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ON THE MULTIPOINT BOUNDARY PROBLEMS OF INTERVAL-VALUED SECOND-ORDER DIFFERENTIAL EQUATIONS UNDER GENERALIZED H-DIFFERENTIABILITY

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Abstract. In this paper we discuss some multipoint boundary value problems (MBVPs) for interval-valued second-order differential equations (ISDEs) under generalized Hukuhara differentiability. Furthermore, we describe an algorithm for solving such MBVPs in $K_C(\mathbb{R})$ using the Hukuhara differentiability method and the real Green's function method. Some examples and computer simulations illustrating our approach are also given.

Keywords: Generalized Hukuhara derivatives, interval-valued second-order differential equations, multipoint boundary value problems, real Green's function method.

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1 Introduction

The interval-valued analysis and interval-valued differential equations (IDEs) are the special cases of the set-valued analysis and set-valued differential equations, respectively. In many cases, when modelling real-world phenomena, information about the behaviour of a dynamical system is uncertain and one has to consider these uncertainties to gain better meaning of full models. The interval-valued differential equations can be used to model dynamical systems subject to uncertainties. The papers [2, 3, 6, 9, 12, 14] are focused on the interval-valued differential equations. These equations can

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be studied with a framework of the classical Hukuhara derivative $D_H X(t)$ (see [1, 5, 10, 13, 16]). However, it causes that the solutions have increasing length of their values. Stefanini and Bede [18] proposed to consider the so-called generalized Hukuhara derivative of interval-valued functions. The interval-valued differential equations with this derivative can have solutions with decreasing length of their values. There are some very important extensions of the interval-valued differential equations; for example, in the papers [6, 12, 14, 15] one can find the studies on interval-valued differential equations under generalized Hukuhara differentiability, *i.e.*, equations of the form

$$D_H^g X(t) = F(t, X(t)), \tag{1.1}$$

where D_H^g is understood as the generalized Hukuhara derivative, with the initial conditions:

$$X(t_0) = X_0 \in K_C(\mathbb{R}), \quad t \in [t_0, t_0 + p]. \tag{1.2}$$

The problem (1.1)–(1.2) is called the initial value problem for an interval-valued differential equation (IVP for IDE).

In [7] and [8] the authors considered the initial value problem for interval-valued second-order differential equations (IVP for ISDEs) of the form:

$$D_H^{2,g} X(t) = F(t, X(t), D_H^g X(t)), \tag{1.3}$$

with the initial conditions:

$$X(t_0) = I_1, \qquad D_H^g X(t_0) = I_2, \qquad t \in [t_0, t_0 + p], \tag{1.4}$$

where $X(t), F(t, X(t)) \in K_C(\mathbb{R})$. And after giving some results on the differentiability of second-order of interval-valued functions, using the method of successive approximations and contraction principle, they proved local and global existence and uniqueness results for the IVP for IDEs (1.3)–(1.4).

In this work, assuming that $t \in [a, b]$, we will study the multipoint boundary value problem for the interval-valued second-order differential equation (1.3) (MBVP for ISDE) with the multipoint boundary conditions:

$$\begin{cases} \alpha_{11} X(a) + \alpha_{12} D_H^g X(a) = \Gamma_1, \\ \alpha_{21} X(b) + \alpha_{22} D_H^g X(b) = \Gamma_2, \end{cases} \tag{1.5}$$

where $\Gamma_1 = [\Gamma_1^-, \Gamma_1^+]$, $\Gamma_2 = [\Gamma_2^-, \Gamma_2^+] \in K_C(\mathbb{R})$, $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \in \mathbb{R}$ with $\alpha_{11}^2 + \alpha_{12}^2 \neq 0$, $\alpha_{21}^2 + \alpha_{22}^2 \neq 0$ and $\Gamma_1, \Gamma_2 \neq [0, 0]$.

It is clear that the MBVP for ISDE (1.3), (1.5) is different from the IVP for ISDE (1.3)–(1.4).

In this paper, we have discussed some MBVPs for ISDEs (1.3), (1.5) under generalized Hukuhara differentiability in $K_C(\mathbb{R})$. The paper is organized as follows. Section 2 is devoted to presenting the preliminaries of interval-valued analysis. In particular, we recall some results concerning first- and second-order generalized Hukuhara derivatives. In Section 3, we formulate some multipoint boundary value problems for interval-valued second-order differential equations under generalized Hukuhara differentiability (MBVPs for ISDEs) in $K_C(\mathbb{R})$. Moreover, we give an algorithm for solving such MBVPs for ISDEs. In the last section, we give some examples illustrating our results.

2 Preliminaries

By $K_C(\mathbb{R})$ we denote the set of all closed intervals (non-empty compact and convex subsets) of $\mathbb{R}$. The addition and scalar multiplication in $K_C(\mathbb{R})$ we define as usual, *i.e.*, for $A, B \in K_C(\mathbb{R})$, $A = [A^-, A^+]$, $B = [B^-, B^+]$, where $A^- \leq A^+$, $B^- \leq B^+$ belong to $\mathbb{R}$, and $\lambda \geq 0$, we have

$$A + B = [A^- + B^-, A^+ + B^+], \quad \lambda A = [\lambda A^-, \lambda A^+], \quad (-\lambda)A = [-\lambda A^+, -\lambda A^-].$$

Furthermore, let $A \in K_C(\mathbb{R})$, $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ and $\lambda_3\lambda_4 \geq 0$. Then we have $\lambda_1(\lambda_2 A) = (\lambda_1\lambda_2)A$ and $(\lambda_3 + \lambda_4)A = \lambda_3 A + \lambda_4 A$.

Definition 1 *Let $A, B \in K_C(\mathbb{R})$. The Hausdorff metric H between the intervals A and B in $K_C(\mathbb{R})$ is defined as follows*:

$$H[A, B] = \max\big\{|A^- - B^-|, |A^+ - B^+|\big\}. \tag{2.1}$$

We define the magnitude and the length of $A \in K_C(\mathbb{R})$ by

$$\|A\| = \max\{|A^-|, |A^+|\} = H[A, \{0\}], \qquad \text{len}(A) = A^+ - A^-,$$

respectively, where $\{0\}$ is the zero element $[0, 0] \in K_C(\mathbb{R})$, which is regarded as one point.

The Hausdorff metric (2.1) has the following properties:

$$\begin{aligned} H[A + C, B + C] &= H[A, B] \quad \text{and} \quad H[A, B] = H[B, A], \\ H[A + B, C + D] &\leq H[A, C] + H[B, D], \\ H[\lambda A, \lambda B] &= |\lambda| H[A, B] \end{aligned}$$

for all $A, B, C, D \in K_C(\mathbb{R})$ and $\lambda \in \mathbb{R}$. It is known that $(K_C(\mathbb{R}), H)$ is a complete, separable and locally compact metric space.

Definition 2 *Let $A, B \in K_C(\mathbb{R})$. If there exists $C \in K_C(\mathbb{R})$ such that $A = B + C$, then we call C the Hukuhara difference of A and B, and denote it by $A \ominus B$, that is, $C = A \ominus B$.*

Let us note that $A \ominus B \neq A + (-1)B$.

Definition 3 *Let $A, B \in K_C(\mathbb{R})$. If there exists an interval $C \in K_C(\mathbb{R})$ such that $A = B + C$ or $B = A + (-1)C$, then we call C the generalized Hukuhara difference of A and B, and denote it by $A \ominus_{gH} B$, that is, $C = A \ominus_{gH} B$.*

It is known that $A \ominus B$ exists when $\text{len}(A) \geq \text{len}(B)$. Moreover, we have the following properties for $A, B, C, D \in K_C(\mathbb{R})$ (see [4, 14]):

- if $A \ominus B$ and $A \ominus C$ do exist, then $H[A \ominus B, A \ominus C] = H[B, C]$;
- if $A \ominus B$ and $C \ominus D$ do exist, then $H[A \ominus B, C \ominus D] = H[A + D, B + C]$;
- if $A \ominus B$ and $A \ominus (B + C)$ do exist, then so does $(A \ominus B) \ominus C$ and $(A \ominus B) \ominus C = A \ominus (B + C)$;

- if $A \ominus B$ and $A \ominus C, C \ominus B$ do exist, then so does $(A \ominus B) \ominus (A \ominus C)$, and $(A \ominus B) \ominus (A \ominus C) = C \ominus B$;
- if $A \ominus B$ exists, then so does the generalized Hukuhara difference $A \ominus_{gH} B$, and $A \ominus_{gH} B = A \ominus B$;
- $A \ominus_{gH} A = 0$;
- $(A + B) \ominus_{gH} B = A$;
- $A \ominus_{gH} B = B \ominus_{gH} A = C$ if and only if $C = 0$ and $A = B$.

Definition 4 *We say that the interval-valued mapping $X\colon [a, b] \subset R^+ \to K_C(\mathbb{R})$ is continuous at a point $t \in [a, b]$, if for every $\varepsilon > 0$ there exists $\delta = \delta(t, \varepsilon) > 0$ such that for all $s \in [a, b]$ such that $|t - s| < \delta$ one has $H[X(t), X(s)] \leq \varepsilon$.*

Definition 5 *Let $X\colon [a, b] \subset R^+ \to K_C(\mathbb{R})$ and $t \in (a, b)$. We say that X is Hukuhara differentiable at t, if there exists $D_H X(t) \in K_C(\mathbb{R})$ such that for all $h > 0$ sufficiently small the Hukuhara differences $X(t + h) \ominus X(t)$ and $X(t) \ominus X(t - h)$ exist and*

$$\lim_{h \searrow 0} H\left[\frac{X(t+h) \ominus X(t)}{h}, D_H X(t)\right] = \lim_{h \searrow 0} H\left[\frac{X(t) \ominus X(t-h)}{h}, D_H X(t)\right] = 0. \qquad (2.2)$$

Definition 6 *Let $X\colon [a, b] \subset R^+ \to K_C(\mathbb{R})$ and $t \in (a, b)$. We say that X is generalized Hukuhara differentiable at t, if there exists $D_H^g X(t) \in K_C(\mathbb{R})$ such that for all $h > 0$ sufficiently small the generalized Hukuhara differences $X(t + h) \ominus_{gH} X(t)$ and $X(t) \ominus_{gH} X(t - h)$ exist and*

$$\lim_{h \searrow 0} H\left[\frac{X(t+h) \ominus_{gH} X(t)}{h}, D_H^g X(t)\right] = \lim_{h \searrow 0} H\left[\frac{X(t) \ominus_{gH} X(t-h)}{h}, D_H^g X(t)\right] = 0. \quad (2.3)$$

At the borders of the domain of definition of X, $t \in \{a, b\}$, only the difference and limit which is expressed in terms of the well-defined $X(t \pm h)$ is required to exist and to be zero, respectively.

The generalized Hukuhara differentiability was introduced in [18] and studied in, for example, [4, 5, 6, 7, 8, 14, 15, 17].

Corollary 1 *Let $X\colon [t_0, T] \to K_C(\mathbb{R})$ such that $X(t) = [X^-(t), X^+(t)] \in K_C(\mathbb{R})$ be Hukuhara differentiable at $t \in (t_0, T)$. Then the bottom function $X^-(t)$ and the top function $X^+(t)$ are differentiable and*

(i) *if $X(t)$ is Hukuhara differentiable, then*

$$D_H X(t) = \left[\min\{(X^-)'(t), (X^+)'(t)\}, \max\{(X^-)'(t), (X^+)'(t)\}\right];$$

(ii) *if $X(t)$ is (H^{g1})-differentiable, then $D_H^{g1} X(t) = [(X^-)'(t), (X^+)'(t)]$;*

(iii) *if $X(t)$ is (H^{g2})-differentiable, then $D_H^{g2} X(t) = [(X^+)'(t), (X^-)'(t)]$.*

For an interval-valued function $X\colon [t_0,T]\to K_C(\mathbb{R})$, if it is H^{g1} one defines the integral (I_1) by the expression

$$\int_{t_0}^{t} X(s)\,\mathrm{d}s = \left[\int_{t_0}^{t} X^-(s)\,\mathrm{d}s, \int_{t_0}^{t} X^+(s)\,\mathrm{d}s\right], \tag{2.4}$$

and if it is H^{g2} one defines the integral (I_2) by the expression

$$\int_{t_0}^{t} X(s)\,\mathrm{d}s = \left[\int_{t_0}^{t} X^+(s)\,\mathrm{d}s, \int_{t_0}^{t} X^-(s)\,\mathrm{d}s\right]. \tag{2.5}$$

In this case, we can write the Newton–Leibniz formula by: $X(t_0) = X(t) + (-1)\int_{t_0}^{t} D_H^g X(s)\,\mathrm{d}s$.

This direction of research is motivated by the results of B. Bede and S. G. Gal [4], Chalco-Cano and Román-Flores [6], V. Lupulescu [12], Marek T. Malinowski [14, 15].

In [8], the authors have defined the second-order generalized Hukuhara differentiability of interval-valued functions X as follows:

Definition 7 *Let $X\colon [t_0,T]\to K_C(\mathbb{R})$ and $t\in[t_0,T]$. We say that $X(t)$ is generalized Hukuhara differentiable in second-order at t, if there exists $D_H^{2,g}X(t)\in K_C(\mathbb{R})$ such that for all $h>0$ sufficiently small the differences $D_H^g X\,(t)\ominus D_H^g X(t+h)$ and $D_H^g X(t)\ominus D_H^g X(t-h)$ exist and the following limits hold* (*in the metric D*)

$$\lim_{h\searrow 0^+} H\left[\frac{D_H^g X\,(t+h)\ominus D_H^g X(t)}{h}, D_H^{2,g}X(t)\right] = \lim_{h\searrow 0^+} H\left[\frac{D_H^g X\,(t)\ominus D_H^g X(t-h)}{h}, D_H^{2,g}X(t)\right] = 0. \tag{2.6}$$

Theorem 1 (see [7, 8]) *Let $X\colon [t_0,T]\to K_C(\mathbb{R})$ and $D_H^g X(t)\colon [t_0,T]\to K_C(\mathbb{R})$ be interval-valued functions, where $X(t)=[X^-(t),X^+(t)]$. If $X(t), D_H^g X(t)$ are (H^{g1})-differentiable* (*or (H^{g2})-differentiable*)*, then $X^-(t), X^+(t)$ and $(X^-(t))', (X^+(t))'$ are differentiable functions and*

(i) $D_H^{2,g}X(t) = [(X^-(t))'', (X^+(t))'']$, *where $X(t)$ and $D_H^g X(t)$ are (H^{g1})-differentiable functions*;

(ii) $D_H^{2,g}X(t) = [(X^+(t))'', (X^-(t))'']$, *where $X(t)$ is a (H^{g1})-differentiable function and $D_H^g X(t)$ is a (H^{g2})-differentiable function*;

(iii) $D_H^{2,g}X(t) = [(X^+(t))'', (X^-(t))'']$, *where $X(t)$ is a (H^{g2})-differentiable function and $D_H^g X(t)$ is a (H^{g1})-differentiable function*;

(iv) $D_H^{2,g}X(t) = [(X^-(t))'', (X^+(t))'']$, *where $X(t)$ and $D_H^g X(t)$ are (H^{g2})-differentiable functions.*

3 Main results

3.1 Some MBVPs for ISDEs

Definition 8 *In the space $K_C(\mathbb{R})$ we consider the interval-valued second-order differential equation under generalized Hukuhara differentiability* (*ISDE*):

$$D_H^{2,g}X(t) = F(t, X(t), D_H^g X(t)), \tag{3.1}$$

with solutions that satisfy the multipoint boundary conditions:

$$\begin{cases} \alpha_{11}X(a) + \alpha_{12}D_H^g X(a) = \Gamma_1, \\ \alpha_{21}X(b) + \alpha_{22}D_H^g X(b) = \Gamma_2, \end{cases} \tag{3.2}$$

where $\Gamma_1 = [\Gamma_1^-, \Gamma_1^+], \Gamma_2 = [\Gamma_2^-, \Gamma_2^+] \in K_C(\mathbb{R})$, $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \in \mathbb{R}$ *with* $\alpha_{11}^2 + \alpha_{12}^2 \neq 0$, $\alpha_{21}^2 + \alpha_{22}^2 \neq 0$ *and* $\Gamma_1, \Gamma_2 \neq [0, 0]$, *and call* (3.1)–(3.2) *the multipoint boundary value problem for the interval-valued second-order differential equation under generalized Hukuhara differentiability* (*MBVP for ISDE*).

Definition 9 *An interval-valued function* $X\colon [a, b] \subset R^+ \to K_C(\mathbb{R})$ *is called a solution of the MBVP for ISDE* (3.1)–(3.2), *if*:

(i) $X(t)$ *and* $D_H^g X(t)$ *are* (H^g)*-differentiable functions such that* $D_H^{2,g}X(t)$ *can be represented in one of the forms described in Theorem* 1;

(ii) $X(t)$ *and* $D_H^g X(t)$ *satisfy the MBVP for ISDE* (3.1)–(3.2).

Definition 10 *By the multipoint boundary value problem for an interval-valued second-order inhomogeneous linear differential equation under generalized Hukuhara differentiability* (*MBVP for ISIDE*) *we mean the equation*

$$D_H^{2,g}X(t) = (-1)[p(t)D_H^g X(t) + q(t)X(t)] + R(t), \tag{3.3}$$

where $X(t) = [X^-(t), X^+(t)], R(t) = [R^-(t), R^+(t)] \in K_C(\mathbb{R})$ *and* $p(t), q(t) \in \mathbb{R}$, *together with the multipoint boundary conditions*

$$\begin{cases} \alpha_{11}X(a) + \alpha_{12}D_H^g X(a) = \Gamma_1, \\ \alpha_{21}X(b) + \alpha_{22}D_H^g X(b) = \Gamma_2, \end{cases} \tag{3.4}$$

where $\Gamma_1 = [\Gamma_1^-, \Gamma_1^+], \Gamma_2 = [\Gamma_2^-, \Gamma_2^+] \in K_C(\mathbb{R})$, $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \in \mathbb{R}$ *and with the condition* $\Gamma_1, \Gamma_2 \neq [0, 0]$.

Definition 11 *An interval-valued function* $X\colon [a, b] \subset R^+ \to K_C(\mathbb{R})$ *is called a solution of the MBVP for ISIDE* (3.3)–(3.4), *if*:

(i) $X(t)$ *and* $D_H^g X(t)$ *are* (H^g)*-differentiable functions such that* $D_H^{2,g}X(t)$ *can be represented in one of the forms described in Theorem* 1;

(ii) $X(t)$ *and* $D_H^g X(t)$ *satisfy the MBVP for ISIDE* (3.3)–(3.4).

Definition 12 *By the multipoint boundary value problem for an interval-valued second-order homogeneous linear differential equation under generalized Hukuhara differentiability* (*MBVP for ISHDE*) *we mean the equation*

$$D_H^{2;g} X(t) = (-1)[p(t) D_H^g X(t) + q(t) X(t)], \tag{3.5}$$

where $X(t) = [X^-(t), X^+(t)], R(t) = [R^-(t), R^+(t)] \in K_C(\mathbb{R})$ *and* $p(t), q(t) \in \mathbb{R}$*, togheter with the multipoint boundary conditions*

$$\begin{cases} \alpha_{11} X(a) + \alpha_{12} D_H^g X(a) = \Gamma_1, \\ \alpha_{21} X(b) + \alpha_{22} D_H^g X(b) = \Gamma_2, \end{cases} \tag{3.6}$$

where $\Gamma_1 = [\Gamma_1^-, \Gamma_1^+], \Gamma_2 = [\Gamma_2^-, \Gamma_2^+] \in K_C(\mathbb{R})$*,* $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \in \mathbb{R}$ *and with the condition* $\Gamma_1, \Gamma_2 \neq [0, 0]$.

Definition 13 *An interval-valued function* $X\colon [a, b] \subset R^+ \to K_C(\mathbb{R})$ *is called a solution of the MBVP for ISHDE* (3.5)–(3.6)*, if*:

(i) $X(t)$ *and* $D_H^g X(t)$ *are* (H^g)*-differentiable functions such that* $D_H^{2;g} X(t)$ *can be represented in one of the forms described in Theorem* 1;

(ii) $X(t)$ *and* $D_H^g X(t)$ *satisfy the MBVP for ISHDE* (3.5)–(3.6).

Remark 1 *The type of the multipoint boundary value problems for interval-valued second-order differential equations under generalized Hukuhara differentiability* (*MBVPs for ISDEs*) *depends on the coefficients* α_{ij} $(i, j = 1, 2)$ *appearing in* (3.2)*. For example, when* $\alpha_{21} = \alpha_{22} = 0$ *we have the initial value problem for the interval-valued second-order differential equation under generalized Hukuhara differentiability* (*IVP for ISDE* (1.1) *– see* [7] *and* [8])*. When* $\alpha_{12} = \alpha_{22} = 0$ (*or when* $\alpha_{11} = \alpha_{21} = 0$) *we have the two point boundary value problem, and when one of the coefficients* α_{ij} *equals* 0 *we have the three point boundary value problem for the interval-valued second-order differential equation under generalized Hukuhara differentiability.*

3.2 Algorithm for solving the MBVPs

3.2.1 The Hukuhara derivatives method

Our strategy of solving (3.3)–(3.4) is based on the choice of the derivative in the interval-valued differential equation. In order to solve (3.3)–(3.4), we have two steps: first, we choose the type of the derivative and change the problem (3.3)–(3.4) to a system of ODEs by using Theorem 1 and considering initial values. In the second step, we solve the obtained system of ODEs.

By Theorem 1, we obtain that four systems of ODEs are possible for the problem (3.3)–(3.4):

Case 1: $X(t)$ and $D_H^g X(t)$ are (H^{g1})-differentiable:

$$\begin{cases} (X^-)''(t) + p(t)(X^-)'(t) + q(t)X^-(t) = R^-(t), \\ (X^+)''(t) + p(t)(X^+)'(t) + q(t)X^+(t) = R^+(t), \\ \alpha_{11}X^-(a) + \alpha_{12}(X^-)'(a) = \Gamma_1^-,\ \alpha_{11}X^+(a) + \alpha_{12}(X^+)'(a) = \Gamma_1^+, \\ \alpha_{21}X^-(b) + \alpha_{22}(X^-)'(b) = \Gamma_2^-,\ \alpha_{21}X^+(b) + \alpha_{22}(X^+)'(b) = \Gamma_2^+, \end{cases} \tag{3.7}$$

Case 2: $X(t)$ is (H^{g1})-differentiable and $D_H^g X(t)$ is (H^{g2})-differentiable:

$$\begin{cases} (X^+)''(t) + p(t)(X^-)'(t) + q(t)X^-(t) = R^-(t), \\ (X^-)''(t) + p(t)(X^+)'(t) + q(t)X^+(t) = R^+(t), \\ \alpha_{11}X^-(a) + \alpha_{12}(X^-)'(a) = \Gamma_1^-,\ \alpha_{11}X^+(a) + \alpha_{12}(X^+)'(a) = \Gamma_1^+, \\ \alpha_{21}X^-(b) + \alpha_{22}(X^-)'(b) = \Gamma_2^-,\ \alpha_{21}X^+(b) + \alpha_{22}(X^+)'(b) = \Gamma_2^+, \end{cases} \tag{3.8}$$

Case 3: $X(t)$ is (H^{g2})-differentiable and $D_H^g X(t)$ is (H^{g1})-differentiable:

$$\begin{cases} (X^+)''(t) + p(t)(X^+)'(t) + q(t)X^-(t) = R^-(t), \\ (X^-)''(t) + p(t)(X^-)'(t) + q(t)X^+(t) = R^+(t), \\ \alpha_{11}X^-(a) + \alpha_{12}(X^+)'(a) = \Gamma_1^-,\ \alpha_{11}X^+(a) + \alpha_{12}(X^-)'(a) = \Gamma_1^+, \\ \alpha_{21}X^-(b) + \alpha_{22}(X^+)'(b) = \Gamma_2^-,\ \alpha_{21}X^+(b) + \alpha_{22}(X^-)'(b) = \Gamma_2^+, \end{cases} \tag{3.9}$$

Case 4: $X(t)$ and $D_H^g X(t)$ are (H^{g2})-differentiable:

$$\begin{cases} (X^-)''(t) + p(t)(X^+)'(t) + q(t)X^-(t) = R^-(t), \\ (X^+)''(t) + p(t)(X^-)'(t) + q(t)X^+(t) = R^+(t), \\ \alpha_{11}X^-(a) + \alpha_{12}(X^+)'(a) = \Gamma_1^-,\ \alpha_{11}X^+(a) + \alpha_{12}(X^-)'(a) = \Gamma_1^+, \\ \alpha_{21}X^-(b) + \alpha_{22}(X^+)'(b) = \Gamma_2^-,\ \alpha_{21}X^+(b) + \alpha_{22}(X^-)'(b) = \Gamma_2^+. \end{cases} \tag{3.10}$$

In the followings, specifically, we give some solving methods for this multipoint boundary value problem for interval-valued second-order differential equations under generalized Hukuhara differentiability (MBVP for ISDEs).

Theorem 2 *Assume that* $F\colon [a,b] \times K_C(\mathbb{R}) \times K_C(\mathbb{R}) \to K_C(\mathbb{R})$ *is continuous. A mapping* $X\colon [a,b] \to K_C(\mathbb{R})$ *is a solution to the problem* (3.1)–(3.2) *if and only if it is* (H^g)*–differentiable on* $[a,b]$ *and* $D_H^g X(a), D_H^g X(b)$ *satisfy* (3.2) *and*

(i) $X(t) = X(a) + D_H^g X(a)(t-a) + \int_a^t \left(\int_a^\gamma F(s, X(s), D_H^g X(s))\,\mathrm{d}s\right)\mathrm{d}\gamma$*, where* $X(t)$ *and* $D_H^g X(t)$ *are* (H^{g1})*-differentiable functions, or*

(ii) $X(t) = X(b) \ominus (-1)D_H^g X(b)(b-t) \ominus (-1)\int_t^b \left(\int_\gamma^b F(s, X(s), D_H^g X(s))\,\mathrm{d}s\right)\mathrm{d}\gamma$*, where* $X(t)$ *is a* (H^{g1})*-differentiable function and* $D_H^g X(t)$ *is a* (H^{g2})*-differentiable function.*

Proof. Since F is continuous, it must be integrable. So (3.1) can be written in each case as follows.

(i) Let X and $D_H^g X(t)$ be (H^{g1})-differentiable functions. Then equation (3.1) can be rewritten equivalently as $D_H^g X(t) = D_H^g X(a) + \int_{t_0}^t F(s, X(s), D_H^g X(s))\,\mathrm{d}s$, and thus

$$X(t) = X(a) + D_H^g X(a)(t-a) + \int_a^t \left(\int_a^\gamma F(s, X(s), D_H^g X(s))\,\mathrm{d}s \right) \mathrm{d}\gamma.$$

(ii) Let X and $D_H^g X(t)$ be (H^{g2})-differentiable functions. Then equation (3.1) can be rewritten equivalently as

$$X(t) = X(b) \ominus (-1) D_H^g X(b)(b-t) \ominus (-1) \int_t^b \left(\int_\gamma^b F(s, X(s), D_H^g X(s))\,\mathrm{d}s \right) \mathrm{d}\gamma.$$

□

Theorem 3 *Let $F\colon [a,b] \times K_C(\mathbb{R}) \times K_C(\mathbb{R}) \to K_C(\mathbb{R})$ be continuous, and suppose that there exist $L_1, L_2 \in \mathbb{R}^+$ such that*

$$H(F(t, X_1, Y_1), F(t, X_2, Y_2)) \le L_1 H(X_1, X_2) + L_2 H(Y_1, Y_2)$$

for all $t \in [a,b]$, $X_1, X_2, Y_1, Y_2 \in K_C(\mathbb{R})$. Then the MBVP for ISDEs (3.1)–(3.2) *has a unique solution on $[a,b]$.*

Proof. Since the proof is similar for all four cases, we only consider the case of (H^{g2})-differentiable interval-valued mapping X. In this case, we consider the complete metric space $\left(C^1([a,b], K_C(\mathbb{R})), H_0^1\right)$, and define the operator

$$\begin{aligned} T\colon C^1([a,b], K_C(\mathbb{R})) &\longrightarrow C^1([a,b], K_C(\mathbb{R})), \\ X &\longmapsto TX, \end{aligned}$$

given by

$$(TX)(t) = X(b) \ominus (-1) D_H^g X(b)(b-t) \ominus (-1) \int_t^b \left(\int_\gamma^b F(s, X(s), D_H^g X(s))\,\mathrm{d}s \right) \mathrm{d}\gamma.$$

Note how we had to change the variable of integration (γ or s) to keep t, the independent variable, as the limit of the last integration. Beside that, we note that

$$D_H^g (TX)(t) = D_H^g X(b) \ominus (-1) \int_a^t F(s, X(s), D_H^g X(s))\,\mathrm{d}s, \quad t \in [a,b].$$

We can prove that T is a contractive mapping with respect to the metric H_1^0. And so, by the Banach fixed point theorem the operator T has a unique fixed point in the space $C^1\big([a,b], K_C(\mathbb{R})\big)$, which is a unique solution for the MBVP (3.1)–(3.2) in the case of (H^{g2})-differentiable functions. □

3.2.2 The real Green's function method

The aim of this subsection is to give the unfamiliar reader some insight toward Green's functions in the space $K_C(\mathbb{R})$, specifically in the applications to multipoint boundary value problems.

We will build the real Green's function method to solve the multipoint boundary value problem for interval-valued second-order differential equations under generalized Hukuhara differentiability (MBVP for ISDEs) (3.1)–(3.2) in the case when interval-valued functions $X(t)$ and $D_H^g X(t)$ are (H^{g1})-differentiable. What we want to say is that we will build the real Green's function for each certain multipoint boundary value problem, because it is very difficult to build an interval-valued Green's function.

First, we will consider the problem (3.3)–(3.4) in a special case when $R(t) = 0$. That means we will consider the multipoint boundary value problem for the interval-valued second-order homogeneous linear differential equation under generalized Hukuhara differentiability (MBVP for ISHDE):

$$D_H^{2,g} X(t) = (-1)\{p(t) D_H^g X(t) + q(t) X(t)\}, \tag{3.11}$$

where $X(t) = [X^-(t), X^+(t)] \in K_C(\mathbb{R})$ and $p(t), q(t) \in \mathbb{R}$, with the multipoint boundary conditions

$$\begin{cases} \alpha_{11} X(a) = (-1)\alpha_{12} D_H^g X(a), \\ \alpha_{21} X(b) = (-1)\alpha_{22} D_H^g X(b), \end{cases} \tag{3.12}$$

where $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \in \mathbb{R}$ with $\alpha_{11}^2 + \alpha_{12}^2 \neq 0,\ \alpha_{21}^2 + \alpha_{22}^2 \neq 0.$

We denote $X(t) = [X^-(t), X^+(t)]$. Hence, from (3.11)–(3.12) we obtain

$$[X''^-(t), X''^+(t)] = (-1)\{p(t)[X'^-(t), X'^+(t)] + q(t)[X^-(t), X^+(t)]\}, \tag{3.13}$$

$$\begin{cases} \alpha_{11}[X^-(a), X^+(a)] = (-1)\alpha_{12}[X'^-(a), X'^+(a)], \\ \alpha_{21}[X^-(b), X^+(b)] = (-1)\alpha_{22}[X'^-(b), X'^+(b)]. \end{cases} \tag{3.14}$$

Put $\bar{x} = \frac{X^-(t)+X^+(t)}{2}$. Thus, we can transform the interval-valued problem (3.13)–(3.14) into the real-valued problem of the following form:

$$\bar{x}''(t) + p(t)\bar{x}'(t) + q(t)\bar{x}(t) = 0, \tag{3.15}$$

$$\begin{cases} \alpha_{11}\bar{x}(a) + \alpha_{12}\bar{x}'(a) = 0, \\ \alpha_{21}\bar{x}(b) + \alpha_{22}\bar{x}'(b) = 0. \end{cases} \tag{3.16}$$

(Note that "0" in (3.15)–(3.16) is zero in the real numbers.)

Therefore, we will build the real Green's function of the real-valued problem (3.15)–(3.16).

Consider the multipoint boundary value problem for the interval-valued second-order nonhomogeneous linear differential equation under generalized Hukuhara differentiability (MBVP for ISIDE):

$$D_H^{2,g} X(t) = (-1)[p(t) D_H^g X(t) + q(t) X(t)] + R(t), \tag{3.17}$$

where $X(t) = [X^-(t), X^+(t)], R(t) = [R^-(t), R^+(t)] \in K_C(\mathbb{R})$ and $p(t), q(t) \in \mathbb{R}$, with the multipoint boundary conditions

$$\begin{cases} \alpha_{11} X(a) + \alpha_{12} D_H^g X(a) = \Gamma_1, \\ \alpha_{21} X(b) + \alpha_{22} D_H^g X(b) = \Gamma_2, \end{cases} \tag{3.18}$$

where $\Gamma_1 = [\Gamma_1^-, \Gamma_1^+], \Gamma_2 = [\Gamma_2^-, \Gamma_2^+] \in K_C(\mathbb{R})$, $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \in \mathbb{R}$ and with the condition $\Gamma_1, \Gamma_2 \neq [0, 0]$.

Theorem 4 *The general solution of the MBVP for ISIDE* (3.17)–(3.18) *is of the form*:

$$X(t) = \int_a^b G(t,s)R(s)\,\mathrm{d}s + Z(t). \tag{3.19}$$

where $R(s)$ *is the interval-valued function appearing in* (3.17) *and* $Z(t)$ *is the general solution of the ISDE* (3.11) *with the real Green's function* $G(t,s)$ *defined by*:

$$G(t,s) = \begin{cases} \dfrac{u_1(s)u_2(t)}{w(s)}, & \text{if } a \leqslant s \leqslant t \leqslant b, \\ \dfrac{u_1(t)u_2(s)}{w(s)}, & \text{if } a \leqslant t \leqslant s \leqslant b, \end{cases} \tag{3.20}$$

where u_1, u_2 *are two linearly independent real solutions of the homogeneous real differential equation of the form* $\bar{x}^{''}(t)+p(t)\bar{x}^{'}(t)+q(t)\bar{x}(t) = 0$ *with* $\alpha_{11}\bar{x}(a)+\alpha_{12}\bar{x}^{'}(a) = 0$ *and* $\alpha_{21}\bar{x}(b)+\alpha_{22}\bar{x}^{'}(b) = 0$, *where* $\bar{x} = \frac{X^-(t)+X^+(t)}{2}$ *and* $w(t)$ *is the Wronskian determinant of* u_1, u_2.

Proof. A solution of the equation (3.17) is of the form $X(t) = Y(t) + Z(t)$, where $Y(t)$ satisfies the following MBVP for ISIDE

$$D_H^{2;g}Y(t) = (-1)[p(t)D_H^gY(t) + q(t)Y(t)] + R(t), \tag{3.21}$$

with the multipoint boundary conditions

$$\begin{cases} \alpha_{11}Y(a) = (-1)\alpha_{12}D_H^gY(a), \\ \alpha_{21}Y(b) = (-1)\alpha_{22}D_H^gY(b), \end{cases} \tag{3.22}$$

and $Z(t)$ satisfies the following MBVP for ISHDE

$$D_H^{2;g}Z(t) = (-1)\{p(t)D_H^gZ(t) + q(t)Z(t)\}, \tag{3.23}$$

with the multipoint boundary conditions

$$\begin{cases} \alpha_{11}Z(a) + \alpha_{12}D_H^gZ(a) = \Gamma_1, \\ \alpha_{21}Z(b) + \alpha_{22}D_H^gZ(b) = \Gamma_2. \end{cases} \tag{3.24}$$

Now, we find the real Green's function for the equation (3.17)–(3.18).

Putting $\bar{y} = \frac{Y^-(t)+Y^+(t)}{2}$, $\bar{f} = \frac{R^-(t)+R^+(t)}{2}$, we can transform the interval-valued problem (3.17)–(3.18) into the real-valued problem similar to (3.15)–(3.16). The real Green's function corresponding to (3.15)–(3.16) must satisfy

$$G''(t,s) + p(t)G'(t,s) + q(t)G(t,s) = \delta(t-s), \tag{3.25}$$

with the multipoint boundary conditions

$$\begin{cases} \alpha_{11}G(a,s) + \alpha_{12}G^{'}(a,s) = 0, \\ \alpha_{21}G(b,s) + \alpha_{22}G^{'}(b,s) = 0. \end{cases} \tag{3.26}$$

The continuity and jump conditions are

$$G(s^-, s) = G(s^+, s),$$
$$G'(s^+, s) - G'(s^-, s) = 1.$$

Let u_1 and u_2 be two linearly independent solutions of the boundary problem for the real homogeneous equation of the form $\bar{y}''(t) + p(t)\bar{y}'(t) + q(t)\bar{y}(t) = 0$ with $\alpha_{11}\bar{y}(a) + \alpha_{12}\bar{y}'(a) = 0$ and $\alpha_{21}\bar{y}(b) + \alpha_{22}\bar{y}'(b) = 0$. The non-vanishing of the Wronskian ensures that these solutions exist. Let $w(t)$ denote the Wronskian of u_1 and u_2. Since the homogeneous equation with homogeneous boundary conditions has only the trivial solution, $w(t)$ is nonzero on $[a, b]$. The real Green's function has the form

$$G(t, s) = \begin{cases} c_1 u_1, & \text{if } a \leqslant s \leqslant t \leqslant b, \\ c_2 u_2, & \text{if } a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{3.27}$$

The continuity and jump conditions for the real Green's function give us the equations

$$c_1 u_1(s) - c_2 u_2(s) = 0,$$
$$c_1 u_1'(s) - c_2 u_2'(s) = -1.$$

By solving this system, we obtain

$$c_1 = \frac{u_2(s)}{w(s)}, \qquad c_2 = \frac{u_1(s)}{w(s)}.$$

Thus, the real Green's function is given by

$$G(t, s) = \begin{cases} \dfrac{u_1(s)u_2(t)}{w(s)}, & \text{if } a \leqslant s \leqslant t \leqslant b, \\ \dfrac{u_1(t)u_2(s)}{w(s)}, & \text{if } a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{3.28}$$

The special solution for the equation (3.17) is

$$Y(t) = \int_a^b G(t, s)R(s)\,\mathrm{d}s.$$

Note that $R(s)$ is the interval-valued function appearing in (3.17), and this special solution does not need to satisfy the boundary condition (3.18). However, it must be one of the solutions of the equation (3.17).

Thus, if there is a unique solution for (3.23)–(3.24), the general solution for (3.17)–(3.18) is

$$X(t) = \int_a^b G(t, s)R(s)\,\mathrm{d}s + Z(t). \tag{3.29}$$

□

In addition, if we want to find a special solution of (3.21)–(3.22), then we have to substitute the general solution (3.29) to the boundary conditions (3.18).

Theorem 5 *Let* $F\colon [a,b]\times K_C(\mathbb{R})\times K_C(\mathbb{R})\to K_C(\mathbb{R})$ *be continuous, and suppose that there exist* $L_1, L_2\in\mathbb{R}^+$ *such that*

$$H[F(t,X(t),D_H^gX(t)),F(t,Y(t),D_H^gY(t))] \\ \le L_1.H[X(t),Y(t)]+L_2.H[D_H^gX(t),D_H^gY(t)] \tag{3.30}$$

for all $t\in[a,b]$, $X(t),D_H^gX(t),Y(t),D_H^gY(t)\in K_C(\mathbb{R})$, *where the real numbers* L_1,L_2 *are such that*

$$\frac{L_1(b-a)^2}{8}+\frac{L_2(b-a)}{2}<1. \tag{3.31}$$

Then the MBVP for ISDEs (3.1)–(3.2) *has a unique solution on* $[a,b]$ *of the form*:

$$X(t)=\int_a^b G(t,s)F(s,X(s),D_H^gX(s))\,\mathrm{d}s+W(t),$$

where the real Green's function $G(t,s)$ *is defined as above and* $W(t)$ *is the general solution of the ISDE* (3.23).

Proof. Consider the operator

$$T\colon C^1([a,b],K_C(\mathbb{R}))\to C^1([a,b],K_C(\mathbb{R}))$$

defined by

$$T(X(t))=\int_a^b G(t,s)F(s,X(s),D_H^gX(s))\,\mathrm{d}s+W(t). \tag{3.32}$$

We have the following estimate

$$\begin{aligned}
&H\left[T(X(t)),T(Y(t))\right]\\
&\leqslant H\left[\int_a^t G(t,s)F(s,X(s),D_H^gX(s))\,\mathrm{d}s+W(t),\int_a^t G(t,s)F(s,Y(s),D_H^gY(s))\,\mathrm{d}s+W(t)\right]\\
&\leqslant \int_a^t |G(t,s)|\,.H[F(s,X(s),D_H^gX(s)),F(s,Y(s),D_H^gY(s))]\,\mathrm{d}s\\
&\leqslant \int_a^b |G(t,s)|\,.L_1.H[X(s),Y(s)]+L_2.H[D_H^gX(s),D_H^gY(s)]\,\mathrm{d}s\\
&\leqslant \int_a^b |G(t,s)|\;\mathrm{d}s.H^*[X,Y],
\end{aligned}$$

where $H^*[X,Y]=\max_{a\leqslant t\leqslant b}\left\{M.H[X(t),Y(t)]+N.H[D_H^gX(t),D_H^gY(t)]\right\}$. So

$$H\left[T(X(t)),T(Y(t))\right]\leqslant\frac{(b-a)^2}{8}H^*[X,Y].$$

Analogously, we have

$$H\left[(TX)'(t),(TY)'(t)\right]\leqslant\int_a^b |G_t(t,s)|\;\mathrm{d}s.H^*[X,Y],$$

$$H\left[(TX)'(t),(TY)'(t)\right]\leqslant\frac{(b-a)}{2}H^*[X,Y].$$

Then $H^*\left[T(X(t)), T(Y(t))\right] \leqslant \left[\frac{(b-a)^2}{8} + \frac{(b-a)}{2}\right] H^*[X,Y]$, and $X(t) \in C^1([a,b])$ is a fixed point of the contractive operator T and this $X(t)$ is the unique solution of the MBVP for ISDE (3.1)–(3.2) of the form

$$X(t) = \int_a^b G(t,s)F(s, X(s), D_H^g X(s))\,\mathrm{d}s + W(t). \qquad \square$$

Remark 2 *When* $\alpha_{12} = \alpha_{22} = 0$ (*or when* $\alpha_{11} = \alpha_{21} = 0$) *we have the two point boundary value problem. We have to use the real Green's function method.*

Particularly, we consider the most simple ISDE:

$$D_H^{2,g} X(t) = Q(t), \tag{3.33}$$

where $X(t) = [X^-(t), X^+(t)], H(t) = [H^-(t), H^+(t)] \in K_C(\mathbb{R})$, with the two point boundary conditions:

$$X(a) = I_1,\ X(b) = I_2, \tag{3.34}$$

where $I_1 = [I_1^-, I_1^+], I_2 = [I_2^-, I_2^+] \in K_C(\mathbb{R})$.

Theorem 6 *The general solution of the MBVP for ISDE* (3.33)–(3.34) *is of the form*:

$$X(t) = \int_a^b G(t,s)Q(s)\,\mathrm{d}s + Z(t), \tag{3.35}$$

where $Q(s)$ *is the interval-valued function appearing in* (3.33), $Z(t)$ *is the general solution of the interval-valued differential equation* (3.33) *in the case* $Q(t) = 0$ *and the real Green's function* $G(t,s)$ *is defined by*:

$$G(t,s) = \begin{cases} \dfrac{(t-b)(s-a)}{b-a}, & \text{if } a \leqslant s \leqslant t \leqslant b, \\ \dfrac{(s-b)(t-a)}{b-a}, & \text{if } a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{3.36}$$

Proof. Similar to the proof of Theorem 4 we will find the real Green's function for the equation of the form

$$D_H^{2,g} Y(t) = Q(t), \qquad Y(a) = Y(b) = [0,0]. \tag{3.37}$$

Put $\bar{y} = \frac{Y^-(t)+Y^+(t)}{2}$ and $\bar{h} = \frac{Q^-(t)+Q^+(t)}{2}$. We can transform the interval-valued problem (3.37) into the real-valued problem of the form

$$\bar{y}'' = \bar{h}(t), \qquad \bar{y}(a) = \bar{y}(b) = 0. \tag{3.38}$$

A pair of solutions to $\bar{y}'' = 0$ is $\bar{y}_1 = 1$ and $\bar{y}_2 = t$.

The real Green's function satisfies

$$G''(t,s) = \delta(t-s), \qquad G(a,s) = G(b,s) = 0. \tag{3.39}$$

The real Green's function has the form

$$G(t,s) = \begin{cases} c_1 + c_2 t, & \text{if } a \leqslant s \leqslant t \leqslant b, \\ d_1 + d_2 t, & \text{if } a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{3.40}$$

Applying the boundary conditions $G(a,s) = G(b,s) = 0$, we see that $c_1 = -c_2 a$ and $d_1 = -d_2 b$. The real Green's function now has the form

$$G(t,s) = \begin{cases} c_2(t-b), & \text{if } a \leqslant s \leqslant t \leqslant b, \\ d_2(t-a), & \text{if } a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{3.41}$$

Since the real Green's function must be continuous,

$$c_2(s-b) = d_2(s-a), \quad d_2 = c_2 \frac{(s-b)}{(s-a)}. \tag{3.42}$$

From the jump condition

$$G'(s^+, s) - G'(s^-, s) = 1, \tag{3.43}$$

we get $c_2 = \frac{(s-a)}{(b-a)}$. Thus, the real Green's function is

$$G(t,s) = \begin{cases} \dfrac{(t-b)(s-a)}{b-a}, & \text{if } a \leqslant s \leqslant t \leqslant b, \\ \dfrac{(s-b)(t-a)}{b-a}, & \text{if } a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{3.44}$$

The special solution for the equation (3.33) is

$$Y(t) = \int_a^b G(t,s)Q(s)\,\mathrm{d}s.$$

Note that $Q(s)$ is the interval-valued function appearing in (3.33), and this special solution does not need to satisfy the boundary condition (3.34).

Thus, if the interval-valued differential equation subject to the inhomogeneous boundary conditions (3.34) has the unique solution $Z(t)$, the general solution for (3.33)–(3.34) is

$$X(t) = \int_a^b G(t,s)Q(s)\,\mathrm{d}s + Z(t). \tag{3.45}$$

□

Remark 3 *In the case, when the interval-valued functions $X(t)$ and $D_H^g X(t)$ are (H^{g2})-differentiable we have solved the multipoint boundary value problem for the interval-valued second-order differential equation under generalized Hukuhara differentiability* (*MBVP for ISDE*) (3.1)–(3.2). *Analogously, we can treat the case, when the interval-valued functions $X(t)$ and $D_H^g X(t)$ are (H^{g1})-differentiable. We have some illustrations for the (H^{g2})-case, for example, in following Case* 2 *of* (b) *of* (4.1)–(4.2).

4 Illustrations

Example 1 *Using the Hukuhara derivatives and the real Green's function methods solve the following MBVP for ISDE*:

$$D_H^{2;g} X(t) = [t^2, e^t], \quad t \in [0,2] \tag{4.1}$$

with the boundary conditions:

$$X(0) = [0,1], \quad X(2) = [-2,4]. \tag{4.2}$$

(a) By the Hukuhara derivatives method solve the MBVP (4.1)–(4.2).

Case 1: From (3.7), we get

$$\begin{cases} (X^-)''(t) = t^2, \\ (X^+)''(t) = e^t, \\ X^-(0) = 0,\ X^+(0) = 1, \\ X^-(2) = -2,\ X^+(2) = 4. \end{cases} \tag{4.3}$$

By solving (4.3), we obtain that $X(t) = \left[\frac{t^4}{12} - \frac{5t}{3},\ e^t - t(\frac{e^2}{2} - 2)\right]$ and $D_H^g X(t)$ are (H^{g1})-differentiable. Moreover, $X(t)$ satisfies the boundary conditions (4.2). Hence, there is a solution in this case. This solution is shown in Figure 1.

$X(t) = \left[\frac{t^4}{12} - \frac{5t}{3},\ e^t - t(\frac{e^2}{2} - 2)\right]$								
t	Exact solution $X^-(t)$	$X^+(t)$	Absolute error degree 2		Absolute error degree 3		Absolute error degree 4	
0.0000	0.0000	1.0000	0.0864	0.1261	0.0096	0.0129	0.4383e-15	0.0009
0.2000	-0.3332	0.8825	0.0096	0.0158	0.0096	0.0120	0.0555e-15	0.0017
0.4000	-0.6645	0.8140	0.0608	0.0886	0.0096	0.0133	0.1110e-15	0.0004
0.6000	-0.9892	0.8054	0.0720	0.1031	0.0016	0.0034	0.0000e-15	0.0011
0.8000	-1.2992	0.8699	0.0512	0.0728	0.0064	0.0079	0.0000e-15	0.0012
1.0000	-1.5833	1.0238	0.0096	0.0136	0.0096	0.0136	0.0000e-15	0.0001
1.2000	-1.8272	1.2867	0.0384	0.0545	0.0064	0.0103	0.4441e-15	0.0012
1.4000	-2.0132	1.6829	0.0752	0.1075	0.0016	0.0010	0.4441e-15	0.0013
1.6000	-2.1205	2.2418	0.0800	0.1158	0.0096	0.0139	0.8882e-15	0.0002
1.8000	-2.1252	2.9995	0.0288	0.0435	0.0096	0.0157	0.0000e-15	0.0019
2.0000	-2.0000	4.0000	0.1056	0.1536	0.0096	0.0147	0.4441e-15	0.0009

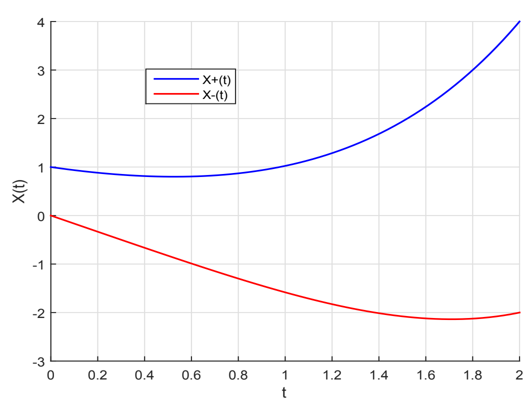

Figure 1: (H^{g1})-solution $X(t)$ (with (H^{g1})-derivative D_H^{g1}) of the MBVP for ISDEs (4.1)–(4.2)

Case 2: From (3.8), we get

$$\begin{cases} (X^+)''(t) = t^2, \\ (X^-)''(t) = e^t, \\ X^-(0) = 0,\ X^+(0) = 1, \\ X^-(2) = -2,\ X^+(2) = 4. \end{cases} \tag{4.4}$$

By solving (4.4), we get $X(t) = \left[e^t - t(\frac{e^2}{2} + \frac{1}{2}) - 1,\ \frac{t^4}{12} + \frac{5t}{6} + 1\right]$. Note that this solution $X(t)$ is (H^{g1})-differentiable and its derivative $D^g_H X(t)$ is (H^{g2})-differentiable. Adding that $X(t)$ satisfies the boundary conditions (4.2), we infer that such a solution is acceptable. This solution is shown in Figure 2.

$X(t) = \left[e^t - t(\frac{e^2}{2} + \frac{1}{2}) - 1,\ \frac{t^4}{12} + \frac{5t}{6} + 1\right]$								
t	Exact solution		Absolute error		Absolute error		Absolute error	
	$X^-(t)$	$X^+(t)$	degree 2		degree 3		degree 4	
0.0000	0.0000	1.0000	0.1261	0.0864	0.0129	0.0096	0.0009	0.0888e-14
0.2000	-0.6175	1.1668	0.0158	0.0096	0.0120	0.0096	0.0017	0.0222e-14
0.4000	-1.1860	1.3355	0.0886	0.0608	0.0133	0.0096	0.0004	0.0222e-14
0.6000	-1.6946	1.5108	0.1031	0.0720	0.0034	0.0016	0.0011	0.0000e-14
0.8000	-2.1301	1.7008	0.0728	0.0512	0.0079	0.0064	0.0012	0.0000e-14
1.0000	-2.4762	1.9167	0.0136	0.0096	0.0136	0.0096	0.0001	0.0000e-14
1.2000	-2.7133	2.1728	0.0545	0.0384	0.0103	0.0064	0.0012	0.0444e-14
1.4000	-2.8171	2.4868	0.1075	0.0752	0.0010	0.0016	0.0013	0.0888e-14
1.6000	-2.7582	2.8795	0.1158	0.0800	0.0139	0.0096	0.0002	0.0000e-14
1.8000	-2.5005	3.3748	0.0435	0.0288	0.0157	0.0096	0.0019	0.0444e-14
2.0000	-2.0000	4.0000	0.1536	0.1056	0.0147	0.0096	0.0009	0.1776e-14

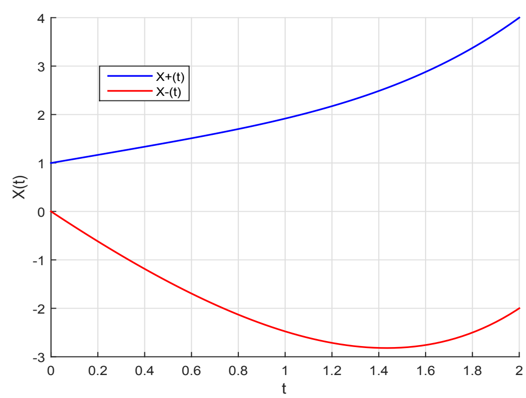

Figure 2: (H^{g1})-solution $X(t)$ (with $D^g_H X(t)$ being (H^{g2})-differentiable) of the MBVP for ISDEs (4.1)–(4.2)

Case 3, Case 4: From (3.9) and (3.10) we have no solutions of the MBVP for ISDEs (4.1)–(4.2).

(b) By the real Green's function method solve the MBVP for ISDE (4.1)–(4.2).

Case 1: By Theorem 6 the general solution of the MBVP for ISDE (4.1)–(4.2) is of the form:

$$X(t) = \int_0^2 G(t,s)[t^2, e^t]\,\mathrm{d}s + W(t), \tag{4.5}$$

where the real Green's function $G(t,s)$ is defined by (3.36) and $W(t)$ is the general solution of the homogeneous interval-valued differential equation.

Thus, the Green's function $G(t,s)$ is defined by:

$$G(t,s) = \begin{cases} \dfrac{(t-2)s}{2}, & \text{if } 0 \leqslant s \leqslant t \leqslant 2, \\ \dfrac{(s-2)t}{2}, & \text{if } 0 \leqslant t \leqslant s \leqslant 2, \end{cases} \tag{4.6}$$

and we get

$$\begin{aligned} X(t) &= \int_0^t G(t,s)[t^2, e^t]\,\mathrm{d}s + \int_t^2 G(t,s)[t^2, e^t]\,\mathrm{d}s + W(t) \\ &= \left[\frac{t^4}{12} - \frac{2t}{3},\ e^t - \frac{te^2}{2} + \frac{t}{2} - 1\right] + \left[C_1 t + C_2,\ C_3 t + C_4\right] \\ &= \left[\frac{t^4}{12} + C_1^* t + C_2,\ e^t + C_3^* t + C_4\right], \end{aligned}$$

where C_1^*, C_2, C_3^*, C_4 are constants. Applying the boundary condition (4.2), we find that the solution is

$$X(t) = \left[\frac{t^4}{12} - \frac{5t}{3},\ e^t - t\Big(\frac{e^2}{2} - 2\Big)\right]$$

(for the numerical simulation and illustrations see Figure 1).

Case 2: This case is similar to Case 1. However, in this case $D_H^g X(t)$ is (H^{g2})-differentiable. Thus, by Theorem 6 the general solution of the MBVP for ISDE (4.1)–(4.2) is of the form:

$$X(t) = \int_0^2 G(t,s)[e^t, t^2]\,\mathrm{d}s + W(t), \tag{4.7}$$

where the real Green's function $G(t,s)$ is defined by (3.36) and $W(t)$ is the general solution of the homogeneous interval-valued differential equation. Note that $\int_0^1 G(t,s)[e^t, t^2]\,\mathrm{d}s$ is a solution of equation (4.1).

Thus, the real Green's function $G(t,s)$ is defined by:

$$G(t,s) = \begin{cases} \dfrac{(t-2)s}{2}, & \text{if } 0 \leqslant s \leqslant t \leqslant 2, \\ \dfrac{(s-2)t}{2}, & \text{if } 0 \leqslant t \leqslant s \leqslant 2, \end{cases} \tag{4.8}$$

and we get

$$\begin{aligned} X(t) &= \int_0^t G(t,s)[e^t, t^2]\,\mathrm{d}s + \int_t^2 G(t,s)[e^t, t^2]\,\mathrm{d}s + W(t) \\ &= \left[e^t - \frac{te^2}{2} + \frac{t}{2} - 1,\ \frac{t^4}{12} - \frac{2t}{3}\right] + \left[C_1 t + C_2,\ C_3 t + C_4\right] \\ &= \left[e^t + C_1^* t + C_2,\ \frac{t^4}{12} + C_3^* t + C_4\right], \end{aligned}$$

where C_1^*, C_2, C_3^*, C_4 are constants. Applying the boundary conditions (4.2), we find that the solution is

$$X(t) = \left[e^t - t\Big(\frac{e^2}{2} + \frac{1}{2}\Big) - 1,\ \frac{t^4}{12} + \frac{5t}{6} + 1\right]$$

(for the numerical simulation and illustrations see Figure 2).

We know that in **Case 3** and **Case 4** from (3.9) and (3.10) we have no solutions of the MBVP for ISDEs (4.1)–(4.2).

Example 2 *By the Hukuhara derivatives and by the real Green's function methods solve the following MBVP for ISDE*:

$$D_H^{2,g} X(t) + \frac{1}{t} D_H^g - \frac{1}{t^2} X(t) = \sin(t)[1,2], \quad t \in \left[\frac{\pi}{2}, \pi\right] \tag{4.9}$$

with the boundary conditions:

$$\begin{cases} X\left(\frac{\pi}{2}\right) - \frac{1}{2} D_H^g X\left(\frac{\pi}{2}\right) = [0,1], \\ \quad X(\pi) - D_H^g X(\pi) = [-1,1]. \end{cases} \tag{4.10}$$

(a) We will find a solution of the MBVP for ISDE (4.9)–(4.10) by the Hukuhara derivatives method.

Case 1: From (3.7), we get

$$\begin{cases} (X^-)''(t) + \frac{1}{t}(X^-)'(t) - \frac{1}{t^2} X^-(t) = \sin(t), \\ (X^+)''(t) + \frac{1}{t}(X^+)'(t) - \frac{1}{t^2} X^+(t) = 2\sin(t), \\ X^-(\frac{\pi}{2}) - \frac{1}{2}(X^-)'(\frac{\pi}{2}) = 0, \ X^+(\frac{\pi}{2}) - \frac{1}{2}(X^+)'(\frac{\pi}{2}) = 1, \\ X^-(\pi) - (X^-)'(\pi) = -1, \ X^+(\pi) - (X^+)'(\pi) = 1. \end{cases} \tag{4.11}$$

By solving (4.11), we obtain that

$$X(t) = \left[\frac{2\pi}{3t} + \frac{1}{3t} - \frac{\cos(t)}{t} - \sin(t) - \frac{2t(3\pi + \pi^2 + 2)}{3\pi^2(\pi - 1)}, \right. \\ \left. \frac{6\pi + 3\pi^2 + 2}{3t(\pi + 1)} - \frac{2\cos(t)}{t} - 2\sin(t) + \frac{2t(6\pi - 3\pi^2 + 4)}{3\pi^2(\pi - 1)}\right]$$

and $D_H^g X(t)$ are (H^{g1})-differentiable. Moreover, $X(t)$ satisfies the boundary conditions (4.10). Hence, there is a solution in this case. The numerical simulation for this solution is shown in Figure 3.

$X(t) = \left[\frac{2\pi}{3t} + \frac{1}{3t} - \frac{\cos(t)}{t} - \sin(t) - \frac{2t(3\pi+\pi^2+2)}{3\pi^2(\pi-1)}, \frac{6\pi+3\pi^2+2}{3t(\pi+1)} - \frac{2\cos(t)}{t} - 2\sin(t) + \frac{2t(6\pi-3\pi^2+4)}{3\pi^2(\pi-1)}\right]$								
t	Exact solution		Absolute error degree 4		Absolute error degree 5		Absolute error degree 6	
	$X^-(t)$	$X^+(t)$						
1.5708	-0.5095	0.9203	0.0714e-3	0.0711e-3	0.1371e-4	0.2090e-4	0.0888e-5	0.1311e-5
1.7279	-0.6526	0.9244	0.1605e-3	0.1693e-3	0.4518e-4	0.6883e-4	0.4159e-5	0.6136e-5
1.8850	-0.7652	0.9821	0.0118e-3	0.0303e-3	0.3101e-4	0.4706e-4	0.6231e-5	0.9182e-5
2.0420	-0.8513	1.0867	0.1067e-3	0.1126e-3	0.2978e-4	0.4559e-4	0.0984e-5	0.1427e-5
2.1991	-0.9148	1.2321	0.0619e-3	0.0442e-3	0.1503e-4	0.2274e-4	0.4780e-5	0.7063e-5
2.3562	-0.9592	1.4119	0.0336e-3	0.0514e-3	0.3360e-4	0.5141e-4	0.0581e-5	0.0832e-5
2.5133	-0.9879	1.6199	0.0826e-3	0.0759e-3	0.0570e-4	0.0893e-4	0.4556e-5	0.6742e-5
2.6704	-1.0047	1.8495	0.0463e-3	0.0202e-3	0.3060e-4	0.4680e-4	0.0163e-5	0.0219e-5
2.8274	-1.0130	2.0938	0.0390e-3	0.0472e-3	0.1978e-4	0.3046e-4	0.5004e-5	0.7422e-5
2.9845	-1.0166	2.3460	0.0780e-3	0.0431e-3	0.3738e-4	0.5731e-4	0.3638e-5	0.5389e-5
3.1416	-1.0189	2.5991	0.0457e-3	0.0318e-3	0.1201e-4	0.1840e-4	0.0804e-5	0.1191e-5

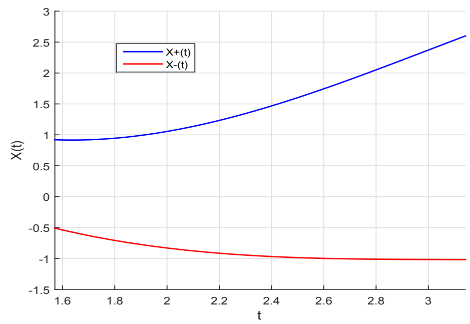

Figure 3: (H^{g1})-solution $X(t)$ (with (H^{g1})-derivative D_H^{g1}) of the MBVP for ISDEs (4.9)–(4.10)

In **Case 2**, **Case 3** and **Case 4** from (3.8), (3.9) and (3.10) we have no solutions of the MBVP for ISDE (4.9)–(4.10).

(b) By the real Green's function method solve the MBVP for ISDE (4.9)–(4.10).

Case 1: By Theorem 4 the general solution of the MBVP for ISDE (4.9)–(4.10) is of the form:

$$X(t) = \int_0^{2\pi} G(t,s)\sin(s)[1,2]\,\mathrm{d}s + Z(t), \tag{4.12}$$

where the real Green's function $G(t,s)$ is defined by (3.20), with $u_1(t) = t$ and $u_2(t) = t^{-1}$ being two linearly independent solutions of homogeneous real differential equation of the form $\bar{x}''(t) + \frac{1}{t}\bar{x}'(t) - \frac{1}{t^2}\bar{x}(t) = 0$ with the homogeneous real boundary conditions, and $Z(t)$ is the general solution of the homogeneous interval-valued differential equation.

Thus, the real Green's function $G(t,s)$ is defined by:

$$G(t,s) = \begin{cases} -\dfrac{s^2 t^{-1}}{2}, & \text{if } \frac{\pi}{2} \leqslant s \leqslant t \leqslant \pi, \\ \dfrac{-t}{2}, & \text{if } \frac{\pi}{2} \leqslant t \leqslant s \leqslant \pi, \end{cases} \tag{4.13}$$

and we have

$$\begin{aligned} X(t) &= \int_0^t G(t,s)\sin(s)[1,2]\,\mathrm{d}s + \int_t^{2\pi} G(t,s)\sin(s)[1,2]\,\mathrm{d}s + Z(t) \\ &= \left[\frac{\pi}{2t} - \frac{\cos(t)}{t} - \sin(t) - \frac{t}{2},\ \frac{\pi}{t} - \frac{2\cos(t)}{t} - 2\sin(t) - t\right] \\ &\quad + \left[C_1 t^{-1} - C_2\frac{t}{2},\ C_3 t^{-1} - C_4\frac{t}{2}\right] \\ &= \left[-\frac{\cos(t)}{t} - \sin(t) + C_1^* t^{-1} + C_2^* t,\ -\frac{2\cos(t)}{t} - 2\sin(t) + C_3^* t^{-1} + C_4^* t\right], \end{aligned} \tag{4.14}$$

where C_1^*, C_2^*, C_3^*, C_4^* are constants. Applying the boundary conditions (4.10), we find that the solution is

$$\begin{aligned} X(t) = \Bigg[&\frac{2\pi}{3t} + \frac{1}{3t} - \frac{\cos(t)}{t} - \sin(t) - \frac{2t(3\pi + \pi^2 + 2)}{3\pi^2(\pi - 1)}, \\ &\frac{6\pi + 3\pi^2 + 2}{3t(\pi + 1)} - \frac{2\cos(t)}{t} - 2\sin(t) + \frac{2t(6\pi - 3\pi^2 + 4)}{3\pi^2(\pi - 1)}\Bigg]; \end{aligned}$$

the illustration of X is shown in Figure 3.

We know that in **Case 2**, **Case 3** and **Case 4** from (3.8), (3.9) and (3.10) we have no solutions of the MBVP for ISDE (4.9)–(4.10).

5 Conclusion

There are some kinds of MBVP for ISDEs. We remark that the solutions of MBVP for ISDEs shrunk much more than the solutions of IVP for ISDEs. The reason for that is that for MBVP for ISDEs there are more binding conditions. We have described an algorithm for solving such types of MBVPs in $K_C(\mathbb{R})$ using, for example, the Hukuhara differentiability method and the real Green's function method. Moreover, we gave some simple examples of solutions of MBVP for ISDEs illustrating our algorithm.

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