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# STRUCTURAL STABILITY OF $p(x)$-LAPLACIAN KIND PROBLEMS WITH NEUMANN TYPE BOUNDARY CONDITION

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**Abstract.** We study the continuous dependence on coefficients of solutions of the nonlinear homogeneous Neumann boundary value problems involving the $p(x)$-Laplace operator.

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## 1 Introduction

Our work has for goal to study the convergence of sequences of solutions of degenerate elliptic problems with variable coercivity and growth exponents $p_n$ of the form

$$(Pb_n) : \begin{cases} b(u_n) - \operatorname{div} a_n(x, \nabla u_n) = f_n & \text{in } \Omega, \\ a_n(x, \nabla u_n) \cdot \eta = 0 & \text{on } \partial\Omega, \end{cases}$$

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where $\Omega$ is an open bounded domain of $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$ and $\eta$ is the outer unit normal to $\partial\Omega$. The function $b : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, onto and non-decreasing such that $b(0) = 0$; $(a_n(x,\xi))_{n\in\mathbb{N}}$ is a family of applications which verify the classical Leray-Lions hypotheses but with a variable summability exponent $p_n(x)$ converging in measure to some exponent $p$ such that $1 < p_- \leq p_n(\cdot) \leq p < +\infty$, $(f_n)_{n\in\mathbb{N}} \subset L^1(\Omega)$. The model problem for our study is so the following:

$$(Pb) : \begin{cases} b(u) - \operatorname{div} a(x, \nabla u) = f & \text{in } \Omega, \\ a(x, \nabla u) \cdot \eta = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is an open bounded domain with smooth boundary $\partial\Omega$ and $\eta$ is the outer unit normal to $\partial\Omega$.

This paper is inspired by recent works of Andreianov, Bendahmane and Ouaro (see [1]) on the structural stability of weak and renormalized solutions $u_n$ of the following nonlinear homogeneous Dirichlet boundary value problem

$$\begin{cases} b(u_n) - \operatorname{div} a_n(x, \nabla u_n) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $(a_n(x,\xi))_{n\in\mathbb{N}}$ verifies the classical Leray-Lions hypotheses with the variable exponents $p_n(x)$ such that $1 < p_- \leq p_n(\cdot) \leq p < +\infty$. In their investigations, the exponent $p_n$ (and thus, the underlying function space for the solution $u_n$) varies with $n$ and the convergence of weak solutions $u_n$ requires some involved assumptions on the convergence of the sequence $f_n$ of the source terms. To bypass this difficulty, they used the technique of renormalized solutions. Indeed, the study of convergence of renormalized solutions of the problem (1.2) permits them to deduce convergence results for the weak solutions under much simpler assumptions on $(f_n)_{n\in\mathbb{N}}$, in particular the weak $L^1$ convergence of $f_n$ to a limit $f$(sufficiently regular) so that to allow for the existence of a weak solution. Moreover, the structural stability result permits them to deduce also new existence results of solutions.

As the boundary value condition is a homogeneous Neumann boundary condition, we cannot work in the space $W_0^{1,p(\cdot)}(\Omega)$ as in [1], but in the space $W^{1,p(\cdot)}(\Omega)$. The $p(x)-$Laplacian operator $\Delta_{p(x)}u$ corresponds to the choice $a(x,\nabla u) := |\nabla u|^{p(x)-2}\nabla u$.

Problems with variable exponents $p(x)$ and $p_n(x)$ were arisen and studied by Zhikov in the pioneering paper [18]. By the introduction of the $p(x)$-Laplacian into models of electrorheological and thermorheological fluids (see [15, 16, 14, 7]), and in the context of image processing (see [6] and [12]), it's important to lead such studies. Concerning the problem (1.1), Bonzi, Nyanquini and Ouaro (see [4]) have proved the existence and uniqueness of a weak solution for $f \in L^\infty(\Omega)$ and the existence and uniqueness of an entropy solution for $L^1$-data $f$. For our study, the data $f$ is in $L^1(\Omega)$ and the common notions of renormalized and entropy solutions are used.

Let us give the outline of the paper. In the section 2, we do some important assumptions and preliminaries for the sequel. In the section 3, we prove the existence and uniqueness of the renormalized solution of (1.1) when the right-hand side $f \in L^1(\Omega)$. In the section 4, we tackle the question of continuous dependence for renormalized solutions.

## 2 Preliminaries

In this section, we do some assumptions on the model problem (1.1) and give some preliminary results.

$$p : \overline{\Omega} \longrightarrow \mathbb{R} \text{ is a continuous function such that } 1 < p_- \leq p < +\infty, \tag{2.1}$$

where $p_- := \inf_{x\in\Omega} p(x)$ and $p := \sup_{x\in\Omega} p(x)$.

$$\begin{cases} b : \mathbb{R} \longrightarrow \mathbb{R} \text{ is a continuous, non-decreasing} \\ \text{and onto function such that } b(0) = 0. \end{cases} \tag{2.2}$$

$a(\cdot,\cdot) : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function with

$$a(x, 0) = 0 \quad \text{ for a.e. } x \in \Omega, \tag{2.3}$$

satisfying, for a.e. $x \in \Omega$, the strict monotonicity assumption

$$(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) > 0 \quad \text{ for all } \xi, \eta \in \mathbb{R}^N, \, \xi \neq \eta, \tag{2.4}$$

and the following growth and coercivity assumptions in $\xi$ :

$$|a(x, \xi)| \leq C_1(\mathcal{M}(x) + |\xi|^{p(x)-1}), \tag{2.5}$$

$$a(x, \xi) \cdot \xi \geq C_2|\xi|^{p(x)}, \tag{2.6}$$

where $C_1$ and $C_2$ are positive constants, $\mathcal{M}$ is a non-negative function in $L^{p'(\cdot)}(\Omega)$ with $1/p(x) + 1/p'(x) = 1$.

For the given exponent $p$, we denote by $p'$ its conjugate exponent such that $1/p(x)+1/p'(x) = 1$ and by $p^*$ its optimal Sobolev embedding exponent such that

$$p^* := \begin{cases} Np/(N - p) & \text{if } p < N, \\ \text{any real value} & \text{if } p = N, \\ \infty & \text{if } p > N. \end{cases}$$

For any given $k > 0$, we define the truncation function $T_k : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$T_k(r) = \max(\min(r, k), -k).$$

We put

$$sign(z) = \begin{cases} 1 & \text{if } z > 0, \\ 0 & \text{if } z = 0, \\ -1 & \text{if } z < 0. \end{cases}$$

The truncation function $T_k$ has so the following properties.

$$|T_k(z)| = \min(|z|, k), \; \lim_{k\to\infty} T_k(z) = z \text{ and } \lim_{k\to 0} \frac{1}{k} T_k(z) = sign(z).$$

For a Lebesgue measurable set $A \subset \Omega$, $\chi_A$ denotes its characteristic function and $meas(A)$ denotes its Lebesgue measure.

Let $u : \Omega \longrightarrow \mathbb{R}$ be a function and $k \in \mathbb{R}$, we write $\{|u| \leq k\}$ for the set $\{x \in \Omega;\ |u(x)| \leq k\}$, (respectively, $\geq, =, <, >$). We will also need to truncate vector-valued functions with the help of the maps

$$\text{for } m > 0,\ h_m : \mathbb{R}^N \longrightarrow \mathbb{R}^N,\ h_m(\lambda) = \begin{cases} \lambda & \text{if } |\lambda| \leq m, \\ \dfrac{m}{|\lambda|}\lambda & \text{if } |\lambda| > m. \end{cases} \tag{2.7}$$

We have the following property (see [1, Lemma 2.1]).

**Lemma 2.1** *Let $h_m(\cdot)$ be defined by (2.7) and $a(x,\cdot)$ be monotone in the sense (2.4). Then, for all $\lambda \in \mathbb{R}^N$, the map $m \longmapsto a(x, h_m(\lambda)) \cdot h_m(\lambda)$ is non-decreasing and converges to $a(x,\lambda)\cdot\lambda$ as $m \to \infty$.*

The exponent $p(\cdot)$ appearing in (2.5) and (2.6) depends on the spatial variable $x$ and then requires so to work with Lebesgue and Sobolev spaces with variable exponents.

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable function $u : \Omega \longrightarrow \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)}\,\mathrm{d}x$$

is finite. If the exponent is bounded, *i.e.*, if $p_+ < \infty$, then the expression

$$\|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0;\ \rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1\right\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $\left(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)}\right)$ is a separable Banach space. Moreover, if $1 < p_- \leq p_+ < \infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$. Moreover, we have the Hölder type inequality

$$\left|\int_\Omega uv\,\mathrm{d}x\right| \leq \left(\frac{1}{p_-} + \frac{1}{(p')_-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}, \tag{2.8}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

$W^{1,p(\cdot)}(\Omega)$ denotes the space of all functions $u \in L^{p(\cdot)}(\Omega)$ such that their gradients $\nabla u$, taken in the sense of distributions, belong to $(L^{p(\cdot)}(\Omega))^N$. This space is a Banach space equipped with the following norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $\left(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)}\right)$ is a separable and reflexive Banach space; for more details on the generalized Lebesgue and Sobolev spaces, see [13]. In the sequel, we will use the same notation $L^{p(\cdot)}(\Omega)$ for the space $(L^{p(\cdot)}(\Omega))^N$ of vector-valued functions.

In manipulating the generalized Lebesgue and Sobolev spaces, the following lemma (cf. [11]) permits to pass from norm to convex modular and vice-versa.

**Lemma 2.2** *If $u_n, u \in L^{p(\cdot)}(\Omega)$ and $p_+ < \infty$, then the following properties hold.*
$(i)$ $\rho_{p(\cdot)}\left(u/\|u\|_{p(\cdot)}\right) = 1$, *if* $u \neq 0$.
$(ii)$ $\rho_{p(\cdot)}(u) < 1$ (*respectively* $= 1; > 1$) $\Longleftrightarrow \|u\|_{p(\cdot)} < 1$ (*respectively* $= 1; > 1$).

$(iii)$ $\rho_{p(\cdot)}(u) \le 1 \Longrightarrow \|u\|_{p(\cdot)}^{p_+} \le \rho_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p_-}$.
$(iv)$ $\rho_{p(\cdot)}(u) \ge 1 \Longrightarrow \|u\|_{p(\cdot)}^{p_-} \le \rho_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p_+}$.
$(v)$ $\|u_n\|_{p(\cdot)} \to 0$ (*respectively* $\to \infty$) $\Longleftrightarrow \rho_{p(\cdot)}(u_n) \to 0$ (*respectively* $\to \infty$).

For a measurable function $u : \Omega \longrightarrow \mathbb{R}$, we introduce the function

$$\rho_{1,p(\cdot)}(u) := \int_\Omega |u|^{p(x)}\,\mathrm{d}x + \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x.$$

Then, we have the following lemma (see [17]).

**Lemma 2.3** *If $u_n, u \in W^{1,p(\cdot)}(\Omega)$ and $p_+ < \infty$, then the following properties are true.*
$(i)$ $\rho_{1,p(\cdot)}(u) < 1$ (*respectively* $= 1; > 1$) $\Longleftrightarrow \|u\|_{1,p(\cdot)} < 1$ (*respectively* $= 1; > 1$).
$(ii)$ $\rho_{1,p(\cdot)}(u) \le 1 \Longrightarrow \|u\|_{1,p(\cdot)}^{p_+} \le \rho_{1,p(\cdot)}(u) \le \|u\|_{1,p(\cdot)}^{p_-}$.
$(iii)$ $\rho_{1,p(\cdot)}(u) \ge 1 \Longrightarrow \|u\|_{1,p(\cdot)}^{p_-} \le \rho_{1,p(\cdot)}(u) \le \|u\|_{1,p(\cdot)}^{p_+}$.
$(iv)$ $\|u_n\|_{1,p(\cdot)} \to 0$ (*respectively* $\to \infty$) $\Longleftrightarrow \rho_{1,p(\cdot)}(u_n) \to 0$ (*respectively* $\to \infty$).

One has below, imbedding result between Lebesgue and Sobolev spaces (see [9, 11]).

**Proposition 2.4** *Let $p, q \in C(\overline{\Omega})$ with $p_- > 1$. Assume that $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$. Then, there is a compact imbedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$. In particular, there is a compact imbedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$.*

The following result (corollary of Lebesgue dominated convergence theorem) is useful to prove strong convergence results.

**Lemma 2.5 (Lebesgue generalized convergence theorem)** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions and $f$ a measurable function such that $f_n \to f$ a.e. in $\Omega$. Let $(g_n)_{n\in\mathbb{N}} \subset L^1(\Omega)$ such that for all $n \in \mathbb{N}$, $|f_n| \le g_n$ a.e. in $\Omega$ and $g_n \to g$ in $L^1(\Omega)$. Then,*

$$\int_\Omega f_n\,\mathrm{d}x \longrightarrow \int_\Omega f\,\mathrm{d}x.$$

For the applications we have in mind, we will need the following theorem in which the results of $(ii)$ and $(iii)$ express convergence in measure of some sequences.

**Theorem 2.6 (Young measures and nonlinear weak-* convergence)**

$(i)$ *Let $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, and $(v_n)_{n\in\mathbb{N}}$ be an equi-integrable sequence in $\Omega$ of functions to values in $\mathbb{R}^d$, $d \in \mathbb{N}$. Then, there exists a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ and a parametrized family $(\nu_x)_{x\in\Omega}$ of probability measures on $\mathbb{R}^d$, weakly measurable in $x$ with respect to the Lebesgue measure on $\Omega$, such that for all Carathéodory function $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^t$, $t \in \mathbb{N}$, we have*

$$\lim_{k\to\infty} \int_\Omega F(x, v_{n_k}(x))\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^d} F(x,\lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x, \tag{2.9}$$

*whenever the sequence $(F(\cdot, v_n(\cdot)))_{n\in\mathbb{N}}$ is equi-integrable in $\Omega$. In particular,*

$$v(x) := \int_{\mathbb{R}^d} \lambda\,\mathrm{d}\nu_x(\lambda) \tag{2.10}$$

*is the weak limit of the sequence $(v_{n_k})_{k\in\mathbb{N}}$ in $L^1(\Omega)$, as $k \to \infty$.*
*The family $(\nu_x)_{x\in\Omega}$ is called the Young measure generated by the subsequence $(v_{n_k})_{k\in\mathbb{N}}$.*

$(ii)$ *If $\Omega$ is of finite measure, and $(\nu_x)_{x\in\Omega}$ is the Young measure generated by a sequence $(v_n)_{n\in\mathbb{N}}$, then*

$$\left(\nu_x = \delta_{v(x)} \ a.e. \ x \in \Omega\right) \Longleftrightarrow (v_n \text{ converges in measure on } \Omega \text{ to } v \text{ as } n \to \infty).$$

$(iii)$ *If $\Omega$ is of finite measure, $(u_n)_{n\in\mathbb{N}}$ generates a Dirac Young measure $\left(\delta_{u(x)}\right)_{x\in\Omega}$ on $\mathbb{R}^{d_1}$, and $(v_n)_{n\in\mathbb{N}}$ generates a Young measure $(\nu_x)_{x\in\Omega}$ on $\mathbb{R}^{d_2}$, then the sequence $((u_n, v_n))_{n\in\mathbb{N}}$ generates the Young measure $\left(\delta_{u(x)} \otimes \nu_x\right)_{x\in\Omega}$ on $\mathbb{R}^{d_1+d_2}$.*

Whenever a sequence $(v_n)_{n\in\mathbb{N}}$ generates a Young measure $(\nu_x)_{x\in\Omega}$, following the terminology of [10] we will say that $(v_n)_{n\in\mathbb{N}}$ nonlinear weak-* converges, and $(\nu_x)_{x\in\Omega}$ is the nonlinear weak-* limit of the sequence $(v_n)_{n\in\mathbb{N}}$. In the case $(v_n)_{n\in\mathbb{N}}$ possesses a nonlinear weak-* convergent subsequence, we will say that it is nonlinear weak-* compact. Theorem 2.6$-(i)$ thus means that any equi-integrable sequence of measurable functions is nonlinear weak-* compact on $\Omega$.

# 3 Renormalized solution

In this part, we define and prove the existence of associated renormalized solution to the problem (1.1).

We define $\mathcal{T}^{1,p(\cdot)}(\Omega)$ as the set of functions $u : \Omega \longrightarrow \mathbb{R}$ measurable such that $T_k(u) \in W^{1,p(\cdot)}(\Omega)$, for any $k > 0$.

The following proposition (see *e.g.* [3]) is useful because it allows us to give a sense to the definition of the renormalized solution for the problem $(1.1)$ (see Definition 3.2 below).

**Proposition 3.1** *Let $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$. Then, there exists a unique measurable function $v : \Omega \longrightarrow \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}}, \quad \textit{for all } k > 0,$$

*where $\chi_E$ is the characteristic function of a measurable set $E$. The function $v$ is a generalized gradient and is denoted by $\nabla T_k(u)$ (weak gradient of $u$). If, moreover, $u$ belongs to $W^{1,p(\cdot)}(\Omega)$, then $v$ belongs to $\left(L^{p(\cdot)}(\Omega)\right)^N$ and coincides with the standard distributional gradient of $u$.*

We define also $\mathbb{S}$ as the set of $W^{1,\infty}$ functions $S : \mathbb{R} \longrightarrow \mathbb{R}$ having a compact support. The following function,

$$\text{for } k > 0, S_k : z \in \Omega \longmapsto \begin{cases} 1 & \text{if } |z| \leq k-1, \\ k - |z| & \text{if } k-1 \leq |z| \leq k, \\ 0 & \text{if } |z| \geq k, \end{cases} \tag{3.1}$$

is an example of function in $\mathbb{S}$ that will be used a lot in the sequel. Note that this function is non-negative with $suppS_k = [-k, k]$ and $suppS'_k$ is contained in $[-k, -k+1] \cup [k-1, k]$ and that the sequences $S_k$ and $S'_k$ are uniformly bounded by one.

Now, we give the definition of renormalized solution of (1.1) under the assumptions $(2.1) - (2.6)$.

**Definition 3.2** *Let $f \in L^1(\Omega)$. A measurable function $u : \Omega \to \mathbb{R}$ is a renormalized solution to the problem* $(1.1)$ *if $T_k(u) \in W^{1,p(\cdot)}(\Omega)$, $b(u) \in L^1(\Omega)$,*

$$\lim_{k\to\infty} \int_{\{k<|u|<k+1\}} a(x, \nabla u) \cdot \nabla u \, \mathrm{d}x = 0, \tag{3.2}$$

*and, for all $S \in \mathbb{S}$ and for all $\phi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, we get*

$$\int_\Omega \Big( S(u)a(x, \nabla u) \cdot \nabla\phi + S'(u)a(x, \nabla u) \cdot (\nabla u)\phi + b(u)S(u)\phi \Big) \, \mathrm{d}x = \int_\Omega f S(u)\phi \, \mathrm{d}x. \tag{3.3}$$

**Remark 3.3** *Since the support of $S$ is compact, we can write supp$S \subset [-k, k]$, and since $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$, then, by Proposition* 3.1*, we can replace the terms $\nabla u$ by $\nabla T_k(u)$ in the equation* $(3.3)$*. Consequently, as $T_k(u) \in W^{1,p(\cdot)}(\Omega)$, then by the growth assumption* $(2.5)$*, the terms $S(u)a(x, \nabla u)$ and $S'(u)a(x, \nabla u) \cdot \nabla u$ both lie in $L^1(\Omega)$. Also, we have $\chi_{\{k<|u|<k+1\}} a(x, \nabla u) \cdot \nabla u \in L^1(\Omega)$. Hence, the Definition* 3.2 *makes good sense.*

## 3.1 Existence of renormalized solution

In this part, we discuss the existence of the renormalized solution to the problem (1.1).

**Theorem 3.4** *Assume that* $(2.1) - (2.6)$ *hold and $f \in L^1(\Omega)$. Then, there exists at least one renormalized solution to the problem* $(1.1)$.

For the proof, we have to consider the notion of weak solution to the problem (1.1).

**Definition 3.5** (*cf.* [4]). *Let $f \in L^\infty(\Omega)$. A measurable function $u : \Omega \to \mathbb{R}$ is a weak solution to the problem* $(1.1)$ *if $u \in W^{1,p(\cdot)}(\Omega)$, $b(u) \in L^\infty(\Omega)$ and*

$$\int_\Omega a(x, \nabla u) \cdot \nabla\phi \, \mathrm{d}x + \int_\Omega b(u)\phi \, \mathrm{d}x = \int_\Omega f\phi \, \mathrm{d}x, \tag{3.4}$$

*for all $\phi \in W^{1,p(\cdot)}(\Omega)$.*

**Proof of Theorem 3.4.** The proof of existence of a renormalized solution of (1.1) is done in three steps: firstly, we introduce approximating problems for which existence of weak solutions $u_n$ is obvious; secondly, we establish some convergence results of this sequence of solutions $u_n$; thirdly, we prove that these approximate solutions $u_n$ tend, as $n$ goes to infinity, to a measurable function $u$ which is a renormalized solution of the problem (1.1).

### 3.1.1 Approximate solutions

Let $f_n = T_n(f)$, then $f_n \in L^\infty(\Omega)$ and converges strongly to $f$ in $L^1(\Omega)$. Moreover, $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$. Now, we consider the problem

$$\begin{cases} b(u_n) - \operatorname{div} a(x, \nabla u_n) = f_n & \text{in } \Omega, \\ a(x, \nabla u_n) \cdot \eta = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Under assumptions $(2.1)-(2.6)$ and since $f_n \in L^\infty(\Omega)$, the problem (3.5) admits a unique weak solution $u_n$ (see [4]), *i.e.* $u_n \in W^{1,p(\cdot)}(\Omega)$, $b(u_n) \in L^\infty(\Omega)$ and, for all $\phi \in W^{1,p(\cdot)}(\Omega)$,

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla\phi\,\mathrm{d}x + \int_\Omega b(u_n)\phi\,\mathrm{d}x = \int_\Omega f_n\phi\,\mathrm{d}x. \tag{3.6}$$

Our goal is to prove that the sequence of these approximated solutions $u_n$ to (3.5) converges to a measurable function $u$ which is a renormalized solution of the limit problem (1.1).

#### 3.1.2 Convergence results

The following proposition regroups convergence results of these approximated solutions (see [2, 4, 5]).

**Proposition 3.6**

$(i)$ *The sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. In particular, there exists a measurable function $u$ and a subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$, such that $u_n \longrightarrow u$ in measure and $u_n \longrightarrow u$ a.e. in $\Omega$.*

$(ii)$ *For all $k>0$, $T_k(u_n) \rightharpoonup T_k(u)$ in $W^{1,p(\cdot)}(\Omega)$ and $T_k(u_n)\to T_k(u)$ in $L^{p(\cdot)}(\Omega)$.*

$(iii)$ *For all $k>0$, $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ in $(L^1(\Omega))^N$.*

$(iv)$ *For all $k>0$, $a(x,\nabla T_k(u_n))$ converges strongly to $a(x,\nabla T_k(u))$ in $(L^1(\Omega))^N$ and weakly in $(L^{p'(\cdot)}(\Omega))^N$.*

Below, we give another result of convergence.

**Lemma 3.7** *For all $k>0$, the sequence $a(x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)$ converges strongly in $(L^1(\Omega))^N$ to $a(x,\nabla T_k(u))\cdot\nabla T_k(u)$.*

*Proof.* We use Vitali's theorem to get this strong convergence in $L^1(\Omega)$. By Proposition 3.6, one has

$$a(x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)\longrightarrow a(x,\nabla T_k(u))\cdot\nabla T_k(u) \text{ a.e. in } \Omega.$$

Moreover, by Hölder type inequality, we get, for $E\subset\Omega$,

$$\int_E a(x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)\,\mathrm{d}x \le 2\|a(x,\nabla T_k(u_n))\|_{L^{p'(\cdot)}(\Omega)}\|\nabla T_k(u_n)\chi_E\|_{L^{p(\cdot)}(\Omega)}.$$

But, the sequence $\big(a(x,\nabla T_k(u_n))\big)_{n\in\mathbb{N}}$ is bounded in $L^{p'(\cdot)}(\Omega)$ because it converges weakly in $L^{p'(\cdot)}(\Omega)$ and $\big(|\nabla T_k(u_n)|^{p(x)}\big)_{n\in\mathbb{N}}$ is equi-integrable in $\Omega$ because $(\nabla T_k(u_n))_{n\in\mathbb{N}}$ converges weakly in $L^{p(\cdot)}(\Omega)$. So,

$$\lim_{meas(E)}\int_E |\nabla T_k(u_n)|^{p(x)}\,\mathrm{d}x = 0.$$

Therefore, by Lemma 2.2, $\|\nabla T_k(u_n)\chi_E\|_{L^{p(\cdot)}(\Omega)} \to 0$ as $meas(E)\to 0$. Hence, one has the sequence $a(x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)$ is equi-integrable in $\Omega$ and so, by Vitali's theorem, one has the result. □

### 3.1.3 Existence of renormalized solution

**Lemma 3.8** *The function $u$ verifies the renormalized formulation* (3.3).

*Proof.* Let $\phi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $S \in \mathbb{S}$. We take $S(u_n)\phi$ as test function in (3.6) to get

$$\int_\Omega S'(u_n)a(x,\nabla u_n)\cdot(\nabla u_n)\phi\,\mathrm{d}x + \int_\Omega S(u_n)a(x,\nabla u_n)\cdot\nabla\phi\,\mathrm{d}x + \int_\Omega b(u_n)S(u_n)\phi\,\mathrm{d}x$$
$$= \int_\Omega f_n S(u_n)\phi\,\mathrm{d}x. \tag{3.7}$$

Since $suppS \subset (-k,k)$ for some real number $k>0$, $\nabla u_n$ can be replaced by $\nabla T_k(u_n)$ in (3.7) and we get

$$\int_\Omega S'(u_n)a(x,\nabla T_k(u_n))\cdot(\nabla T_k(u_n))\phi\,\mathrm{d}x + \int_\Omega S(u_n)a(x,\nabla T_k(u_n))\cdot\nabla\phi\,\mathrm{d}x$$
$$+\int_\Omega b(u_n)S(u_n)\phi\,\mathrm{d}x = \int_\Omega f_n S(u_n)\phi\,\mathrm{d}x. \tag{3.8}$$

By definition, the functions $b$ and $S$ are continuous and $suppS$ is compact. So, both sequences $b(u_n)S(u_n)$ and $S(u_n)$ are bounded. Moreover, $b(u_n)S(u_n)$ and $S(u_n)$ converge almost everywhere to $b(u)S(u)$ and $S(u)$ respectively. So, by Lebesgue dominated convergence theorem, they converge to $b(u)S(u)$ and $S(u)$, respectively, strongly in $L^1(\Omega)$. One has so

$$\lim_{n\to\infty}\int_\Omega b(u_n)S(u_n)\phi\,\mathrm{d}x = \int_\Omega b(u)S(u)\phi\,\mathrm{d}x.$$

By Proposition 3.6-$(iv)$, one can see that $a(x,\nabla T_k(u_n))$ converges weakly to $a(x,\nabla T_k(u))$ in $L^{p'(\cdot)}(\Omega)$, and as $S(u_n)\nabla\phi$ converges strongly to $S(u)\nabla\phi$ in $L^{p(\cdot)}(\Omega)$, we deduce that

$$\lim_{n\to\infty}\int_\Omega S(u_n)a(x,\nabla T_k(u_n))\cdot\nabla\phi\,\mathrm{d}x = \int_\Omega S(u)a(x,\nabla T_k(u))\cdot\nabla\phi\,\mathrm{d}x.$$

According to Lemma 3.7, $a(x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)$ converges strongly to $a(x,\nabla T_k(u))\cdot\nabla T_k(u)$ in $L^1(\Omega)$. So,

$$\lim_{n\to\infty}\int_\Omega S'(u_n)a(x,\nabla T_k(u_n))\cdot(\nabla T_k(u_n))\phi\,\mathrm{d}x = \int_\Omega S'(u)a(x,\nabla T_k(u))\cdot(\nabla T_k(u))\phi\,\mathrm{d}x.$$

Now, we are interested in the term of the right-hand side of (3.8). Since $T_n(f)$ converges strongly to $f$ in $L^1(\Omega)$, we conclude that

$$\lim_{n\to\infty}\int_\Omega f_n S(u_n)\phi\,\mathrm{d}x = \int_\Omega fS(u)\phi\,\mathrm{d}x.$$

Thus, passing to the limit in (3.8), we get that $u$ verifies equality (3.3). □

**Lemma 3.9** *The function $u$ respects the estimate* (3.2).

*Proof.* Let's take $T_{k+1}(u_n) - T_k(u_n)$ as test function in (3.6) to get

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla (T_{k+1}(u_n) - T_k(u_n))\, \mathrm{d}x + \int_\Omega b(u_n)(T_{k+1}(u_n) - T_k(u_n))\, \mathrm{d}x$$
$$= \int_\Omega f_n (T_{k+1}(u_n) - T_k(u_n))\, \mathrm{d}x. \tag{3.9}$$

One has $T_{k+1}(z) - T_k(z) = \begin{cases} sign(z) & \text{if } |z| > k+1, \\ 0 & \text{if } |z| < k, \\ z - ksign(z) & \text{if } k \leq |z| \leq k+1. \end{cases}$

The test function $T_{k+1}(u_n) - T_k(u_n)$ has a support contained in the set $\{|u_n| \geq k\}$, is bounded by one and has the same sign as $u_n$ which has the same sign as $b(u_n)$ when $b(0) = 0$ and $b$ is non-decreasing. So, $b(u_n)(T_{k+1}(u_n) - T_k(u_n)) \geq 0$. We have also $\nabla(T_{k+1}(u_n) - T_k(u_n)) = \nabla u_n \chi_{\{k<|u_n|<k+1\}}$ and the equality (3.9) becomes

$$\int_{\{k<|u_n|<k+1\}} a(x, \nabla u_n) \cdot \nabla u_n \, \mathrm{d}x \leq \int_{\{|u_n| \geq k\}} f_n \, \mathrm{d}x. \tag{3.10}$$

Recalling that $f_n := T_n(f)$ converges strongly to $f$ in $L^1(\Omega)$, one can see that the sequence $(f_n)_{n\in\mathbb{N}}$ is equi-integrable on $\Omega$. It is sufficient so to prove that $meas(\{|u_n| \geq k\})$ converges to zero as $k$ goes to infinity uniformly in $n$. Indeed, we take $T_k(u_n)$ as test function in the weak formulation (3.6) to get

$$\int_\Omega a(x, \nabla T_k(u_n)) \cdot \nabla T_k(u_n)\, \mathrm{d}x + \int_\Omega b(u_n) T_k(u_n)\, \mathrm{d}x = \int_\Omega f_n T_k(u_n)\, \mathrm{d}x.$$

Since $a(x, \nabla T_k(u_n)) \cdot \nabla T_k(u_n)$ is positive by (2.6), we get

$$\int_\Omega b(u_n) T_k(u_n)\, \mathrm{d}x \leq \int_\Omega f_n T_k(u_n)\, \mathrm{d}x, \tag{3.11}$$

which becomes

$$\int_{\{|u_n| \geq k\}} b(u_n) T_k(u_n)\, \mathrm{d}x \leq \int_\Omega f_n T_k(u_n)\, \mathrm{d}x, \tag{3.12}$$

because $b(u_n) T_k(u_n) \geq 0$.

Since $b$ is non-decreasing and $b(0) = 0$, one has $|b(u_n)| \geq \min(b(k), |b(-k)|)$ on the set $\{|u_n| \geq k\}$, and the inequality above becomes

$$\min(b(k), |b(-k)|) \int_{\{|u_n| \geq k\}} |T_k(u_n)|\, \mathrm{d}x \leq \int_\Omega f_n T_k(u_n)\, \mathrm{d}x.$$

Therefore, since $|T_k(u_n)| = k$ on $\{|u_n| \geq k\}$ then, one gets

$$k \cdot \min(b(k), |b(-k)|) \cdot meas(\{|u_n| \geq k\}) \leq k \int_\Omega f_n \, \mathrm{d}x \leq k \|f\|_{L^1(\Omega)}$$

which leads to

$$meas(\{|u_n| \geq k\}) \leq \frac{\|f\|_{L^1(\Omega)}}{\min(b(k), |b(-k)|)} \longrightarrow 0, \text{ as } k \longrightarrow \infty, \tag{3.13}$$

since $b$ is non-decreasing and onto so, $b$ has an infinity limit at infinity. Hence, by equi-integrability of $f_n$, the right-hand side of (3.10) tends to zero uniformly in $n$ as $k \to \infty$. And so, by monotonicity (2.4), the inequality (3.10) becomes

$$\lim_{k\to\infty} \sup_n \int_{\{k<|u_n|<k+1\}} a(x, \nabla u_n) \cdot \nabla u_n \,\mathrm{d}x = 0$$

or again

$$\lim_{k\to\infty} \lim_{n\to\infty} \int_\Omega a(x, \nabla T_{k+1}(u_n)) \cdot \nabla T_{k+1}(u_n) \chi_{\{k<|u_n|<k+1\}} \,\mathrm{d}x = 0. \tag{3.14}$$

Let

$$D_{n,k} := a(x, \nabla T_{k+1}(u_n)) \cdot \nabla T_{k+1}(u_n).$$

According to Lemma 3.7, $D_{n,k} \to a(x, \nabla T_{k+1}(u)) \cdot \nabla T_{k+1}(u)$ strongly in $L^1(\Omega)$. Moreover, since $u_n$ converges a.e. to $u$ by Proposition 3.6, then by the continuity of $\chi_{(k,k+1)\cup(-k-1,-k)}(\cdot)$ on the image of $\Omega$ by $u(\cdot)$, we conclude that, as $n \to \infty$,

$$\chi_{\{k<|u_n|<k+1\}} = \chi_{(k,k+1)\cup(-k-1,-k)}(u_n) \to \chi_{(k,k+1)\cup(-k-1,-k)}(u) = \chi_{\{k<|u|<k+1\}} \text{ a.e. in } \Omega.$$

Indeed, $\chi_{(k,k+1)\cup(-k-1,-k)}(\cdot)$ is continuous if $meas\,(\{|u| = k\}) = 0$ for a.e. $k > 0$. But, for all $n$, one has,

$$\left\{|T_k(u)| \geq k - \frac{1}{2}\right\} \subset \{|u_n| \geq k-1\} \cup \left\{|T_k(u_n) - T_k(u)| > \frac{1}{2}\right\}$$

and so

$$meas\left(\left\{|T_k(u)| \geq k - \frac{1}{2}\right\}\right) \leq meas\,(\{|u_n| \geq k-1\}) + meas\left(\left\{|T_k(u_n) - T_k(u)| > \frac{1}{2}\right\}\right).$$

From (3.13) and as $T_k(u_n)$ converges to $T_k(u)$ in measure in $\Omega$, one gets, as $n \to \infty$,

$$meas\,(\{|u| = k\}) \leq meas\left(\left\{|T_k(u)| \geq k - \frac{1}{2}\right\}\right) \leq 0 \Longrightarrow meas\,(\{|u| = k\}) = 0.$$

Now, since

$$\begin{cases} D_{n,k}\chi_{\{k<|u_n|<k+1\}} \to a(x, \nabla T_{k+1}(u)) \cdot \nabla T_{k+1}(u) \chi_{\{k<|u|<k+1\}} \text{ a.e. in } \Omega. \\ |D_{n,k}\chi_{\{k<|u_n|<k+1\}}| \leq D_{n,k} \in L^1(\Omega) \text{ a.e. in } \Omega, \text{ for all } n \in \mathbb{N}, \text{ and} \\ D_{n,k} \to a(x, \nabla T_{k+1}(u)) \cdot \nabla T_{k+1}(u) \text{ in } L^1(\Omega), \end{cases}$$

then, by the Lebesgue generalized convergence theorem, we can write

$$\lim_{n\to\infty} \int_\Omega D_{n,k}\chi_{\{k<|u_n|<k+1\}} \,\mathrm{d}x = \int_\Omega a(x, \nabla T_{k+1}(u)) \cdot \nabla T_{k+1}(u) \chi_{\{k<|u|<k+1\}} \,\mathrm{d}x. \tag{3.15}$$

Now, coming back to the equality (3.14), we get the equality

$$\lim_{k\to\infty} \int_{\{k<|u|<k+1\}} a(x, \nabla u) \cdot \nabla u \,\mathrm{d}x = 0, \tag{3.16}$$

which proves the Lemma 3.9. □

**Lemma 3.10** *$u$ is a renormalized solution to the problem* (1.1).

*Proof.* By the Proposition 3.6$-(ii)$, one has $T_k(u) \in W^{1,p(\cdot)}(\Omega)$. Moreover, one can show that $\|b(u_n)\|_{L^1(\Omega)}$ is uniformly bounded. Indeed, by (3.12), one has

$$\int_{\{|u_n|\geq k\}} |b(u_n)|\,\mathrm{d}x \leq \int_\Omega f_n\,\mathrm{d}x.$$

Then,

$$\begin{aligned}\int_\Omega |b(u_n)|\,\mathrm{d}x &= \int_{\{|u_n|<k\}} |b(u_n)|\,\mathrm{d}x + \int_{\{|u_n|\geq k\}} |b(u_n)|\,\mathrm{d}x \\ &\leq \max(b(k), |b(-k)|)\cdot meas(\Omega) + \int_\Omega f_n\,\mathrm{d}x.\end{aligned}$$

Since $f_n$ converges strongly, then the right-hand side is bounded. Therefore, $\|b(u_n)\|_{L^1(\Omega)}$ is uniformly bounded. One has also, by the continuity of $b$, $b(u_n) \longrightarrow b(u)$ a.e. in $\Omega$. So, Fatou's lemma gives us

$$\int_\Omega |b(u)|\,\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega |b(u_n)|\,\mathrm{d}x.$$

Hence, $b(u) \in L^1(\Omega)$. Also, thanks to the lemmas 3.8 and 3.9, we conclude that $u$ is a renormalized solution to the problem (1.1). This is the end of the proof of Theorem 3.4. □

## 3.2 Uniqueness of renormalized solution

Now, let's go to the uniqueness of the solution of problem (1.1).

**Theorem 3.11** *Assume that* $(2.1)-(2.6)$ *hold and* $f \in L^1(\Omega)$. *Then, there is uniqueness of the renormalized solution to the problem* (1.1).

*Proof.* Let $k, h > 0$ and $u_1$ and $u_2$ be two renormalized solutions of problem (1.1) associated to the same data $f \in L^1(\Omega)$. As $T_h(u_2) \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, then one has $T_k(u_1 - T_h(u_2)) \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ which can be taken as test function in (3.3) for $u_1$. Similarly, we can take $T_k(u_2 - T_h(u_1))$ as test function in (3.3) for $u_2$. By addition, we get

$$\begin{aligned}&\int_{\{|u_1-T_h(u_2)|\leq k\}} S_M(u_1)a(x,\nabla u_1)\cdot\nabla(u_1 - T_h(u_2))\,\mathrm{d}x \\ &\quad + \int_{\{|u_2-T_h(u_1)|\leq k\}} S_M(u_2)a(x,\nabla u_2)\cdot\nabla(u_2 - T_h(u_1))\,\mathrm{d}x \\ &\quad + \int_\Omega S'_M(u_1)a(x,\nabla u_1)\cdot(\nabla u_1)T_k(u_1 - T_h(u_2))\,\mathrm{d}x \\ &\quad + \int_\Omega S'_M(u_2)a(x,\nabla u_2)\cdot(\nabla u_2)T_k(u_2 - T_h(u_1))\,\mathrm{d}x \\ &\quad + \int_\Omega S_M(u_1)b(u_1)T_k(u_1 - T_h(u_2))\,\mathrm{d}x \;+\; \int_\Omega S_M(u_2)b(u_2)T_k(u_2 - T_h(u_1))\,\mathrm{d}x \\ &= \int_\Omega f\Big(S_M(u_1)T_k(u_1 - T_h(u_2)) + S_M(u_2)T_k(u_2 - T_h(u_1))\Big)\,\mathrm{d}x, \qquad (3.17)\end{aligned}$$

where $(S_M)$ is the sequence of functions in $\mathbb{S}$ defined in (3.1). While $M$ and $k$ are fixed, $h$ can be sent to infinity. Define the sets

$$E_1 := \{|u_1 - u_2| \leq k, |u_2| \leq h\},\ E_2 := E_1 \cap \{|u_1| \leq h\} \text{ and } E_3 := E_1 \cap \{|u_1| > h\}.$$

We start with the first integral in (3.17). By (2.4), we have

$$\begin{aligned}
&\int_{\{|u_1 - T_h(u_2)| \leq k\}} S_M(u_1) a(x, \nabla u_1) \cdot \nabla(u_1 - T_h(u_2)) \, \mathrm{d}x \\
&= \int_{\{|u_1 - T_h(u_2)| \leq k, |u_2| \leq h\}} S_M(u_1) a(x, \nabla u_1) \cdot \nabla(u_1 - T_h(u_2)) \, \mathrm{d}x \\
&\quad + \int_{\{|u_1 - T_h(u_2)| \leq k, |u_2| > h\}} S_M(u_1) a(x, \nabla u_1) \cdot \nabla(u_1 - T_h(u_2)) \, \mathrm{d}x \\
&= \int_{\{|u_1 - T_h(u_2)| \leq k, |u_2| \leq h\}} S_M(u_1) a(x, \nabla u_1) \cdot \nabla(u_1 - u_2) \, \mathrm{d}x \\
&\quad + \int_{\{|u_1 - T_h(u_2)| \leq k, |u_2| > h\}} S_M(u_1) a(x, \nabla u_1) \cdot \nabla u_1 \, \mathrm{d}x \\
&\geq \int_{\{|u_1 - T_h(u_2)| \leq k, |u_2| \leq h\}} S_M(u_1) a(x, \nabla u_1) \cdot \nabla(u_1 - u_2) \, \mathrm{d}x \\
&= \int_{E_2} S_M(u_1) a(x, \nabla u_1) \cdot \nabla(u_1 - u_2) \, \mathrm{d}x + \int_{E_3} S_M(u_1) a(x, \nabla u_1) \cdot \nabla(u_1 - u_2) \, \mathrm{d}x \\
&= \int_{E_2} S_M(u_1) a(x, \nabla u_1) \cdot \nabla(u_1 - u_2) \, \mathrm{d}x + \int_{E_3} S_M(u_1) a(x, \nabla u_1) \cdot \nabla u_1 \, \mathrm{d}x \\
&\quad - \int_{E_3} S_M(u_1) a(x, \nabla u_1) \cdot \nabla u_2 \, \mathrm{d}x \\
&\geq \int_{E_2} S_M(u_1) a(x, \nabla u_1) \cdot \nabla(u_1 - u_2) \, \mathrm{d}x - \int_{E_3} S_M(u_1) a(x, \nabla u_1) \cdot \nabla u_2 \, \mathrm{d}x. \qquad (3.18)
\end{aligned}$$

Using (2.5) and Hölder type inequality, the last integral in (3.18) gives

$$\begin{aligned}
&\left| \int_{E_3} S_M(u_1) a(x, \nabla u_1) \cdot \nabla u_2 \, \mathrm{d}x \right| \\
&\quad \leq C_1 \left( \|\mathcal{M}\|_{p'(\cdot)} + \||\nabla u_1|^{p(x)-1}\|_{L^{p'(\cdot)}(\{h < |u_1| \leq h+k\})} \right) \|\nabla u_2\|_{L^{p(\cdot)}(\{h-k < |u_2| \leq h\})}. \qquad (3.19)
\end{aligned}$$

Now, we take $\phi = T_k(u_1 - T_h(u_1))$ as test function in (3.3) for $u_1$ and $S \in \mathbb{S}$ such that $S = S_{h+k+1}$. We get

$$\begin{aligned}
&\int_\Omega S(u_1) a(x, \nabla u_1) \cdot \nabla T_k(u_1 - T_h(u_1)) \, \mathrm{d}x + \int_\Omega S'(u_1) a(x, \nabla u_1) \cdot (\nabla u_1) T_k(u_1 - T_h(u_1)) \, \mathrm{d}x \\
&\quad + \int_\Omega b(u_1) S(u_1) T_k(u_1 - T_h(u_1)) \, \mathrm{d}x \; = \; \int_\Omega f S(u_1) T_k(u_1 - T_h(u_1)) \, \mathrm{d}x.
\end{aligned}$$

Since the third is non-negative, then one has

$$\int_{\{h < |u_1| \leq h+k\}} a(x, \nabla u_1) \cdot \nabla u_1 \, \mathrm{d}x - k \int_{\{h+k < |u_1| \leq h+k+1\}} a(x, \nabla u_1) \cdot \nabla u_1 \, \mathrm{d}x \leq k \int_{\{|u_1| > h\}} |f| \, \mathrm{d}x.$$

By using (2.6), we get

$$C_2 \int_{\{h<|u_1|\le h+k\}} |\nabla u_1|^{p(x)}\,\mathrm{d}x \le k\left(\int_{\{|u_1|>h\}} |f|\,\mathrm{d}x + \int_{\{h+k<|u_1|\le h+k+1\}} a(x,\nabla u_1)\cdot\nabla u_1\,\mathrm{d}x\right).$$

By (3.2) and since $meas(\{|u_1|>h\}) \to 0$ as $h\to\infty$, and since $f\in L^1(\Omega)$, we deduce that

$$\lim_{h\to\infty}\int_{\{h<|u_1|\le h+k\}} |\nabla u_1|^{p(x)}\,\mathrm{d}x = 0, \text{ for any fixed number } k>0,$$

and so, by Lemma 2.2, we get $\lim\limits_{h\to\infty} \||\nabla u_1|^{p(x)-1}\|_{L^{p'(\cdot)}(\{h<|u_1|\le h+k\})} = 0.$

Similarly, taking $\phi = T_k(u_2 - T_h(u_2))$ as test function in (3.3) for $u_2$ with the same $S$ in $\mathbb{S}$, we get

$$\lim_{h\to\infty}\int_{\{h<|u_2|\le h+k\}} |\nabla u_2|^{p(x)}\,\mathrm{d}x = 0, \text{ for any fixed number } k>0.$$

Hence,

$$\lim_{h\to\infty}\int_{\{h-k<|u_2|\le h\}} |\nabla u_2|^{p(x)}\,\mathrm{d}x = \lim_{l\to\infty}\int_{\{l<|u_2|\le l+k\}} |\nabla u_2|^{p(x)}\,\mathrm{d}x = 0,$$

for any fixed number $k>0$ with $l = h-k$.

So, by Lemma 2.2, $\|\nabla u_2\|_{L^{p(\cdot)}(\{h-k<|u_2|\le h\})} \to 0$ as $h\to\infty$, for any fixed number $k>0$. Therefore, from (3.18) and (3.19), we obtain

$$\begin{aligned}\int_{\{|u_1-T_h(u_2)|\le k\}} & S_M(u_1)a(x,\nabla u_1)\cdot\nabla(u_1 - T_h(u_2))\,\mathrm{d}x \\ &\ge I_h + \int_{E_2} S_M(u_1)a(x,\nabla u_1)\cdot\nabla(u_1-u_2)\,\mathrm{d}x,\end{aligned} \tag{3.20}$$

where $I_h$ converges to zero as $h\to\infty$.

We may adopt the same procedure to treat the second term in (3.17) to obtain

$$\begin{aligned}\int_{\{|u_2-T_h(u_1)|\le k\}} & S_M(u_2)a(x,\nabla u_2)\cdot\nabla(u_2 - T_h(u_1))\,\mathrm{d}x \\ &\ge J_h - \int_{E_2} S_M(u_2)a(x,\nabla u_2)\cdot\nabla(u_1-u_2)\,\mathrm{d}x,\end{aligned} \tag{3.21}$$

where $J_h$ converges to zero as $h\to\infty$.

Now, for all $h,k>0$, we set

$$\begin{aligned} K_h &= \int_\Omega S_M(u_1)b(u_1)T_k(u_1-T_h(u_2))\,\mathrm{d}x + \int_\Omega S_M(u_2)b(u_2)T_k(u_2-T_h(u_1))\,\mathrm{d}x,\\ R_h &= \int_\Omega S'_M(u_1)a(x,\nabla u_1)\cdot(\nabla u_1)T_k(u_1-T_h(u_2))\,\mathrm{d}x \\ &\quad + \int_\Omega S'_M(u_2)a(x,\nabla u_2)\cdot(\nabla u_2)T_k(u_2-T_h(u_1))\,\mathrm{d}x \end{aligned}$$

and

$$F_h = \int_\Omega f\Big(S_M(u_1)T_k(u_1-T_h(u_2)) + S_M(u_2)T_k(u_2-T_h(u_1))\Big)\,\mathrm{d}x.$$

We have

$$S_M(u_1)b(u_1)T_k(u_1 - T_h(u_2)) \to S_M(u_1)b(u_1)T_k(u_1 - u_2) \text{ a.e. in } \Omega, \text{ as } h \to \infty,$$

and

$$|S_M(u_1)b(u_1)T_k(u_1 - T_h(u_2))| \leq k|b(u_1)| \in L^1(\Omega).$$

Then, by Lebesgue dominated convergence theorem, we deduce that

$$\lim_{h\to\infty} \int_\Omega S_M(u_1)b(u_1)T_k(u_1 - T_h(u_2))\,\mathrm{d}x = \int_\Omega S_M(u_1)b(u_1)T_k(u_1 - u_2)\,\mathrm{d}x. \tag{3.22}$$

Similarly, we have

$$\lim_{h\to\infty} \int_\Omega S_M(u_2)b(u_2)T_k(u_2 - T_h(u_1))\,\mathrm{d}x = \int_\Omega S_M(u_2)b(u_2)T_k(u_2 - u_1)\,\mathrm{d}x. \tag{3.23}$$

Using (3.22) and (3.23), we get

$$\lim_{h\to\infty} K_h = \int_\Omega \big(S_M(u_1)b(u_1) - S_M(u_2)b(u_2)\big)T_k(u_1 - u_2)\,\mathrm{d}x. \tag{3.24}$$

By the same procedure as above, we use the Lebesgue dominated convergence theorem to obtain

$$\lim_{h\to\infty} R_h = \int_\Omega \big(S'_M(u_1)a(x,\nabla u_1)\cdot\nabla u_1 - S'_M(u_2)a(x,\nabla u_2)\cdot\nabla u_2\big)T_k(u_1 - u_2)\,\mathrm{d}x \tag{3.25}$$

and

$$\lim_{h\to\infty} F_h = \int_\Omega f\big(S_M(u_1) - S_M(u_2)\big)T_k(u_1 - u_2)\,\mathrm{d}x. \tag{3.26}$$

Using (3.20), (3.21), (3.24) − (3.26), we get from (3.17) the following inequality as $h \to \infty$.

$$\begin{aligned}
&\int_{\{|u_1-u_2|\leq k\}} \big(S_M(u_1)a(x,\nabla u_1) - S_M(u_2)a(x,\nabla u_2)\big)\cdot\nabla(u_1 - u_2)\,\mathrm{d}x \\
&\qquad + \int_\Omega \big(S'_M(u_1)a(x,\nabla u_1)\cdot\nabla u_1 - S'_M(u_2)a(x,\nabla u_2)\cdot\nabla u_2\big)T_k(u_1 - u_2)\,\mathrm{d}x \\
&\qquad + \int_\Omega \big(S_M(u_1)b(u_1) - S_M(u_2)b(u_2)\big)T_k(u_1 - u_2)\,\mathrm{d}x \\
&\quad \leq \int_\Omega f\big(S_M(u_1) - S_M(u_2)\big)T_k(u_1 - u_2)\,\mathrm{d}x.
\end{aligned} \tag{3.27}$$

Now, we fix $k > 0$ and we pass to the limit in (3.27), as $M$ tends to infinity.

The second term of the left-hand side of (3.27) is, in absolute value, smaller than

$$k\left(\int_{\{M-1\leq|u_1|\leq M\}} a(x,\nabla u_1)\cdot\nabla u_1\,\mathrm{d}x + \int_{\{M-1\leq|u_2|\leq M\}} a(x,\nabla u_2)\cdot\nabla u_2\,\mathrm{d}x\right)$$

which converges to zero, as $M \to \infty$, thanks to relation (3.2) for $u_1$ and $u_2$. Therefore, the second integral of (3.27) converges to zero as $M \to \infty$.

Since $S_M \to 1$ as $M \to \infty$, then $\big(S_M(u_1)a(x,\nabla u_1) - S_M(u_2)a(x,\nabla u_2)\big)\cdot\big(\nabla u_1 - \nabla u_2\big)$ converges a.e. to $\big(a(x,\nabla u_1) - a(x,\nabla u_2)\big)\cdot\big(\nabla u_1 - \nabla u_2\big)$ and moreover, thanks to (2.5) and to Hölder type inequality, one has, a.e. in $\{|u_1 - u_2| \leq k\}$,

$$\begin{aligned}&\big|\big(S_M(u_1)a(x,\nabla u_1) - S_M(u_2)a(x,\nabla u_2)\big)\cdot\big(\nabla u_1 - \nabla u_2\big)\big|\\&\quad\leq \big(|a(x,\nabla u_1)| + |a(x,\nabla u_2)|\big)\cdot\big(|\nabla u_1| + |\nabla u_2|\big) \in L^1(\Omega).\end{aligned}$$

Thus, by the Lebesgue dominated convergence theorem, the first integral in (3.27) converges to the integral of $\big(a(x,\nabla u_1) - a(x,\nabla u_2)\big)\cdot\nabla(u_1 - u_2)$ in $\{|u_1 - u_2| \leq k\}$.

Similarly, by the Lebesgue dominated convergence theorem, the third integral in (3.27) converges to the integral of $\big(b(u_1) - b(u_2)\big)T_k(u_1 - u_2)$ in $\Omega$.

We next examine the right-hand side of (3.27). For all $k > 0$,

$$f\big(S_M(u_1) - S_M(u_2)\big)T_k(u_1 - u_2) \to 0 \text{ a.e. in } \Omega \text{ as } M \to \infty,$$

and

$$|f\big(S_M(u_1) - S_M(u_2)\big)T_k(u_1 - u_2)| \leq 2k|f| \in L^1(\Omega).$$

The Lebesgue dominated convergence theorem allows us to write

$$\lim_{M\to\infty}\int_\Omega f\big(S_M(u_1) - S_M(u_2)\big)T_k(u_1 - u_2)\,\mathrm{d}x = 0.$$

Thus, as $M \to \infty$, (3.27) gives

$$\begin{aligned}&\int_{\{|u_1-u_2|\leq k\}}\big(a(x,\nabla u_1) - a(x,\nabla u_2)\big)\cdot\nabla(u_1 - u_2)\,\mathrm{d}x\\&+\int_\Omega\big(b(u_1) - b(u_2)\big)T_k(u_1 - u_2)\,\mathrm{d}x \leq\ 0,\end{aligned}\tag{3.28}$$

for all $k > 0$. From (3.28), since all terms are non-negative, then one deduces

$$\int_{\{|u_1-u_2|<k\}}\big(a(x,\nabla u_1) - a(x,\nabla u_2)\big)\cdot\nabla(u_1 - u_2)\,\mathrm{d}x = 0\tag{3.29}$$

and

$$\int_\Omega\big(b(u_1) - b(u_2)\big)T_k(u_1 - u_2)\,\mathrm{d}x = 0.\tag{3.30}$$

From (3.29) and the strict monotonicity assumption (2.4), one has

$$u_1 - u_2 = c \text{ a.e. in } \Omega, \text{ where } c \text{ is a real constant.}\tag{3.31}$$

From (3.32), one has

$$\int_\Omega|b(u_1) - b(u_2)|\,\mathrm{d}x = \lim_{k\to\infty}\int_\Omega\big(b(u_1) - b(u_2)\big)\frac{1}{k}T_k(u_1 - u_2)\,\mathrm{d}x = 0\tag{3.32}$$

which gives

$$b(u_1) = b(u_2).\tag{3.33}$$

Thus, from (3.31) and (3.33), the uniqueness of the renormalized solution follows in the sense of $b$.

□

## 4 Continuous dependence for renormalized solutions

We consider elliptic problems

$$(Pb_n): \begin{cases} b(u_n) - \operatorname{div} a_n(x, \nabla u_n) = f_n & \text{in } \Omega, \\ a_n(x, \nabla u_n) \cdot \eta = 0 & \text{on } \partial\Omega, \end{cases}$$

where the assumptions $(2.1) - (2.6)$ are verified, for all $n \in \mathbb{N}$, with the diffusion flux functions $a_n(\cdot, \cdot)$, the exponents $p_n : \overline{\Omega} \longrightarrow [p_-, p_+]$ and the non-negative functions $\mathcal{M}_n$ in $L^{p'_n(\cdot)}(\Omega)$ such that the sequence $\left(\|\mathcal{M}_n\|_{L^{p'_n(\cdot)}(\Omega)}\right)_{n\in\mathbb{N}}$ is uniformly bounded, and with $C_1$, $C_2$, $p_+$ and $p_-$ independent of $n$.

Note that, for $f_n \in L^1(\Omega)$ and under assumptions $(2.1) - (2.6)$, the problem $(Pb_n)$ admits a unique renormalized solution $u_n$.

The purpose of this section is to prove that the sequence of solutions $(u_n)_{n\in\mathbb{N}}$ to problems $(Pb_n)$ converges to a function $u$ which is a solution of limit problem (1.1) with the exponent $p$, when we have the following convergence assumption:

$$\left[ \begin{array}{l} \text{for all bounded subset } K \text{ of } \mathbb{R}^N, \\ \sup\limits_{\xi\in K} |a_n(\cdot, \xi) - a(\cdot, \xi)| \text{ converges to zero in measure on } \Omega, \end{array} \right. \tag{4.1}$$

where $a(x, \xi)$ verifies the assumptions $(2.3) - (2.6)$ with the exponent $p$ verifying (2.1) such that

$$p_n \text{ converges to } p \text{ in measure on } \Omega. \tag{4.2}$$

We assume also that

$$f_n \text{ converges to } f \text{ weakly in } L^1(\Omega). \tag{4.3}$$

We further assume that the exponents $p$ and $p_n$ verify log-Hölder continuity assumption:

$$\exists c > 0, \forall x, y \in \overline{\Omega}, x \neq y, -(\log |x - y|)|p(x) - p(y)| \leq c. \tag{4.4}$$

**Remark 4.1** *Note that several regularity results for Sobolev spaces with variable exponents can be obtained thanks to log-Hölder continuity condition* $(4.4)$*; in particular,* $C^\infty(\overline{\Omega})$ *is dense in* $W^{1,p(\cdot)}(\Omega)$ *(for more details, see* [8]*).*

Now, through the theorem below, we establish a structural stability result for the renormalized solutions.

**Theorem 4.2** *Under the assumptions* $(4.1) - (4.3)$*, let* $(u_n)_{n\in\mathbb{N}}$ *be the sequence of renormalized solutions of the problems* $(Pb_n)$ *associated to* $a_n(\cdot, \cdot)$*,* $f_n$ *and the exponents* $p_n$ *with* $a(\cdot, \cdot)$*,* $f$ *and* $p$ *the respective limits of* $a_n$*,* $f_n$ *and* $p_n$ *in* $(4.1) - (4.3)$*. Assume that the exponents* $p$*,* $p_n$ *satisfy the log-Hölder continuity assumption* $(4.4)$*. Then there exists a measurable function* $u$ *on* $\Omega$ *such that* $u_n$*,* $\nabla u_n$ *converge to* $u$*,* $\nabla u$*, respectively, a.e. in* $\Omega$*, as* $n \to \infty$*. The function* $u$ *is, in fact, a renormalized solution of the problem* $(1.1)$ *associated to the diffusion flux* $a(\cdot, \cdot)$ *and the source term* $f$*.*

**Proof of Theorem 4.2.** We shall divide the proof into several steps. Throughout the proof, we reason up to an extracted subsequence of $(u_n)_{n\in\mathbb{N}}$ (still denoted by $(u_n)_{n\in\mathbb{N}}$) and every positive constant independent of $n$ will be denoted by $C$.

**Lemma 4.3** $(i)$ *For all $k>0$, the sequence $\left(\|T_k(u_n)\|_{1,p_n(\cdot)}\right)_{n\in\mathbb{N}}$ is bounded.*

$(ii)$ *The sequence of renormalized solutions $(u_n)_{n\in\mathbb{N}}$ of the problems $(Pb_n)$ verifies, for $k>0$ large enough,*

$$meas(\{|u_n|>k\}) \leq \frac{C\|f_n\|_{L^1}}{\min\left(b(k),|b(-k)|\right)}, \tag{4.5}$$

$$\sup_n meas(\{|u_n|>k\}) \to 0, \text{ as } k\to\infty, \tag{4.6}$$

*and*

$$\lim_{k\to\infty}\sup_n \int_{\{k<|u_n|<k+1\}} |\nabla u_n|^{p_n(x)}\,\mathrm{d}x = 0. \tag{4.7}$$

$(iii)$ *There exists a measurable function $u$ on $\Omega$ such that, up to a subsequence, $T_k(u_n) \rightharpoonup T_k(u)$ in $W^{1,p_-}(\Omega)$, for all $k>0$. Moreover, $u_n\to u$ a.e. in $\Omega$, $\nabla T_k(u_n)$ converges to a Young measure $(\nu_x^k)_{x\in\Omega}$ on $\mathbb{R}^N$ in the sense of the nonlinear weak-$^*$ convergence and*

$$\nabla T_k(u) = \int_{\mathbb{R}^N} \lambda\,\mathrm{d}\nu_x^k(\lambda). \tag{4.8}$$

$(iv)$ *For all $k>0$, $|\lambda|^{p(x)}$ is integrable with respect to the measure $\mathrm{d}\nu_x^k(\lambda)\,\mathrm{d}x$ on $\mathbb{R}^N\times\Omega$ and $T_k(u)\in W^{1,p(\cdot)}(\Omega)$.*

$(v)$ *One has*

$$\lim_{k\to\infty}\int_\Omega |\nabla(T_{k+1}(u)-T_k(u))|^{p(x)}\,\mathrm{d}x = 0. \tag{4.9}$$

*Proof.* $(i)$ In the renormalized formulation (3.3) of the problem $(Pb_n)$, we choose $S=S_{h+k}\in\mathbb{S}$ defined in (3.1) with $h,k>0$, $h$ large enough. Because $u_n$ is a renormalized solution of the problem $(Pb_n)$, we have $T_k(u_n)\in W^{1,p_n(\cdot)}(\Omega)\cap L^\infty(\Omega)$ and so, we can take $\phi=T_k(u_n)$ as test function in the renormalized formulation (3.3) to obtain, the term $\int_\Omega b(u_n)S(u_n)T_k(u_n)\,\mathrm{d}x$ being non-negative,

$$\int_\Omega a_n(x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)\,\mathrm{d}x + \int_\Omega S'(u_n)a_n(x,\nabla u_n)\cdot(\nabla u_n)T_k(u_n)\,\mathrm{d}x \leq k\int_\Omega |f_n|\,\mathrm{d}x.$$

While $k$ is fixed, $h$ can be sent to infinity. By (3.2), the second integral vanishes, as $h\to\infty$, because $\left|\int_\Omega S'(u_n)a_n(x,\nabla u_n)\cdot(\nabla u_n)T_k(u_n)\,\mathrm{d}x\right| \leq k\int_{\{h+k-1<|u_n|<h+k\}} a_n(x,\nabla u_n)\cdot\nabla u_n\,\mathrm{d}x$. So, by using coercivity condition (2.6), as $h\to\infty$ we have

$$C\int_\Omega |\nabla T_k(u_n)|^{p_n(x)}\,\mathrm{d}x \leq k\int_\Omega |f_n|\,\mathrm{d}x.$$

Since the sequence $(f_n)_{n\in\mathbb{N}}$ converges weakly to $f$ in $L^1(\Omega)$, then the right-hand side of this last inequality is uniformly bounded. So, we obtain

$$\int_\Omega |\nabla T_k(u_n)|^{p_n(x)}\,\mathrm{d}x \leq Ck. \tag{4.10}$$

Moreover,

$$\int_\Omega |T_k(u_n)|^{p_n(x)}\,\mathrm{d}x \le \int_\Omega k^{p_n(x)}\,\mathrm{d}x \le \max(k^{p_+},k^{p_-})meas(\Omega). \tag{4.11}$$

From (4.10) and (4.11), we deduce that the sequence $\rho_{1,p_n(\cdot)}(T_k(u_n))$ is uniformly bounded. By the Lemma 2.3 and the fact that $p_n(\cdot)\in[p_-,p_+]$, one has

$$\|T_k(u_n)\|_{1,p_n(\cdot)} \le \max\left(\rho_{1,p_n(\cdot)}(T_k(u_n))^{1/p_-},\rho_{1,p_n(\cdot)}(T_k(u_n))^{1/p_+}\right).$$

We conclude that the sequence $\|T_k(u_n)\|_{1,p_n(\cdot)}$ is uniformly bounded.

$(ii)$ In the renormalized formulation (3.3) of problem $(Pb_n)$, we take $S=S_k\in\mathbb{S}$ and $\phi=T_{\frac{1}{k}}(u_n)$ as test function, with $k>0$ large enough. We obtain

$$\int_\Omega S(u_n)a_n(x,\nabla u_n)\cdot\nabla T_{\frac{1}{k}}(u_n)\,\mathrm{d}x + \int_\Omega S'(u_n)a_n(x,\nabla u_n)\cdot(\nabla u_n)T_{\frac{1}{k}}(u_n)\,\mathrm{d}x$$
$$+\int_\Omega b(u_n)S(u_n)T_{\frac{1}{k}}(u_n)\,\mathrm{d}x = \int_\Omega f_nS(u_n)T_{\frac{1}{k}}(u_n)\,\mathrm{d}x,$$

which gives

$$\int_\Omega a_n\left(x,\nabla T_{\frac{1}{k}}(u_n)\right)\cdot\nabla T_{\frac{1}{k}}(u_n)\,\mathrm{d}x + \int_\Omega S'(u_n)a_n(x,\nabla u_n)\cdot(\nabla u_n)T_{\frac{1}{k}}(u_n)\,\mathrm{d}x$$
$$+\int_\Omega b(u_n)S(u_n)T_{\frac{1}{k}}(u_n)\,\mathrm{d}x \le \frac{1}{k}\|f_n\|_{L^1(\Omega)},$$

or again

$$k\int_\Omega S'(u_n)a_n(x,\nabla u_n)\cdot(\nabla u_n)T_{\frac{1}{k}}(u_n)\,\mathrm{d}x + \int_\Omega b(u_n)S(u_n)kT_{\frac{1}{k}}(u_n)\,\mathrm{d}x \le \|f_n\|_{L^1(\Omega)}.$$

The term $k\int_\Omega S'(u_n)a_n(x,\nabla u_n)\cdot(\nabla u_n)T_{\frac{1}{k}}(u_n)\,\mathrm{d}x$ vanishes, as $k\to\infty$, due to (3.2). Also, one has $kT_{\frac{1}{k}}(u_n)\to sign(u_n)$ as $k\to\infty$. So, by using Fatou's lemma, we get, as $k\to\infty$,

$$\int_\Omega |b(u_n)|\,\mathrm{d}x \le \|f_n\|_{L^1(\Omega)}. \tag{4.12}$$

Therefore, for $k>0$,

$$\int_{\{|u_n|>k\}} |b(u_n)|\,\mathrm{d}x \le \|f_n\|_{L^1(\Omega)}. \tag{4.13}$$

Since $|b(u_n)|\ge\min(b(k),|b(-k)|)$ on $\{|u_n|>k\}$, then the relation (4.13) gives

$$\min(b(k),|b(-k)|)meas(\{|u_n|>k\}) \le \|f_n\|_{L^1(\Omega)}$$

or again

$$meas(\{|u_n|>k\}) \le \frac{\|f_n\|_{L^1(\Omega)}}{\min(b(k),|b(-k)|)}. \tag{4.14}$$

Being weakly convergent in $L^1(\Omega)$, the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded, so the right-hand side of (4.14) tends to zero as $k\to\infty$, then $meas(\{|u_n|>k\})$ tends to zero as $k\to\infty$ uniformly in $n$ and (4.6) is proved.

For the proof of (4.7), let's take $\phi = T_{k+1}(u_n) - T_k(u_n)$ as test function and $S \in \mathbb{S}$ such that $S = S_{k+2}$ in the renormalized formulation (3.3). The function $T_{k+1}(u_n) - T_k(u_n)$ has a support contained in the set $\{|u_n| \geq k\}$ and is bounded by one. We obtain by (2.6)

$$C\int_{\{k<|u_n|<k+1\}} |\nabla u_n|^{p_n(x)}\,\mathrm{d}x + \int_\Omega S'(u_n)a_n(x,\nabla u_n)\cdot(\nabla u_n)\phi\,\mathrm{d}x \leq \int_{\{|u_n|\geq k\}} |f_n|\,\mathrm{d}x. \quad (4.15)$$

By the property (3.2) and by equi-integrability of $f_n$, and because of (4.6), for $k \to \infty$, one deduces, from (4.15), the estimate (4.7).

$(iii)$ From Lemma $4.3 - (i)$, one gets

$$\begin{aligned}
\|T_k(u_n)\|^{p_-}_{W^{1,p_-}(\Omega)} &= \int_\Omega |T_k(u_n)|^{p_-}\,\mathrm{d}x + \int_\Omega |\nabla T_k(u_n)|^{p_-}\,\mathrm{d}x \\
&\leq \int_\Omega \left(1 + |T_k(u_n)|^{p_n(x)}\right)\mathrm{d}x + \int_\Omega \left(1 + |\nabla T_k(u_n)|^{p_n(x)}\right)\mathrm{d}x \\
&\leq 2meas(\Omega) + \rho_{1,p_n(\cdot)}(T_k(u_n)) \\
&\leq const(k).
\end{aligned}$$

And so, the sequence $T_k(u_n)$ is uniformly bounded in $W^{1,p_-}(\Omega)$. Therefore, up to a subsequence, we can assume that the sequence $T_k(u_n)$ converges to a certain function $\sigma_k$ weakly in $W^{1,p_-}(\Omega)$, and by the compact imbedding theorem of $W^{1,p_-}(\Omega)$ in $L^{p_-}(\Omega)$, one can see that $T_k(u_n)$ converges strongly to $\sigma_k$ in $L^{p_-}(\Omega)$ and so a.e. in $\Omega$. Now, we have to prove that $\sigma_k = T_k(u)$ a.e. in $\Omega$ where $u_n \to u$ a.e. in $\Omega$.

Let $s > 0$ and define the sets

$$E_n := \{|u_n| > k\}, \;\; E_m := \{|u_m| > k\} \;\text{ and }\; E_{n,m} := \{|T_k(u_n) - T_k(u_m)| > s\},$$

with $k > 0$. One has $\{|u_n - u_m| > s\} \subset E_n \cup E_m \cup E_{n,m}$ which gives

$$meas(\{|u_n - u_m| > s\}) \leq meas(E_n) + meas(E_m) + meas(E_{n,m}).$$

Let $\varepsilon > 0$. According to (4.6), we can choose $k = k(\varepsilon)$ to get

$$meas(E_n) \leq \frac{\varepsilon}{3} \;\text{ and }\; meas(E_m) \leq \frac{\varepsilon}{3}.$$

Since $T_k(u_n)$ converges strongly in $L^{p_-}(\Omega)$, then it is a Cauchy sequence in $L^{p_-}(\Omega)$. Hence, there exists $n_0 = n_0(\varepsilon, s) \in \mathbb{N}$ such that for all $n, m \geq n_0$,

$$meas(E_{n,m}) \leq \frac{1}{s^{p_-}}\int_\Omega |T_k(u_n) - T_k(u_m)|^{p_-}\,\mathrm{d}x \leq \frac{\varepsilon}{3}.$$

So, we deduce that

$$meas(\{|u_n - u_m| > s\}) \leq \varepsilon, \text{ for all } n, m \geq n_0.$$

Finally, the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Hence, by extraction of subsequence, there exists a measurable function $u$ such that $u_n \to u$ a.e. in $\Omega$. Since $T_k$ is continuous, we have $T_k(u_n) \to T_k(u)$ a.e. in $\Omega$ and, by the uniqueness of the limit, one has $\sigma_k = T_k(u)$ a.e. in $\Omega$ because $T_k(u_n) \to \sigma_k$ a.e. in $\Omega$.

Also, the weak convergence of $T_k(u_n)$ to $T_k(u)$ in $W^{1,p_-}(\Omega)$ leads to the weak convergence of $\nabla T_k(u_n)$ to $\nabla T_k(u)$ in $L^{p_-}(\Omega)$. Thanks to Theorem 2.6$-(i)$, $\nabla T_k(u_n)$ nonlinear weak-* converges to a Young measure $\left(\nu_x^k\right)_{x\in\Omega}$ and since its weak limit is $\nabla T_k(u)$, then $\nabla T_k(u)$ verifies the equality (4.8) according to (2.10).

$(iv)$ By assumption (4.2), $p_n \to p$ in measure on $\Omega$, and since $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ in $L^{p_-}(\Omega)$, then according to Theorem 2.6$-(ii)$,$(iii)$, for all $k \in \mathbb{N}$, the sequence $\left(p_n, \nabla T_k(u_n)\right)_n$ converges to the Young measure $\delta_{p(x)} \otimes d\nu_x^k$ on $\mathbb{R}\times\mathbb{R}^N$.

Let us now consider the Carathéodory function

$$F_m : (x,(\lambda_0,\lambda)) \in \Omega\times(\mathbb{R}\times\mathbb{R}^N) \longmapsto |h_m(\lambda)|^{\lambda_0},\ m\in\mathbb{N},$$

where $h_m$ is defined by (2.7). The sequence $(F_m(\cdot,(p_n(\cdot),\nabla T_k(u_n))))_{n\in\mathbb{N}}$ is equi-integrable in $\Omega$ since it is uniformly bounded in $L^1(\Omega)$ according to (4.10). Then, we apply the nonlinear weak-* convergence property (2.9) to the function $F_m$ to get

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega F_m(x,(p_n(x),\nabla T_k(u_n)(x)))\,\mathrm{d}x &= \int_\Omega\int_{\mathbb{R}\times\mathbb{R}^N} F_m(x,(\lambda_0,\lambda))\,\mathrm{d}\delta_{p(x)}(\lambda_0)\,\mathrm{d}\nu_x^k(\lambda)\,\mathrm{d}x\\ &= \int_\Omega\int_{\mathbb{R}^N} F_m(x,(p(x),\lambda))\,\mathrm{d}\nu_x^k(\lambda)\,\mathrm{d}x\\ &= \int_{\Omega\times\mathbb{R}^N} |h_m(\lambda)|^{p(x)}\,\mathrm{d}\nu_x^k(\lambda)\,\mathrm{d}x.\end{aligned}$$

Moreover,

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega F_m(x,(p_n(x),\nabla T_k(u_n)(x)))\,\mathrm{d}x &= \lim_{n\to\infty}\int_\Omega |h_m(\nabla T_k(u_n))|^{p_n(x)}\,\mathrm{d}x\\ &\le \lim_{n\to\infty}\int_\Omega |\nabla T_k(u_n)|^{p_n(x)}\,\mathrm{d}x\\ &\le Ck,\end{aligned}$$

according to (4.10). So,

$$\int_{\Omega\times\mathbb{R}^N} |h_m(\lambda)|^{p(x)}\,\mathrm{d}\nu_x^k(\lambda)\,\mathrm{d}x \le Ck.$$

Since the sequence $(|h_m|)_{m\in\mathbb{N}}$ is increasing and $h_m(\lambda) \longrightarrow \lambda$ as $m\to\infty$, then by the monotone convergence theorem, we deduce that

$$\int_{\Omega\times\mathbb{R}^N} |\lambda|^{p(x)}\,\mathrm{d}\nu_x^k(\lambda)\,\mathrm{d}x \le Ck.$$

By the formula (4.8) and Jensen inequality, one has

$$\int_\Omega |\nabla T_k(u)|^{p(x)}\,\mathrm{d}x = \int_\Omega\left|\int_{\mathbb{R}^N}\lambda\,\mathrm{d}\nu_x^k(\lambda)\right|^{p(x)}\mathrm{d}x \le \int_{\Omega\times\mathbb{R}^N}|\lambda|^{p(x)}\,\mathrm{d}\nu_x^k(\lambda)\,\mathrm{d}x < Ck.$$

Hence, we deduce that $\nabla T_k(u)\in L^{p(\cdot)}(\Omega)$ and thus $T_k(u)\in W^{1,p(\cdot)}(\Omega)$.

$(v)$ Up to a subsequence, by $(iii)$, $T_{k+1}(u_n)-T_k(u_n)$ converges to $T_{k+1}(u)-T_k(u)$ a.e. in $\Omega$ and weakly in $W^{1,p_-}(\Omega)$. By arguing as in $(iv)$, we get $\nabla(T_{k+1}(u)-T_k(u))\in L^{p(\cdot)}(\Omega)$ and its modular is upper bounded by

$$\sup_n\int_\Omega |\nabla(T_{k+1}(u)-T_k(u))|^{p_n(x)}\,\mathrm{d}x = \sup_n\int_{\{k<|u_n|<k+1\}}|\nabla u_n|^{p_n(x)}\,\mathrm{d}x \to 0,\ \text{as } k\to\infty,$$

by (4.7). Thus, (4.9) follows. □

**Lemma 4.4** *$(i)$ For all $k > 0$, the sequence $(\mathcal{Y}_n^k)_{n\in\mathbb{N}}$, $\mathcal{Y}_n^k(x) := a_n(x, \nabla T_k(u_n(x)))$, is equi-integrable in $\Omega$ and its weak limit $\mathcal{Y}^k \in L^{p'(\cdot)}(\Omega)$ is such that*

$$\mathcal{Y}^k(x) := \int_{\mathbb{R}^N} a(x,\lambda)\,\mathrm{d}\nu_x^k(\lambda),\ a.e.\ x \in \Omega. \tag{4.16}$$

*$(ii)$ For all $\widehat{k} > k > 0$, one has $\mathcal{Y}^k = \mathcal{Y}^{\widehat{k}}\chi_{\{|u|<k\}}$.*

*Proof.* $(i)$ We first show that the sequence $(\mathcal{Y}_n^k)_{n\in\mathbb{N}}$, $\mathcal{Y}_n^k := a_n(x, \nabla T_k(u_n))$, is equi-integrable in $\Omega$. The assumption (2.5) applied on $a_n(\cdot,\cdot)$ with exponent $p_n(x)$ implies, for all measurable subset $E \subset \Omega$,

$$\begin{aligned}\int_E |\mathcal{Y}_n^k|\,\mathrm{d}x \leq & C\int_E \Big(1+\mathcal{M}_n + |\nabla T_k(u_n)|^{p_n(x)-1}\Big)\,\mathrm{d}x\\ \leq & C\int_E (1+\mathcal{M}_n)\,\mathrm{d}x + 2C\||\nabla T_k(u_n)|^{p_n(x)-1}\|_{L^{p_n'(\cdot)}}\|\chi_E\|_{L^{p_n(\cdot)}}\\ \leq & C\int_E (1+\mathcal{M}_n)\,\mathrm{d}x + C'\max\Big(\big(\rho_{p_n}(\chi_E)\big)^{1/p_+}, \big(\rho_{p_n}(\chi_E)\big)^{1/p_-}\Big)\\ \leq & C\int_E (1+\mathcal{M}_n)\,\mathrm{d}x + C'\max\big(meas(E)^{1/p_+}, meas(E)^{1/p_-}\big)\end{aligned} \tag{4.17}$$

by using Hölder type inequality and Lemma 2.2, where $2C\||\nabla T_k(u_n)|^{p_n(x)-1}\|_{L^{p_n'(\cdot)}}$ is upper bounded by $C'$ by (4.10).

The whole right-hand side of (4.17) tends to zero when $meas(E)$ tends to zero because the sequence $(\mathcal{M}_n)_{n\in\mathbb{N}}$ is equi-integrable in $\Omega$. And so, the sequence $(\mathcal{Y}_n^k)_{n\in\mathbb{N}}$ is equi-integrable in $\Omega$. By Theorem 2.6$-(i)$, there exists a weak limit $\mathcal{Y}^k$ for the sequence $\mathcal{Y}_n^k$ in $L^1(\Omega)$.

In the following lines, we prove that the weak limit $\mathcal{Y}^k$ verifies the formula (4.16) and belongs to $L^{p'(\cdot)}(\Omega)$. We put the set

$$R_n := \{x \in \Omega; |p(x) - p_n(x)| < 1/2\}$$

and we consider auxiliary functions $\tilde{\mathcal{Y}}_n^k := a\big(x, (\nabla T_k(u_n))\chi_{R_n}\big)$. Let's show that the sequence $\Big(\tilde{\mathcal{Y}}_n^k\Big)_{n\in\mathbb{N}}$ is equi-integrable in $\Omega$. Indeed, we apply (2.5) with the exponent $p(\cdot)$ on $a(\cdot,\cdot)$ to get

$$\int_E \tilde{\mathcal{Y}}_n^k\,\mathrm{d}x \leq C\int_E (1+\mathcal{M})\,\mathrm{d}x + C\int_{E\cap R_n} |\nabla T_k(u_n)|^{p(x)-1}\,\mathrm{d}x. \tag{4.18}$$

The first term of the right-hand side of (4.18) tends to zero when $meas(E)$ tends to zero. Also, for $x \in R_n$, one has $p(x) \leq p_n(x) + 1/2$ and, by using Hölder type inequality, we have

$$\begin{aligned}\int_{E\cap R_n} |\nabla T_k(u_n)|^{p(x)-1}\,\mathrm{d}x \leq & \int_E \big(1+|\nabla T_k(u_n)|^{p_n(x)-1/2}\big)\,\mathrm{d}x\\ \leq & meas(E) + C\||\nabla T_k(u_n)|^{p_n(x)-1/2}\|_{L^{(2p_n(\cdot))'}}\|\chi_E\|_{L^{2p_n(\cdot)}}.\end{aligned} \tag{4.19}$$

But, by (4.10), one has

$$\rho_{(2p_n)'}\Big(|\nabla T_k(u_n)|^{p_n(x)-1/2}\Big) = \rho_{p_n}\left(\nabla T_k(u_n)\right) \leq Ck. \tag{4.20}$$

Also, by Proposition 2.3, one has

$$\begin{aligned}\|\chi_E\|_{L^{2p_n(\cdot)}} &\leq \max\left( (\rho_{2p_n}(\chi_E))^{1/(2p)_+}, (\rho_{2p_n}(\chi_E))^{1/(2p)_-} \right) \\ &\leq \max\left( (meas(E))^{1/(2p)_+}, (meas(E))^{1/(2p)_-} \right). \end{aligned} \tag{4.21}$$

From $(4.19) - (4.21)$, the second term of the right-hand side of (4.18) is uniformly small for $meas(E)$ small, and the equi-integrability of $\left(\tilde{\mathcal{Y}}_n^k\right)_{n\in\mathbb{N}}$ follows. Now, we assert that, by extraction of a subsequence, the sequence $\tilde{\mathcal{Y}}_n^k$ converges weakly to some function $\tilde{\mathcal{Y}}^k$ in $L^1(\Omega)$ as $n\to\infty$ thanks to Theorem 2.6$-(i)$.

It remains to prove that $\mathcal{Y}^k = \tilde{\mathcal{Y}}^k$. For that, it is sufficient to prove that $\mathcal{Y}_n^k - \tilde{\mathcal{Y}}_n^k$ converges strongly to zero in $L^1(\Omega)$. Indeed, let $\varepsilon > 0$. By the Chebyshev inequality, one has

$$\begin{aligned} meas(\{|\nabla T_k(u_n)| > L\}) &\leq \left(\int_\Omega |\nabla T_k(u_n)|\, dx\right) \Big/ L \\ &\leq \int_\Omega \left(1 + |\nabla T_k(u_n)|^{p_n(x)}\right) \mathrm{d}x \Big/ L \\ &\leq (meas(\Omega) + Ck)/L, \end{aligned}$$

by inequality (4.10). It follows that $\sup_n (meas(\{|\nabla T_k(u_n)| > L\}) \to 0$ as $L\to\infty$. The sequence $\mathcal{Y}_n^k - \tilde{\mathcal{Y}}_n^k$ is equi-integrable in $\Omega$, so there exists $L_0 = L_0(\varepsilon)$ such that for $L > L_0$, one has

$$\int_{\{|\nabla T_k(u_n)|>L\}} |\mathcal{Y}_n^k - \tilde{\mathcal{Y}}_n^k|\, \mathrm{d}x \leq \varepsilon/4, \quad \text{for all } n\in\mathbb{N}. \tag{4.22}$$

By the assumption (4.1), one has for all $\sigma > 0$,

$$\lim_{n\to\infty} meas\left(\left\{x\in\Omega;\ \sup_{|\lambda|\leq L} |a_n(x,\lambda) - a(x,\lambda)| \geq \sigma\right\}\right) = 0.$$

Hence, by equi-integrability of $\mathcal{Y}_n^k - \tilde{\mathcal{Y}}_n^k$ on $\Omega$, there exists $n_0 = n_0(\sigma, L_0)\in\mathbb{N}$ such that for all $n > n_0$,

$$\int_{\left\{x\in\Omega;\ \sup_{|\lambda|\leq L} |a_n(x,\lambda) - a(x,\lambda)| \geq \sigma\right\}} |\mathcal{Y}_n^k - \tilde{\mathcal{Y}}_n^k|\, \mathrm{d}x \leq \varepsilon/4. \tag{4.23}$$

By the definition, one has $\tilde{\mathcal{Y}}_n^k = a(x, \nabla T_k(u_n))$ on the set $R_n$ and we consider the following set

$$R_n^{L,\sigma} := \left\{x\in R_n;\ \sup_{|\lambda|\leq L} |a_n(x,\lambda) - a(x,\lambda)| < \sigma, |\nabla T_k(u_n)| \leq L\right\}.$$

Since $|\nabla T_k(u_n)| \leq L$ on $R_n^{L,\sigma}$, then one has

$$|a_n(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u_n))| < \sigma \ \text{ on } \ R_n^{L,\sigma},$$

and so, for all $n$,

$$\int_{R_n^{L,\sigma}} |\mathcal{Y}_n^k - \tilde{\mathcal{Y}}_n^k|\, \mathrm{d}x \leq \sigma\, meas(\Omega) \leq \varepsilon/4, \tag{4.24}$$

by taking $\sigma = \sigma(\varepsilon) < \varepsilon/(4meas(\Omega))$. Also, by (4.22) and (4.23), we have

$$\int_{R_n \setminus R_n^{L,\sigma}} |\mathcal{Y}_n^k - \tilde{\mathcal{Y}}_n^k| \, \mathrm{d}x \leq \varepsilon/2, \text{ for all } n > n_0(\sigma(\varepsilon), L(\varepsilon)). \tag{4.25}$$

Since $p_n$ converges to $p$ in measure on $\Omega$, one has $meas(\Omega \setminus R_n) = meas\big(\{|p - p_n| \geq 1/2\}\big)$ converges to zero as $n \to \infty$; and the equi-integrability of $\mathcal{Y}_n^k$ gives, for sufficiently large $n$,

$$\int_{\Omega \setminus R_n} |\mathcal{Y}_n^k - \tilde{\mathcal{Y}}_n^k| \, \mathrm{d}x = \int_{\Omega \setminus R_n} |\mathcal{Y}_n^k| \, \mathrm{d}x \leq \varepsilon/4. \tag{4.26}$$

Now, by using (4.24), (4.25) and (4.26), we get, for $n > n_0(\sigma(\varepsilon), L(\varepsilon))$,

$$\int_\Omega |\mathcal{Y}_n^k - \tilde{\mathcal{Y}}_n^k| \, \mathrm{d}x \leq \varepsilon.$$

Hence, the sequence $\mathcal{Y}_n^k - \tilde{\mathcal{Y}}_n^k$ converges strongly to zero in $L^1(\Omega)$, as $n$ goes to infinity, and so, $\mathcal{Y}^k = \tilde{\mathcal{Y}}^k$.

Let us show the representation formula (4.16) for $\mathcal{Y}^k$. Since $meas(\Omega \setminus R_n) \to 0$ as $n \to \infty$, so, by the equi-integrability of $\nabla T_k(u_n)$ in $\Omega$, one can see that $\nabla T_k(u_n)(1 - \chi_{R_n})$ converges to zero as $n \to \infty$. Therefore, the sequence $\nabla T_k(u_n)\chi_{R_n}$ converges to the same Young measure $\nu_x^k$ as the sequence $\nabla T_k(u_n)$. Now, fix $\psi \in \mathcal{D}(\Omega)$ and let's consider the Carathéodory function $a(\cdot, \cdot) \cdot \psi$. Since the sequence $\tilde{\mathcal{Y}}_n^k = a(x, \nabla T_k(u_n)\chi_{R_n})$ is equi-integrable in $\Omega$, then we can use the nonlinear weak-* convergence property (2.9) to get

$$\lim_{n\to\infty} \int_\Omega a(x, \nabla T_k(u_n)\chi_{R_n}) \cdot \psi \, \mathrm{d}x = \int_{\Omega \times \mathbb{R}^N} a(x, \lambda) \cdot \psi \, \mathrm{d}\nu_x^k(\lambda) \, \mathrm{d}x. \tag{4.27}$$

Since $a(x, \nabla T_k(u_n)\chi_{R_n})$ converges weakly to $\tilde{\mathcal{Y}}^k$, (4.27) becomes

$$\int_\Omega \tilde{\mathcal{Y}}^k \cdot \psi \, \mathrm{d}x = \int_{\Omega \times \mathbb{R}^N} a(x, \lambda) \cdot \psi \, \mathrm{d}\nu_x^k(\lambda) \, \mathrm{d}x = \int_\Omega \left( \int_{\mathbb{R}^N} a(x, \lambda) \, \mathrm{d}\nu_x^k(\lambda) \right) \cdot \psi \, \mathrm{d}x$$

which means that

$$\mathcal{Y}^k = \tilde{\mathcal{Y}}^k = \int_{\mathbb{R}^N} a(x, \lambda) \, \mathrm{d}\nu_x^k(\lambda) \text{ in } \mathcal{D}'(\Omega) \text{ and so, a.e. in } \Omega.$$

Now, we end the proof with $\mathcal{Y}^k \in L^{p'(\cdot)}(\Omega)$. One uses Jensen inequality, the assumption (2.5) and Lemma 4.3$-(v)$ to obtain

$$\begin{aligned}
\int_\Omega |\mathcal{Y}^k(x)|^{p'(x)} \, \mathrm{d}x &= \int_\Omega \left| \int_{\mathbb{R}^N} a(x, \lambda) \, \mathrm{d}\nu_x^k(\lambda) \right|^{p'(x)} \mathrm{d}x \\
&\leq \int_{\Omega \times \mathbb{R}^N} |a(x, \lambda)|^{p'(x)} \, \mathrm{d}\nu_x^k(\lambda) \, \mathrm{d}x \\
&\leq \int_{\Omega \times \mathbb{R}^N} C(\mathcal{M}(x) + |\lambda|^{p(x)}) \, \mathrm{d}\nu_x^k(\lambda) \, \mathrm{d}x < \infty.
\end{aligned}$$

$(ii)$ Let $\widehat{k} > k > 0$ and let's put $g_n^k := a_n(x, \nabla T_{\widehat{k}}(u_n))\chi_{\{|u|<k\}}$. According to $(i)$, $\left(g_n^k\right)_{n\in\mathbb{N}}$ converges weakly to $\mathcal{Y}^{\widehat{k}}\chi_{\{|u|<k\}}$ in $L^1(\Omega)$. If we prove that this sequence converges weakly to $\mathcal{Y}^k$ in $L^1(\Omega)$, we can conclude that $\mathcal{Y}^k = \mathcal{Y}^{\widehat{k}}\chi_{\{|u|<k\}}$, by uniqueness of the limit. Let's put

$$h_n^k := a_n(x, \nabla T_{\widehat{k}}(u_n))\chi_{\{|u_n|<k\}}.$$

Since $\widehat{k} > k$, one has

$$T_k(u_n) \equiv T_k(T_{\widehat{k}}(u_n)),$$

and so

$$\nabla T_k(u_n) = \nabla T_{\widehat{k}}(u_n)\chi_{\left\{|T_{\widehat{k}}(u_n)|<k\right\}} = \nabla T_{\widehat{k}}(u_n)\chi_{\{|u_n|<k\}}.$$

Moreover, from assumption (2.3), one has $a_n(x, 0) = 0$ a.e. $x \in \Omega$. Hence,

$$a_n(x, \nabla T_{\widehat{k}}(u_n))\chi_{\{|u_n|<k\}} \equiv a_n(x, \nabla T_k(u_n))$$

and the sequence $\left(h_n^k\right)_{n\in\mathbb{N}}$ converges weakly to $\mathcal{Y}^k$ in $L^1(\Omega)$, according to $(i)$.

Consider the sequence $\left(d_n^k\right)_{n\in\mathbb{N}}$ such that

$$d_n^k := g_n^k - h_n^k = a_n(x, \nabla T_{\widehat{k}}(u_n))\left(\chi_{\{|u|<k\}} - \chi_{\{|u_n|<k\}}\right).$$

The function $\chi_{(-k,k)}(\cdot)$ is continuous on the image of $\Omega$ by $u(\cdot)$ for a.e. $k > 0$. Indeed, one has $meas\left(\{|u| = k\}\right) = 0$ for a.e. $k > 0$ by arguing as in the proof of Lemma 3.9. Therefore, since $u_n$ converges to $u$ a.e. in $\Omega$, then

$$\chi_{\{|u_n|<k\}} = \chi_{(-k,k)}(u_n) \to \chi_{(-k,k)}(u) = \chi_{\{|u|<k\}} \text{ a.e. in } \Omega \text{ as } n \to \infty.$$

So,

$$d_n^k \to 0 \text{ a.e. in } \Omega.$$

Moreover, by $(i)$, the sequence $\left(d_n^k\right)_{n\in\mathbb{N}}$ is equi-integrable in $\Omega$. Hence, by Vitali's theorem, the sequence $\left(d_n^k\right)_{n\in\mathbb{N}}$ converges strongly to zero in $L^1(\Omega)$. Therefore, $g_n^k = h_n^k + d_n^k$ tends to $\mathcal{Y}^k$ weakly in $L^1(\Omega)$. So, this ends the proof of $(ii)$. □

**Lemma 4.5** $(i)$ *For all* $k > 0$*,*

$$\int_\Omega \mathcal{Y}^k \cdot \nabla T_k(u)\,\mathrm{d}x \geq \liminf_{n\to\infty} \int_\Omega \mathcal{Y}_n^k \cdot \nabla T_k(u_n)\,\mathrm{d}x \tag{4.28}$$

*and the "div-curl" inequality*

$$\int_{\Omega\times\mathbb{R}^N} \left(a(x,\lambda) - a(x, \nabla T_k(u))\right) \cdot \left(\lambda - \nabla T_k(u)\right) \mathrm{d}\nu_x^k(\lambda)\,\mathrm{d}x \leq 0 \tag{4.29}$$

*holds.*

$(ii)$ *For all* $k > 0$*,*

$$\mathcal{Y}^k(x) = a(x, \nabla T_k(u(x))) \textit{ for a.e. } x \in \Omega, \tag{4.30}$$

*and* $\nabla T_k(u_n)$ *converges to* $\nabla T_k(u)$ *in measure in* $\Omega$ *as* $n \to \infty$*.*

*Proof.* $(i)$ Let $\psi \in C^{\infty}(\overline{\Omega})$. Since $p_n(\cdot)$ is log-Hölder continuous, then $C^{\infty}(\overline{\Omega})$ is dense in $W^{1,p_n(\cdot)}(\Omega)$. So, we can take $\psi$ as test function in the renormalized formulation (3.3) for $u_n$. We get

$$\begin{aligned}&\left|\int_\Omega \Big(b(u_n)S(u_n)\psi + S(u_n)\mathcal{Y}_n^M\cdot\nabla\psi - f_nS(u_n)\psi\Big)\,\mathrm{d}x\right|\\ &\leq \|\psi\|_{L^\infty}\int_\Omega |S'(u_n)|\mathcal{Y}_n^M\cdot\nabla T_M(u_n)\,\mathrm{d}x, \end{aligned}\tag{4.31}$$

where $S \in \mathbb{S}$ with $suppS \subset [-M, M]$, $M > 0$. We are going to pass to the limit in (4.31), as $n$ tends to infinity. By Lemma 4.3$-(iii)$, $u_n$ converges to $u$ a.e. in $\Omega$. By the continuity of $b$ and $S$, the term $b(u_n)S(u_n)$ converges a.e. in $\Omega$ to $b(u)S(u)$. Also, $|b(u_n)S(u_n)\psi| \leq \|S\|_{L^\infty}\max(b(M), |b(-M)|)|\psi| \in L^1(\Omega)$ and so, by the Lebesgue dominated convergence theorem,

$$\int_\Omega b(u_n)S(u_n)\psi\,\mathrm{d}x \longrightarrow \int_\Omega b(u)S(u)\psi\,\mathrm{d}x,\ \text{as } n\to\infty. \tag{4.32}$$

Let's prove now that

$$\int_\Omega f_nS(u_n)\psi\,\mathrm{d}x \longrightarrow \int_\Omega fS(u)\psi\,\mathrm{d}x,\ \text{as } n\to\infty. \tag{4.33}$$

One has

$$\int_\Omega f_nS(u_n)\psi\,\mathrm{d}x = \int_\Omega f_nS(u)\psi\,\mathrm{d}x + \int_\Omega f_n(S(u_n)-S(u))\psi\,\mathrm{d}x. \tag{4.34}$$

On the one hand, one has $\int_\Omega f_nS(u)\psi\,\mathrm{d}x \longrightarrow \int_\Omega fS(u)\psi\,\mathrm{d}x$ since $f_n \rightharpoonup f$ in $L^1(\Omega)$. On the other hand, one has, for $R > 0$,

$$\begin{aligned}&\int_\Omega |f_n(S(u_n)-S(u))\psi|\,\mathrm{d}x\\ &= \int_{\{|f_n|>R\}} |f_n(S(u_n)-S(u))\psi|\,\mathrm{d}x + \int_{\{|f_n|\leq R\}} |f_n(S(u_n)-S(u))\psi|\,\mathrm{d}x\\ &\leq 2\|\psi\|_{L^\infty}\|S\|_{L^\infty}\int_{\{|f_n|>R\}}|f_n|\,\mathrm{d}x + R\|\psi\|_{L^\infty}\int_\Omega |S(u_n)-S(u)|\,\mathrm{d}x. \end{aligned}\tag{4.35}$$

For $R > 0$ fixed, the second term of the right-hand side of the inequality (4.35) tends to zero as $n\to\infty$. Indeed, because of the continuity of $S$ and the compactness of $suppS$, $S(u_n)$ converges strongly to $S(u)$ in $L^1(\Omega)$ by the Lebesgue dominated convergence theorem. By the Chebyshev inequality and since $f_n$ is bounded in $L^1(\Omega)$, one has

$$\sup_n meas(\{|f_n| > R\}) \leq \frac{\sup_n \|f_n\|_1}{R} \leq \frac{C}{R} \longrightarrow 0\ \text{ as } R\longrightarrow\infty.$$

Since the sequence $f_n$ is equi-integrable in $\Omega$, then the first term in the right-hand side of (4.35) can be made as small as desired by the choice of $R$. Hence, the second term of the right-hand side of (4.34) tends to zero. And so, we deduce the convergence result (4.33).

Next, we prove that

$$\int_\Omega S(u_n)\mathcal{Y}_n^M\cdot\nabla\psi\,\mathrm{d}x \to \int_\Omega S(u)\mathcal{Y}^M\cdot\nabla\psi\,\mathrm{d}x,\ \text{as } n\to\infty. \tag{4.36}$$

Indeed, for $R > 0$,

$$\int_\Omega S(u_n)\mathcal{Y}_n^M \cdot \nabla\psi \,\mathrm{d}x = \int_{\{|\nabla\psi|<R\}} S(u_n)\mathcal{Y}_n^M \cdot \nabla\psi \,\mathrm{d}x + \int_{\{|\nabla\psi|\geq R\}} S(u_n)\mathcal{Y}_n^M \cdot \nabla\psi \,\mathrm{d}x. \quad (4.37)$$

For the first term of the right-hand side of (4.37), one has

$$\int_{\{|\nabla\psi|<R\}} S(u_n)\mathcal{Y}_n^M \cdot \nabla\psi \,\mathrm{d}x = \int_{\{|\nabla\psi|<R\}} S(u)\mathcal{Y}_n^M \cdot \nabla\psi \,\mathrm{d}x + \int_{\{|\nabla\psi|<R\}} (S(u_n) - S(u))\mathcal{Y}_n^M \cdot \nabla\psi \,\mathrm{d}x. \quad (4.38)$$

Since $\mathcal{Y}_n^M \rightharpoonup \mathcal{Y}^M$ in $L^{p'(\cdot)}(\Omega)$ by Lemma 4.4$-(i)$, then the first term of the right-hand side of (4.38) tends to $\int_{\{|\nabla\psi|<R\}} S(u)\mathcal{Y}^M \cdot \nabla\psi \,\mathrm{d}x$ as $n \to \infty$.

For $\alpha > 0$ fixed, we can rewrite the second term of the right-hand side of (4.38) as follows.

$$\begin{aligned}
&\int_{\{|\nabla\psi|<R\}} |(S(u_n) - S(u))\mathcal{Y}_n^M \cdot \nabla\psi| \,\mathrm{d}x \\
&= \int_{\{|\nabla\psi|<R\}\cap\{|\mathcal{Y}_n^M|\leq\alpha\}} |(S(u_n) - S(u))\mathcal{Y}_n^M \cdot \nabla\psi| \,\mathrm{d}x \\
&\quad + \int_{\{|\nabla\psi|<R\}\cap\{|\mathcal{Y}_n^M|>\alpha\}} |(S(u_n) - S(u))\mathcal{Y}_n^M \cdot \nabla\psi| \,\mathrm{d}x \\
&\leq \alpha R \int_\Omega |S(u_n) - S(u)| \,\mathrm{d}x + 2R\|S\|_{L^\infty} \int_{\{|\mathcal{Y}_n^M|>\alpha\}} |\mathcal{Y}_n^M| \,\mathrm{d}x. \quad (4.39)
\end{aligned}$$

The sequence $\mathcal{Y}_n^M$ is equi-integrable in $\Omega$ and is bounded in $L^1(\Omega)$ as it converges weakly in $L^1(\Omega)$, so using the same argument which leads to assert that the right-hand side of (4.35) tends to zero, as $n \to \infty$, in the inequality (4.39), then the second term of the right-hand side of (4.38) tends to zero as $n \to \infty$. Thus, the first term of the right-hand side of (4.37) converges to $\int_{\{|\nabla\psi|<R\}} S(u)\mathcal{Y}^M \cdot \nabla\psi \,\mathrm{d}x$ as $n \to \infty$.

For the second term of the right-hand side of (4.37), we note that, by Hölder type inequality,

$$\left| \int_{\{|\nabla\psi|\geq R\}} \mathcal{Y}_n^M \cdot (\nabla\psi S(u_n)) \,\mathrm{d}x \right| \leq C\|S\|_{L^\infty} \|\mathcal{Y}_n^M\|_{L^{p_n'(\cdot)}(\Omega)} \|\chi_{\{|\nabla\psi|\geq R\}} \nabla\psi\|_{L^{p_n(\cdot)}(\Omega)}. \quad (4.40)$$

One has $\|\mathcal{Y}_n^M\|_{L^{p_n'(\cdot)}(\Omega)} \leq C$ by Lemma 4.3 and the growth assumption (2.5). Since $\psi \in C^\infty(\overline{\Omega})$, one clearly has $meas(\{|\nabla\psi| \geq R\}) \to 0$ as $R \to 0$ because $|\nabla\psi|$ is bounded. Therefore,

$$\int_{\{|\nabla\psi|\geq R\}} |\nabla\psi|^{p_n(\cdot)} \,\mathrm{d}x \leq C\, meas(\{|\nabla\psi| \geq R\}),$$

where $C$ is independent of $R$. So, by Lemma 2.2$-(iii), (iv)$, $\sup_n \|\chi_{\{|\nabla\psi|\geq R\}} \nabla\psi\|_{L^{p_n(\cdot)}(\Omega)}$ tends to zero as $R \to \infty$. Therefore, the second term of right-hand side of (4.37) tends to zero as $R \to \infty$. Hence, as $n \to \infty$ and $R \to \infty$ in the equality (4.37), we deduce (4.36).

Thanks to convergences (4.32), (4.33) and (4.36), we deduce for $n$ large enough,

$$\begin{aligned}
&\left| \int_\Omega \Big( b(u)S(u)\psi + S(u)\mathcal{Y}^M \cdot \nabla\psi - fS(u)\psi \Big) \,\mathrm{d}x \right| \\
&\quad \leq \|\psi\|_{L^\infty} \sup_n \int_\Omega |S'(u_n)| a_n(x, \nabla T_M(u_n)) \cdot \nabla T_M(u_n) \,\mathrm{d}x. \quad (4.41)
\end{aligned}$$

Now, fix $k > 0$. By Lemma 4.3$-(iv)$, one has $T_k(u) \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. So, by the density of $C^\infty(\overline{\Omega})$ in $W^{1,p(\cdot)}(\Omega)$, we can replace $\psi$ by $T_k(u)$ in (4.41).

Consider the sequence $(S_M)_M \subset \mathbb{S}$ such that :

- $S_M$ and $S'_M$ are uniformly bounded;
- $S_M = 1$ on $[-M+1, M-1]$, $suppS_M \subset [-M, M]$, for all $M \in \mathbb{N}^*$;
- the map $M \longmapsto b(z)S_M(z)$ is non-decreasing, for all $z \in \mathbb{R}$.

From now on, we replace $S$ by $S_M$ in (4.41). According to Lemma 4.4$-(ii)$, for $M > k$, one has $\mathcal{Y}^k = \mathcal{Y}^M \chi_{\{|u|<k\}}$. Since $\nabla T_k(u) = 0$ outside $\{|u| < k\}$, then we can replace $\mathcal{Y}^M \cdot \nabla T_k(u)$ by $\mathcal{Y}^k \cdot \nabla T_k(u)$. Also, one has $suppS'_M \subset [-M, -M+1] \cup [M-1, M]$ and the sequence $S'_M$ is uniformly bounded *i.e.*$\|S'_M\|_{L^\infty(\Omega)} \leq C$, where $C$ is a positive constant independent of $M$. So, the term of the right-hand side of (4.41) is bounded by

$$\begin{aligned}
&C \sup_n \int_{\{M-1 \leq |u_n| \leq M\}} a_n(x, \nabla T_M(u_n)) \cdot \nabla T_M(u_n)\,\mathrm{d}x \\
&\quad \leq C \sup_n \int_{\{M-1 \leq |u_n| \leq M\}} \Big(\mathcal{M}_n |\nabla T_M(u_n)| + |\nabla T_M(u_n)|^{p_n(x)}\Big)\,\mathrm{d}x \\
&\quad \leq C \sup_n \|\mathcal{M}_n \chi_{\{|u_n| \geq M-1\}}\|_{L^{p'_n(\cdot)}(\Omega)} \|\nabla T_M(u_n) \chi_{\{M-1 \leq |u_n| \leq M\}}\|_{L^{p_n(\cdot)}(\Omega)} \\
&\qquad + C \sup_n \int_{\{M-1 \leq |u_n| \leq M\}} |\nabla T_M(u_n)|^{p_n(x)}\,\mathrm{d}x,
\end{aligned} \tag{4.42}$$

by using the growth condition on $a_n(\cdot,\cdot)$ and the Hölder type inequality. Thanks to Lemma 2.2, the estimates (4.6) and (4.7), and the fact that $\mathcal{M}_n$ is equi-integrable, one can see that the term of the right-hand side of (4.41) tends to zero when $M \to \infty$ in (4.42). By the monotone convergence theorem, since $b(u)S_M(u)$ is non-decreasing and converges a.e. in $\Omega$ to $b(u)$, then $b(u)S_M(u)\psi$ converges strongly to $b(u)\psi$ in $L^1(\Omega)$. Moreover, by the Lebesgue dominated convergence theorem, the terms $S_M(u)\mathcal{Y}^k \cdot \nabla\psi$ and $fS_M(u)\psi$ converge, respectively, strongly to $\mathcal{Y}^k \cdot \nabla\psi$ and to $f\psi$ in $L^1(\Omega)$. Hence, the inequality (4.41) becomes, with $\psi$ replaced by $T_k(u)$,

$$\int_\Omega \Big(b(u)T_k(u) + \mathcal{Y}^k \cdot \nabla T_k(u) - fT_k(u)\Big)\,\mathrm{d}x = 0. \tag{4.43}$$

Now, we consider the renormalized formulation (3.3) for $u_n$ where we take $T_k(u_n)$ as test function and $S \in \mathbb{S}$ with $S = S_h$,

$$\begin{aligned}
&\int_\Omega \Big(S_h(u_n)\mathcal{Y}_n^k \cdot \nabla T_k(u_n) + S'_h(u_n)a_n(x, \nabla u_n) \cdot (\nabla u_n)T_k(u_n) + b(u_n)S_h(u_n)T_k(u_n)\Big)\,\mathrm{d}x \\
&\quad = \int_\Omega f_n S_h(u_n)T_k(u_n)\,\mathrm{d}x.
\end{aligned} \tag{4.44}$$

We are going to pass to the limit in (4.44), as $h \to \infty$. We use the property (3.2) to pass to the limit, as $h \to \infty$, in the term containing the factor $S'_h(u_n)$ and, since $S_h$ is monotone in $h$, we use monotone convergence theorem to pass to the limit in the terms containing the factor $S_h(u_n)$. We get then

$$\int_\Omega \Big(\mathcal{Y}_n^k \cdot \nabla T_k(u_n) + b(u_n)T_k(u_n)\Big)\,\mathrm{d}x = \int_\Omega f_n T_k(u_n)\,\mathrm{d}x, \tag{4.45}$$

as $h \to \infty$ in (4.44). Since $u_n$ converges to $u$ a.e. in $\Omega$, and also because $f_n \rightharpoonup f$ in $L^1(\Omega)$ and $\|T_k\| < \infty$, arguing as in (4.34) and (4.35), we have

$$\int_\Omega f_n T_k(u_n)\,\mathrm{d}x = \int_\Omega f_n T_k(u)\,\mathrm{d}x + \int_\Omega f_n \left(T_k(u_n) - T_k(u)\right)\,\mathrm{d}x \to \int_\Omega f T_k(u)\,\mathrm{d}x, \text{ as } n \to \infty.$$

In the sequel, since $b(u_n)T_k(u_n) \geq 0$, by Fatou's lemma, one deduces

$$\int_\Omega \big(b(u)T_k(u) - fT_k(u)\big)\,\mathrm{d}x \leq \liminf_{n\to\infty} \left(\int_\Omega \big(b(u_n)T_k(u_n) - f_n T_k(u_n)\big)\,\mathrm{d}x\right).$$

And so, from the inequality above and by using (4.45) and (4.43), we get (4.28).

Now, let's go to the proof of the "div-curl" inequality (4.29). Thanks to Lemma 2.1, we know that the sequence

$$\Big(a_n(x, h_m(\nabla T_k(u_n))) \cdot h_m(\nabla T_k(u_n))\Big)_{m>0}$$

is upper bounded by $\mathcal{Y}_n^k \cdot \nabla T_k(u_n)$ because it converges while growing to $\mathcal{Y}_n^k \cdot \nabla T_k(u_n)$, as $m \to \infty$. So, one has, by (4.28),

$$\int_\Omega \mathcal{Y}^k \cdot \nabla T_k(u)\,\mathrm{d}x \geq \liminf_{n\to\infty} \int_\Omega a_n(x, h_m(\nabla T_k(u_n))) \cdot h_m(\nabla T_k(u_n))\,\mathrm{d}x, \text{ for all } m > 0.$$

Since $\int_\Omega \lambda\,\mathrm{d}\nu_x^k(\lambda)$ and $\int_\Omega a(x,\lambda)\,\mathrm{d}\nu_x^k(\lambda)$ are, respectively, the weak limits of $\nabla T_k(u_n)$ and $a_n(x, \nabla T_k(u_n))$, then using the nonlinear weak-* convergence property (2.9), we get

$$\lim_{n\to\infty} \int_\Omega a_n(x, h_m(\nabla T_k(u_n))) \cdot h_m(\nabla T_k(u_n))\,\mathrm{d}x = \int_{\Omega\times\mathbb{R}^N} a(x, h_m(\lambda)) \cdot h_m(\lambda)\,\mathrm{d}\nu_x^k(\lambda)\,\mathrm{d}x,$$

and so

$$\int_\Omega \mathcal{Y}^k \cdot \nabla T_k(u)\,\mathrm{d}x \geq \int_{\Omega\times\mathbb{R}^N} a(x, h_m(\lambda)) \cdot h_m(\lambda)\,\mathrm{d}\nu_x^k(\lambda)\,\mathrm{d}x.$$

Now, thanks to Lemma 2.1, we can apply the monotone convergence theorem on the sequence $(a(x, h_m(\lambda)) \cdot h_m(\lambda))_m$ to deduce that, as $m \to \infty$,

$$\int_\Omega \mathcal{Y}^k \cdot \nabla T_k(u)\,\mathrm{d}x \geq \int_{\Omega\times\mathbb{R}^N} a(x,\lambda) \cdot \lambda\,\mathrm{d}\nu_x^k(\lambda)\,\mathrm{d}x. \tag{4.46}$$

Now using the representation formulas (4.8) and (4.16), and the fact that $\nu_x^k(\lambda)$ is a probability measure on $\mathbb{R}^N$ for a.e. $x \in \Omega$, we find

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^N} \big(a(x,\lambda) - a(x,\nabla T_k(u))\big) \cdot \big(\lambda - \nabla T_k(u)\big)\,\mathrm{d}\nu_x^k(\lambda)\,\mathrm{d}x \\
&= \int_{\Omega\times\mathbb{R}^N} a(x,\lambda) \cdot \lambda\,\mathrm{d}\nu_x^k(\lambda)\,\mathrm{d}x - \int_\Omega \left(\int_{\mathbb{R}^N} a(x,\lambda)\,\mathrm{d}\nu_x^k(\lambda)\right) \nabla T_k(u)\,\mathrm{d}x \\
&\quad - \int_\Omega a(x, \nabla T_k(u)) \left(\int_{\mathbb{R}^N} \lambda\,\mathrm{d}\nu_x^k(\lambda)\right)\,\mathrm{d}x \\
&\quad + \int_\Omega (a(x,\nabla T_k(u)) \cdot \nabla T_k(u)) \left(\int_{\mathbb{R}^N} \mathrm{d}\nu_x^k(\lambda)\right)\,\mathrm{d}x \\
&= \int_{\Omega\times\mathbb{R}^N} a(x,\lambda) \cdot \lambda\,\mathrm{d}\nu_x^k(\lambda)\,\mathrm{d}x - \int_\Omega \left(\int_{\mathbb{R}^N} a(x,\lambda)\,\mathrm{d}\nu_x^k(\lambda)\right)\left(\int_{\mathbb{R}^N} \lambda\,\mathrm{d}\nu_x^k(\lambda)\right)\,\mathrm{d}x \\
&= \int_{\Omega\times\mathbb{R}^N} a(x,\lambda) \cdot \lambda\,\mathrm{d}\nu_x^k(\lambda)\,\mathrm{d}x - \int_\Omega \mathcal{Y}^k \cdot \nabla T_k(u)\,\mathrm{d}x.
\end{aligned} \tag{4.47}$$

From (4.47) and (4.46), we deduce (4.29).

$(ii)$ We prove (4.30) *i.e.* $\mathcal{Y}^k = a(x, \nabla T_k(u))$ a.e. in $\Omega$.
Thanks to the "div-curl" inequality (4.29) and the strict monotonicity assumption (2.4) on $a(x, \cdot)$, one has

$$\Big(a(x,\lambda) - a(x, \nabla T_k(u))\Big) \cdot \Big(\lambda - \nabla T_k(u)\Big)\, \mathrm{d}\nu_x^k(\lambda) = 0 \ \text{ for a.e. } x \in \Omega,$$

and, subsequently for a.e. $x \in \Omega$, $\lambda = \nabla T_k(u)$ wrt the measure $\nu_x^k$ on $\mathbb{R}^N$. Since, by the representation formula (4.8), $\nabla T_k(u) = \displaystyle\int_\Omega \lambda\, \mathrm{d}\nu_x^k(\lambda)$, then the measure $\nu_x^k$ reduces to the Dirac measure $\delta_{\nabla T_k(u)}$. Now, from the representation formula (4.16) we can deduce (4.30). Indeed, one has

$$\mathcal{Y}^k(x) = \int_{\mathbb{R}^N} a(x,\lambda)\, \mathrm{d}\nu_x^k(\lambda) = \int_{\mathbb{R}^N} a(x,\lambda)\, \mathrm{d}\delta_{\nabla T_k(u(x))}(\lambda) = a(x, \nabla T_k(u(x))).$$

Moreover, the sequence $\nabla T_k(u_n)$ generates the Young measure $\nu_x^k = \delta_{\nabla T_k(u)}$ a.e. in $\Omega$. So, from Theorem 2.6$-(ii)$, $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ in measure on $\Omega$ as $n \to \infty$. □

**Lemma 4.6** *For a.e.* $k > 0$, $a_n(x, \nabla T_k(u_n)) \cdot \nabla T_k(u_n)$ *converges to* $a(x, \nabla T_k(u)) \cdot \nabla T_k(u)$ *strongly in* $L^1(\Omega)$.

*Proof.* By Lemma $4.5 - (ii)$ and (4.1), up to a subsequence, we have $a_n(x, \nabla T_k(u_n)) \cdot \nabla T_k(u_n)$ converges to $a(x, \nabla T_k(u)) \cdot \nabla T_k(u)$ a.e. in $\Omega$. Since $a_n(x, \nabla T_k(u_n)) \cdot \nabla T_k(u_n) \geq 0$, by Fatou's lemma, one has

$$\int_\Omega a(x, \nabla T_k(u)) \cdot \nabla T_k(u)\, \mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega a_n(x, \nabla T_k(u_n)) \cdot \nabla T_k(u_n)\, \mathrm{d}x$$

and so, by (4.28), we have

$$\liminf_{n\to\infty} \int_\Omega a_n(x, \nabla T_k(u_n)) \cdot \nabla T_k(u_n)\, \mathrm{d}x = \int_\Omega a(x, \nabla T_k(u)) \cdot \nabla T_k(u)\, \mathrm{d}x.$$

Thus, by the Scheffé's theorem (see [19]), up to subsequence, one has $a_n(x, \nabla T_k(u_n)) \cdot \nabla T_k(u_n)$ converges to $a(x, \nabla T_k(u)) \cdot \nabla T_k(u)$ strongly in $L^1(\Omega)$. □

**Lemma 4.7** $u$ *is a renormalized solution of* (1.1).

*Proof.* By Lemma $4.3 - (iv)$, one has $T_k(u) \in W^{1,p(\cdot)}(\Omega)$. Now, we prove that $b(u) \in L^1(\Omega)$. Indeed, from (4.43), one has

$$\int_\Omega b(u) \frac{1}{k} T_k(u)\, \mathrm{d}x \leq \|f\|_{L^1(\Omega)}$$

which becomes by Fatou's lemma, for $k \to 0$,

$$\int_\Omega |b(u)|\, \mathrm{d}x \leq \|f\|_{L^1(\Omega)}.$$

Next, we prove (3.2) with the diffusion flux $a(\cdot, \cdot)$. By (2.5) and Hölder type inequality, we get

$$\begin{aligned}\int_{\{k<|u|<k+1\}} a(x, \nabla u) \cdot \nabla u\, \mathrm{d}x \leq & C \int_{\{k<|u|<k+1\}} \Big(\mathcal{M}|\nabla u| + |\nabla u|^{p(x)}\Big)\, \mathrm{d}x \\ \leq & C \|\mathcal{M}\chi_{\{|u|>k\}}\|_{L^{p'(\cdot)}(\Omega)} \|(\nabla u)\chi_{\{k<|u|<k+1\}}\|_{L^{p(\cdot)}(\Omega)} \\ & + C \int_{\{k<|u|<k+1\}} |\nabla u|^{p(x)}\, \mathrm{d}x. \end{aligned} \tag{4.48}$$

Thus, (3.2) follows from (4.9).

It remains to prove (3.3) for $u$. Because $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$ and in $W^{1,p_n(\cdot)}(\Omega)$, $p$ and $p_n$ verify (4.4), we can take test functions in $C^\infty(\overline{\Omega})$. So, let $\psi \in C^\infty(\overline{\Omega})$ be a test function for the renormalized formulation (3.3) for $u_n$. One has

$$\begin{aligned}&\int_\Omega \Big(S(u_n)a_n(x,\nabla u_n)\cdot\nabla\psi + S'(u_n)a_n(x,\nabla u_n)\cdot\nabla u_n\psi + b(u_n)S(u_n)\psi\Big)\,\mathrm{d}x\\ &= \int_\Omega f_n S(u_n)\psi\,\mathrm{d}x,\end{aligned} \tag{4.49}$$

where $S \in \mathbb{S}$ with $suppS \subset [-M, M]$. As $n \to \infty$ in (4.49), reasoning as above to pass from (4.31) to (4.41), we get the different limits given in (4.32), (4.33), and (4.36). So, we should direct especially our attention to the term

$$\int_\Omega S'(u_n)a_n(x,\nabla u_n)\cdot(\nabla u_n)\psi\,\mathrm{d}x = \int_\Omega S'(u_n)\mathcal{Y}_n^M\cdot(\nabla T_M(u_n))\psi\,\mathrm{d}x.$$

The sequence $S'(u_n)$ is uniformly bounded and converges to $S'(u)$ a.e. in $\Omega$. Thanks to Lemma 4.6$-(i)$ and by using Lebesgue generalized convergence theorem, this term converges to

$$\int_\Omega S'(u)\mathcal{Y}^M\cdot\nabla T_M(u)\psi\,\mathrm{d}x = \int_\Omega S'(u)a(x,\nabla u)\cdot\nabla u\,\psi\,\mathrm{d}x.$$

We deduce the renormalized formulation (3.3) for $u$ with all test function in $C^\infty(\overline{\Omega})$, which ends the proof of Theorem 4.2. □

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