21}^{0}(x,t)$ of (2.1): $z=\frac{\lambda_2-x_1}{(t-\tau)^{1/3}}$, $\theta'=\frac{x'-\xi'}{(t-\tau)^{1/3}}$. Then the integral $I_{21}^{0}(x,t)$ takes the form:

$$I_{21}^{0}(x,t)=\int\limits_{\frac{\lambda_2-x_1}{t^{1/3}}}^{+\infty} f(-z)\,\mathrm{d}z\int\limits_{\widetilde{K'}} f^{*}(\theta')\left[\alpha\left(x'-\theta'\frac{(\lambda_2-x_1)}{z},t-\frac{(\lambda_2-x_1)^3}{z^3}\right)-\alpha(x',t)\right]\mathrm{d}\theta'.$$

Further, since the function $\alpha(x',t)$ is continuous on the point (x',t), for any $\varepsilon>0$ there can be found $\delta(\varepsilon)$ such that

$$\left|\alpha(x',t)-\alpha(x'-h_1,t-h_2)\right|<\varepsilon,$$

as $\max\{h_1,h_2\}<\delta(\varepsilon)$.

For the fixed $t>0$ one can always take $0<\delta(\varepsilon)<t^{1/3}$. Then

$$\frac{\lambda_2-x_1}{t^{1/3}}<\frac{\lambda_2-x_1}{\delta(\varepsilon)}<\infty$$

and the following representation

$$I_{21}^{0}(x,t)=\int\limits_{\frac{\lambda_2-x_1}{t^{1/3}}}^{\frac{\lambda_2-x_1}{\delta(\varepsilon)}} f(-z)\,\mathrm{d}z\int\limits_{\widetilde{K'}} f^{*}(\theta')[\alpha(x'-\theta'\frac{(\lambda_2-x_1)}{z},t-\frac{(\lambda_2-x_1)^3}{z^3})-\alpha(x',t)]\,\mathrm{d}\theta'+$$

$$+\int\limits_{\frac{\lambda_2-x_1}{\delta(\varepsilon)}}^{+\infty} f(-z)\,\mathrm{d}z\int\limits_{\widetilde{K'}} f^{*}\left(\theta'\right)[\alpha(x'-\theta'\frac{(\lambda_2-x_1)}{z},t-\frac{(\lambda_2-x_1)^3}{z^3})-\alpha(x',t)]\,\mathrm{d}\theta'=$$

$$=I_{211}^{0}(x,t)+I_{212}^{0}(x,t)$$

is valid.

Considering continuity of the function $\alpha(x',t)$ and properties of the function $f(z)$, we infer:

$$I_{211}^{0}(x,t)\le 2\max_{\theta',\tau}\left|\alpha(\theta',\tau)\right|\int\limits_{\frac{\lambda_2-x_1}{t^{1/3}}}^{\frac{\lambda_2-x_1}{\delta(\varepsilon)}}\left|f(-z)\right|\mathrm{d}z\int\limits_{\widetilde{K'}}\left|f^{*}(\theta')\right|\mathrm{d}\theta'.$$

Moreover, for the fixed $\delta(\varepsilon)$ and $t > 0$, by virtue of validity of the relation (1.10) for the function $f(-z)$, we have

$$\lim_{(x_1,x_2,\dots,x_n,t)\to(\lambda_2-0,x_2,\dots,x_n,t)} I^0_{211}(x,t) = 0.$$

Further, noting that for

$$0 \le \frac{\lambda_2 - x_1}{\delta(\varepsilon)} \le p < \infty$$

the following inequality

$$0 \le \frac{\lambda_2 - x_1}{p} \le \delta(\varepsilon) < \infty$$

takes place, we get

$$I^0_{212}(x,t) \le \max_{h_1\le\delta(\varepsilon),h_2\le\delta(\varepsilon)} \left|\alpha(\xi',\tau) - \alpha(\xi'-h_1,\tau-h_2)\right| \int\limits_0^{+\infty} |f(-z)|\,\mathrm{d}z \int\limits_{\widetilde{K'}} \left|f^*(\theta')\right| \,\mathrm{d}\theta'.$$

Hence, considering (1.10), relations

$$\int\limits_0^\infty x^q \sin(ax+b)\,\mathrm{d}x = a^{\frac{1}{q+1}}\Gamma(1+q)\cos(b+\frac{q\pi}{2}), a>0, -1<q<0,$$

$$\int\limits_0^\infty x^q \cos(ax+b)\,\mathrm{d}x = a^{\frac{1}{q+1}}\Gamma(1+q)\sin(b+\frac{q\pi}{2}), a>0, -1<q<0,$$

and also continuity of $\alpha(x',t)$, and arbitrariness of ε, we obtain

$$\lim_{(x_1,x_2,\dots,x_n,t)\to(\lambda_2-0,x_2,\dots,x_n,t)} I^0_{212}(x,t) = 0.$$

Finally

$$\lim_{(x_1,x_2,\dots,x_n,t)\to(\lambda_2-0,x_2,\dots,x_n,t)} I^0_{2}(x,t) = \frac{\pi^n}{3}\alpha(x',t).$$

□

Lemma 2 *Let the function $\alpha(x',t) \in C\left(\overline{K'}\times[0,T]\right)$ and be satisfied to the Hölder inequality with the index $\beta > \frac{3}{4}$ by t. Then*

$$\lim_{(x_1,x',t)\to(x_1^0+0,x',t)} \iint\limits_{0\,\Omega'}^{t} \frac{\partial^2}{\partial\xi_1^2} U_0(x_1-x_1^0, x'-\xi'; t-\tau)\alpha(\xi',\tau)\,\mathrm{d}\xi'\,\mathrm{d}\tau = -\frac{2\pi^n}{3}\alpha(x',t), \quad (2.3)$$

$$\lim_{(x_1,x',t)\to(x_1^0+0,x',t)} \iint\limits_{0\,\Omega'}^{t} \frac{\partial^2}{\partial\xi_1^2} U_1(x_1-x_1^0, x'-\xi'; t-\tau)\alpha(\xi',\tau)\,\mathrm{d}\xi'\,\mathrm{d}\tau = 0, \quad (2.4)$$

$$\lim_{(x_1,x',t)\to(x_1^0+0,x',t)} \iint\limits_{0\,\Omega'}^{t} \frac{\partial^2}{\partial\xi_1^2} V(x_1-x_1^0, x'-\xi'; t-\tau)\alpha(\xi',\tau)\,\mathrm{d}\xi'\,\mathrm{d}\tau = 0, \quad (2.5)$$

where $K' \subset \Omega'$ is an arbitrary bounded domain.

Proof. Let $K = \{x : \lambda_i \le x_i \le \lambda_{i+1}\}$, and the hyperplane $(x_1^0 = \lambda_1, x_2, \dots, x_n, t) \subset K$ be the boundary of K. We consider the case of $(x_1, x_2, \dots, t) \to (\lambda_1 + 0, x_2, \dots, x_n, t)$.

Using the relation (1.8), transform the expression of $I_2^0(x,t)$. Then we obtain

$$I_2^0(x,t) = -\int_0^t \frac{x_1-\lambda_1}{3(t-\tau)^{1/3}} f\left(\frac{x_1-\lambda_1}{(t-\tau)^{1/3}}\right) d\tau \int_{K'} \frac{1}{(t-\tau)^{1/3}} f^*\left(\frac{x'-\xi'}{(t-\tau)^{1/3}}\right) \alpha(\xi',\tau)\, d\xi'.$$

Transforming this integral, i.e. adding and subtracting the function $\alpha(x',t)$, we obtain

$$\begin{aligned} I_2^0(x,t) = &-\int_0^t \frac{x_1-\lambda_2}{3(t-\tau)^{1/3}} f\left(\frac{x_1-\lambda_2}{(t-\tau)^{1/3}}\right) d\tau \times \\ &\times \int_{K'} \frac{1}{(t-\tau)^{1/3}} f^*\left(\frac{x'-\xi'}{(t-\tau)^{1/3}}\right) \big(\alpha(\xi',\tau) - \alpha(x',t)\big)\, d\xi' - \\ &- \alpha(x',t) \int_0^t \frac{x_1-\lambda_1}{3(t-\tau)^{1/3}} f\left(\frac{x_1-\lambda_1}{(t-\tau)^{1/3}}\right) d\tau \int_{K'} \frac{1}{(t-\tau)^{1/3}} f^*\left(\frac{x'-\xi'}{(t-\tau)^{1/3}}\right) d\xi' \\ = &\; I_{21}^0(x,t) + \alpha(x',t) I_{22}^0(x,t)\,. \end{aligned} \tag{2.6}$$

Now we consider each integral in (2.6) separately. Then, if in $I_{22}^0(x,t)$ we make change of variables: $z = \frac{x_1-\lambda_1}{(t-\tau)^{1/3}}$ and $\theta' = \frac{x'-\xi'}{(t-\tau)^{1/3}}$, we get

$$I_{22}(x,t) = \int_{\frac{x_1-\lambda_1}{t^{1/3}}}^{+\infty} f(z)\, dz \int_{\widetilde{K'}} f^*(\theta')\, d\theta'.$$

Hence, passing to the limit $(x_1, x_2, \dots, x_n, t) \to (\lambda_1 + 0, x_2, \dots, x_n, t)$ and taking (1.11) into account, we have

$$\lim_{(x_1,x_2,\dots,x_n,t)\to(\lambda_1+0,x_2,\dots,x_n,t)} I_{22}^0(x,t) = -\frac{2\pi^n}{3}.$$

Now we make change of variables in $I_{21}^0(x,t)$ of (1.2): $z = \frac{x_1-\lambda_1}{(t-\tau)^{1/3}}$ and $\theta' = \frac{x'-\xi'}{(t-\tau)^{1/3}}$. Then the integral $I_{21}^0(x,t)$ takes the following form

$$I_{21}(x,t) = -\int_{\frac{x_1-\lambda_1}{t^{1/3}}}^{+\infty} f(z)\, dz \int_{\widetilde{K'}} f^*(\theta') \left[\alpha\left(x' - \theta'\frac{(x_1-\lambda_1)}{z}, t - \frac{(x_1-\lambda_1)^3}{z^3}\right) - \alpha(x',t)\right] d\theta'.$$

If we consider relations (1.9) and (1.10) and conditions of lemma 2, we obtain

$$\lim_{(x_1,x_2,\dots,x_n,t)\to(\lambda_1+0,x_2,\dots,x_n,t)} I_{21}(x,t) = 0.$$

Hence, finally we have

$$\lim_{(x_1,x_2,\dots,x_n,t)\to(\lambda_1+0,x_2,\dots,x_n,t)} I_2(x,t) = -\frac{2\pi^n}{3}\alpha(x',t).$$

Taking (1.11) into account, one can analogously prove the relations (2.4) and (2.5). □

Note, similar statement as (2.5) can be proved for functions $U_2, U_3, \dots, U_n$.

2.2 First boundary value problem

It is necessary to find in the domain $D = \Omega \times (0, T)$, where $\Omega = \{x : \lambda_i \leq x_i \leq \chi_i\}$, $x = (x_1, x_2, \dots, x_n)$, regular solutions $u(x,t) \in C^{2n+1,1}_{x,t}(D) \cap C^{2n,0}_{x,t}(\overline{D})$ of the equation (1.7), satisfying the following boundary conditions:

$$u(x,0) = 0, \tag{2.7}$$

$$u(x,t) = \varphi_j(x', t), \quad \text{at} \quad \partial\Omega \times [0,T], \;\; j = 1, 2, \dots, 2n, \tag{2.8}$$

$$\frac{\partial}{\partial x} u(\lambda_i, t) = \psi_i(x', t), \;\; i = 1, 2, \dots, n, \tag{2.9}$$

where $x' = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, and $\varphi_i(x', t), \psi_j(x', t)$ are smooth given functions satisfying natural agreement conditions at corner points of the parallelepiped D.

To prove the uniqueness for the solution of boundary value problems (1.7), (2.7)–(2.9), we use the method of integral energy, i.e. integrating the identity

$$\int\limits_0^t \int\limits_\Omega u \left(\sum_{i=1}^n \frac{\partial^3 u}{\partial x_i^3} - \frac{\partial u}{\partial t} \right) \mathrm{d}x \, \mathrm{d}t = 0 \tag{2.10}$$

by parts, we will prove that $u(x,t) = 0$ in $\overline{D}$.

We find solutions of the problem (1.7), (2.7)–(2.9) in the following form:

$$u(x,t) = \sum_{i=1}^n \int\limits_0^t \int\limits_{\Omega'} \frac{\partial}{\partial \xi_i} U_0 \left(x_1 - \xi_1, x_2 - \xi_2, \dots, x_n - \xi_n; t - \tau \right) \Big|_{\xi_i = \lambda_i} \alpha_i \left(\xi', \tau \right) \, \mathrm{d}\xi' \, \mathrm{d}\tau +$$

$$+ \sum_{i=1}^n \int\limits_0^t \int\limits_{\Omega'} \frac{\partial}{\partial \xi_i} U_0 \left(x_1 - \xi_1, x_2 - \xi_2, \dots, x_n - \xi_n; t - \tau \right) \Big|_{\xi_i = \chi_i} \beta_i \left(\xi', \tau \right) \, \mathrm{d}\xi' \, \mathrm{d}\tau +$$

$$+ \sum_{i=1}^n \int\limits_0^t \int\limits_{\Omega'} \frac{\partial}{\partial \xi_i} U_i \left(x_1 - \xi_1, x_2 - \xi_2, \dots, x_n - \xi_n; t - \tau \right) \Big|_{\xi_i = \lambda_i} \gamma_i \left(\xi', \tau \right) \, \mathrm{d}\xi' \, \mathrm{d}\tau,$$

where $\Omega' = \{x' : \lambda' < x' < \chi'\}$, $x' = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$.

Now, using Lemmas 1 and 2, and applying Abel's transformation [4] , we obtain a system of Volterra integral equations of the second kind with respect to unknown functions $\alpha_i(x', t)$, $\beta_j(x', t)$ and $\gamma_i(x', t)$. This system is solvable in the class of continuous functions.

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