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NON-LOCAL BOUNDARY CONDITIONS FOR NONLINEAR ELLIPTIC PROBLEMS WITH BOUNDED DATA AND GENERAL FUNCTIONS

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Abstract. In this article, we study the existence and uniqueness of solutions for nonlinear elliptic problems with non-local boundary conditions. In order to get the unique solution, we study first an auxiliary problem, for which we deduce useful a priori estimates. The study of the auxiliary problem gives us the equivalence between this kind of problem and a nonlinear problem with very large diffusion around the boundary.

Keywords: Leray–Lions type operator, non-local boundary conditions, operator of type M, standard monotonicity arguments.

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1 Introduction and assumptions

Let Ω be an open bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ such that $\partial\Omega$ is Lipschitz and $\partial\Omega = \Gamma_D \cup \Gamma_{Ne}$ with $\Gamma_D \cap \Gamma_{Ne} = \emptyset$. Our aim is to study the following problem

$$P(\beta,\rho,f,d)\begin{cases} \beta(u) - \nabla\cdot a(x,\nabla u) = f & \text{in } \Omega,\\ u = 0 & \text{on } \Gamma_D,\\ \rho(u) + \displaystyle\int_{\Gamma_{Ne}} a(x,\nabla u).\eta = d & \text{on } \Gamma_{Ne},\\ u \equiv \text{constant} & \text{on } \Gamma_{Ne}, \end{cases}$$

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where η is the unit outward normal vector on $\partial\Omega$, β and ρ are two continuous and non-decreasing functions on $\mathbb{R}$ such that

$$\mathcal{D}(\beta) = \mathcal{D}(\rho) = \mathbb{R},\ \ \mathrm{Im}(\beta) = \mathrm{Im}(\rho) = \mathbb{R}\ \text{ and }\ \beta(0) = \rho(0) = 0,$$

a is a Leray–Lions type operator, f is a function in $L^\infty(\Omega)$ and $d \in \mathbb{R}$.

Recall that a Leray–Lions type operator is a Carathéodory function $a(x,\xi)\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ (*i.e.*, $a(x,\xi)$ is continuous in ξ for a.e. $x \in \Omega$ and measurable in x for every $\xi \in \mathbb{R}^N$) and there exists $p \in (1,+\infty)$ such that

- there exists a positive constant C with

$$|a(x,\xi)| \le C\big(j(x) + |\xi|^{p-1}\big) \tag{1.1}$$

  for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, where j is a non-negative function in $L^{p'}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$;

- the following inequalities hold

$$\big(a(x,\xi) - a(x,\eta)\big).(\xi - \eta) > 0 \tag{1.2}$$

  for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$, and there exists $C' > 0$ such that

$$\frac{1}{C'}|\xi|^p \le a(x,\xi).\xi \tag{1.3}$$

  for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$.

Boundary value problems involving PDEs arise in physical sciences and applied mathematics. In some of these problems, subsidiary conditions are imposed locally. In some other cases, non-local conditions are imposed. It is sometimes better to impose non-local conditions since the measurements needed by a non-local condition may be more precise than the measurement given by a local condition. Indeed, in the problem $P(\beta,\rho,f,d)$, in contrast to the standard case where the condition on the boundary is given on the local values of the flux, non-local boundary conditions act on the average of the flux on the boundary. More precisely, in addition to the Dirichlet boundary condition on Γ_D, *i.e.*,

$$u = 0 \text{ on } \Gamma_D, \tag{1.4}$$

u is asked to satisfy the following non-local condition

$$\rho(u) + \int_{\Gamma_{Ne}} a(x,\nabla u).\eta = d \text{ on } \Gamma_{Ne}. \tag{1.5}$$

It is well-known that under only conditions (1.4) and (1.5), the problem $P(\beta,\rho,f,d)$ is ill-posed. To make the problem $P(\beta,\rho,f,d)$ well-posed, we ask the unknown function u to be constant on Γ_{Ne}. Beside the mathematical interest of non-local conditions, it seems that this type of boundary condition appears in petroleum engineering model for well modelling in a 3D stratified petroleum reservoir with arbitrary geometry; this kind of boundary condition also arises in petroleum engineering, in the simulation of wells performance, since a nonlinear relation exists between the performance pressure tangential gradient and the fluid velocity along the well (see [1, 2] and [4] for details).

In the sequel, we consider the following spaces:

$$W_D^{1,p}(\Omega) = \{\varphi \in W^{1,p}(\Omega) : \varphi = 0 \text{ on } \Gamma_D\}$$

and

$$W_{Ne}^{1,p}(\Omega) = \{\varphi \in W_D^{1,p}(\Omega) : \varphi \equiv \text{constant on } \Gamma_{Ne}\}.$$

For any $v \in W_{Ne}^{1,p}(\Omega)$, we set $v_{Ne} := v|_{\Gamma_{Ne}}$.

The concept of a solution for $P(\beta, \rho, f, d)$ is given as follow.

Definition 1 *A measurable function $u \colon \Omega \to \mathbb{R}$ is a solution of $P(\beta, \rho, f, d)$ if*

$$\begin{cases} u \in W_{Ne}^{1,p}(\Omega),\ \beta(u) \in L^1(\Omega) \text{ and for every } \varphi \in W_{Ne}^{1,p}(\Omega) \cap L^\infty(\Omega), \\ \displaystyle\int_\Omega a(x, \nabla u).\nabla\varphi \, \mathrm{d}x + \int_\Omega \beta(u)\varphi \, \mathrm{d}x = \int_\Omega f\varphi \, \mathrm{d}x + \big(d - \rho(u)_{Ne}\big)\varphi_{Ne}. \end{cases} \tag{1.6}$$

Our main result in this paper is the following

Theorem 1 *For any $f \in L^\infty(\Omega)$ the problem $P(\beta, \rho, f, d)$ admits a unique solution u.*

Before proving Theorem 1, we study an auxiliary problem, from which we deduce useful a priori estimates.

The paper is organized as follow. In Section 2, we study the auxiliary problem and in Section 3, we prove the existence and uniqueness of solutions to the problem $P(\beta, \rho, f, d)$.

2 The approximated problem corresponding to $P(\beta, \rho, f, d)$

We define a new bounded domain $\widetilde{\Omega}$ in $\mathbb{R}^N$ as follow. We fix $\delta > 0$ and we set $\widetilde{\Omega} = \Omega \cup \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Gamma_{Ne}) < \delta\}$. Then, $\partial\widetilde{\Omega} = \Gamma_D \cup \widetilde{\Gamma}_{Ne}$ is Lipschitz with $\Gamma_D \cap \widetilde{\Gamma}_{Ne} = \emptyset$.

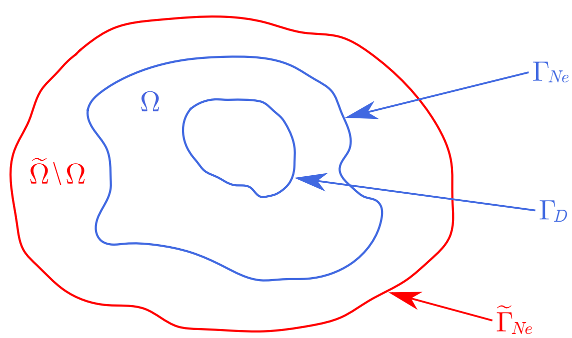

Figure 1: Domains representation

Let us consider a Leray–Lions type operator $\widetilde{a}(x, \xi) \colon \widetilde{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying (1.1), (1.2) and (1.3). We consider the problem

$$P(\widetilde{\beta}, \widetilde{\rho}, \widetilde{f}, \widetilde{d}) \begin{cases} \widetilde{\beta}(x, u) - \nabla \cdot \widetilde{a}(x, \nabla u) = \widetilde{f} & \text{in } \widetilde{\Omega}, \\ u = 0 & \text{on } \Gamma_D, \\ \widetilde{\rho}(u) + \widetilde{a}(x, \nabla u).\eta = \widetilde{d} & \text{on } \widetilde{\Gamma}_{Ne}, \end{cases}$$

where the functions $\widetilde{\beta}$, $\widetilde{\rho}$, $\widetilde{f}$ and $\widetilde{d}$ are defined as follow:

- $\widetilde{\beta}(x,s) = \beta(s)\chi_{\Omega}(x)$ for all $(x,s) \in \widetilde{\Omega} \times \mathbb{R}$;
- $\widetilde{\rho}(s) = \frac{1}{|\widetilde{\Gamma}_{Ne}|}\rho(s)$ for all $s \in \mathbb{R}$, where $|\widetilde{\Gamma}_{Ne}|$ denotes the Hausdorff measure of $\widetilde{\Gamma}_{Ne}$;
- $\widetilde{f}(x) = (f\chi_{\Omega})(x)$ for all $x \in \widetilde{\Omega}$;
- $\widetilde{d}$ is a function in $L^{\infty}(\widetilde{\Gamma}_{Ne})$ such that

$$\int_{\widetilde{\Gamma}_{Ne}} \widetilde{d}\,\mathrm{d}\sigma = d. \tag{2.1}$$

Obviously, we have $\widetilde{f} \in L^{\infty}(\widetilde{\Omega})$.

The following definition gives the notion of a solution for the problem $P(\widetilde{\beta},\widetilde{\rho},\widetilde{f},\widetilde{d})$.

Definition 2 *A measurable function $u\colon \widetilde{\Omega} \to \mathbb{R}$ is a solution for $P(\widetilde{\beta},\widetilde{\rho},\widetilde{f},\widetilde{d})$ if*

$$\begin{cases} u \in W_D^{1,p}(\widetilde{\Omega}),\ \beta(u) \in L^1(\Omega) \textit{ and for every } \widetilde{\varphi} \in W_D^{1,p}(\widetilde{\Omega}) \cap L^{\infty}(\Omega), \\ \displaystyle\int_{\Omega} \widetilde{a}(x,\nabla u).\nabla\widetilde{\varphi}\,\mathrm{d}x + \int_{\Omega} \beta(u)\widetilde{\varphi}\,\mathrm{d}x = \int_{\Omega} f\widetilde{\varphi}\,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} \big(\widetilde{d} - \widetilde{\rho}(u)\big)\widetilde{\varphi}\,\mathrm{d}\sigma. \end{cases} \tag{2.2}$$

We have the following existence result for the problem $P(\widetilde{\beta},\widetilde{\rho},\widetilde{f},\widetilde{d})$.

Theorem 2 *Assume the functions $\widetilde{\beta},\widetilde{\rho},\widetilde{f}$ and $\widetilde{d}$ are as above. Then, the problem $P(\widetilde{\beta},\widetilde{\rho},\widetilde{f},\widetilde{d})$ admits at least one solution in the sense of Definition 2.*

Before proving Theorem 2, we study an existence result to the following problem. For any $k > 0$ we consider

$$P_k(\widetilde{\beta},\widetilde{\rho},\widetilde{f},\widetilde{d}) \begin{cases} T_k\big(\widetilde{\beta}(x,u_k)\big) - \nabla\cdot\widetilde{a}(x,\nabla u_k) = \widetilde{f} & \text{in } \widetilde{\Omega}, \\ u_k = 0 & \text{on } \Gamma_D, \\ T_k\left(\widetilde{\rho}(u_k)\right) + \widetilde{a}(x,\nabla u_k).\eta = \widetilde{d} & \text{on } \widetilde{\Gamma}_{Ne}, \end{cases}$$

where the truncation function $T_k\colon \mathbb{R} \to \mathbb{R}$ is defined as

$$T_k(s) = \begin{cases} -k, & \text{if } s < -k, \\ s, & \text{if } |s| \le k, \\ k, & \text{if } s > k. \end{cases}$$

We next prove the following theorem.

Theorem 3 *Assume the functions $\widetilde{\beta},\widetilde{\rho},\widetilde{f}$ and $\widetilde{d}$ are as above. Then, for any $k > 0$ the problem $P_k(\widetilde{\beta},\widetilde{\rho},\widetilde{f},\widetilde{d})$ admits at least one solution u_k in the sense*

$$\begin{cases} u_k \in W_D^{1,p}(\widetilde{\Omega}) \textit{ and for all } \widetilde{\varphi} \in W_D^{1,p}(\widetilde{\Omega}), \\ \displaystyle\int_{\widetilde{\Omega}} \widetilde{a}(x,\nabla u_k).\nabla\widetilde{\varphi}\,\mathrm{d}x + \int_{\Omega} T_k\left(\beta(u_k)\right)\widetilde{\varphi}\,\mathrm{d}x = \int_{\Omega} f\widetilde{\varphi}\,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} \big(\widetilde{d} - T_k\left(\widetilde{\rho}(u_k)\right)\big)\widetilde{\varphi}\,\mathrm{d}\sigma. \end{cases} \tag{2.3}$$

Furthermore, for any k large enough

$$\begin{aligned} |\beta(u_k)| &\le \theta_1 := \max\big\{\|f\|_\infty, (\beta\circ\rho_0^{-1})\big(|\widetilde{\Gamma}_{Ne}|\|\widetilde{d}\|_\infty\big)\big\} \text{ a.e. in } \Omega, \\ |\widetilde{\rho}(u_k)| &\le \theta_2 := \max\big\{\|\widetilde{d}\|_\infty, (\widetilde{\rho}\circ\beta_0^{-1})\big(\|f\|_\infty\big)\big\} \text{ a.e. in } \widetilde{\Gamma}_{Ne}. \end{aligned} \tag{2.4}$$

Proof. For any $k>0$ let us introduce the operator $\Lambda_k\colon W_D^{1,p}(\widetilde{\Omega}) \to \big(W_D^{1,p}(\widetilde{\Omega})\big)'$ such that for any $(u,v)\in W_D^{1,p}(\widetilde{\Omega})\times W_D^{1,p}(\widetilde{\Omega})$,

$$\langle \Lambda_k(u), v\rangle = \int_{\widetilde{\Omega}} \widetilde{a}(x,\nabla u).\nabla v \,\mathrm{d}x + \int_\Omega T_k\left(\beta(u)\right) v \,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} T_k\left(\widetilde{\rho}(u)\right) v \,\mathrm{d}\sigma. \tag{2.5}$$

We need to prove that for any $k>0$ the operator Λ_k is bounded, coercive, of type M and hence, surjective.

(i) Boundedness of Λ_k. For every $(u,v)\in W_D^{1,p}(\widetilde{\Omega})\times W_D^{1,p}(\widetilde{\Omega})$ we have

$$\begin{aligned} \big|\langle \Lambda_k(u), v\rangle\big| &\le \int_{\widetilde{\Omega}} |\widetilde{a}(x,\nabla u)||\nabla v| \,\mathrm{d}x + k\int_\Omega |v| \,\mathrm{d}x + k\int_{\widetilde{\Gamma}_{Ne}} |v| \,\mathrm{d}\sigma \\ &\le \int_{\widetilde{\Omega}} |\widetilde{a}(x,\nabla u)||\nabla v| \,\mathrm{d}x + kC_1(\mathrm{meas}(\Omega),p)\|v\|_{L^p(\Omega)} + kC_2(|\widetilde{\Gamma}_{Ne}|,p)\|v\|_{L^p(\widetilde{\Gamma}_{Ne})} \\ &\le \int_{\widetilde{\Omega}} C\big(j(x)+|\nabla u|^{p-1}\big)|\nabla v| \,\mathrm{d}x + kC_1(\mathrm{meas}(\Omega),p)\|v\|_{L^p(\Omega)} \\ &\quad + kC_2(|\widetilde{\Gamma}_{Ne}|,p)\|v\|_{L^p(\widetilde{\Gamma}_{Ne})} \quad \text{thanks to assumption (1.1)} \\ &\le \int_{\widetilde{\Omega}} Cj(x)|\nabla v| \,\mathrm{d}x + \int_{\widetilde{\Omega}} C|\nabla u|^{p-1}|\nabla v| \,\mathrm{d}x + kC_1(\mathrm{meas}(\Omega),p)\|v\|_{L^p(\Omega)} \\ &\quad + kC_2(|\widetilde{\Gamma}_{Ne}|,p)\|v\|_{L^p(\widetilde{\Gamma}_{Ne})} \\ &\le C_3(j,p)\|\nabla v\|_{L^p(\widetilde{\Omega})} + C_4\|\nabla u\|^{p-1}_{L^p(\widetilde{\Omega})}\|\nabla v\|_{L^p(\widetilde{\Omega})} + kC_1(\mathrm{meas}(\Omega),p)\|v\|_{L^p(\Omega)} \\ &\quad + kC_2(|\widetilde{\Gamma}_{Ne}|,p)\|v\|_{L^p(\widetilde{\Gamma}_{Ne})}. \end{aligned}$$

Thanks to Theorem 1 in [3], we have $\|v\|_{L^p(\widetilde{\Gamma}_{Ne})} \le C\|v\|_{W_D^{1,p}(\widetilde{\Omega})}$.

Taking into account the fact that $\|\widetilde{\varphi}\|_{L^p(\widetilde{\Omega})} \le \|\widetilde{\varphi}\|_{W_D^{1,p}(\widetilde{\Omega})}$ and $\|\nabla\widetilde{\varphi}\|_{L^p(\widetilde{\Omega})} \le \|\widetilde{\varphi}\|_{W_D^{1,p}(\widetilde{\Omega})}$ for any $\widetilde{\varphi}\in W_D^{1,p}(\widetilde{\Omega})$, we get

$$\begin{aligned} \big|\langle \Lambda_k(u), v\rangle\big| &\le C_3(j,p)\|v\|_{W_D^{1,p}(\widetilde{\Omega})} + C_4\|u\|^{p-1}_{W_D^{1,p}(\widetilde{\Omega})}\|v\|_{W_D^{1,p}(\widetilde{\Omega})} + kC_1(\mathrm{meas}(\Omega),p)\|v\|_{W_D^{1,p}(\widetilde{\Omega})} \\ &\quad + kC_2(|\widetilde{\Gamma}_{Ne}|,p)\|v\|_{W_D^{1,p}(\widetilde{\Omega})} \\ &\le \Big(C_3(j,p) + C_4\|u\|^{p-1}_{W_D^{1,p}(\widetilde{\Omega})} + kC_1(\mathrm{meas}(\Omega),p) + kC_2(|\widetilde{\Gamma}_{Ne}|,p)\Big)\|v\|_{W_D^{1,p}(\widetilde{\Omega})}. \end{aligned}$$

From this inequality one sees that Λ_k maps bounded subsets of $W_D^{1,p}(\widetilde{\Omega})$ into bounded subsets of $\big(W_D^{1,p}(\widetilde{\Omega})\big)'$. Therefore, Λ_k is bounded on $W_D^{1,p}(\widetilde{\Omega})$.

(ii) Coerciveness of Λ_k. We have to show that for any $k>0$ we have $\frac{\langle \Lambda_k(u),u\rangle}{\|u\|_{W_D^{1,p}(\widetilde{\Omega})}} \to +\infty$ as $\|u\|_{W_D^{1,p}(\widetilde{\Omega})} \to +\infty$.

For any $u \in W_D^{1,p}(\widetilde{\Omega})$ we have

$$\langle \Lambda_k(u), u\rangle = \int_{\widetilde{\Omega}} a(x, \nabla u).\nabla u \,\mathrm{d}x + \int_{\Omega} T_k\left(\beta(u)\right) u \,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} T_k\left(\widetilde{\rho}(u)\right) u \,\mathrm{d}\sigma. \tag{2.6}$$

The last two terms on the right-hand side of (2.6) are non-negative. Using the assumption (1.3), we deduce that

$$\int_{\widetilde{\Omega}} a(x, \nabla u).\nabla u \,\mathrm{d}x \geq \frac{1}{C'}\|\nabla u\|^p_{L^p(\widetilde{\Omega})}.$$

Therefore, from (2.6) we get

$$\langle \Lambda_k(u), u\rangle \geq \frac{1}{C'}\|\nabla u\|^p_{L^p(\widetilde{\Omega})}.$$

Since $u \in W_D^{1,p}(\widetilde{\Omega})$, by the Poincaré inequality, we have $\|u\|^p_{L^p(\widetilde{\Omega})} \leq C\|\nabla u\|^p_{L^p(\widetilde{\Omega})}$. Then, $\|u\|_{W_D^{1,p}(\widetilde{\Omega})} \to +\infty$ implies $\|\nabla u\|_{L_D^p(\widetilde{\Omega})} \to +\infty$. Hence, Λ_k is coercive.

(iii) The operator Λ_k is of type M. For the proof of (iii), we need the following lemma.

Lemma 1 (cf. [5]) *Let $\mathcal{A}$ and $\mathcal{B}$ be two operators. If $\mathcal{A}$ is of type M and $\mathcal{B}$ is monotone and weakly continuous, then $\mathcal{A} + \mathcal{B}$ is of type M.*

Now, we set $\langle \mathcal{A}u, v\rangle := \int_{\widetilde{\Omega}} \widetilde{a}(x, \nabla u).\nabla v \,\mathrm{d}x$ and $\langle \mathcal{B}_k u, v\rangle := \int_{\Omega} T_k\left(\beta(u)\right) v \,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} T_k\left(\widetilde{\rho}(u)\right) v \,\mathrm{d}\sigma$. Then, for every $k > 0$ we have $\Lambda_k = \mathcal{A} + \mathcal{B}_k$. We now have to show that for every $k > 0$ the operator $\mathcal{B}_k$ is monotone and weakly continuous, because it is well-known that the operator $\mathcal{A}$ is of type M. For the monotonicity of $\mathcal{B}_k$, we have to show that $\langle \mathcal{B}_k u - \mathcal{B}_k v, u - v\rangle \geq 0$ for all $(u, v) \in W_D^{1,p}(\widetilde{\Omega}) \times W_D^{1,p}(\widetilde{\Omega})$. We have

$$\begin{aligned}&\langle \mathcal{B}_k u - \mathcal{B}_k v, u - v\rangle \\ &\quad = \int_{\Omega} \big(T_k\left(\beta(u)\right) - T_k\left(\beta(v)\right)\big)(u - v) \,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} \big(T_k\left(\widetilde{\rho}(u)\right) - T_k\left(\widetilde{\rho}(v)\right)\big)(u - v) \,\mathrm{d}\sigma.\end{aligned}$$

From the monotonicity of β, $\widetilde{\rho}$ and the map T_k, we conclude that

$$\langle \mathcal{B}_k u - \mathcal{B}_k v, u - v\rangle \geq 0. \tag{2.7}$$

We need now to prove that for each $k > 0$ the operator $\mathcal{B}_k$ is weakly continuous, that is, for all sequences $(u_n)_{n\in\mathbb{N}} \subset W_D^{1,p}(\widetilde{\Omega})$ such that $u_n \rightharpoonup u$ in $W_D^{1,p}(\widetilde{\Omega})$, we have $\mathcal{B}_k u_n \rightharpoonup \mathcal{B}_k u$ as $n \to +\infty$. For all $\phi \in W_D^{1,p}(\widetilde{\Omega})$ we have

$$\langle \mathcal{B}_k u_n, \phi\rangle := \int_{\Omega} T_k\left(\beta(u_n)\right) \phi \,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} T_k\left(\widetilde{\rho}(u_n)\right) \phi \,\mathrm{d}\sigma. \tag{2.8}$$

Passing to the limit in (2.8) as n goes to $+\infty$ and using the Lebesgue dominated convergence theorem, for the first term on the right-hand side of (2.8), we obtain

$$\lim_{n\to+\infty} \int_{\Omega} T_k\left(\beta(u_n)\right) \phi \,\mathrm{d}x = \int_{\Omega} T_k\left(\beta(u)\right) \phi \,\mathrm{d}x. \tag{2.9}$$

Furthermore, since $u_n \rightharpoonup u$ in $W_D^{1,p}(\widetilde{\Omega})$, up to a subsequence, we have $u_n \to u$ in $L^p(\partial\widetilde{\Omega})$ and a.e. on $\partial\widetilde{\Omega}$, and we deduce using again the Lebesgue dominated convergence theorem that

$$\lim_{n\to+\infty} \int_{\widetilde{\Gamma}_{Ne}} T_k\left(\widetilde{\rho}(u_n)\right) \phi \,\mathrm{d}\sigma = \int_{\widetilde{\Gamma}_{Ne}} T_k\left(\widetilde{\rho}(u)\right) \phi \,\mathrm{d}\sigma. \tag{2.10}$$

From (2.9) and (2.10) we conclude that for every $k > 0$, $\lim_{n\to+\infty}\langle \mathcal{B}_k u_n, \phi\rangle = \langle \mathcal{B}_k u, \phi\rangle$, which means that $\mathcal{B}_k u_n \rightharpoonup \mathcal{B}_k u$.

The operator $\mathcal{A}$ is type M and as $\mathcal{B}_k$ is monotone and weakly continuous, thanks to Lemma 1, we conclude that the operator Λ_k is of type M. Then, for any $L \in \big(W_D^{1,p}(\widetilde{\Omega})\big)'$, there exists $u_k \in W_D^{1,p}(\widetilde{\Omega})$ such that $\Lambda_k(u_k) = L$. We consider $L \in \big(W_D^{1,p}(\widetilde{\Omega})\big)'$ defined by $L(v) := \int_\Omega f v \,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} \widetilde{d} v \,\mathrm{d}\sigma$ for $v \in W_D^{1,p}(\widetilde{\Omega})$ and we obtain (2.3).

To end the proof of Theorem 3, we prove inequalities (2.4). For any $\epsilon > 0$ let us introduce the function $H_\epsilon \colon \mathbb{R} \to \mathbb{R}$:

$$H_\epsilon(s) = \begin{cases} 0, & \text{if } s < 0, \\ \frac{s}{\epsilon}, & \text{if } 0 \le s \le \epsilon, \\ 1, & \text{if } s > \epsilon. \end{cases}$$

In (2.3) we set $\widetilde{\varphi} = H_\epsilon(u_k - M)$, $\epsilon > 0$, where $M > 0$ is to be fixed later. We get

$$\begin{aligned} &\int_{\widetilde{\Omega}} \widetilde{a}(x, \nabla u_k).\nabla H_\epsilon(u_k - M)\,\mathrm{d}x + \int_\Omega T_k\left(\beta(u_k)\right) H_\epsilon(u_k - M)\,\mathrm{d}x \\ &\quad = \int_\Omega f H_\epsilon(u_k - M)\,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} \big(\widetilde{d} - T_k\left(\widetilde{\rho}(u_k)\right)\big) H_\epsilon(u_k - M)\,\mathrm{d}\sigma. \end{aligned} \tag{2.11}$$

The first term in (2.11) is non-negative. Indeed,

$$\int_{\widetilde{\Omega}} \widetilde{a}(x, \nabla u_k).\nabla H_\epsilon(u_k - M)\,\mathrm{d}x = \frac{1}{\epsilon}\int_{\{0\le u_k - M \le \epsilon\}} \widetilde{a}(x, \nabla u_k).\nabla u_k\,\mathrm{d}x \ge 0.$$

From (2.11) we obtain

$$\int_\Omega T_k\left(\beta(u_k)\right) H_\epsilon(u_k - M)\,\mathrm{d}x \le \int_\Omega f H_\epsilon(u_k - M)\,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} \big(\widetilde{d} - T_k\left(\widetilde{\rho}(u_k)\right)\big) H_\epsilon(u_k - M)\,\mathrm{d}\sigma.$$

Then, one has

$$\begin{aligned} &\int_\Omega \big(T_k\left(\beta(u_k)\right) - T_k\left(\beta(M)\right)\big) H_\epsilon(u_k - M)\,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} \big(T_k\left(\widetilde{\rho}(u_k)\right) - T_k\left(\widetilde{\rho}(M)\right)\big) H_\epsilon(u_k - M)\,\mathrm{d}\sigma \\ &\quad \le \int_\Omega \big(f - T_k\left(\beta(M)\right)\big) H_\epsilon(u_k - M)\,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} \big(\widetilde{d} - T_k\left(\widetilde{\rho}(M)\right)\big) H_\epsilon(u_k - M)\,\mathrm{d}\sigma. \end{aligned}$$

Letting ε go to 0 in the above inequality, we get

$$\begin{aligned} &\int_\Omega \big(T_k\left(\beta(u_k)\right) - T_k\left(\beta(M)\right)\big) \operatorname{sign}_0^+(u_k - M)\,\mathrm{d}x \\ &\qquad + \int_{\widetilde{\Gamma}_{Ne}} \big(T_k\left(\widetilde{\rho}(u_k)\right) - T_k\left(\widetilde{\rho}(M)\right)\big) \operatorname{sign}_0^+(u_k - M)\,\mathrm{d}\sigma \\ &\quad \le \int_\Omega \big(f - T_k\left(\beta(M)\right)\big) \operatorname{sign}_0^+(u_k - M)\,\mathrm{d}x \\ &\qquad + \int_{\widetilde{\Gamma}_{Ne}} \big(\widetilde{d} - T_k\left(\widetilde{\rho}(M)\right)\big) \operatorname{sign}_0^+(u_k - M)\,\mathrm{d}\sigma, \end{aligned}$$

which is equivalent to say

$$\int_\Omega \big(T_k\left(\beta(u_k)\right)-T_k\left(\beta(M)\right)\big)^+\,\mathrm{d}x+\int_{\widetilde{\Gamma}_{Ne}}\big(T_k\left(\widetilde{\rho}(u_k)\right)-T_k\left(\widetilde{\rho}(M)\right)\big)^+\,\mathrm{d}\sigma$$
$$\leq \int_\Omega \big(f-T_k\left(\beta(M)\right)\big)\,\mathrm{sign}_0^+(u_k-M)\,\mathrm{d}x+\int_{\widetilde{\Gamma}_{Ne}}\big(\widetilde{d}-T_k\left(\widetilde{\rho}(M)\right)\big)\,\mathrm{sign}_0^+(u_k-M)\,\mathrm{d}\sigma.$$

As $\mathrm{Im}(\beta)=\mathrm{Im}(\rho)=\mathbb{R}$, we can fix $M=M_0=\max\big\{\beta_0^{-1}(\|f\|_\infty),\rho_0^{-1}\big(|\widetilde{\Gamma}_{Ne}|\|\widetilde{d}\|_\infty\big)\big\}$.

From the above inequality we obtain

$$\int_\Omega \big(T_k\left(\beta(u_k)\right)-T_k\left(\beta(M_0)\right)\big)^+\,\mathrm{d}x+\int_{\widetilde{\Gamma}_{Ne}}\big(T_k\left(\widetilde{\rho}(u_k)\right)-T_k\left(\widetilde{\rho}(M_0)\right)\big)^+\,\mathrm{d}\sigma$$
$$\leq \int_\Omega \big(f-T_k\left(\|f\|_\infty\right)\big)\,\mathrm{sign}_0^+(u_k-M_0)\,\mathrm{d}x+\int_{\widetilde{\Gamma}_{Ne}}\big(\widetilde{d}-T_k\big(\|\widetilde{d}\|_\infty\big)\big)\,\mathrm{sign}_0^+(u_k-M_0)\,\mathrm{d}\sigma.$$

For $k>k_0:=\max\big\{\|f\|_\infty,\|\widetilde{d}\|_\infty\big\}$, it follows that

$$\int_\Omega \big(T_k\left(\beta(u_k)\right)-T_k\left(\beta(M_0)\right)\big)^+\,\mathrm{d}x+\int_{\widetilde{\Gamma}_{Ne}}\big(T_k\left(\widetilde{\rho}(u_k)\right)-T_k\left(\widetilde{\rho}(M_0)\right)\big)^+\,\mathrm{d}\sigma\leq 0. \tag{2.12}$$

Then, it yields

$$\int_\Omega \big(T_k\left(\beta(u_k)\right)-T_k\left(\beta(M_0)\right)\big)^+\,\mathrm{d}x\leq 0,$$
$$\int_{\widetilde{\Gamma}_{Ne}}\big(T_k\left(\widetilde{\rho}(u_k)\right)-T_k\left(\widetilde{\rho}(M_0)\right)\big)^+\,\mathrm{d}\sigma\leq 0.$$

So

$$\begin{aligned}\int_\Omega \big(T_k\left(\beta(u_k)\right)-T_k\left(\beta(M_0)\right)\big)^+\,\mathrm{d}x&=0,\\ \int_{\widetilde{\Gamma}_{Ne}}\big(T_k\left(\widetilde{\rho}(u_k)\right)-T_k\left(\widetilde{\rho}(M_0)\right)\big)^+\,\mathrm{d}\sigma&=0.\end{aligned} \tag{2.13}$$

From (2.13) we have

$$\big(T_k\left(\beta(u_k)\right)-T_k\left(\beta(M_0)\right)\big)^+=0 \text{ a.e. in } \Omega,$$
$$\big(T_k\left(\widetilde{\rho}(u_k)\right)-T_k\left(\widetilde{\rho}(M_0)\right)\big)^+=0 \text{ a.e. on } \widetilde{\Gamma}_{Ne}.$$

This means that for any $k>k_0:=\max\big\{\|f\|_\infty,\|\widetilde{d}\|_\infty\big\}$ we have

$$\begin{aligned}T_k\left(\beta(u_k)\right)&\leq T_k\left(\beta(M_0)\right) \text{ a.e. in } \Omega,\\ T_k\left(\widetilde{\rho}(u_k)\right)&\leq T_k\left(\widetilde{\rho}(M_0)\right) \text{ a.e. on } \widetilde{\Gamma}_{Ne}.\end{aligned} \tag{2.14}$$

From (2.14) we deduce that for every $k>k_1:=\max\big\{\|f\|_\infty,\|\widetilde{d}\|_\infty,\beta(M_0),\widetilde{\rho}(M_0)\big\}$ we have

$$\beta(u_k)\leq\beta(M_0) \text{ a.e. in } \Omega,$$
$$\widetilde{\rho}(u_k)\leq\widetilde{\rho}(M_0) \text{ a.e. on } \widetilde{\Gamma}_{Ne}.$$

Note that with the choice of M_0 and the fact that $\mathcal{D}(\beta)=\mathcal{D}(\rho)=\mathbb{R}$, for every $k>k_1:=\max\big\{\|f\|_\infty,\|\widetilde{d}\|_\infty,\beta(M_0),\widetilde{\rho}(M_0)\big\}$ we have

$$\begin{aligned}\beta(u_k)&\leq\max\big\{\|f\|_\infty,(\beta\circ\rho_0^{-1})\big(|\widetilde{\Gamma}_{Ne}|\|\widetilde{d}\|_\infty\big)\big\} \text{ a.e. in } \Omega,\\ \widetilde{\rho}(u_k)&\leq\max\big\{\|\widetilde{d}\|_\infty,(\widetilde{\rho}\circ\beta_0^{-1})\left(\|f\|_\infty\right)\big\} \text{ a.e. on } \widetilde{\Gamma}_{Ne}.\end{aligned} \tag{2.15}$$

We need to show that for any k large enough

$$\begin{aligned}\beta(u_k) &\geq -\max\big\{\|f\|_\infty, (\beta\circ\rho_0^{-1})\big(|\widetilde{\Gamma}_{Ne}|\|\widetilde{d}\|_\infty\big)\big\} \text{ a.e. in } \Omega,\\ \widetilde{\rho}(u_k) &\geq -\max\big\{\|\widetilde{d}\|_\infty, (\widetilde{\rho}\circ\beta_0^{-1})\,(\|f\|_\infty)\big\} \text{ a.e. on } \widetilde{\Gamma}_{Ne}.\end{aligned} \tag{2.16}$$

It is easy to see that if u_k is a solution of $P_k(\widetilde{\beta}, \widetilde{\rho}, \widetilde{f}, \widetilde{d})$, then $(-u_k)$ is a solution of

$$P_k(\widehat{\beta}, \widehat{\rho}, \widehat{f}, \widehat{d}\,)\begin{cases} T_k\big(\widehat{\beta}(x,u)\big) - \nabla\cdot\widehat{a}(x,\nabla u) = \widehat{f} & \text{in } \widetilde{\Omega},\\ u = 0 & \text{on } \Gamma_D,\\ T_k\left(\widehat{\rho}(u)\right) + \widehat{a}(x,\nabla u).\eta = \widehat{d} & \text{on } \widetilde{\Gamma}_{Ne},\end{cases}$$

where

$$\widehat{a}(x,\xi) = -\widetilde{a}(x,-\xi),\;\; \widehat{\beta}(x,s) = -\widetilde{\beta}(x,-s),\;\; \widehat{\rho}(s) = -\widetilde{\rho}(-s),\;\; \widehat{f} = -\widetilde{f} \text{ and } \widehat{d} = -\widetilde{d}.$$

Then, for every $k > k_2 := \max\big\{\|f\|_\infty, \|\widetilde{d}\|_\infty, -\beta(-M_0), -\widetilde{\rho}(-M_0)\big\}$ we have

$$\begin{aligned}-\beta(u_k) &\leq \max\big\{\|f\|_\infty, (\beta\circ\rho_0^{-1})\big(|\widetilde{\Gamma}_{Ne}|\|\widetilde{d}\|_\infty\big)\big\} \text{ a.e. in } \Omega,\\ -\widetilde{\rho}(u_k) &\leq \max\big\{\|\widetilde{d}\|_\infty, (\widetilde{\rho}\circ\beta_0^{-1})\,(\|f\|_\infty)\big\} \text{ a.e. on } \widetilde{\Gamma}_{Ne},\end{aligned}$$

which implies (2.16).

From the relations (2.15) and (2.16), we deduce (2.4). □

Since u_k is a solution of $P_k(\widetilde{\beta}, \widetilde{\rho}, \widetilde{f}, \widetilde{d})$, thanks to (2.4) and the fact that Ω is bounded, we have $\beta(u_k) \in L^1(\Omega)$. For $k = 1 + \max\{\theta_1, \theta_2\}$ fixed, by (2.4), one sees that the problem $P(\widetilde{\beta}, \widetilde{\rho}, \widetilde{f}, \widetilde{d})$ admits at least one solution u.

Remark 1 *Using the relations* (2.4) *and the fact that the functions* β *and* ρ *are non-decreasing, one sees that for k large enough the solution* u *of the problem* $P(\widetilde{\beta}, \widetilde{\rho}, \widetilde{f}, \widetilde{d})$ *belongs to* $L^\infty(\Omega)\cap L^\infty(\widetilde{\Gamma}_{Ne})$ *and* $|u| \leq C(\beta, \theta_1)$ *a.e. in* Ω *and* $|u| \leq C(\rho, \theta_2)$ *a.e. on* $\widetilde{\Gamma}_{Ne}$.

Now, we set

$$\widetilde{a}(x,\xi) := a(x,\xi)\chi_\Omega(x) + \frac{1}{\epsilon^p}|\xi|^{p-2}\xi\chi_{\widetilde{\Omega}\setminus\Omega}(x) \text{ for all } (x,\xi)\in\widetilde{\Omega}\times\mathbb{R}^N$$

and we consider the following problem

$$P_\epsilon(\widetilde{\beta}, \widetilde{\rho}, \widetilde{f}, \widetilde{d})\begin{cases}\widetilde{\beta}(x,u_\epsilon) - \nabla.\Big(a(x,\nabla u_\epsilon) + \dfrac{1}{\epsilon^p}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon\chi_{\widetilde{\Omega}\setminus\Omega}(x)\Big) = \widetilde{f} & \text{in } \widetilde{\Omega},\\ u_\epsilon = 0 & \text{on } \Gamma_D,\\ \widetilde{\rho}(u_\epsilon) + \big(a(x,\nabla u_\epsilon)\big).\eta = \widetilde{d} & \text{on } \widetilde{\Gamma}_{Ne}.\end{cases}$$

Thanks to Theorem 2, $P_\epsilon(\widetilde{\beta}, \widetilde{\rho}, \widetilde{f}, \widetilde{d})$ has at least one solution. So, there exists at least one measurable function $u_\epsilon\colon\widetilde{\Omega}\to\mathbb{R}$ such that

$$\begin{cases} u_\epsilon \in W_D^{1,p}(\widetilde{\Omega}),\;\beta(u_\epsilon)\in L^1(\Omega) \text{ and for every } \widetilde{\varphi}\in W_D^{1,p}(\widetilde{\Omega})\cap L^\infty(\Omega),\\ \displaystyle\int_\Omega a(x,\nabla u_\epsilon).\nabla\widetilde{\varphi}\,\mathrm{d}x + \int_{\widetilde{\Omega}\setminus\Omega}\frac{1}{\epsilon^p}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon.\nabla\widetilde{\varphi}\,\mathrm{d}x + \int_\Omega\beta(u_\epsilon)\widetilde{\varphi}\,\mathrm{d}x\\ \qquad\qquad = \displaystyle\int_\Omega f\widetilde{\varphi}\,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}}\big(\widetilde{d} - \widetilde{\rho}(u_\epsilon)\big)\widetilde{\varphi}\,\mathrm{d}\sigma.\end{cases} \tag{2.17}$$

Thanks to Remark 1, we have $u_\epsilon\in L^\infty(\Omega)\cap L^\infty(\widetilde{\Gamma}_{Ne})$.

Remark 2 *If u_ϵ is a solution of $P_\epsilon(\widetilde{\beta}, \widetilde{\rho}, \widetilde{f}, \widetilde{d})$, then using test functions $\widetilde{\varphi} \in W^{1,p}_{Ne}(\Omega) \cap L^\infty(\Omega)$ such that $\widetilde{\varphi} \equiv$ constant on $\widetilde{\Omega} \setminus \Omega$, we see that the second term in the equality in* (2.17) *is equal to zero and the last term is equal to $\big(d - \int_{\widetilde{\Gamma}_{Ne}} \widetilde{\rho}(u_\epsilon)\,\mathrm{d}\sigma\big)\widetilde{\varphi}_{Ne}$, so that one has*

$$\int_\Omega a(x, \nabla u_\epsilon).\nabla\widetilde{\varphi}\,\mathrm{d}x + \int_\Omega \beta(u_\epsilon)\widetilde{\varphi}\,\mathrm{d}x = \int_\Omega f\widetilde{\varphi}\,\mathrm{d}x + \left(d - \int_{\widetilde{\Gamma}_{Ne}} \widetilde{\rho}(u_\epsilon)\,\mathrm{d}\sigma\right)\widetilde{\varphi}_{Ne}. \tag{2.18}$$

The next result gives us a priori estimates on the solution u_ϵ of the problem $P_\epsilon(\widetilde{\beta}, \widetilde{\rho}, \widetilde{f}, \widetilde{d})$.

Proposition 1 *Let u_ϵ be a solution of $P_\epsilon(\widetilde{\beta}, \widetilde{\rho}, \widetilde{f}, \widetilde{d})$. Then, the following statements hold true*:

(i) $\int_\Omega |\nabla u_\epsilon|^p\ \mathrm{d}x + \int_{\widetilde{\Omega}\setminus\Omega} \frac{1}{\epsilon^p}|\nabla u_\epsilon|^p\,\mathrm{d}x \le C \times \big(\|\widetilde{d}\|_{L^1(\widetilde{\Gamma}_{Ne})} + \|f\|_{L^1(\Omega)}\big)$, *where C is a positive constant independent of ϵ*;

(ii) $\int_\Omega |\beta(u_\epsilon)|\ \mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} |\widetilde{\rho}(u_\epsilon)|\,\mathrm{d}\sigma \le \|\widetilde{d}\|_{L^1(\widetilde{\Gamma}_{Ne})} + \|f\|_{L^1(\Omega)}$.

Proof. We set $\widetilde{\varphi} = u_\epsilon$ in (2.17) to get

$$\begin{aligned}\int_\Omega a(x, \nabla u_\epsilon).\nabla u_\epsilon\,\mathrm{d}x + \int_{\widetilde{\Omega}\setminus\Omega} \frac{1}{\epsilon^p}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon.\nabla u_\epsilon\,\mathrm{d}x + \int_\Omega \beta(u_\epsilon)u_\epsilon\,\mathrm{d}x \\ = \int_\Omega f u_\epsilon\,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} \big(\widetilde{d} - \widetilde{\rho}(u_\epsilon)\big)u_\epsilon\,\mathrm{d}\sigma.\end{aligned} \tag{2.19}$$

(i) Obviously, we have $\int_\Omega \beta(u_\epsilon)u_\epsilon\,\mathrm{d}x \ge 0$, $\int_{\widetilde{\Omega}\setminus\Omega} \frac{1}{\epsilon^p}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon.\nabla u_\epsilon\,\mathrm{d}x \ge 0$ and $\int_\Omega f u_\epsilon\,\mathrm{d}x \le C(\beta, \theta_1)\|f\|_{L^1(\Omega)}$. For the last term in (2.19), we have

$$\begin{aligned}\int_{\widetilde{\Gamma}_{Ne}} \big(\widetilde{d} - \widetilde{\rho}(u_\epsilon)\big)u_\epsilon\,\mathrm{d}\sigma &= \int_{\widetilde{\Gamma}_{Ne}} \widetilde{d}u_\epsilon\,\mathrm{d}\sigma - \int_{\widetilde{\Gamma}_{Ne}} \widetilde{\rho}(u_\epsilon)u_\epsilon\,\mathrm{d}\sigma \\ &\le \int_{\widetilde{\Gamma}_{Ne}} \widetilde{d}u_\epsilon\,\mathrm{d}\sigma \\ &\le \int_{\widetilde{\Gamma}_{Ne}} |\widetilde{d}||u_\epsilon|\,\mathrm{d}\sigma \\ &\le C(\rho, \theta_2)\|\widetilde{d}\|_{L^1(\widetilde{\Gamma}_{Ne})}.\end{aligned}$$

Having in mind the relation $a(x, \xi).\xi \ge \frac{1}{C'}|\xi|^p$, we get

$$\int_\Omega a(x, \nabla u_\epsilon).\nabla u_\epsilon\,\mathrm{d}x \ge \frac{1}{C'}\int_\Omega |\nabla u_\epsilon|^p\ \mathrm{d}x.$$

Using the inequalities above, one gets

$$\begin{aligned}\int_\Omega |\nabla u_\epsilon|^p\ \mathrm{d}x &\le C'\big(C(\beta, \theta_1)\|f\|_{L^1(\Omega)} + C(\rho, \theta_2)\|\widetilde{d}\|_{L^1(\widetilde{\Gamma}_{Ne})}\big) \\ &\le C \times \big(\|\widetilde{d}\|_{L^1(\widetilde{\Gamma}_{Ne})} + \|f\|_{L^1(\Omega)}\big)\end{aligned} \tag{2.20}$$

with $C \ge \max\{C'C(\beta, \theta_1), C'C(\rho, \theta_2)\}$.

In (2.19), the terms $\int_\Omega a(x,\nabla u_\epsilon).\nabla u_\epsilon \,\mathrm{d}x$ and $\int_\Omega \beta(u_\epsilon)u_\epsilon \,\mathrm{d}x$ are non-negative so that we obtain

$$\int_{\widetilde{\Omega}\setminus\Omega} \frac{1}{\epsilon^p}|\nabla u_\epsilon|^p \,\mathrm{d}x \leq C \times \left(\|\widetilde{d}\|_{L^1(\widetilde{\Gamma}_{Ne})} + \|f\|_{L^1(\Omega)}\right). \tag{2.21}$$

Adding (2.20) and (2.21), we obtain (i).

(ii) We set $\widetilde{\varphi} = T_k(u_\epsilon)$, $k > 0$, in (2.17) to get

$$\begin{aligned}\int_\Omega a(x,\nabla u_\epsilon).\nabla T_k(u_\epsilon)\,\mathrm{d}x + \int_{\widetilde{\Omega}\setminus\Omega} \frac{1}{\epsilon^p}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon.\nabla T_k(u_\epsilon)\,\mathrm{d}x + \int_\Omega \beta(u_\epsilon)T_k(u_\epsilon)\,\mathrm{d}x\\ + \int_{\widetilde{\Gamma}_{Ne}} \widetilde{\rho}(u_\epsilon)T_k(u_\epsilon)\,\mathrm{d}\sigma = \int_\Omega fT_k(u_\epsilon)\,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} \widetilde{d}T_k(u_\epsilon)\,\mathrm{d}\sigma.\end{aligned} \tag{2.22}$$

The first two terms in (2.22) are non-negative. For the terms on the right-hand side of (2.22), we have

$$\begin{aligned}\int_\Omega fT_k(u_\epsilon)\,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} \widetilde{d}T_k(u_\epsilon)\,\mathrm{d}\sigma &\leq k\left(\int_\Omega |f|\,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} \widetilde{d}\,\mathrm{d}\sigma\right)\\ &= k\Big(\|\widetilde{d}\|_{L^1(\widetilde{\Gamma}_{Ne})} + ||f||_{L^1(\Omega)}\Big).\end{aligned}$$

Then, from (2.22), it yields

$$\int_\Omega \beta(u_\epsilon)T_k(u_\epsilon)\,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} \widetilde{\rho}(u_\epsilon)T_k(u_\epsilon)\,\mathrm{d}\sigma \leq k\Big(\|\widetilde{d}\|_{L^1(\widetilde{\Gamma}_{Ne})} + ||f||_{L^1(\Omega)}\Big).$$

We divide the above inequality by k and let k go to zero to get

$$\begin{aligned}\int_\Omega \beta(u_\epsilon)\,\mathrm{sign}(u_\epsilon)\,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} \widetilde{\rho}(u_\epsilon)\,\mathrm{sign}(u_\epsilon)\,\mathrm{d}\sigma &= \int_\Omega |\beta(u_\epsilon)|\,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} |\widetilde{\rho}(u_\epsilon)|\,\mathrm{d}\sigma\\ &\leq \left(\|\widetilde{d}\|_{L^1(\widetilde{\Gamma}_{Ne})} + ||f||_{L^1(\Omega)}\right).\end{aligned}$$

□

The following result states useful convergences results.

Proposition 2 *As $\epsilon \to 0$, we have*

(i) $u_\epsilon \to u$ *a.e. in* Ω *and a.e. on* $\widetilde{\Gamma}_{Ne}$;

(ii) $\beta(u_\epsilon) \to \beta(u)$ *in* $L^1(\Omega)$;

(iii) $\nabla u_\epsilon \rightharpoonup \nabla u$ *in* $\left(L^p(\widetilde{\Omega}\setminus\Omega)\right)^N$ *and* $\nabla u = 0$ *in* $\widetilde{\Omega}\setminus\Omega$;

(iv) $\widetilde{\rho}(u_\epsilon) \to \widetilde{\rho}(u)$ *in* $L^1(\widetilde{\Gamma}_{Ne})$;

(v) $a(x,\nabla u_\epsilon) \rightharpoonup a(x,\nabla u)$ *in* $\left(L^{p'}(\Omega)\right)^N$.

Proof. (i) For any $0 < \varepsilon < 1$ we have

$$\int_{\widetilde{\Omega}} |\nabla u_\epsilon|^p \,\mathrm{d}x = \int_\Omega |\nabla u_\epsilon|^p \,\mathrm{d}x + \int_{\widetilde{\Omega}\setminus\Omega} |\nabla u_\epsilon|^p \,\mathrm{d}x \leq \int_\Omega |\nabla u_\epsilon|^p \,\mathrm{d}x + \int_{\widetilde{\Omega}\setminus\Omega} \frac{1}{\epsilon^p}|\nabla u_\epsilon|^p \,\mathrm{d}x.$$

Then, thanks to Proposition 1 (i), the sequence $(|\nabla u_\epsilon|)_{\epsilon>0}$ is bounded in $L^p(\widetilde{\Omega})$. Since $u_\epsilon \in W_D^{1,p}(\widetilde{\Omega})$, the sequence $(u_\epsilon)_{\epsilon>0}$ is bounded in $W_D^{1,p}(\widetilde{\Omega})$. Up to a subsequence, we get as $\varepsilon \to 0$ that

$$u_\epsilon \rightharpoonup u \text{ in } W_D^{1,p}(\widetilde{\Omega}),\ u_\epsilon \to u \text{ a.e. in } \widetilde{\Omega} \text{ and a.e. on } \widetilde{\Gamma}_{Ne}.$$

We conclude (i) using the fact that $\Omega \subset \widetilde{\Omega}$.

(ii) As $u_\epsilon \to u$ a.e. in Ω and β is continuous, we deduce that $\beta(u_\epsilon) \to \beta(u)$ a.e. in Ω. Using Fatou's lemma and Proposition 1 (ii), we obtain $\beta(u) \in L^1(\Omega)$. Having in mind that $|\beta(u_\epsilon)| \leq \theta_1$ a.e. in Ω, by using Lebesgue dominated convergence theorem (since Ω is bounded), we see that $\beta(u_\epsilon) \to \beta(u)$ in $L^1(\Omega)$.

(iii) The fact that $u_\epsilon \rightharpoonup u$ in $W_D^{1,p}(\widetilde{\Omega})$ implies that $\nabla u_\epsilon \rightharpoonup \nabla u$ in $\left(L^p(\widetilde{\Omega})\right)^N$ and then $\nabla u_\epsilon \rightharpoonup \nabla u$ in $\left(L^p(\widetilde{\Omega} \setminus \Omega)\right)^N$. By Proposition 1 (i) we can assert that $\left(\frac{1}{\epsilon^p}|\nabla u_\epsilon|^p\right)_{\epsilon>0}$ is bounded in $L^1(\widetilde{\Omega} \setminus \Omega)$. Then, $\left(\frac{1}{\epsilon}\nabla u_\epsilon\right)_{\epsilon>0}$ is bounded in $\left(L^p(\widetilde{\Omega} \setminus \Omega)\right)^N$. Therefore, there exists $\Theta \in \left(L^p(\widetilde{\Omega} \setminus \Omega)\right)^N$ such that

$$\frac{1}{\epsilon}\nabla u_\epsilon \rightharpoonup \Theta \text{ in } \left(L^p(\widetilde{\Omega} \setminus \Omega)\right)^N \text{ as } \varepsilon \to 0.$$

For any $v \in \left(L^{p'}(\widetilde{\Omega} \setminus \Omega)\right)^N$ we have

$$\int_{\widetilde{\Omega}\setminus\Omega} \nabla u_\epsilon . v \,\mathrm{d}x = \int_{\widetilde{\Omega}\setminus\Omega} \epsilon \left(\frac{1}{\epsilon}\nabla u_\epsilon\right). v \,\mathrm{d}x = \int_{\widetilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon}\nabla u_\epsilon - \Theta\right).(\epsilon v) \,\mathrm{d}x + \epsilon \int_{\widetilde{\Omega}\setminus\Omega} \Theta . v \,\mathrm{d}x. \quad (2.23)$$

As $(\epsilon v)_{\epsilon>0}$ converges strongly to zero in $\left(L^{p'}(\widetilde{\Omega}\setminus\Omega)\right)^N$, passing to the limit in (2.23) as $\epsilon \to 0$, we get

$$\nabla u_\epsilon \rightharpoonup 0 \text{ in } \left(L^p(\widetilde{\Omega} \setminus \Omega)\right)^N.$$

Hence, one has $\nabla u_\epsilon \rightharpoonup \nabla u = 0$ in $\left(L^p(\widetilde{\Omega} \setminus \Omega)\right)^N$.

(iv) As $u_\epsilon \to u$ a.e. on $\widetilde{\Gamma}_{Ne}$ and $\widetilde{\rho}$ is continuous, we get $\widetilde{\rho}(u_\epsilon) \to \widetilde{\rho}(u)$ a.e. on $\widetilde{\Gamma}_{Ne}$. Using Fatou's lemma and Proposition 1 (ii), we obtain $\widetilde{\rho}(u) \in L^1(\widetilde{\Gamma}_{Ne})$. By the estimate $|\widetilde{\rho}(u_\epsilon)| \leq \theta_2$ a.e. in $\widetilde{\Gamma}_{Ne}$ and the Lebesgue dominated convergence theorem, we get (iv).

(v) The sequence $(a(x, \nabla u_\epsilon))_{\varepsilon>0}$ is bounded in $\left(L^{p'}(\Omega)\right)^N$ according to (1.1). We can extract a subsequence such that $a(x, \nabla u_\epsilon) \rightharpoonup \Phi$ in $\left(L^{p'}(\Omega)\right)^N$. We have to show that $\Phi = a(x, \nabla u)$ a.e. in Ω. The proof consists of two steps.

Step 1: We prove that

$$\limsup_{\epsilon \to 0} \int_\Omega a(x, \nabla u_\epsilon). \nabla \left(u_\epsilon - u\right)\, \mathrm{d}x \leq 0. \quad (2.24)$$

Let us take $\widetilde{\varphi} = u_\epsilon - u$ as a test function in (2.17). We get

$$\begin{aligned} &\int_\Omega a(x, \nabla u_\epsilon).\nabla(u_\epsilon - u) \,\mathrm{d}x + \int_{\widetilde{\Omega}\setminus\Omega} \frac{1}{\epsilon^p}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon.\nabla(u_\epsilon - u) \,\mathrm{d}x + \int_\Omega \beta(u_\epsilon)(u_\epsilon - u) \,\mathrm{d}x \\ &\quad + \int_{\widetilde{\Gamma}_{Ne}} \widetilde{\rho}(u_\epsilon)(u_\epsilon - u) \,\mathrm{d}\sigma = \int_\Omega f(u_\epsilon - u) \,\mathrm{d}x + \int_{\widetilde{\Gamma}_{Ne}} \widetilde{d}(u_\epsilon - u) \,\mathrm{d}\sigma. \end{aligned} \quad (2.25)$$

Note that

$$\int_\Omega \beta(u_\epsilon)(u_\epsilon - u) \,\mathrm{d}x = \int_\Omega \left(\beta(u_\epsilon) - \beta(u)\right)(u_\epsilon - u) \,\mathrm{d}x + \int_\Omega \beta(u)(u_\epsilon - u) \,\mathrm{d}x.$$

Since $\int_\Omega (\beta(u_\epsilon) - \beta(u))(u_\epsilon - u)\,\mathrm{d}x \geq 0$, we get

$$\int_\Omega \beta(u_\epsilon)(u_\epsilon - u)\,\mathrm{d}x \geq \int_\Omega \beta(u)(u_\epsilon - u)\,\mathrm{d}x.$$

Using the Lebesgue dominated convergence theorem we deduce that $\lim_{\varepsilon\to 0}\int_\Omega \beta(u)(u_\epsilon - u)\,\mathrm{d}x = 0$, and then

$$\limsup_{\varepsilon\to 0}\int_\Omega \beta(u_\epsilon)(u_\epsilon - u)\,\mathrm{d}x \geq 0. \tag{2.26}$$

As $\nabla u = 0$ in $\widetilde{\Omega}\setminus\Omega$, we obtain

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\int_{\widetilde{\Omega}\setminus\Omega}\frac{1}{\epsilon^p}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon.\nabla(u_\epsilon - u)\,\mathrm{d}x\\ &\qquad = \limsup_{\varepsilon\to 0}\int_{\widetilde{\Omega}\setminus\Omega}\frac{1}{\epsilon^p}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon.\nabla u_\epsilon\,\mathrm{d}x \geq 0.\end{aligned} \tag{2.27}$$

We also have

$$\begin{aligned}\int_{\widetilde{\Gamma}_{Ne}}\widetilde{\rho}(u_\epsilon)(u_\epsilon - u)\,\mathrm{d}\sigma &= \int_{\widetilde{\Gamma}_{Ne}}\Big(\widetilde{\rho}(u_\epsilon) - \widetilde{\rho}(u)\Big)(u_\epsilon - u)\,\mathrm{d}\sigma + \int_{\widetilde{\Gamma}_{Ne}}\widetilde{\rho}(u)(u_\epsilon - u)\,\mathrm{d}\sigma\\ &\geq \int_{\widetilde{\Gamma}_{Ne}}\widetilde{\rho}(u)(u_\epsilon - u)\,\mathrm{d}\sigma.\end{aligned}$$

As $u_\epsilon \to u$ a.e. on $\widetilde{\Gamma}_{Ne}$, by the Lebesgue dominated convergence theorem, we get

$$\lim_{\varepsilon\to 0}\int_{\widetilde{\Gamma}_{Ne}}\widetilde{\rho}(u)(u_\epsilon - u)\,\mathrm{d}\sigma = 0.$$

Hence,

$$\limsup_{\varepsilon\to 0}\int_{\widetilde{\Gamma}_{Ne}}\widetilde{\rho}(u_\epsilon)(u_\epsilon - u)\,\mathrm{d}\sigma \geq 0. \tag{2.28}$$

Applying the Lebesgue dominated convergence theorem again, we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega f(u_\epsilon - u)\,\mathrm{d}x = 0 \tag{2.29}$$

and

$$\lim_{\varepsilon\to 0}\int_{\widetilde{\Gamma}_{Ne}}\widetilde{d}(u_\epsilon - u)\,\mathrm{d}\sigma = 0. \tag{2.30}$$

Letting ε go to zero in (2.25) and using (2.26)–(2.30), we get (2.24).

Step 2: By standard monotonicity arguments we prove that $\Phi = a(x,\nabla u)$ a.e. in Ω.

Let $\varphi \in \mathcal{D}(\Omega)$ and $\lambda \in \mathbb{R}^*$. Using (2.24) and (1.2), we get

$$\begin{aligned}\lambda\lim_{\epsilon\to 0}\int_\Omega a(x,\nabla u_\epsilon).\nabla\varphi\,\mathrm{d}x &\geq \limsup_{\epsilon\to 0}\int_\Omega a(x,\nabla u_\epsilon).\nabla\big(u_\epsilon - u + \lambda\varphi\big)\,\mathrm{d}x\\ &\geq \limsup_{\epsilon\to 0}\int_\Omega a(x,\nabla(u-\lambda\varphi)).\nabla\big(u_\epsilon - u + \lambda\varphi\big)\,\mathrm{d}x.\end{aligned}$$

Hence,

$$\lambda\lim_{\epsilon\to 0}\int_\Omega a(x,\nabla u_\epsilon).\nabla\varphi\,\mathrm{d}x \geq \ \lambda\int_\Omega a(x,\nabla(u-\lambda\varphi)).\nabla\varphi\,\mathrm{d}x. \tag{2.31}$$

Dividing (2.31) by $\lambda > 0$ and by $\lambda < 0$ respectively, and passing to the limit with $\lambda \to 0$ we obtain

$$\lim_{\epsilon \to 0} \int_\Omega a(x, \nabla u_\epsilon).\nabla\varphi \, \mathrm{d}x = \int_\Omega a(x, \nabla u).\nabla\varphi \, \mathrm{d}x.$$

This means that $\int_\Omega \Phi.\nabla\varphi \, \mathrm{d}x = \int_\Omega a(x,\nabla u).\nabla\varphi \, \mathrm{d}x$ and so $\mathrm{div}(\Phi) = \mathrm{div}\, a(x,\nabla u)$ in $\mathcal{D}'(\Omega)$. Hence, $\Phi = a(x,\nabla u)$ a.e. in Ω and we have $a(x,\nabla u_\epsilon) \rightharpoonup a(x,\nabla u)$ in $\left(L^{p'}(\Omega)\right)^N$ as $\varepsilon \to 0$.

□

3 Existence and uniqueness of solutions to $P(\beta, \rho, f, d)$

We are now able to prove Theorem 1.

Proof. The fact that $\nabla u = 0$ in $L^p(\widetilde{\Omega} \setminus \Omega)$ ensure that $u \equiv$ constant on $\widetilde{\Omega} \setminus \Omega$, so that $u \in W^{1,p}_{Ne}(\Omega)$. Moreover, in the proof of Proposition 2 (ii) we have already seen that $\beta(u) \in L^1(\Omega)$.

To show that u is a solution of $P(\beta, \rho, f, d)$, we only have to prove the equality in (1.6). For any $\varphi \in W^{1,p}_{Ne}(\Omega) \cap L^\infty(\Omega)$, we consider the function $\widetilde{\varphi} \in W^{1,p}_D(\widetilde{\Omega}) \cap L^\infty(\Omega)$ such that $\widetilde{\varphi} = \varphi\chi_\Omega + \varphi_{Ne}\chi_{\widetilde{\Omega}\setminus\Omega}$. Then, $\widetilde{\varphi} \equiv$ constant on $\widetilde{\Omega} \setminus \Omega$. Such function $\widetilde{\varphi}$ in the equality in (2.17) gives us, thanks to Remark 2,

$$\int_\Omega a(x,\nabla u_\epsilon).\nabla\varphi \, \mathrm{d}x + \int_\Omega \beta(u_\epsilon)\varphi \, \mathrm{d}x = \int_\Omega f\varphi \, \mathrm{d}x + \left(d - \int_{\widetilde{\Gamma}_{Ne}} \widetilde{\rho}(u_\epsilon) \, \mathrm{d}\sigma\right) \varphi_{Ne}. \tag{3.1}$$

Passing to the limit in (3.1) as $\epsilon \to 0$ and using the convergences in Proposition 2, one has

$$\begin{aligned}
\int_\Omega a(x,\nabla u).\nabla\varphi \, \mathrm{d}x + \int_\Omega \beta(u)\varphi \, \mathrm{d}x &= \int_\Omega f\varphi \, \mathrm{d}x + d\varphi_{Ne} - \left(\lim_{\varepsilon \to 0} \int_{\widetilde{\Gamma}_{Ne}} \widetilde{\rho}(u_\varepsilon) \, \mathrm{d}\sigma\right) \varphi_{Ne} \\
&= \int_\Omega f\varphi \, \mathrm{d}x + d\varphi_{Ne} - \left(\int_{\widetilde{\Gamma}_{Ne}} \widetilde{\rho}(u) \, \mathrm{d}\sigma\right) \varphi_{Ne} \\
&= \int_\Omega f\varphi \, \mathrm{d}x + d\varphi_{Ne} - \left(\int_{\widetilde{\Gamma}_{Ne}} \widetilde{\rho}(u)_{Ne} \, \mathrm{d}\sigma\right) \varphi_{Ne} \\
&= \int_\Omega f\varphi \, \mathrm{d}x + \big(d - \rho(u)_{Ne}\big)\varphi_{Ne},
\end{aligned}$$

which means that u is a solution of $P(\beta, \rho, f, d)$.

Let us prove now the uniqueness part of Theorem 1. This proof is a straightforward consequence of the following lemma.

Lemma 2 *Assume that u_1 and u_2 are two solutions for the problems $P(\beta, \rho, f_1, d_1)$ and $P(\beta, \rho, f_2, d_2)$, respectively. Then,*

$$\big(\rho(u_1)_{Ne} - \rho(u_2)_{Ne}\big)^+ + \int_\Omega \Big(\beta(u_1) - \beta(u_2)\Big)^+ \mathrm{d}x \le \|f_1 - f_2\|_{L^1(\Omega)} + |d_1 - d_2|\,. \tag{3.2}$$

Proof. For any $\varphi \in W^{1,p}_{Ne}(\Omega) \cap L^\infty(\Omega)$ we have

$$\int_\Omega a(x, \nabla u_1).\nabla\varphi \,\mathrm{d}x + \int_\Omega \beta(u_1)\varphi \,\mathrm{d}x = \int_\Omega f_1\varphi \,\mathrm{d}x + \big(d_1 - \rho(u_1)_{Ne}\big)\varphi_{Ne} \tag{3.3}$$

and

$$\int_\Omega a(x, \nabla u_2).\nabla\varphi \,\mathrm{d}x + \int_\Omega \beta(u_2)\varphi \,\mathrm{d}x = \int_\Omega f_2\varphi \,\mathrm{d}x + \big(d_2 - \rho(u_2)_{Ne}\big)\varphi_{Ne}. \tag{3.4}$$

Subtracting (3.4) from (3.3), one has

$$\begin{aligned} &\int_\Omega \big(a(x, \nabla u_1) - a(x, \nabla u_2)\big).\nabla\varphi \,\mathrm{d}x + \int_\Omega (\beta(u_1) - \beta(u_2))\,\varphi \,\mathrm{d}x \\ &\qquad + \big(\rho(u_1)_{Ne} - \rho(u_2)_{Ne}\big)\varphi_{Ne} = \int_\Omega (f_1 - f_2)\,\varphi \,\mathrm{d}x + (d_1 - d_2)\,\varphi_{Ne}. \end{aligned} \tag{3.5}$$

In (3.5) we take $\varphi = H_\varepsilon(u_1 - u_2)$ to get

$$\begin{aligned} &\int_\Omega \big(a(x, \nabla u_1) - a(x, \nabla u_2)\big).\nabla H_\varepsilon(u_1 - u_2) \,\mathrm{d}x \\ &\quad + \int_\Omega (\beta(u_1) - \beta(u_2))\, H_\varepsilon(u_1 - u_2) \,\mathrm{d}x \\ &\quad + \big(\rho(u_1)_{Ne} - \rho(u_2)_{Ne}\big)\,(H_\varepsilon(u_1 - u_2))_{Ne} \\ &\quad = \int_\Omega (f_1 - f_2)\, H_\varepsilon(u_1 - u_2) \,\mathrm{d}x + (d_1 - d_2)\,(H_\varepsilon(u_1 - u_2))_{Ne}\,. \end{aligned} \tag{3.6}$$

Since $|H_\epsilon(r)| \leq 1$ for all $r \in \mathbb{R}$, it follows that

$$\int_\Omega (f_1 - f_2)\, H_\varepsilon(u_1 - u_2) \,\mathrm{d}x + (d_1 - d_2)\,(H_\varepsilon(u_1 - u_2))_{Ne} \leq \|f_1 - f_2\|_{L^1(\Omega)} + |d_1 - d_2|\,.$$

Thanks to (1.2), the first term in (3.6) is non-negative. Indeed, we have

$$\begin{aligned} &\int_\Omega \big(a(x, \nabla u_1) - a(x, \nabla u_2)\big).\nabla H_\varepsilon(u_1 - u_2) \,\mathrm{d}x \\ &\quad = \frac{1}{\varepsilon}\int_{\{0<u_1-u_2<\varepsilon\}} \big(a(x, \nabla u_1) - a(x, \nabla u_2)\big).\,(\nabla u_1 - \nabla u_2)\,\mathrm{d}x \geq 0. \end{aligned}$$

Hence, from (3.6) we obtain

$$\begin{aligned} &\int_\Omega (\beta(u_1) - \beta(u_2))\, H_\varepsilon(u_1 - u_2) \,\mathrm{d}x + \big(\rho(u_1)_{Ne} - \rho(u_2)_{Ne}\big)\,(H_\varepsilon(u_1 - u_2))_{Ne} \\ &\quad \leq \|f_1 - f_2\|_{L^1(\Omega)} + |d_1 - d_2|\,. \end{aligned} \tag{3.7}$$

Letting ε go to zero in (3.7), we deduce (3.2). □

Inequality (3.2) allows us to write

$$\begin{aligned} &\big(\rho(u_1)_{Ne} - \rho(u_2)_{Ne}\big)^+ \leq \|f_1 - f_2\|_{L^1(\Omega)} + |d_1 - d_2|\,, \\ &\int_\Omega (\beta(u_1) - \beta(u_2))^+ \,\mathrm{d}x \leq \|f_1 - f_2\|_{L^1(\Omega)} + |d_1 - d_2|\,. \end{aligned} \tag{3.8}$$

For $f_1 = f_2$ and $d_1 = d_2$, by (3.8) one has

$$\big(\rho(u_1)_{Ne} - \rho(u_2)_{Ne}\big)^+ = 0,$$
$$\int_\Omega (\beta(u_1) - \beta(u_2))^+ \, \mathrm{d}x = 0.$$

Therefore,

$$\big(\rho(u_1)_{Ne} - \rho(u_2)_{Ne}\big)^+ = 0 \text{ and } (\beta(u_1) - \beta(u_2))^+ = 0 \text{ a.e. in } \Omega.$$

This means that

$$\rho(u_1)_{Ne} \leq \rho(u_2)_{Ne} \text{ and } \beta(u_1) \leq \beta(u_2) \text{ a.e. in } \Omega.$$

Since β and ρ are non-decreasing continuous functions on $\mathbb{R}$, we have

$$(u_1)_{Ne} \leq (u_2)_{Ne} \text{ and } u_1 \leq u_2 \text{ a.e. in } \Omega.$$

By changing the roles of u_1 and u_2, we obtain

$$(u_2)_{Ne} \leq (u_1)_{Ne} \text{ and } u_2 \leq u_1 \text{ a.e. in } \Omega.$$

Hence, we get

$$(u_1)_{Ne} = (u_2)_{Ne} \text{ and } u_1 = u_2 \text{ a.e. in } \Omega$$

and the uniqueness part is proved. □

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