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EXISTENCE OF SOLUTIONS FOR A CLASS OF SEMILINEAR EVOLUTION EQUATIONS WITH IMPULSES AND DELAYS

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Abstract. We prove the existence and uniqueness of solutions for the following class of semilinear evolution equations with impulses and delays:

$$\begin{cases} z' = -Az + F(t, z_t), & z \in Z,\ t \in (0, \tau],\ t \neq t_k, \\ z(s) = \phi(s), & s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(z(t_k)), & k = 1, 2, 3, \dots, p, \end{cases}$$

where $0 < t_1 < t_2 < t_3 < \cdots < t_p < \tau$, Z is a Banach space, z_t is defined as a function from $[-r, 0]$ to Z by $z_t(s) = z(t + s)$, $-r \leq s \leq 0$, and $J_k \colon Z^\alpha \to Z^\alpha$, $F \colon [0, \tau] \times C(-r, 0; Z^\alpha) \to Z$. In the above problem, $A \colon D(A) \subset Z \to Z$ is a sectorial operator in Z with $-A$ being the generator of a strongly continuous compact semigroup $\{T(t)\}_{t \geq 0}$, and $Z^\alpha = D(A^\alpha)$. The novelty of this work is that our class of evolution equations contains nonlinear terms that involve spatial derivatives. Our framework includes several important partial differential equations such as the Burgers equation with impulses and delays.

Keywords: Delays, fractional power spaces, impulses, Karakostas fixed point theorem, sectorial operators, semigroups, semilinear evolution equations.

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1 Introduction

In the context of semilinear evolution equations in function spaces, difficulties arise when the nonlinear term consists of a composition operator, usually called Nemytskii's operator, which almost never maps a function space into itself unless the generator is affine. For example, the composition of two functions of $L^2(\Omega)$ does not necessarily belong to $L^2(\Omega)$.

This work was motivated primarily by the Burgers equation, which involves a nonlinear term with spatial derivatives. This greatly complicates the problem when one tries to study the approximate controllability of this equation on a fixed interval $[0,\tau]$, because for each control we need to have a corresponding solution defined on the same fixed time interval. To address this problem we must use the fact that the Laplacian operator generates an analytic semigroup which is compact, and use the fractional power spaces to formulate the problem as an abstract evolution equation in a suitable Hilbert space. The fundamental problem is that the composition operator associated to the nonlinear term is well-defined only from adequate fractional power spaces to the $L^2(\Omega)$ space. We spent a lot of time looking for good results that can be applied to the Burgers equations with impulses and delays, but we did not find any. In fact, the presented examples do not involve nonlinear terms with spatial derivatives. Therefore, the novelty of this work lies in the fact that we allow nonlinear terms involving spatial derivative and we use fractional power spaces and the Karakostas fixed point theorem [7]. Moreover, our technique can be applied to other equations like the Navier–Stokes equation.

In this regards we study the existence and uniqueness of solutions for the following semilinear evolution equation with impulses and delays

$$\begin{cases} z' = -Az + F(t, z_t), & z \in Z,\ t \in (0,\tau],\ t \neq t_k, \\ z(s) = \phi(s), & s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(z(t_k)), & k = 1, 2, 3, \dots, p, \end{cases} \tag{1.1}$$

where $0 < t_1 < t_2 < t_3 < \cdots < t_p < \tau$, and Z is a Banach space. With $r > 0$ given, let z_t denote the function from $[-r, 0]$ to Z defined by $z_t(s) = z(t+s)$, $-r \leq s \leq 0$. Moreover, $Z^\alpha := D(A^\alpha)$, $J_k \colon Z^\alpha \to Z^\alpha$ and $F \colon [0,\tau] \times C(-r, 0; Z^\alpha) \to Z$ are smooth functions, and $A \colon D(A) \subset Z \to Z$ is a sectorial operator in Z, and $-A$ generates a strongly continuous compact semigroup $\{T(t)\}_{t \geq 0} \subset Z$.

There are many practical examples of impulsive systems with delays, *e.g.*, chemical reactor systems, financial systems with two state variables (namely, the amount of money in a market and the savings rate of a central bank), and the growth of population diffusing in its habitat modelled by a reaction-diffusion equation. One may easily visualize situations in these examples where abrupt changes such as disasters, meltdowns and instantaneous shocks may occur. These problems are modelled by impulsive differential equations, cf. *e.g.* Lakshmikantham [8] and Samoilenko and Perestyuk [11].

The existence and asymptotic behaviour of solutions of functional differential equations without impulses have been studied by S. M. Rankin III in [10] using fractional power spaces. The existence of solutions for impulsive abstract partial differential equations with state dependent delay has been studied by E. Hernandez, M. Pierri and G. Goncalves [6] without using fractional power spaces, since the nonlinear term does not involve spatial derivative. Likewise, the existence of solutions for semilinear differential evolution equations with impulses and delay has been studied by N. Abada, M. Benchohra and H. Hammouche in [1] and by N. Abada and M. Benchohra in [1] without using fractional power spaces. The existence and stability properties of partial functional differential equations have been studied by C. C. Travis and G. F. Webb in [14]. On the other hand, the existence and the asymptotic behaviour of a functional differential equations without impulses have been studied by S. M. Rankin III in [10] using fractional power spaces. The approximate controllability of semilinear partial neutral functional differential systems has been studied by Xianlong Fu and Kaidong Mei in [3] using also fractional power spaces. In the latter work, since the nonlinear terms involve spatial derivative, spaces of fractional exponents are used.

Our results will be applied to the following impulsive semilinear Burgers equation with impulses and delays

$$\begin{cases} \dfrac{\partial z(t,x)}{\partial t} = \nu z_{xx}(t,x) - z(t-r,x)z_x(t-r,x) + f(t,z(t-r)), \\ z(t,0) = z(t,1) = 0, \quad t \in [0,\tau], \\ z(s,x) = \phi(s,x), \quad s \in [-r,0],\ x \in [0,1], \\ z(t_k^+,x) = z(t_k^-,x) + J_k(z(t_k,x)), \quad x \in \Omega,\ k = 1,2,3,\ldots,p, \end{cases} \tag{1.2}$$

where $\phi \in \mathcal{C}([-r,0];H_0^1) = \mathcal{C}([-r,0];Z^{1/2})$ with $Z = L_2[0,1]$, $Z^{1/2} = D((-\Delta)^{1/2})$, and the functions f, J_k are globally Lipschitz.

The following Burgers equation with delay

$$\begin{cases} \dfrac{\partial z(t,x)}{\partial t} = \nu z_{xx}(t,x) - z(t,x)z_x(t-r,x), \\ z(t,0) = z(t,1) = 0, \quad t \in [0,\tau], \\ z(s,x) = \phi(s,x), \quad s \in [-r,0],\ x \in [0,1], \end{cases} \tag{1.3}$$

has been studied by Weijiu Liu in [9], Yanbin Tang and Ming Wang in [12] and Yanbin Tang in [13], where the existence and uniqueness of global solutions has been proved.

2 Preliminaries

Throughout this paper, the operator $A\colon D(A) \subset Z \to Z$ is sectorial and $-A$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{T(t)\}_{t\geq 0} \subset Z$, with $0 \in \rho(A)$. Therefore, fractional power operators A^α, $0 < \alpha \leq 1$, are well-defined. Since A^α is a closed operator, its domain $D(A^\alpha)$ is a Banach space endowed with the graph norm

$$\|z\|_\alpha = \|A^\alpha z\|, \ \ z \in D(A^\alpha).$$

This Banach space is denoted by $Z^\alpha = D(A^\alpha)$ and is dense in Z. Moreover, for $0 < \beta < \alpha \leq 1$ the embedding $Z^\alpha \hookrightarrow Z^\beta$ is compact whenever the resolvent operator of A is compact.

For the semigroup the following properties will be used: there are constants $\eta > 0$, $M \geq 1$, $M_\alpha \geq 0$ and $C_{1-\alpha}$ such that

$$\|T(t)\| \leq M, \quad t \geq 0, \tag{2.1}$$

$$\|A^\alpha T(t)\| \leq \frac{M_\alpha}{t^\alpha} e^{-\eta t}, \quad t > 0, \tag{2.2}$$

$$A^{\alpha}T(t)z = T(t)A^{\alpha}z, \quad \forall z \in Z^{\alpha}, \tag{2.3}$$

$$\|(T(t)-I)z\| \le \frac{C_{1-\alpha}}{\alpha}t^{\alpha}\|A^{\alpha}z\|, \quad t>0,\ \forall z \in Z^{\alpha}. \tag{2.4}$$

For more properties of sectorial operators and strongly continuous semigroups we refer the reader to the book by D. Henry [5] and the book by Jerome A. Goldstein [4].

The functions $J_k\colon Z^{\alpha} \to Z^{\alpha}$ are continuous and the function $F\colon [0,\infty) \times \mathcal{D}_{\alpha} \to Z$ is smooth, where the set D_{α} denotes the space

$$\mathcal{D}_{\alpha} = \{\phi : [-r,0] \to Z^{\alpha} : \phi \text{ is continuous}\}$$

endowed with the norm

$$\|\phi\|_d = \sup_{-r\le s\le 0} \|\phi(s)\|_{\alpha}.$$

A natural space to work with evolution equations with delay and impulses is the following Banach space: with the notation $J := [-r,\tau]$ and $J' = [-r,\tau]\backslash\{t_1,t_2,\dots,t_p\}$, define

$$\begin{aligned} PC_{\alpha} &= PC(J;Z^{\alpha}) \\ &:= \{z\colon J \to Z^{\alpha} : z \in C(J';Z^{\alpha}) : \forall k=1,2,\dots,p,\ z(t_k^+), z(t_k^-) \text{ exist, and } z(t_k)=z(t_k^-)\} \end{aligned}$$

endowed with the norm

$$\|z\| = \sup_{t\in[-r,\tau]} \|z(t)\|_{\alpha}.$$

For a function $y \in PC([-r,\tau];Z^{\alpha})$ and $i = 1,2,\dots,p$, we define the function $\tilde{y}_i \in C([t_i,t_{i+1}];Z^{\alpha})$ by the formula

$$\tilde{y}_i(t) = \begin{cases} y(t) & \text{for } t \in (t_i,t_{i+1}], \\ y(t_i^+) & \text{for } t = t_i. \end{cases} \tag{2.5}$$

For $W \subset PC([-r,\tau];Z^{\alpha})$ and $i = 1,2,\dots,p$, we define $\tilde{W}_i = \{\tilde{y}_i : y \in W\}$. Following the classical Arzelà–Ascoli theorem one gets a characterization of compactness in $PC([-r,\tau];Z^{\alpha})$.

Lemma 1 *A set $W \subset PC([-r,\tau];Z^{\alpha})$ is relatively compact in $PC([-r,\tau];Z^{\alpha})$ if and only if each set $\tilde{W}_i$, $i=1,2,\dots,p$, with $t_0=0$ and $t_{p+1}=\tau$, is relatively compact in $C([t_i,t_{i+1}];Z^{\alpha})$.*

Theorem 1 (G. L. Karakostas [7]) *Let Z and Y be Banach spaces and let D be a closed convex subset of Z. Moreover, let $\mathcal{B}\colon D \to Y$ be a continuous operator such that $\mathcal{B}(D)$ is a precompact subset of Y, and let*

$$\mathcal{T}\colon D \times \overline{\mathcal{B}(D)} \to D \tag{2.6}$$

be a continuous operator such that the family $\{\mathcal{T}(\cdot,y) : y \in \overline{\mathcal{B}(D)}\}$ is equicontractive. Then the operator equation

$$\mathcal{T}(z,\mathcal{B}(z)) = z \tag{2.7}$$

admits a solution in D.

Lemma 2 (Generalized Gronwall–Bellman inequality [8, 11]) *Let a non-negative function $z \in PC([-r,\infty);\mathbb{R})$ satisfy for $t \geq t_0$ the inequality*

$$z(t) \leq C + \int_{t_0}^{t} v(s)z(s)\,\mathrm{d}s + \sum_{t_0<t_k<t} \beta_k u(t_k),$$

where $C \geq 0$, $\beta_k \geq 0$, $v(s) > 0$, and t_k's are the discontinuity points of first type for the function z. Then we have

$$z(t) \leq C \prod_{t_0<t_k<t} (1+\beta_k) e^{\int_{t_0}^{t} v(s)\,\mathrm{d}s}.$$

3 Existences of mild solutions

In this section we shall prove the main result of this work, which concerns the existence of mild solutions of problem (1.1).

Definition 1 *A function $z \in PC_\alpha$ is said to be a mild solution of problem* (1.1) *if it satisfies the integral equation*

$$\begin{aligned} z(t) &= T(t)\phi(0) + \int_0^t T(t-s)F(s,z_s)\,\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)J_k(z(t_k)), \quad t \in [0,\tau], \\ z(t) &= \phi(t), \quad t \in [-r,0]. \end{aligned} \tag{3.1}$$

Let us consider the following hypotheses.

(H1) There exist constants $d_k > 0$, $k = 1,2,\dots,p$, such that

$$M\sum_{k=1}^{p} d_k < \frac{1}{2}, \quad \|J_k(y) - J_k(z)\|_\alpha \leq d_k\|y-z\|_\alpha, \quad y,z \in Z^\alpha,$$

where M is as in (2.1).

(H2) The function $F\colon [0,\tau] \times \mathcal{D}_\alpha \to Z$ satisfies the following conditions:

$$\begin{aligned} \|F(t,\phi_1) - F(t,\phi_2)\| &\leq \mathcal{K}(\|\phi_1\|_d, \|\phi_2\|_d)\|\phi_1 - \phi_2\|_d, \quad \phi_1,\phi_2 \in \mathcal{D}_\alpha, \\ \|F(t,\phi)\| &\leq \Psi(\|\phi\|), \quad \phi \in \mathcal{D}_\alpha, \end{aligned}$$

where $\mathcal{K}\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\Psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and non-decreasing functions of their arguments.

(H3) Assume that the following relation holds for ρ,τ:

$$\frac{\tau^{1-\alpha}}{1-\alpha} M_\alpha \Psi(\|\tilde{\phi}\| + \rho) + M\sum_{k=1}^{p} d_k(\|\tilde{\phi}| + \rho) \leq \rho,$$

where the function $\tilde{\phi}$ is defined as follows:

$$\tilde{\phi}(t) = \begin{cases} T(t)\phi(0), & t \in [0,\tau], \\ \phi(t), & t \in [-r,0]. \end{cases} \tag{3.2}$$

(H4) Assume that the following relation holds for ρ, τ:

$$\frac{\tau^{1-\alpha}}{1-\alpha} M_\alpha \mathcal{K}(\|\tilde{\phi}\|_d + \rho, \|\tilde{\phi}\|_d + \rho) + M \sum_{k=1}^{p} d_k < 1.$$

Theorem 2 *Suppose that* (H1)–(H3) *hold. Then, problem* (1.1) *has at least one mild solution on* $[-r, \tau]$.

Proof. We shall transform problem (1.1) into a fixed point problem. Define the following two operators

$$\mathcal{T}\colon PC([-r,\tau]; Z^\alpha) \times PC([-r,\tau]; Z^\alpha) \to PC([-r,\tau]; Z^\alpha)$$

and

$$\mathcal{B}\colon PC([-r,\tau]; Z^\alpha) \to PC([-r,\tau]; Z^\alpha)$$

by

$$\mathcal{T}(z,y)(t) = \begin{cases} y(t) + \sum_{0<t_k<t} T(t-t_k) J_k(z(t_k)), & t \in [0,\tau], \\ \phi(t), & t \in [-r, 0], \end{cases}$$

$$\mathcal{B}(y)(t) = \begin{cases} T(t)\phi(0) + \int_0^t T(t-s) F(s, y_s)\, \mathrm{d}s, & t \in [0,\tau], \\ \phi(t), & t \in [-r, 0]. \end{cases}$$

The problem of finding the solution of problem (1.1) is reduced to the problem of finding the solution of the operator equation $\mathcal{T}(z, \mathcal{B}(z)) = z$. First, we shall prove that the operator $\mathcal{B}$ is compact. After that, we shall prove that the family $\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{B}(D)}\}$ is equicontractive, where D is the closed convex set given by (3.3). So, by applying Theorem 1 we get the result. The proof of this theorem will be given by claims.

Claim 1: The operator $\mathcal{B}$ is continuous. In fact, consider $z, y \in PC([-r,\tau]; Z^\alpha)$ and the following estimate

$$\begin{aligned} \|\mathcal{B}(z)(t) - \mathcal{B}(y)(t)\|_\alpha &\le \int_0^t \|A^\alpha T(t-s)(F(s,z_s) - F(s,y_s))\|\, \mathrm{d}s \\ &\le \int_0^t \frac{M_\alpha}{(t-s)^\alpha} \|F(s,z_s) - F(s,y_s)\|\, \mathrm{d}s \\ &\le \int_0^t \frac{M_\alpha}{(t-s)^\alpha} \mathcal{K}(\|z_s\|_d, \|y_s\|_d) \|z_s - y_s\|_d\, \mathrm{d}s \\ &\le M_\alpha \frac{\tau^{1-\alpha}}{1-\alpha} \mathcal{K}(\|z\|, \|y\|) \|z - y\|. \end{aligned}$$

Therefore,

$$\|\mathcal{B}(z) - \mathcal{B}(y)\| \le M_\alpha \frac{\tau^{1-\alpha}}{1-\alpha} \mathcal{K}(\|z\|, \|y\|) \|z - y\|.$$

So, $\mathcal{B}$ is continuous. Moreover, $\mathcal{B}$ is locally Lipschitz.

Claim 2: The operator $\mathcal{B}$ maps bounded sets into bounded sets of $PC([-r,\tau];Z^\alpha)$. It is enough to show that for any $q>0$ there exists $l>0$ such that for each $y\in B_q=\{z\in PC_\alpha:\|z\|\le q\}$ we have $\|\mathcal{B}y\|\le l$. In fact, choose $y\in B_q$; then the following estimate holds

$$\begin{aligned}
\|\mathcal{B}(y)(t)\|_\alpha &\le \|A^\alpha T(t)\phi(0)\| + \int_0^t \|A^\alpha F(s,y_s))\|\,\mathrm{d}s\\
&\le M\|A^\alpha\phi(0)\| + \int_0^t \frac{M_\alpha}{(t-s)^\alpha}\|F(s,y_s)\|\,\mathrm{d}s\\
&\le M\|\phi(0)\|_\alpha + M_\alpha\frac{\tau^{1-\alpha}}{1-\alpha}\Psi(\|y\|)\\
&\le M\|\phi(0)\|_\alpha + M_\alpha\frac{\tau^{1-\alpha}}{1-\alpha}\Psi(q)=l.
\end{aligned}$$

Claim 3: The operator $\mathcal{B}$ maps bounded sets into equicontinuous sets of $PC([-r,\tau];Z^\alpha)$. In fact, consider B_q as in the foregoing claim. Then we shall prove that the family of functions $\mathcal{B}(B_q)$ is equicontinuous on the interval $[-r,\tau]$. Clearly, it is sufficient to prove this on $(0,\tau]$. Let $0<\tau_1<\tau_2<\tau$ and consider the following estimate for $y\in B_q$:

$$\begin{aligned}
\|\mathcal{B}(y)(\tau_2)-\mathcal{B}(y)(\tau_1)\|_\alpha &\le \|A^\alpha T(\tau_2)\phi(0)-A^\alpha T(\tau_1)\phi(0)\|\\
&\quad+\int_0^{\tau_1-\epsilon}\|(A^\alpha T(\tau_2-s)-A^\alpha T(\tau_1-s))F(s,y_s)\|\,\mathrm{d}s\\
&\quad+\int_{\tau_1-\epsilon}^{\tau_1}\|(A^\alpha T(\tau_2-s)-A^\alpha T(\tau_1-s))F(s,y_s)\|\,\mathrm{d}s\\
&\quad+\int_{\tau_1}^{\tau_2}\|A^\alpha T(\tau_2-s)F(s,y_s)\|\,\mathrm{d}s\\
&\le \|T(\tau_2)-T(\tau_1)\|\|\phi(0)\|_\alpha\\
&\quad+\|T(\tau_2-\tau_1+\epsilon)-T(\epsilon)\|\int_0^{\tau_1-\epsilon}\|A^\alpha T(\tau_1-s-\epsilon)F(s,y_s)\|\,\mathrm{d}s\\
&\quad+\frac{M_\alpha\Psi(\|y\|)}{1-\alpha}\big\{(\tau_2-\tau_1+\epsilon)^{1-\alpha}-(\tau_2-\tau_1)^{1-\alpha}+(\epsilon)^{1-\alpha}\big\}\\
&\quad+\frac{M_\alpha\Psi(\|y\|)}{1-\alpha}(\tau_2-\tau_1)^{1-\alpha}\\
&\le \|T(\tau_2)-T(\tau_1)\|\|\phi(0)\|_\alpha\\
&\quad+\|T(\tau_2-\tau_1+\epsilon)-T(\epsilon)\|\frac{M_\alpha\Psi(q)}{1-\alpha}(\tau_1-\epsilon)^{1-\alpha}\\
&\quad+\frac{M_\alpha\Psi(q)}{1-\alpha}\big\{(\tau_2-\tau_1+\epsilon)^{1-\alpha}-(\tau_2-\tau_1)^{1-\alpha}+\epsilon^{1-\alpha}\big\}\\
&\quad+\frac{M_\alpha\Psi(q)}{1-\alpha}(\tau_2-\tau_1)^{1-\alpha}.
\end{aligned}$$

Since $T(t)$ is a compact operator for $t>0$, $\{T(t)\}_{t\ge0}$ is a uniformly continuous semigroup. This implies that $\|\mathcal{B}(y)(\tau_2)-\mathcal{B}(y)(\tau_1)\|_\alpha$ goes to zero uniformly with respect to y as $\tau_2-\tau_1\to0$, and therefore $\mathcal{B}(B_q)$ is equicontinuous.

Claim 4: The set $W=\{\mathcal{B}(y):y\in B_q\}$ is relatively compact in $PC([-r,\tau];Z^\alpha)$. To prove this it is enough to prove that the corresponding sets $\tilde{W}_i$ are relatively compact in $C([t_i,t_{i+1}];Z^\alpha)$ for $i=0,1,2,\ldots,p$ with $t_0=0$ and $t_{p+1}=\tau$. According to the Arzelà–Ascoli theorem in infinite dimensional Banach spaces it is sufficient to prove that $\tilde{W}_i(t) = \{\tilde{\mathcal{B}(y)}_i(t) : y \in B_q\}$ is relatively compact in Z^α for each $t \in [t_i, t_{i+1}]$.

In fact, the case $t \in [-r, 0]$ is trivial, since $W(t) = \phi(t)$. Now, suppose that $t \in [t_i, t_{i+1}]$. Then

$$\tilde{W}_i(t) = T(t)\phi(0) + \tilde{V}_i(t),$$

where

$$\tilde{V}_i(t) = \left\{ v_i(t) = \int_0^t T(t-s)F(s, \tilde{y}_{i,s})\,\mathrm{d}s : y \in B_q \right\}.$$

It is sufficient to prove that $\tilde{V}_i(t)$ is relatively compact in Z^α. Observe that for $0 < \alpha < \beta < 1$, we have the following estimate

$$\begin{aligned}
\|A^\beta v_i(t)\| &\le \int_0^t \|A^\beta T(t-s)F(s, \tilde{y}_{is})\|\,\mathrm{d}s \\
&\le \int_0^t \frac{M_\beta}{(t-s)^\beta}\|F(s, \tilde{y}_{is})\|\,\mathrm{d}s \\
&\le \frac{M_\beta \Psi(q)}{1-\beta}\tau^{1-\beta},
\end{aligned}$$

which implies that $\{A^\beta \tilde{V}_i(t)\}$ is bounded in Z. On the other hand, we know that $A^{-\beta} : Z \to Z^\alpha$ is a compact operator, since the imbedding $Z^\beta \hookrightarrow Z^\alpha$ is compact. Therefore, $\{\tilde{V}_i(t)\}$ is compact in Z^α, and consequently $W = \{\mathcal{B}(y) : y \in B_q\}$ is relatively compact in $PC([-r, \tau]; Z^\alpha)$. Hence, the operator $\mathcal{B}$ is compact.

Claim 5: The family $\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{B}(D)}\}$ is equicontractive and the conditions of Theorem 1 are satisfied for the following closed and convex set

$$D = D(\rho, \tau, \phi) = \{y \in PC([-r, \tau]; Z^\alpha) : \|y - \tilde{\phi}\| \le \rho\}, \tag{3.3}$$

where the function $\tilde{\phi}$ is defined as follows:

$$\tilde{\phi}(t) = \begin{cases} T(t)\phi(0), & t \in [0, \tau], \\ \phi(t), & t \in [-r, 0]. \end{cases}$$

In fact, for $z, x \in PC([-r, \tau]; Z^\alpha)$ and $t \in [0, \tau]$ we have the following estimate:

$$\begin{aligned}
\|\mathcal{T}(z, \mathcal{B}(y))(t) - \mathcal{T}(x, \mathcal{B}(y))(t)\|_\alpha &\le \sum_{0<t_k<t} \|A^\alpha T(t-t_k)(J_k(z(t_k)) - J_k(x(t_k)))\| \\
&\le M \sum_{k=1}^p \|A^\alpha (J_k(z(t_k)) - J_k(x(t_k)))\| \\
&\le M \sum_{k=1}^p d_k \|z(t_k) - x(t_k)\|_\alpha \\
&\le M \sum_{k=1}^p d_k \|z - x\|.
\end{aligned}$$

Hence,

$$\|\mathcal{T}(z, \mathcal{B}(y)) - \mathcal{T}(x, \mathcal{B}(y))\| \le \left(M \sum_{k=1}^p d_k \right) \|z - x\|,$$

which shows that $\mathcal{T}(\cdot, y)$ is a contraction independently of $y \in \overline{\mathcal{B}(D)}$, since $M\sum_{k=1}^{p} d_k < 1$.

Finally, we shall prove that

$$\mathcal{T}(\cdot, \mathcal{B})D(\rho, \tau, \phi) \subset D(\rho, \tau, \phi).$$

In fact, let us consider $z \in D(\rho, \tau, \phi)$ and

$$\mathcal{T}(z, \mathcal{B}(z))(t) = \begin{cases} T(t)\phi(0) + \displaystyle\int_0^t T(t-s)F(s, z_s)\,\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)J_k(z(t_k)), & t \in [0, \tau], \\ \phi(t), & t \in [-r, 0]. \end{cases}$$

Therefore,

$$\begin{aligned} \|\mathcal{T}(z, \mathcal{B}(z))(t) - \tilde{\phi}(t)\|_\alpha &\le \int_0^t \|A^\alpha T(t-s)F(s, z_s)\|\,\mathrm{d}s + \sum_{0<t_k<t} \|A^\alpha T(t-t_k)J_k(z(t_k))\| \\ &\le \int_0^t \frac{M_\alpha}{(t-s)^\alpha}\|F(s, z_s)\|\,\mathrm{d}s + M\sum_{k=1}^{p} \|A^\alpha(J_k(z(t_k)) - J_k(0))\| \\ &\le \frac{\tau^{1-\alpha}}{1-\alpha}M_\alpha\Psi(\|z\|) + M\sum_{k=1}^{p} d_k\|z(t_k)\|_\alpha \\ &\le \frac{\tau^{1-\alpha}}{1-\alpha}M_\alpha\Psi(\|\phi\|_d + \rho) + \left(M\sum_{k=1}^{p} d_k\right)(\|\phi\|_d + \rho) \le \rho. \end{aligned}$$

From the hypothesis (H3) we get that $\mathcal{T}(\cdot, \mathcal{B})D(\rho, \tau, \phi) \subset D(\rho, \tau, \phi)$. Hence, as a consequence of Theorem 1 it follows that the equation $\mathcal{T}(z, \mathcal{B}(z)) = z$ has a solution, which is a mild solution of problem (1.1). □

Theorem 3 *In addition to the conditions of Theorem* 2*, suppose that* (H4) *holds. Then problem* (1.1) *has only one mild solution on* $[-r, \tau]$.

Proof. Let z_1 and z_2 be two solutions of problem (1.1). Then consider the following estimate:

$$\begin{aligned} \|z_1(t) - z_2(t)\|_\alpha &\le \int_0^t \|A^\alpha T(t-s)(F(s, z_{1s}) - F(s, z_{2s}))\|\,\mathrm{d}s \\ &\quad + \sum_{0<t_k<t} \|A^\alpha T(t-t_k)(J_k(z_1(t_k)) - J_k(z_2(t_k)))\| \\ &\le \int_0^t \frac{M_\alpha}{(t-s)^\alpha}\|F(s, z_{1s}) - F(s, z_{2s})\|\,\mathrm{d}s \\ &\quad + M\sum_{k=1}^{p} \|A^\alpha(J_k(z_1(t_k)) - J_k(z_2(t_k)))\| \\ &\le \frac{M_\alpha\tau^{1-\alpha}}{1-\alpha}\mathcal{K}(\|z_1\|, \|z_2\|)\|z_1 - z_2\| + M\sum_{k=1}^{p} d_k\|z_1 - z_2\| \end{aligned}$$

$$\leq \left(\frac{M_\alpha \tau^{1-\alpha}}{1-\alpha} \mathcal{K}(\|\phi\|_d + \rho, \|\phi\|_d + \rho) + M \sum_{k=1}^{p} d_k \right) \|z_1 - z_2\|.$$

From the hypotheses (H4) we know that $\frac{M_\alpha \tau^{1-\alpha}}{1-\alpha} \mathcal{K}(\|\phi\|_d + \rho, \|\phi\|_d + \rho) + M \sum_{k=1}^{p} d_k < 1$, which implies that $z_1 = z_2$. □

Now, we shall consider the following subset $\tilde{D}_\alpha$ of Z^α:

$$\tilde{D}_\alpha = \{y \in Z^\alpha : \|y\|_\alpha \leq q\} \quad \text{with} \quad q = \|\tilde{\phi}\| + \rho. \tag{3.4}$$

Therefore, for all $z \in D$ we have $z(t) \in \tilde{D}_\alpha$ for $-r \leq t \leq \tau$.

Theorem 4 *Suppose that the conditions of Theorem* 3 *hold. If z is a solution of problem* (1.1) *on $[-r, \tau_1)$ and τ_1 is maximal, so there is no solution of* (1.1) *on $[-r, \tau_2)$ if $\tau_2 > \tau_1$, then either $\tau_1 = +\infty$ or else there exists a sequence $\tau_n \to \tau_1$ as $n \to \infty$ such that $z(\tau_n) \to \partial \tilde{D}_\alpha$.*

Proof. Suppose that $\tau_1 < \infty$ and $z(t)$ does not enter a neighbourhood N of $\tilde{D}_\alpha$ for $0 < \tau_2 \leq t < \tau_1$. Let us take $N = \tilde{D}_\alpha \backslash B$, where B is a closed subset of $\tilde{D}_\alpha$, and $z(t) \in B$ for $0 < \tau_2 \leq t < \tau_1$. We shall prove the existence of $z_1 \in B$ such that $z(t) \to z_1$ in Z^α as $t \to \tau_1^-$, which together with Theorem 3 would imply that the solution may be extended beyond time τ_1, contradicting the maximality of τ_1.

In fact, if we consider $0 < t_p < \tau_2 \leq \tau < t < \tau_1$, then

$$\begin{aligned}
\|z(t) - z(\tau)\|_\alpha \leq\ & \|A^\alpha T(t)\phi(0) - A^\alpha T(\tau)\phi(0)\| \\
& + \int_0^{\tau-\epsilon} \|(A^\alpha T(t-s) - A^\alpha T(\tau - s))F(s, z_s)\| \,\mathrm{d}s \\
& + \int_{\tau-\epsilon}^{\tau} \|(A^\alpha T(t-s) - A^\alpha T(\tau - s))F(s, z_s)\| \,\mathrm{d}s \\
& + \int_{\tau}^{t} \|A^\alpha T(t-s) F(s, z_s)\| \,\mathrm{d}s \\
& + \|T(t - \tau + \epsilon) - T(\epsilon)\| \sum_{k=1}^{p} \|T(\tau - t_k - \epsilon) A^\alpha J_k(z(t_k))\| \\
\leq\ & \|T(t) - T(\tau)\| \|\phi(0)\|_\alpha \\
& + \|T(t - \tau + \epsilon) - T(\epsilon)\| \int_0^{\tau-\epsilon} \|A^\alpha T(\tau - s - \epsilon) F(s, z_s)\| \,\mathrm{d}s \\
& + \frac{M_\alpha \Psi(q)}{1-\alpha} \big\{ (t - \tau + \epsilon)^{1-\alpha} - (t-\tau)^{1-\alpha} + (\epsilon)^{1-\alpha} \big\} \\
& + \frac{M_\alpha \Psi(q)}{1-\alpha} (t-\tau)^{1-\alpha} + M \|T(t - \tau + \epsilon) - T(\epsilon)\| \sum_{k=1}^{p} \|J_k(z(t_k)\|_\alpha.
\end{aligned}$$

Since $T(t)$ is a compact operator for $t > 0$, $\{T(t)\}_{t \geq 0}$ is a uniformly continuous semigroup. This implies that $\|z(t) - z(\tau)\|_\alpha$ goes to zero. Therefore, $\lim_{t \to \tau_1} z(t) = z_1$ exists in Z^α, and since B is closed, z_1 belongs to B. This completes the proof. □

Corollary 1 *Under the conditions of Theorem* 3*, if the second part of hypothesis* (H2) *is changed to*

$$\|F(t,\phi)\| \le h(t)(1+\|\phi(0)\|_\alpha), \quad \phi \in \mathcal{D}_\alpha,$$

where $h(\cdot)$ is a continuous function on $[-r,\infty)$, then a unique solution of problem (1.1) *exists on* $[-r,\infty)$.

Proof. We have

$$\begin{aligned}
\|z(t)\|_\alpha &\le M\|\phi(0)\|_\alpha + \int_0^t \|A^\alpha T(t-s)F(s,z_s)\|\,\mathrm{d}s + \sum_{0<t_k<t} \|A^\alpha T(t-t_k)J_k(z(t_k))\| \\
&\le M\|\phi(0)\|_\alpha + \int_0^t \frac{M_\alpha}{(t-s)^\alpha} e^{-\eta(t-s)}\|F(s,z_s)\|\,\mathrm{d}s + M\sum_{k=1}^p \|A^\alpha(J_k(z(t_k)) - J_k(0))\| \\
&\le M\|\phi(0)\|_\alpha + \int_0^t \frac{M_\alpha}{(t-s)^\alpha} e^{-\eta(t-s)}(1+\|z(s)\|_\alpha)\,\mathrm{d}s + M\sum_{k=1}^p d_k\|z(t_k)\|_\alpha \\
&\le M\|\phi(0)\|_\alpha + \frac{\Gamma(1-\alpha)}{\eta^{1-\alpha}}M_\alpha + \int_0^t \frac{M_\alpha}{(t-s)^\alpha} e^{-\eta(t-s)}\|z(s)\|_\alpha\,\mathrm{d}s + M\sum_{k=1}^p d_k\|z(t_k)\|_\alpha.
\end{aligned}$$

Then, applying Lemma 2, we get the following estimate

$$\|z(t)\|_\alpha \le \left(M\|\phi(0)\|_\alpha + \frac{\Gamma(1-\alpha)}{\eta^{1-\alpha}}M_\alpha\right) \prod_{t_0<t_k<t}(1+Md_k)e^{\frac{\Gamma(1-\alpha)}{\eta^{1-\alpha}}M_\alpha}.$$

This implies that $\|z(t)\|_\alpha$ remains bounded as $t \to \tau_1$. Applying Theorem 4 we get the result. □

Theorem 5 *Under the conditions of Theorem* 2*, if z is a solution of problem* (1.1) *on* $[-r,\infty)$ *with* $\|z(t)\|_\alpha$ *bounded as $t \to \infty$, then $\{z(t,\phi)\}_{t>0}$ is a compact set in Z^α.*

Proof. Observe that for $0<\alpha<\beta<1$, we have the following estimate for $t>t_p$:

$$\begin{aligned}
\|A^\beta z(t)\| &\le \|A^{\beta-\alpha}T(t)A^\alpha\phi(0)\|_\alpha + \int_0^t \|A^\beta T(t-s)F(s,z_s)\|\,\mathrm{d}s \\
&\quad + \sum_{0<t_k<t} \|A^{\beta-\alpha}T(t-t_k)A^\alpha J_k(z(t_k))\| \\
&\le \frac{M_\beta}{t^{\beta-\alpha}}\|\phi(0)\|_\alpha + \int_0^t \frac{M_\beta}{(t-s)^\beta}\|F(s,z_s)\|\,\mathrm{d}s \\
&\quad + \sum_{k=1}^p \frac{M_\beta}{(t-t_k)^{\beta-\alpha}}\|A^\alpha(J_k(z(t_k)) - J_k(0))\| \\
&\le \frac{M_\beta}{t^{\beta-\alpha}}\|\phi(0)\|_\alpha + \int_0^t \frac{M_\beta}{(t-s)^\beta}\|F(s,z_s)\|\,\mathrm{d}s + \sum_{k=1}^p \frac{M_\beta}{(t-t_k)^{\beta-\alpha}} d_k\|z(t_k)\|_\alpha \\
&\le \frac{M_\beta}{t^{\beta-\alpha}}\|\phi(0)\|_\alpha + \frac{t^{1-\beta}}{1-\beta}M_\beta\Psi(\|z||) + \frac{M_\beta}{(t-t_p)^{\beta-\alpha}}\sum_{k=1}^p d_k\|z(t_k)\|_\alpha,
\end{aligned}$$

which implies that $\{A^\beta z(t) : t \in [-r,\infty)\}$ is bounded in Z. On the other hand, we know that $A^{-\beta}\colon Z \to Z^\alpha$ is a compact operator, since the imbedding $Z^\beta \hookrightarrow Z^\alpha$ is compact. Therefore, $\{z(t) : t \in [-r,\infty)\}$ is compact in Z^α. □

4 Application to the Burgers equation

In this section we shall apply our previous results to the Burgers equation with impulses and delay (1.2). To this end, we make the following hypotheses. The nonlinear functions $f, J_k \colon \mathbb{R} \to \mathbb{R}$ are smooth enough and there exist constants $L > 0$, L_k such that

$$|f(t,z) - f(t,w)| \le L|z - w|, \quad t \in [0,\tau],\ z, w \in \mathbb{R}, \tag{4.1}$$

$$|J_k(z) - J_k(w)| \le L_k|z - w|, \quad t \in [0,\tau],\ z, w \in \mathbb{R},\ k = 1, 2, \dots, p, \tag{4.2}$$

$$|f(t,z,u)| \le a(t)|z| + b(t), \quad t \in [0,\tau] \text{ and } z, u \in \mathbb{R}, a(\cdot), b(\cdot) \in L_\infty[0,\tau]. \tag{4.3}$$

Let $\Omega = [0,1]$. By $\mathcal{C}$ we shall denote the space of continuous functions

$$\mathcal{C} = \{\phi \colon [-r, 0] \to H_0^1(\Omega) = Z^{1/2} : \phi \text{ is continuous}\},$$

endowed with the norm

$$\|\phi\| = \sup_{-r \le s \le 0} \|\phi(s)\|_{Z^{1/2}}, \text{ and } \phi(s)(x) = \phi(s,x),\ x \in \Omega = [0,1].$$

Now, we choose a Hilbert space where system (1.2) can be written as an abstract differential equation (see [2]); to this end, we consider the following notations.

Let us consider the Hilbert space $Z = L_2(\Omega)$ and $0 < \lambda_1 < \lambda_2 < \dots < \lambda_j \longrightarrow \infty$ the eigenvalues of the operator $A\phi = -\nu\phi_{xx}$. Then we have the following well-known properties.

(i) There exists a complete orthonormal set $\{\phi_j\}$ of eigenvectors of A.

(ii) For all $z \in D(A)$ we have

$$Az = \sum_{j=1}^{\infty} \lambda_j \langle \xi, \phi_j \rangle \phi_j = \sum_{j=1}^{\infty} \lambda_j E_j z, \tag{4.4}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in Z and

$$E_j z = \langle z, \phi_j \rangle \phi_j. \tag{4.5}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in Z and $z = \sum_{j=1}^{\infty} E_j z$, $z \in Z$.

(iii) $-A$ generates an analytic semigroup $\{T(t)\}$ given by

$$T(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j z \text{ and } \|T(t)\| \le e^{-\lambda_1 t}, \quad t \ge 0. \tag{4.6}$$

Consequently, systems (1.2) can be written as an abstract functional differential equation with memory in Z:

$$\begin{cases} z' = -Az + f^e(t, z_t(-r)), & z \in Z,\ t \ge 0, \\ z(s) = \phi(s), & s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k^e(z(t_k)), & k = 1, 2, 3, \dots, p, \end{cases} \tag{4.7}$$

where $z_t \in \mathcal{C}([-r,0]; Z^{1/2})$ is defined by $z_t(s) = z(t+s)$, $-r \le s \le 0$, and the functions $J_k^e\colon Z \to Z$, $f^e\colon [0,\tau] \times \mathcal{C} \to Z$ are defined for $k = 1, 2, \dots, p$ by

$$J_k^e(z)(x) = J_k(z(x)), \quad \forall x \in \Omega$$

and

$$f^e(t,\phi)(x) = \phi(-r,x)\phi_x(-r,x) + f(t,\phi(-r,x)), \quad \forall x \in \Omega.$$

Proposition 1 *The function f^e is locally Lipschitz in the second variable. Moreover, the following estimates hold*:

$$\|f^e(t,\phi_1) - f^e(t,\phi_2)\| \le \{\|\phi_1 - \phi_2\|_{\mathcal{C}} + L\}\|\phi_1 - \phi_2\|_{\mathcal{C}}, \tag{4.8}$$

$$\begin{aligned} \|f^e(t,\phi)\| &\le \|\phi(-r)\|^2 + 4\|a\|_{L_\infty}\|\phi(-r)\| + 4\|b\|_{L_\infty}\sqrt{\mu(\Omega)} \\ &\le \|\phi\|_{\mathcal{C}}^2 + 4\|a\|_{L_\infty}\|\phi\|_{\mathcal{C}} + 4\|b\|_{L_\infty}\sqrt{\mu(\Omega)}. \end{aligned} \tag{4.9}$$

Proof. Clearly, the following estimate holds:

$$\|f^e(t,\phi) - f^e(t,\psi)\|_Z \le \|\phi(-r)\phi_x(-r) - \psi(-r)\psi_x(-r)\|_Z + L\|\phi(-r) - \psi(-r)\|_Z. \tag{4.10}$$

On the other hand,

$$\begin{aligned} &\|\phi(-r)\phi_x(-r) - \psi(-r)\psi_x(-r)\|_Z \\ &\quad \le \|\phi(-r)[\phi_x(-r) - \psi_x(-r)]\|_Z + \|[\phi(-r) - \psi(-r)]\psi_x(-r)\|_Z \\ &\quad \le \|\phi(-r)\|_{L_\infty}\|[\phi_x(-r) - \psi_x(-r)]\|_Z + \|[\phi(-r) - \psi(-r)]\|_{L_\infty}\|\psi_x(-r)\|_Z. \end{aligned}$$

Then, for all $z \in Z^{1/2} = H_0^1(\Omega)$, by the Sobolev theorem and the Poincaré inequality we have

$$\|z\|_{L_\infty}^2 \le 2\|z\|_Z\|z_x\|_Z \le \|z\|_Z^2 + \|z_x\|_Z^2 = \|z\|_{Z^{1/2}}^2$$

and

$$\begin{aligned} &\|\phi(-r)\phi_x(-r) - \psi(-r)\psi_x(-r)\|_Z \\ &\quad \le \|\phi(-r)\|_{Z^{1/2}}\|[\phi(-r) - \psi(-r)]\|_{Z^{1/2}} + \|[\phi(-r) - \psi(-r)]\|_{Z^1}\|\psi(-r)\|_{Z^{1/2}}. \end{aligned}$$

Using this estimate and (4.10) we get the result. □

In this case the functions $\mathcal{K}\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\Psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ are given by

$$\mathcal{K}(v,w) = v + w + L \ \text{ and } \ \Psi(v) = v^2 + 4\|a\|_\infty v + 4\|b\|_\infty,$$

since $\sqrt{\mu(\Omega)} = 1$. Therefore, we have the following result for the impulsive Burgers equation with delay.

Theorem 6 *For d_k small enough there exists $\tau > 0$ such that the system* (1.2) *has only one mild solution defined on* $[-r,\tau]$.

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