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APPROXIMATE CONTROLLABILITY WITH CONSTRAINT ON THE CONTROL FOR THE HEAT EQUATION

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Abstract. In this work, we study an approximate controllability problem with constraint on the control. This problem appears naturally in the notion of discriminating sentinel with instantaneous observation. The main tool is a theorem of uniqueness of the solution of ill-posed Cauchy problem for the heat equation.

Keywords: Heat equation, approximate controllability, discriminating sentinels.

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1 Statement of the problem

1.1 Problem formulation

For $d \in \mathbb{N}^*$, let Ω be a bounded open subset of $\mathbf{R}^d$ with boundary Γ of class C^2, $T > 0$, and let ω be an open non empty subset of Ω. Set $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$. We consider the parabolic evolution equation:

$$\left\{ \begin{array}{rcll} -q' - \Delta q &=& 0 & \text{in} \quad Q, \\ q &=& 0 & \text{on} \quad \Sigma, \\ q(T) &=& h + k\chi_\omega & \text{in} \quad \Omega, \end{array} \right. \tag{1.1}$$

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where $(\cdot)'$ is the partial derivative with respect to time t, $h \in L^2(Q)$, $k \in L^2(\omega)$ and χ_ω denotes the characteristic function of ω. It is well known that problem (1.1) admits a unique solution $q \in H^{2,1}(Q)$ (see for instance [8], [9]).

Remark 1 *System (1.1) is a backward parabolic problem. It appears under this form in J.L. Lions' sentinels theory as the associated adjoint state. (cf. [10, p. 22]. See also below in Section 3).*

We will use the notation

$$q = q(x, t; k) \tag{1.2}$$

to mean that the solution q of (1.1) depends on the control k which plays a particular role. More precisely, let $\varepsilon > 0$; we would like to choose k in order to achieve the following objective: let h be a given function in $L^2(\Omega)$, $l_o \in L^2(\omega)$ and

$$\mathcal{M} \text{ a real closed vector subspace of } L^2(\omega). \tag{1.3}$$

Denoting by $\mathcal{M}^\perp$ the orthogonal subspace of $\mathcal{M}$ in $L^2(\omega)$ we look for a control variable $k \in L^2(\omega)$ such that

$$k \in \mathcal{M}^\perp, \tag{1.4}$$

and such that if $q = q(x, t; k)$ is the unique solution of (1.1), then

$$\|q(., 0; k) + l_o\, \chi_\omega\|_{L^2(\Omega)} \leq \varepsilon \quad \text{in} \quad \Omega, \tag{1.5}$$

and

$$\|k\|_{L^2(\omega)} \quad = \quad \text{minimum} \tag{1.6}$$

to mean that k is the control of minimal norm in $L^2(\omega)$. The role of k is to guarantee the approximate controllability property (1.5) in the presence of the forcing term h and under the restriction (1.4). The approximate controllability problem (1.1), (1.4) and (1.5) is by now well understood in the case $\mathcal{M} = \{0\}$. It has been studied by several authors using different methods. We refer to , C. Fabre *and al.* [2], J. P Puel [18], E. Zuazua [21], and references therein for other related controllability problems.

To the best of our knowledge, this paper is the first one dealing with the case $\mathcal{M} \neq \{0\}$. We study the case when $\mathcal{M}$ is of finite dimension. In this case, some compatibility conditions are required for controllability to hold. We shall return to this matter later on. We encounter this problem in the notion of instantaneous discriminating sentinel with the observation at $t = 0$ and $t = T$.

1.2 The main result

The main result is the following

Theorem 2 *For $\varepsilon > 0$, $h \in L^2(\Omega)$ and $l_o \in L^2(\omega)$, there exist some control k and some state q such that (1.1), (1.4) and (1.5) hold. Moreover, there exists a unique pair $(\hat{k}_\varepsilon, \hat{q}_\varepsilon)$ with $\hat{k}_\varepsilon$ of minimal norm in $L^2(\omega)$, i.e. such that (1.1), (1.4),(1.5) and (1.6) hold.*

The optimality system satisfied by $(\hat{k}_\varepsilon, \hat{q}_\varepsilon)$ is established as follows. Set

$$P = \text{the orthogonal projection operator from } L^2(\omega) \text{ onto } \mathcal{M}, \tag{1.7}$$

and for $\varphi \in L^2(Q)$

$$P\varphi = \text{the orthogonal projection of } \varphi\chi_\omega. \tag{1.8}$$

Let $\varphi^o \in L^2(\Omega)$ and φ the associated solution of

$$\begin{cases} \varphi' - \Delta\varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(0) = \varphi^o & \text{in } \Omega. \end{cases} \tag{1.9}$$

We now introduce the functional J_ε defined by

$$\begin{aligned} J_\varepsilon(\varphi^o) &= \frac{1}{2}\int_\omega |\varphi(T) - P(\varphi(T)\chi_\omega)|^2 \, dx + \sqrt{\varepsilon}\, \|\varphi^o\|_{L^2(\Omega)} \\ &\quad + \int_\Omega (\varphi(T)h + \varphi^o\, l_o\, \chi_\omega) \, dx. \end{aligned} \tag{1.10}$$

Consider the following unconstrained problem:

$$(\mathcal{P}_\varepsilon) \quad \begin{cases} \min J_\varepsilon(\varphi^o) \\ \varphi^o \in L^2(\Omega) \end{cases}. \tag{1.11}$$

Then, we have

Theorem 3 *Problem* (1.11) *has a unique solution* $\widehat{\varphi}^o \in L^2(\Omega)$. *Furthermore, if* $\widehat{\varphi}$ *is the solution of* (1.9) *associated to* $\widehat{\varphi}^o$, *then* $\left(\widehat{k} = \widehat{\varphi}(T)\chi_\omega - P\widehat{\varphi}(T), q\right)$ *is solution such that (1.1), (1.4), (1.5) and (1.6) hold.*

The paper is organized as follows: Section 2 is devoted to prove Theorem 2, the main tool being Lemma 4. In Section 3, we prove Theorem 3 using the result of Fenchel-Rockafellar. In Section 4, we give an application of the above results to the approximate instantaneous discriminating sentinels theory of J. L. Lions.

2 Approximate controllability with constraints on the control

Lemma 4 *Let* $m \in \mathcal{M}$. *Then there is no* $\varphi \in L^2(Q)$ / $\varphi \neq 0$ *such that* φ *satisfies*

$$\begin{cases} \varphi' - \Delta\varphi = 0 & \textit{in } Q, \\ \varphi = 0 & \textit{on } \Sigma, \\ \varphi(T)\chi_\omega = m\chi_\omega. \end{cases} \tag{2.1}$$

Proof. If the problem (2.1) admits a solution, then it is given by

$$\varphi(x,t) = \sum_{j=1}^{\infty} \alpha_j(T) w_j(x), \tag{2.2}$$

where w_j are eigenfunctions of

$$\left\{\begin{array}{rcll} -\Delta w &=& \lambda w & \text{in } \Omega, \\ w &=& 0 & \text{on } \Gamma. \end{array}\right. \tag{2.3}$$

Differentiate the solution (2.2) once with respect to t and twice with respect to x and substitute these derivatives into the first equation of (2.1). We then obtain

$$\sum_{j=1}^{\infty}\left(\alpha_j'(t)+\lambda_j\alpha_j(T)\right)w_j(x)=0. \tag{2.4}$$

Thus,

$$\alpha_j'(T)+\lambda_j\alpha_j(T)=0, \tag{2.5}$$

because (w_j) form an orthonormal base of $L^2(\Omega)$. Furthermore, the function φ satisfies the boundary conditions if and only if

$$\sum_{j=1}^{\infty}\alpha_j(T)w_j(x)=m\chi_\omega. \tag{2.6}$$

As $m\chi_\omega\in L^2(\Omega)$ then

$$m\chi_\omega=\sum_{j=1}^{\infty}\langle m\chi_\omega,w_j\rangle_{L^2(\Omega)}w_j(x). \tag{2.7}$$

Consequently

$$\alpha_j(T)=\langle m\chi_\omega,w_j\rangle_{L^2(\Omega)}. \tag{2.8}$$

Finally, we have

$$\left\{\begin{array}{rcll} \alpha_j'(T)+\lambda_j\alpha_j(T) &=& 0 & \text{in } (0,T) \\ \alpha_j(T) &=& \langle m\chi_\omega,w_j\rangle_{L^2(\Omega)} & \end{array}\right. \tag{2.9}$$

Then the solution of the first order linear is given by

$$\alpha_j(T)=\langle m\chi_\omega,w_j\rangle_{L^2(\Omega)}e^{\lambda_j(T-t)}. \tag{2.10}$$

Consequently, if the problem (2.1) admits a solution, it is necessarily in the form:

$$\varphi(x,t)=\sum_{j=1}^{\infty}\langle m\chi_\omega,w_j\rangle_{L^2(\Omega)}e^{\lambda_j(T-t)}w_j(x).$$

We prove now that $\varphi\notin L^2(Q)$. Indeed,

$$\begin{aligned} \int_0^T|\alpha_j(T)|^2\,\mathrm{d}t &= \left|\langle m\chi_\omega,w_j\rangle_{L^2(\Omega)}\right|^2\int_0^T e^{2\lambda_j(T-t)}\,\mathrm{d}t \\ &= \left|\langle m\chi_\omega,w_j\rangle_{L^2(\Omega)}\right|^2\left[\frac{-1}{2\lambda_j}+\frac{1}{2\lambda_j}e^{2\lambda_jT}\right] \end{aligned} \tag{2.11}$$

But, λ_j is the eigenvalue of Problem (2.3), then $\lambda_j \underset{j\longrightarrow\infty}{\longrightarrow} \infty$. Consequently,

$$\int_0^T|\alpha_j(T)|^2\,\mathrm{d}t \underset{j\longrightarrow\infty}{\longrightarrow} \infty. \tag{2.12}$$

which means that the series whose general term $\alpha_j(t)$ is not normally convergent. So, Problem (2.1) admits no solution. □

2.1 Proof of theorem 2

Let q be a solution of the system (1.1) and q_1 a solution of the following system:

$$\left\{\begin{array}{rcll} L^*q_1 & = & 0 & \text{in} \quad Q, \\ q_1 & = & 0 & \text{on} \quad \Sigma, \\ q_1(T) & = & h & \text{in} \quad \Omega, \end{array}\right. \tag{2.13}$$

where L^* is the adjoint differential operator defined by

$$L^* = -\frac{\partial}{\partial t} - \Delta. \tag{2.14}$$

We put

$$z = q - q_1. \tag{2.15}$$

Then, z is the solution of the following problem:

$$\left\{\begin{array}{rcll} L^*z & = & 0 & \text{in} \quad Q, \\ z & = & 0 & \text{on} \quad \Sigma, \\ z(T) & = & k\chi_\omega & \text{in} \quad \Omega. \end{array}\right. \tag{2.16}$$

We now introduce the set of states reachable at time 0 defined by:

$$R(0) = \{z(k,0)\ ,\ \ k \in \mathcal{M}^\perp\}. \tag{2.17}$$

We give now the proof of Theorem 2.

It is clear that $R(0)$ is a vector subspace of $L^2(\Omega)$. According to the Hahn-Banach theorem, it will be dense in $L^2(\Omega)$ if and only if its orthogonal in $L^2(\Omega)$ is reduced to zero. As $\{0\} \subset R(0)^\perp$, it remains to show that $R(0)^\perp \subset \{0\}$. Let $\varphi^o \in R(0)^\perp$, then

$$\langle \varphi^o, z(0)\rangle_{L^2(\Omega)} = \int\limits_\Omega \varphi^o z(0)\,\mathrm{d}x = 0, \tag{2.18}$$

where z is solution of (2.16). It is therefore natural to define the adjoint φ of z, this is the solution of the following problem:

$$\left\{\begin{array}{rcll} L\varphi & = & 0 & \text{in} \quad Q, \\ \varphi & = & 0 & \text{on} \quad \Sigma, \\ \varphi(0) & = & \varphi^o & \text{in} \quad \Omega, \end{array}\right.$$

where L is the differential operator defined by

$$L = \frac{\partial}{\partial t} - \Delta. \tag{2.19}$$

The system (1.9) is a classical problem of heat equation which has a unique solution $\varphi \in C\left([0,T]\,; L^2(\Omega)\right) \cap L^2(0,T;H_0^1(\Omega))$.

Now multiply the first equation of system (1.9) by z. After integration by parts on Q, it comes

$$\begin{array}{rcl} 0 & = & \displaystyle\int_0^T\int_\Omega \varphi L^* z\ \ \mathrm{d}x\,\mathrm{d}t + \int_\Omega \varphi(T)z(T)\,\mathrm{d}x - \int_\Omega \varphi(0)\,z(0)\ \mathrm{d}x \\ & & \displaystyle -\int_0^T\int_\Gamma \frac{\partial\varphi}{\partial\nu} z\ \ \mathrm{d}\gamma\,\mathrm{d}t + \int_0^T\int_\Gamma \varphi\frac{\partial z}{\partial\nu}\ \ \mathrm{d}\gamma\,\mathrm{d}t. \end{array} \tag{2.20}$$

Since z and φ are solutions of (2.16) and (1.9) respectively, (2.20) becomes

$$\int_{\Omega} \varphi(T) k \chi_{\omega} \, \mathrm{d}x - \int_{\Omega} \varphi^{o} z(0) \, \mathrm{d}x = 0. \tag{2.21}$$

This is equivalent to

$$\int_{\Omega} \varphi(T) k \chi_{\omega} \, \mathrm{d}x = 0 \quad \forall k \in \mathcal{M}^{\perp}, \tag{2.22}$$

because, $\varphi^{o} \in R(0)^{\perp}$ and $z(0) \in R(0)$. Finally, we have

$$\varphi(T) \chi_{\omega} \in \mathcal{M}. \tag{2.23}$$

Therefore, φ satisfies (1.9) and (2.23) and by applying Lemma 4, we deduce that

$$\varphi = 0 \quad \text{in} \quad \Omega \times (0, T).$$

As a consequence, $\varphi^{o} = 0$ which shows that $R(0)^{\perp} = \{0\}$.

3 Characterization of optimal control

In this section, we will characterize the optimal control using a result of Fenchel-Rockafellar duality (cf. [4]).

Proposition 5 *The functional J_{ε} defined in* (1.10) *is coercive.*

Proof. To prove that J_{ε} is coercive, it suffices to show the following relation:

$$\lim_{\|\varphi^{o}\|_{L^2(\Omega)} \longrightarrow \infty} \frac{J_{\varepsilon}(\varphi^{o})}{\|\varphi^{o}\|_{L^2(\Omega)}} \geq \sqrt{\varepsilon}. \tag{3.1}$$

Let $\left(\varphi_j^{o}\right) \subset L^2(\Omega)$ be a sequence of initial data for the adjoint system (1.9) with $\left\|\varphi_j^{o}\right\|_{L^2(\Omega)} \to \infty$. We normalize them as follows

$$\widetilde{\varphi}_j^{o} = \frac{\varphi_j^{o}(T) - P\left(\varphi_j^{o}(T)\chi_{\omega}\right)}{\left\|\varphi_j^{o}\right\|_{L^2(\Omega)}}, \tag{3.2}$$

so that $\left\|\widetilde{\varphi}_j^{o}\right\|_{L^2(\Omega)} \leq 1$. On the other hand, let $\widetilde{\varphi}_j$ be the solution of (1.9) with initial data $\widetilde{\varphi}_j^{o}$. Then, we have

$$\begin{aligned} \frac{J_{\varepsilon}\left(\varphi_j^{o}\right)}{\left\|\varphi_j^{o}\right\|_{L^2(\Omega)}} &= \frac{1}{2}\left\|\varphi_j^{o}\right\|_{L^2(\Omega)} \int_{\omega} \left|\widetilde{\varphi}_j(T)\right|^2 \, \mathrm{d}x + \sqrt{\varepsilon} \\ &\quad + \int_{\Omega} \widetilde{\varphi}_j(T) h \, \mathrm{d}x + \int_{\Omega} \frac{P\varphi_j(T) h + \varphi_j^{o}\, l_o \, \chi_{\omega}}{\left\|\varphi_j^{o}\right\|_{L^2(\Omega)}} \, \mathrm{d}x. \end{aligned} \tag{3.3}$$

We now show that the last integral in Equation (3.3) is bounded. Indeed, we know that φ_j is the solution of the problem

$$\left\{\begin{array}{rcll} L\varphi_j &=& 0 & \text{in} \quad Q, \\ \varphi_j &=& 0 & \text{on} \quad \Sigma, \\ \varphi_j\,(0) &=& \varphi_j^o & \text{in} \quad \Omega. \end{array}\right. \tag{3.4}$$

Multiplying the first equation of system (3.4) by φ_j then integrating by parts on Q , yields

$$0 = \int_0^T \int_\Omega L\varphi_j\,\varphi_j\;\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\left\|\varphi_j(T)\right\|^2_{L^2(\Omega)} - \frac{1}{2}\left\|\varphi_j^o\right\|^2_{L^2(\Omega)} + \left\|\nabla\varphi_j\right\|^2_{L^2(Q)}. \tag{3.5}$$

By the Poincaré inequality, (3.5) becomes,

$$C_o\left\|\varphi_j\right\|^2_{L^2(Q)} \le \left\|\nabla\varphi_j\right\|^2_{L^2(Q)} \le \frac{1}{2}\left\|\varphi_j^o\right\|^2_{L^2(\Omega)}. \tag{3.6}$$

Now, by Cauchy Schwartz inequality, one finds

$$\int_\Omega \frac{P\varphi_j(T)h_{-}\varphi_j^o\,l_o\,\chi_\omega}{\left\|\varphi_j^o\right\|_{L^2(\Omega)}}\,\mathrm{d}x \le C_1\frac{\left\|\varphi_j\right\|_{L^2(Q)}}{\left\|\varphi_j^o\right\|_{L^2(\Omega)}} + \left\|l_o\,\chi_\omega\right\|_{L^2(\Omega)}. \tag{3.7}$$

From (3.6), (3.7) and the fact that $l_o\,\chi_\omega \in L^2(\Omega)$, we conclude that

$$\int_\Omega \frac{P\varphi_j(T)h + \varphi_j^o\,l_o\,\chi_\omega}{\left\|\varphi_j^o\right\|_{L^2(\Omega)}}\,\mathrm{d}x \le C. \tag{3.8}$$

Returning to relation (3.3), two cases can occur:

1. $\int_\omega \left|\widetilde{\varphi}_j(T)\right|^2\,\mathrm{d}x > 0$. In this case, we immediately obtain

$$\frac{J_\varepsilon\left(\varphi_j^o\right)}{\left\|\varphi_j^o\right\|_{L^2(\Omega)}} \underset{\left\|\varphi_j^o\right\|_{L^2(\Omega)}\longrightarrow+\infty}{\longrightarrow} +\infty. \tag{3.9}$$

2. $\int_\omega \left|\widetilde{\varphi}_j\,(T)\right|^2\,\mathrm{d}x = 0$. In this case, since $\left(\widetilde{\varphi}_j^o\right)_j$ is bounded in $L^2(\Omega)$, we can extract a subsequence $\left(\widetilde{\varphi}_j^o\right)_j$ such that:

$$\left\{\begin{array}{l} \widetilde{\varphi}_j^o \rightharpoonup \psi^o \;\text{ weakly in }\; L^2(\Omega), \\ \widetilde{\varphi}_j \rightharpoonup \psi \;\text{ weakly in } L^2(0,T;H_o^1(\Omega)), \end{array}\right. \tag{3.10}$$

where ψ is solution of system (1.9) with initial data ψ^o. Moreover, by lower semi continuity of the norm, it comes

$$\int_\omega \left|\psi(T)\right|^2\,\mathrm{d}x \le \liminf \int_\omega \left|\widetilde{\varphi}_j(T)\right|^2\,\mathrm{d}x = 0. \tag{3.11}$$

Therefore,

$$\psi(T) = 0 \quad \text{in} \quad \omega. \tag{3.12}$$

And as ψ is solution of (1.9), and in view of (3.12), we have

$$\psi = 0 \ \text{ in } \Omega \times (0, T).$$

Thus,

$$\widetilde{\varphi}_j \rightharpoonup 0 \ \text{ weakly in } L^2(0, T; H_o^1(\Omega)). \tag{3.13}$$

Moreover, from inequality (3.6), we deduce that $\left(\dfrac{\varphi_j}{\left\| \varphi_j^o \right\|_{L^2(\Omega)}} \right)_j$ is bounded in $L^2(0, T; H_o^1(\Omega))$. Hence

$$\frac{\varphi_j}{\left\| \varphi_j^o \right\|_{L^2(\Omega)}} \rightharpoonup \xi \ \text{ in } L^2(0, T; H_o^1(\Omega)). \tag{3.14}$$

This implies that

$$\frac{\varphi_j(T)}{\left\| \varphi_j^o \right\|_{L^2(\Omega)}} \rightharpoonup \xi(T) \quad \text{in } L^2(\Omega).$$

And as P is a compact operator, then

$$\frac{P\varphi_j(T)}{\left\| \varphi_j^o \right\|_{L^2(\Omega)}} \longrightarrow P\xi(T) \ \text{ strongly in } L^2(\omega), \tag{3.15}$$

But,

$$\widetilde{\varphi}_j = \frac{\varphi_j - P(\varphi_j)}{\left\| \varphi_j^o \right\|_{L^2(\Omega)}} \rightharpoonup 0. \tag{3.16}$$

From (3.14), (3.15) and (3.16), we conclude that

$$P\xi(T) = \xi(T). \tag{3.17}$$

Therefore,

$$\xi(T) \in \mathcal{M} \quad \text{and} \quad L\xi = 0 \quad \text{in} \quad L^2(Q). \tag{3.18}$$

So by Lemma 4, it comes

$$\xi = 0 \quad \text{in } Q \tag{3.19}$$

As a consequence,

$$\frac{P\varphi_j(T)}{\left\| \varphi_j^o \right\|_{L^2(\Omega)}} \longrightarrow 0. \tag{3.20}$$

But,

$$\frac{J_\varepsilon\left(\varphi_j^o\right)}{\left\| \varphi_j^o \right\|_{L^2(\Omega)}} = \left[\sqrt{\varepsilon} + \int_\Omega \widetilde{\varphi}_j(T)\, h \,\mathrm{d}x + \int_\Omega \frac{P\varphi_j(T)\, h + \varphi_j^o\, l_o\, \chi_\omega}{\left\| \varphi_j^o \right\|_{L^2(\Omega)}} \,\mathrm{d}x \right]. \tag{3.21}$$

Thus,

$$\liminf_{j\to+\infty}\frac{J_\varepsilon\left(\varphi_j^o\right)}{\left\|\varphi_j^o\right\|_{L^2(\Omega)}}\geq\sqrt{\varepsilon}. \qquad (3.22)$$

Hence relation (3.1) is satisfied.

□

3.1 Proof of theorem 3

Here, we give the proof of Theorem 3.

As J_ε attains its minimum value at $\widehat{\varphi}^o\in L^2(\Omega)$, then, for any $\psi^o\in L^2(\Omega)$ and any $s\in\mathbb{R}$ we have

$$J_\varepsilon\left(\widehat{\varphi}^o\right)\leq J_\varepsilon\left(\widehat{\varphi}^o+s\,\psi^o\right). \qquad (3.23)$$

On the other hand,

$$\begin{aligned}J_\varepsilon\left(\widehat{\varphi}^o+s\,\psi^o\right) &= \frac{1}{2}\int\limits_\omega\left|\widehat{\varphi}(T)-P\widehat{\varphi}(T)\right|^2\,\mathrm{d}x+\frac{s^2}{2}\int\limits_\omega\left|\psi(T)-P\psi(T)\right|^2\,\mathrm{d}x\\ &+\sqrt{\varepsilon}\left\|\widehat{\varphi}^o+s\,\psi^o\right\|_{L^2(\Omega)}+s\int\limits_\omega\left(\widehat{\varphi}(T)-P\widehat{\varphi}(T)\right)\left(\psi(T)-P\psi(T)\right)\,\mathrm{d}x\\ &+\int\limits_\Omega\left[\left(\widehat{\varphi}\left(T\right)+s\,\psi(T)\right)h-\left(\widehat{\varphi}^o+s\,\psi^o\right)l_o\,\chi_\omega\right]\,\mathrm{d}x. \qquad (3.24)\end{aligned}$$

Substituting (3.24) in (3.23) and after simplifications, we find

$$\begin{aligned}0 &\leq \sqrt{\varepsilon}\left[\left\|\widehat{\varphi}^o+s\,\psi^o\right\|_{L^2(\Omega)}-\left\|\widehat{\varphi}^o\right\|_{L^2(\Omega)}\right]+\frac{s^2}{2}\int\limits_\omega\left|\psi(T)-P\psi(T)\right|^2\,\mathrm{d}x\\ &+s\left[\int\limits_\omega\left(\widehat{\varphi}(T)-P\widehat{\varphi}(T)\right)\left(\psi(T)-P\psi(T)\right)\,\mathrm{d}x\right. \qquad (3.25)\\ &\left.+\int\limits_\Omega\left(\psi(T)h+\psi^o\,l_o\,\chi_\omega\right)\,\mathrm{d}x\right]. \qquad (3.26)\end{aligned}$$

On the other hand

$$\left\|\widehat{\varphi}^o+s\,\psi^o\right\|_{L^2(\Omega)}-\left\|\widehat{\varphi}^o\right\|_{L^2(\Omega)}\leq|s|\,\left\|\psi^o\right\|_{L^2(\Omega)}. \qquad (3.27)$$

From (3.26) and (3.27), we obtain

$$\begin{aligned} 0 \;\leq\;& \sqrt{\varepsilon}\,|s|\,\|\psi^o\|_{L^2(\Omega)} + \frac{s^2}{2}\int_\omega |\psi(T) - P\psi\,(T)|^2 \,\mathrm{d}x \\ &+ s\left[\int_\omega (\widehat{\varphi}(T) - P\widehat{\varphi}(T))\,(\psi(T) - P\psi(T))\,\mathrm{d}x \right.\\ &\left.+ \int_\Omega (\psi(T)h + \psi^o\, l_o\, \chi_\omega)\,\mathrm{d}x\right] \quad \forall \psi^o \in L^2(\Omega) \text{ and } s \in \mathbb{R}. \end{aligned}$$

Dividing by $s > 0$ and by passing to the limit $s \longrightarrow 0$, we obtain

$$\begin{aligned} 0 \;\leq\;& \sqrt{\varepsilon}\,\|\psi^o\|_{L^2(\Omega)} + \int_\omega (\widehat{\varphi}(T) - P\widehat{\varphi}(T))\,(\psi(T) - P\psi(T))\,\mathrm{d}x \\ &+ \int_\Omega (\psi(T)h + \psi^o\, l_o\, \chi_\omega)\,\mathrm{d}x. \end{aligned}$$

The same calculations with $s < 0$ give

$$\begin{aligned} &\left| \int_\omega (\widehat{\varphi}(T) - P\widehat{\varphi}(T))\,\psi(T)\,\mathrm{d}x + \int_\Omega (\psi(T)h + \psi^o\, l_o\, \chi_\omega)\,\mathrm{d}x \right| \\ &\qquad\qquad\qquad \leq \sqrt{\varepsilon}\,\|\psi^o\|_{L^2(\Omega)} \qquad \forall\, \psi^o \in L^2(\Omega). \end{aligned}$$

Also if we take $\widehat{k} = \widehat{\varphi}(T)\chi_\omega - P\widehat{\varphi}(T)$ in (1.1) and we multiply the first equation of the system (1.1) by ψ solution of (1.9) and we get after integration by parts over Q,

$$\int_\Omega q(0)\psi^o\,\mathrm{d}x = \int_\Omega h\psi\,(T)\,\mathrm{d}x + \int_\omega (\widehat{\varphi}(T) - P\widehat{\varphi}(T))\,\psi(T)\,\mathrm{d}x. \tag{3.28}$$

It comes from the last two relations:

$$\left| \int_\Omega (q(0) + l_o\, \chi_\omega)\,\psi^o\,\mathrm{d}x \right| \leq \sqrt{\varepsilon}\,\|\psi^o\|_{L^2(\Omega)} \quad \forall \psi^o \in L^2(\Omega).$$

Consequently,

$$\|q(0) + l_o\, \chi_\omega\|_{L^2(\Omega)} \leq \sqrt{\varepsilon}. \tag{3.29}$$

4 Approximate instantaneous discriminating sentinels

The notion of sentinel was introduced by J. L. Lions to study systems of incomplete data [10]. It is based on the following considerations:

− a state equation with incomplete data,

− a system of observations,

− a functional called a sentinel and who allows to distinguish two types of missing data.

More precisely, in the first step, we consider the semilinear parabolic equation:

$$\begin{cases} y' - \Delta y &= \xi + \lambda \hat{\xi} & \text{in} \quad Q, \\ y &= 0 & \text{on} \quad \Sigma, \\ y(0) &= y^0 + \tau \hat{y}^0 & \text{in} \quad \Omega. \end{cases} \tag{4.1}$$

We are interested in systems with data that are not completely known. The functions ξ and y^0 are known with ξ in $L^2(Q)$ and y^0 in $L^2(\Omega)$. However, the terms $\lambda\hat{\xi}$ and $\tau\hat{y}^0$ are unknown, but are such that

$$\begin{cases} \|\hat{\xi}\|_{L^2(Q)} \leq 1, \quad \|\hat{y}^0\|_{L^2(\Omega)} \leq 1 \\ \text{and that the reals } \lambda \text{ and } \tau \text{ are small enough.} \end{cases} \tag{4.2}$$

The problem (4.1) admits a unique solution in $C([0,T], L^2(\Omega))$. For the sake of simplicity, we denote

$$y(x,t;\lambda,\tau) = y(\lambda,\tau)$$

the unique solution of (4.1). Therefore, the map

$$\begin{cases} (\lambda,\tau) \longmapsto y(\lambda,\tau) \\ \text{is in } C^1(R \times R; C([0,T], L^2(\Omega)). \end{cases} \tag{4.3}$$

In the second step, we consider the observation. More precisely, we give an observation system, i.e. an open subset $\mathcal{O}$ nonempty of Ω, called an observatory and observation y_{obs} of the state y on $\mathcal{O} \times \{0,T\}$. If y is a representation "faithful" of the phenomenon studied, then $y(\lambda,\tau) = y_{obs}$ on $\mathcal{O} \times \{0,T\}$.

Assume that the observation at $t = 0$ and $t = T$ was noisy i.e.

$$y_{obs}(t=0) = e_o + \sum_{i=1}^{N} \alpha_i e_i \tag{4.4}$$

$$y_{obs}(t=T) = m_o + \sum_{j=1}^{M} \beta_j m_j, \tag{4.5}$$

where the functions $e_o, e_1, ..., e_N$ and $m_0, m_1, ..., m_M$ are given measurements of y in $L^2(\mathcal{O})$, but where the real coefficients α_i, β_j are unknown. We assume that α_i, β_j are *small*. We refer to the terms $\alpha_i e_i$ and $\beta_j m_j$ as the *interference terms*. We can assume without lossing of generality that

the functions e_i and m_j are linearly independent on $\mathcal{O}$.

Finally, we introduce now the notion of *sentinel*. However, we give now $h_o \in L^2(\mathcal{O})$, $q^o \in L^2(\Omega)$, and we consider also ω an open subset nonempty of Ω such that $\omega \subset \overline{\omega} \subset \mathcal{O}$. For a control variable $(u,v) \in L^2(\omega) \times L^2(\omega)$, we set:

$$\mathcal{S}(\lambda,\tau) = \int_\Omega (h_o \chi_{\mathcal{O}} + v\chi_\omega) y(x,T;\lambda,\tau)\ \mathrm{d}x + \int_\Omega (q^o + u\chi_\omega)\, y(x,0;\lambda,\tau)\ \mathrm{d}x, \tag{4.6}$$

where $\chi_{\mathcal{O}}$ and χ_ω are the characteristic functions of $\mathcal{O}$ and ω respectively.

More precisely, We seek (u, v) such that

(i) S is stationary at first order with respect to the missing terms $\tau\hat{y}^0$, that is

$$\frac{\partial S}{\partial \tau}(0,0) = 0, \quad \forall \hat{y}^0 \tag{4.7}$$

(ii) S is stationary with respect to the interference terms $\alpha_i e_i$, that is

$$\int_\Omega q^o e_i \; \mathrm{d}x + \int_\omega u e_i \; \mathrm{d}x = 0, \quad 1 \leq i \leq N. \tag{4.8}$$

(iii) S is stationary with respect to the interference terms $\beta_j m_j$, that is

$$\int_{\mathcal{O}} h_0 m_j \; \mathrm{d}x + \int_\omega v m_j \; \mathrm{d}x = 0, \quad 1 \leq j \leq M. \tag{4.9}$$

and

(iv) (u, v) is of minimal norm in $L^2(\omega) \times L^2(\omega)$ among control functions in $L^2(\omega) \times L^2(\omega)$ which satisfy the above conditions, that is

$$\| (u,v) \|_{L^2(\omega)\times L^2(\omega)} \quad = \quad \min. \tag{4.10}$$

Any S such that (4.7), (4.23), (4.24) and (4.10) hold, is called instantaneous discriminating sentinel.

4.1 Equivalence to the null-controllability

4.1.1 Interpretation of (i)

The condition (4.7) is equivalent to:

$$\int\limits_\Omega (h_o\,\chi_{\mathcal{O}} + v\chi_\omega) y_\tau(T) \; \mathrm{d}x + \int\limits_\Omega (q^o + u\chi_\omega)\, y_\tau(0) \; \mathrm{d}x = 0, \tag{4.11}$$

where $y_\tau = \dfrac{\partial y}{\partial \tau}(0,0)$ is the solution of system:

$$\left\{ \begin{array}{rcll} L y_\tau & = & 0 & \text{in} \quad Q, \\ y_\tau & = & 0 & \text{on} \quad \Sigma, \\ y_\tau(0) & = & \widehat{y}^o & \text{in} \quad \Omega, \end{array} \right. \tag{4.12}$$

and where L is the differential operator defined in (2.19).

To transform the condition (4.11), we introduce the adjoint state q solution of the following retrograde problem:

$$\left\{ \begin{array}{rcll} L^* q & = & 0 & \text{in} \quad Q, \\ q & = & 0 & \text{in} \quad \Sigma, \\ q(T) & = & h_o\,\chi_{\mathcal{O}} + v\,\chi_\omega & \text{on} \quad \Omega, \end{array} \right. \tag{4.13}$$

with L^* is the adjoint differential operator defined in (2.14). The problem (4.13) has a unique solution $q \in H^{2,1}(Q)$.

We multiply the first equation of system (4.13) by y_τ and we integrate by parts on Q. We find

$$\int_\Omega (h_o\,\chi_{\mathcal{O}} + v\,\chi_\omega) y_\tau(T)\ \mathrm{d}x\,\mathrm{d}t = \int_\Omega q(0)\widehat{y}^o\,\mathrm{d}x \qquad \forall \widehat{y}^o \in L^2(\Omega). \tag{4.14}$$

Thus, the condition (4.7) (or (4.11)) is satisfied if and only if

$$q(0) = -\left(q^o + u\chi_\omega\right). \tag{4.15}$$

4.1.2 Interpretation of (ii)

Now consider condition (4.8). Let $\mathcal{E}$ be the vector subspace of $L^2(\omega)$ generated by the N linearly independent functions $e_i\chi_\omega$. Then, condition (4.8) is satisfied if and only if there exists a unique $l_o \in \mathcal{E}$ such that

$$u = l_o + l \quad /\ l \in \mathcal{E}^\perp, \tag{4.16}$$

where $\mathcal{E}^\perp$ is orthogonal to $\mathcal{E}$.

4.1.3 Interpretation of (iii)

To transform the condition (4.9) we introduce $\mathcal{M}$ the vector subspace of $L^2(\Omega)$ generated by the M linearly independent functions $m_i\chi_\omega$. Then, the condition (4.9) is satisfied if and only if there exists a unique $k_o \in \mathcal{M}$ such that

$$v = k_o + k \quad /\ k \in \mathcal{M}^\perp, \tag{4.17}$$

where $\mathcal{M}^\perp$ is the orthogonal of $\mathcal{M}$.

Therefore, the problem of finding a control $(u, v) \in L^2(\omega) \times L^2(\omega)$ such that $(\mathcal{S}, u, v)$ verifies (4.7)–(4.10) is equivalent to find the control pair (l, k) such that:

$$l \in \mathcal{E}^\perp,\ k \in \mathcal{M}^\perp, \tag{4.18}$$

and if q is the solution of the system:

$$\begin{cases} L^*q &= 0 & \text{in} & Q, \\ q &= 0 & \text{on} & \Sigma, \\ q(T) &= h + k\chi_\omega & \text{in} & \Omega, \end{cases} \tag{4.19}$$

we have

$$q(0) = -\left(q^o + l_0 + l\chi_\omega\right), \tag{4.20}$$

with

$$\|\,(l, k)\,\|_{L^2(\omega)\times L^2(\omega)} \quad = \quad \min. \tag{4.21}$$

Remark 6 *We cannot hope to solve the problem of exact controllability* (4.18)–(4.20)*. Indeed, for* $k \in L^2(\omega)$ *the solution of* (4.18)–(4.19) *be very regular and of regularity difficult to characterize. This is due to the regulating effect of the heat equation. So, if we take* $(q^o + l_0 + l\chi_\omega)$ *in a classical Sobolev space (here* $L^2(\Omega)$ *), it will be useless to look* k *such that* $q(0) = -\left(q^o + l_0 + l\chi_\omega\right)$*. That is why, it is reasonable to content ourselves with the approximate controllability, which brings naturally to the notion of approximate sentinel.*

Definition 7 *Let $\varepsilon > 0$. The functional $\mathcal{S}$ defined in* (4.6) *assumed non-zero is called an approximate discriminating sentinel if there is a control $(u, v) \in L^2(\omega)$ such that the triplet $(\mathcal{S}, u, v)$ satisfies the following conditions:*

(i) S is insensitive in an approximate way with respect to missing terms $\tau\hat{y}^0$, that is

$$\left|\frac{\partial \mathcal{S}}{\partial \tau}(0,0)\right| \leq \varepsilon \tag{4.22}$$

(ii) S is stationary with respect to the interference terms $\alpha_i e_i$, that is

$$\int_\Omega q^o e_i \, \mathrm{d}x + \int_\omega u e_i \, \mathrm{d}x = 0, \quad 1 \leq i \leq N. \tag{4.23}$$

(iii) S is stationary with respect to the interference terms $\beta_j m_j$, that is

$$\int_{\mathcal{O}} h_0 m_j \, \mathrm{d}x + \int_\omega v m_j \, \mathrm{d}x = 0, \quad 1 \leq j \leq M. \tag{4.24}$$

and

(iv) (u, v) is of minimal norm in $L^2(\omega) \times L^2(\omega)$ among control functions in $L^2(\omega) \times L^2(\omega)$ which satisfy the above conditions, that is

$$\| (u,v) \|_{L^2(\omega)\times L^2(\omega)} \quad = \quad \min . \tag{4.25}$$

Finally, the problem of existence of an approximate instantaneous discriminating sentinel is equivalent to the following problem: Find (l, k) such that:

$$l \in \mathcal{E}^\perp, \; k \in \mathcal{M}^\perp, \tag{4.26}$$

and if q is the solution of the system:

$$\begin{cases} L^* q &= 0 & \text{in} \quad Q, \\ q &= 0 & \text{on} \quad \Sigma, \\ q(T) &= h + k\chi_\omega & \text{in} \quad \Omega, \end{cases} \tag{4.27}$$

we have

$$\|q(0) + q^o + l_0 + l\chi_\omega\| \leq \varepsilon, \tag{4.28}$$

with

$$\| (l,k) \|_{L^2(\omega)\times L^2(\omega)} \quad = \quad \min . \tag{4.29}$$

Remark 8 *There exists $l = 0 \in \mathcal{E}^\perp$ satisfying* (4.26) *and* (4.28)*, so the problem* (4.26)–(4.29) *is reduced to a problem* (1.1) *which was studied in Section 1.*

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