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APPLICATIONS OF HOMOTOPY PERTURBATION METHOD AND ELZAKI TRANSFORM FOR SOLVING NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

A. NEAMATY*
Department of Mathematics, University of Mazandaran, Babolsar, Iran

B. AGHELI†
Department of Mathematics, Qaemshahr Branch,
Islamic Azad University, Qaemshahr, Iran

R. DARZI‡
Department of Mathematics, Neka Branch, Islamic Azad University, Neka, Iran

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Abstract. In this paper a reliable combination of the homotopy perturbation method and the Elzaki transform has been presented for the purpose of investigating some nonlinear partial differential equations of fractional order.

We have shown that using the homotopy perturbation method, we can handle the nonlinear terms. Further, we have applied this suggested homotopy perturbation method in order to reformulate the initial value problem. The application of this method results in the solution regarding transformed variables. Besides, the series solution is obtained through using the inverse transformation. The obtained results show the efficiency of the proposed method.

Keywords: Elzaki transform, He's polynomials, homotopy perturbation method, time-fractional partial differential equation.

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*e-mail address: namaty@umz.ac.ir
†e-mail address: b.agheli@qaemshahriau.ac.ir
‡e-mail address: r.darzi@iauneka.ac.ir

1 Introduction

The homotopy perturbation method (HPM) was first proposed by He (1998) to deal with linear and nonlinear problems [11]. Various studies have investigated this suggested method since it was proposed. These studies have applied it to solve various linear and nonlinear initial value problems [1, 2, 11]. A major advantage of HPM is that it addresses a problem directly without the need for any form of transformation, linearization, discrimination, or any other unrealistic assumption.

This is in contrast with the classical HPM and other series solution methods which form a recurrent scheme of the solution using only one type of the problem conditions: either the initial conditions or the boundary conditions.

Since fractional differential equations can be extensively used in fluid mechanics, mathematical biology, electro-chemistry, physics, and in other similar fields and because of their various applications, they have been given a lot of attention by researchers in recent years. For instance, fractional derivatives can be used to model the nonlinear oscillation of earthquake. Also, through applying fractional derivatives in fluid-dynamic traffic model, one can eliminate the problems and deficiency resulting from the assumption of continuum traffic flow [17]. As a result of the works done by researchers in the related area in recent years, a number of fractional differential equations have been investigated and consequently solutions for these equations have been proposed; among the equations that have been investigated are: impulsive fractional differential equations [15], fractional advection-dispersion equations [12], certain types of time-fractional diffusion equations [19], fractional generalized Burgers' fluid [20], fractional KdV-type equations [8], space-time fractional Whitham-Broer-Kaupand equations [9], fractional heat- and wave-like equations [14], and space fractional backward Kolmogorov equations [13].

In a study, Tarig M. Elzaki and Sailh M. Elzaki [4–6], indicated that the modified Sumudu transform [3, 10, 21] or the Elzaki transform can be successfully applied to partial differential equations, ordinary differential equations, system of ordinary and partial differential equations and integral equations. The Elzaki transform is a very efficient tool that can be applied to solve some differential equations that we cannot solve using the Sumudu transform [7].

The present paper is organized as follows. In Section 2, we have presented some fundamental definitions of fractional calculus, the Elzaki transform of fractional derivative and the classical HPM. Then, in Section 3, we have introduced and elaborated on the homotopy perturbation Elzaki transform method (HPETM). And finally, in Section 4, we have offered some examples to solve in order to show the validity and efficiency of this approach.

2 Preliminaries

In this part of the paper, we present and define fractional equations, the Elzaki transform and we obtain the Elzaki transform of the Caputo fractional derivative. Furthermore, the fractional homotopy perturbation method is introduced and explained in detail.

2.1 Fractional calculus

In this subsection of the paper, we present and define the Riemann–Liouville fractional integral and the Caputo fractional derivative (see also [18]).

Definition 1 *A real function $f(x)$, $x > 0$, is considered to be in the space C_ν, ($\nu \in \mathbb{R}$), if there exists a real number $n(> \nu)$ so that $f(x) = x^n f_1(x)$, where $f_1(x) \in C[0, +\infty)$, and it is said to be in the space C_ν^k if and only if $f^{(k)} \in C_\nu$, $k \in \mathbb{N}$.*

Definition 2 *The Riemann–Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in C_\nu$, $\nu \geq -1$, is given by*

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-r)^{\alpha-1} f(r)\, \mathrm{d}r,$$
$$I^\alpha f(x) = I_0^\alpha f(x), \qquad I^0 f(x) = f(x).$$

Definition 3 *The Caputo fractional derivative of f is defined as*

$$D^\alpha f(x) = I^{k-\alpha} D^k f(x) = \frac{1}{\Gamma(k-\alpha)} \int_0^x (x-r)^{k-\alpha-1} f^{(k)}(r)\, \mathrm{d}r, \quad x > 0,$$

where $f \in C_{-1}^k$, $k-1 < \alpha \leq k$ and $k \in \mathbb{N}$.

Property 1 *For $k-1 < \alpha \leq k$, $k \in \mathbb{N}$, $f \in C_\nu^k$, $\nu \geq -1$ and $x > 0$, the following properties are satisfied*:

(i) $D_a^\alpha I_a^\alpha f(x) = f(x)$;

(ii) $I_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{j=0}^{k-1} f^{(j)}(a^+) \frac{(x-a)^j}{j!}$.

2.2 Elzaki transform

The basic definition of the modified Sumudu transform or the Elzaki transform is given as follows. The Elzaki transform of the function $f(t)$ is:

$$T(v) = E\{f(t), v\} = v \int_0^\infty f(t) e^{-\frac{t}{v}}\, \mathrm{d}t, \quad t > 0. \tag{2.1}$$

Tarig M. Elzaki and Sailh M. Elzaki in [4, 5] presented a modified version of the Sumudu transform or the Elzaki transform. They used this transform to solve partial differential equations, ordinary differential equations, systems of ordinary and partial differential equations and integral equations. The Elzaki transform is a very efficient and powerful tool that can be used to solve some differential equations which cannot be solved by the Sumudu transform (see [7]).

In order to obtain the Elzaki transform of partial derivatives, we have used integration of parts, and then the result is (cf. [7]):

1) $E\{\frac{\partial f(x,t)}{\partial t}\} = \frac{1}{v}T(x,v) - vf(x,0)$;

2) $E\{\frac{\partial^2 f(x,t)}{\partial t^2}\} = \frac{1}{v^2}T(x,v) - f(x,0) - v\frac{\partial f(x,0)}{\partial t}$;

3) $E\{\frac{\partial f(x,t)}{\partial x}\} = \frac{\mathrm{d}}{\mathrm{d}x}T(x,v)$;

4) $E\{\frac{\partial^2 f(x,t)}{\partial x^2}\} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}T(x,v)$.

2.3 Elzaki transform of the Caputo fractional derivative

In order to obtain the Elzaki transform of the Caputo fractional derivative, we apply the Laplace transform formula for the Caputo fractional derivative (see [18]):

$$L\{D^\alpha f(x), s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \qquad n-1 < \alpha \leq n. \tag{2.2}$$

Theorem 1 *Let $T(v)$ be the Elzaki transform of $f(t)$, that is, $T(v) = E\{f(t), v\}$ and let*

$$g(t) = \begin{cases} f(t-\tau), & t \geq \tau, \\ 0, & t < \tau. \end{cases}$$

Then $E\{g(t)\} = e^{\frac{-\tau}{v}} T(v)$.

Proof. See [6]. □

The Elzaki transform can certainly be used to deal with all problems that are usually treated by the well-known and widely applied Laplace transform.

In fact, as we will see in the next theorem, the Elzaki transform is closely related to the Laplace transform $F(s)$.

Theorem 2 ([6]) *Let*

$$A = \left\{ f(t) |\ \textit{there exist}\ M, k_1, k_2 > 0\ \textit{such that}\ |f(t)| < Me^{|t|/k_i},\ \textit{if}\ t \in (-1)^j \times [0, \infty) \right\}$$

and let $f(t) \in A$. If $F(s)$ is the Laplace transform of $f(t)$, then the Elzaki transform $T(v)$ of $f(t)$ is given by

$$T(v) = vF\left(\frac{1}{v}\right). \tag{2.3}$$

Theorem 3 *Suppose $T(v)$ is the Elzaki transform of the function $f(t)$. Then*

$$E\{D^\alpha f(t), v\} = \frac{T(v)}{v^\alpha} - \sum_{k=0}^{n-1} v^{k-\alpha+2} f^{(k)}(0).$$

Proof. From Theorem 2 we have

$$E\{D^{\alpha}f(t),v\} = vL\left\{D^{\alpha}f(t),\frac{1}{v}\right\}.$$

Now, applying equation (2.2), we get

$$E\{D^{\alpha}f(t),v\} = v\left(\left(\frac{1}{v}\right)^{\alpha}F\left(\frac{1}{v}\right)-\sum_{k=0}^{n-1}\left(\frac{1}{v}\right)^{\alpha-k-1}f^{(k)}(0)\right) = \frac{vF(\frac{1}{v})}{v^{\alpha}}-\sum_{k=0}^{n-1}v^{k-\alpha+2}f^{(k)}(0).$$

□

2.4 Homotopy perturbation method

In order to understand the basic idea of He's homotopy perturbation method, let us consider the following general nonlinear differential equation

$$A(u)-f(r)=0, \quad r\in\Omega, \tag{2.4}$$

with the boundary conditions

$$B\left(u,\partial u/\partial n\right), \quad r\in\Gamma, \tag{2.5}$$

in which A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω.

Now, we can divide the operator A into two parts: L and N, where L is linear, and N is nonlinear. Thus, equation (2.4) can be rewritten as:

$$L(u)+N(u)-f(r)=0. \tag{2.6}$$

Then, by using a homotopy technique, we can form a homotopy $v(r,p)\colon \Omega\times[0,1]\to\mathbb{R}$ which satisfies:

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[A(v)-f(r)]=0, \quad p\in[0,1], \tag{2.7}$$

or

$$H(v,p)=L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)]=0, \tag{2.8}$$

in which $p\in[0,1]$ is considered as an embedding parameter, and u_0 is the initial approximation of equation (2.4). This will meet the boundary conditions. Then, the result will be:

$$H(v,0)=L(v)-L(u_0)=0, \tag{2.9}$$
$$H(v,1)=A(v)-f(r)=0. \tag{2.10}$$

The process of deformation involves the changing process of p from zero to unity, which includes the process of $v(r,p)$ changing from $u_0(r)$ to $u(r)$. Also, $L(v)-L(u_0)$ and $A(v)-f(r)$ are called homotopic in topology. If we suppose the embedding parameter p $(0\le p\le 1)$ is small, using the classical perturbation technique, we can thus suppose that the solution of equations (2.9) and (2.10) can be given as a power series in p, *i.e.*,

$$v=v_0+pv_1+p^2v_2+p^3v_3+\ldots, \tag{2.11}$$

and assuming $p = 1$ leads to the approximate solution of equation (2.7) as:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \dots. \tag{2.12}$$

It is interesting to mention that the most important advantage of He's homotopy perturbation method is that one can construct the perturbation equation in various ways, through homotopy in topology, that is this method is problem dependent. Moreover, the initial approximation can be freely selected.

3 Homotopy perturbation Elzaki transform method (HPETM)

In this section, we introduce the fractional homotopy perturbation Elzaki transform method. We consider the following general nonlinear problem, say in two independent variables x and t:

$$D_t^\alpha u(x,t) = \Re\, u(x,t) + f(x,t), \tag{3.1}$$

where D_t^α is the fractional Caputo derivative with respect to t, $\alpha > 0$, $\Re$ is an operator in x and t which might include derivatives with respect to 'x', $u(x,t)$ is an unknown function, and $f(x,t)$ is the source in homogeneous term.

Taking Elzaki transforms on both sides of equation (3.1), we get

$$E\{D_t^\alpha u(x,t)\} = E\{\Re\, u(x,t)\} + E\{f(x,t)\}. \tag{3.2}$$

Using the differentiation property of the Elzaki transform, we have

$$E\{u(x,t)\} = \sum_{k=0}^{n-1} v^{k+2} u^{(k)}(x,0) + v^\alpha (E\{\Re\, u(x,t)\} + E\{f(x,t)\}). \tag{3.3}$$

Applying the inverse Elzaki transform to both sides of equation (3.3), we find that

$$u(x,t) = G(x,t) - E^{-1}\big(v^\alpha \big(E\{\Re\, u(x,t)\} + E\{f(x,t)\}\big)\big), \tag{3.4}$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions.

Now, we apply the homotopy perturbation method:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t). \tag{3.5}$$

And the nonlinear term can be decomposed as

$$\Re\, u(x,t) = \sum_{n=0}^{\infty} p^n H_n(u), \tag{3.6}$$

where $H_n(u)$ are given by

$$H_n(u_0, u_1, \cdots, u_n) = \frac{1}{n!} \frac{\partial}{\partial p^n} \left[\Re\left(\sum_{i=0}^{\infty} p^i u_i \right) \right]. \tag{3.7}$$

Substituting equations (3.5) and (3.6) into equation (3.4), we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) + p\Bigg\{E^{-1}\Bigg(v^{\alpha}\Bigg(E\Bigg\{\Re\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\Bigg\}\Bigg)\Bigg)\Bigg\}. \quad (3.8)$$

This is a mixture of the Elzaki transform and the homotopy perturbation method for solving nonlinear partial differential equations of fractional order.

Comparing the coefficient of like powers of p, the following approximations are obtained:

$$\begin{aligned}
p^0 &: \quad u_0(x,t) = G(x,t),\\
p^1 &: \quad u_1(x,t) = E^{-1}\big(v^{\alpha}\big(E\{\Re u_0(x,t) + H_0(u)\}\big)\big),\\
p^2 &: \quad u_2(x,t) = E^{-1}\big(v^{\alpha}\big(E\{\Re u_1(x,t) + H_1(u)\}\big)\big),\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .
\end{aligned}$$

Then the solution is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots .$$

4 Applications

In this section, we apply the Elzaki transform and the homotopy perturbation method to solve a nonlinear time-fractional advection partial differential equation, a time-fractional hyperbolic equation and a time-fractional Fisher's equation. All of the plots and computing for these equations have been done on a PC applying some programs written in *Maple*.

Example 1 Consider the nonlinear time-fractional advection partial differential equation

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}} u(x,t) + u(x,t)\, u_x(x,t) = x(1+t^2), \quad t > 0,\ x \in \mathbb{R},\ 0 < \alpha \leq 1, \quad (4.1)$$

with the initial condition

$$u(x,0) = 0. \quad (4.2)$$

Taking the Elzaki transform of equation (4.1) subjected to the initial condition, we have

$$E\{u(x,t)\} = xv^{\alpha+2} + 2xv^{\alpha+4} - v^{\alpha}E\{u(x,t)u_x(x,t)\}. \quad (4.3)$$

The inverse Elzaki transform implies that

$$u(x,t) = \frac{xt^{\alpha}}{\Gamma(\alpha+1)} + \frac{2xt^{\alpha+2}}{\Gamma(\alpha+3)} - E^{-1}\left\{v^{\alpha}E\{u(x,t)u_x(x,t)\}\right\}. \quad (4.4)$$

Now, applying the homotopy perturbation method, we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \frac{xt^{\alpha}}{\Gamma(\alpha+1)} + \frac{2xt^{\alpha+2}}{\Gamma(\alpha+3)} + p\Bigg\{E^{-1}\Bigg(v^{\alpha}\Bigg(\sum_{n=0}^{\infty} p^n H_n(u)\Bigg)\Bigg)\Bigg\}, \quad (4.5)$$

where $H_n(u)$ are He's polynomials that represent the nonlinear terms. The first few components of He's polynomials are given by:

$$
\begin{aligned}
&H_0(u) = -u_0 u_{0x},\\
&H_1(u) = -(u_0 u_{1x} + u_1 u_{0x}),\\
&H_2(u) = -(u_2 u_{0x} + u_{2x} u_0 + u_1 u_{1x}),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .
\end{aligned}
$$

Comparing the coefficients of the same powers of p , we get:

$$
\begin{aligned}
p^0: \quad & u_0(x,t) = x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right),\\
p^1: \quad & u_1(x,t) = -x\left(\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} + \frac{4\Gamma(2\alpha+3)t^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)}\right.\\
& \qquad\qquad \left. + \frac{4\Gamma(2\alpha+5)t^{3\alpha+4}}{\Gamma(\alpha+3)^3\Gamma(3\alpha+5)\Gamma(5\alpha+7)}\right),\\
p^2: \quad & u_2(x,t) = 2x\left(\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)t^{5\alpha}}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} + \frac{8\Gamma(2\alpha+5)\Gamma(4\alpha+7)t^{5\alpha+6}}{\Gamma(\alpha+3)^3\Gamma(3\alpha+5)\Gamma(5\alpha+7)}\right)\\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .
\end{aligned}
$$

Then, the third-order term approximate solution for equation (4.1) is given by

$$
\begin{aligned}
u(x,t) = {} & x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right) - x\left(\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}\right.\\
& \left. + \frac{4\Gamma(2\alpha+3)t^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} + \frac{4\Gamma(2\alpha+5)t^{3\alpha+4}}{\Gamma(\alpha+3)^3\Gamma(3\alpha+5)\Gamma(5\alpha+7)}\right)\\
& + 2x\left(\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)t^{5\alpha}}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)}\frac{8\Gamma(2\alpha+5)\Gamma(4\alpha+7)t^{5\alpha+6}}{\Gamma(\alpha+3)^3\Gamma(3\alpha+5)\Gamma(5\alpha+7)}\right).
\end{aligned}
$$

The solution that we have found is equivalent to the exact solution in a closed form $u(x,t) = xt$, which is the same third-order term approximate solution for equations (4.1)–(4.2) obtained from [16] using VIM. We can also solve the nonlinear time-fractional advection partial differential equation (4.1) in [16] through applying ADM.

In Table 1, we can see the approximate solutions for $\alpha = 1$, which is derived for different values of x and t using HPETM, HPM and VIM.

t	x	u_{VIM}	u_{HPM}	u_{HPETM}	u_{Exact}
0.2	0.25	0.050309	0.0499876	0.050000	0.050000
	0.50	0.100619	0.099978	0.100000	0.100000
	0.75	0.150928	0.149968	0.150001	0.150000
	1.0	0.201237	0.199957	0.200001	0.200000
0.4	0.25	0.101894	0.099645	0.100023	0.100000
	0.50	0.203787	0.199290	0.200046	0.200000
	0.75	0.305681	0.298935	0.300069	0.300000
	1.0	0.407575	0.398580	0.400092	0.400000
0.6	0.25	0.153094	0.147158	0.150411	0.150000
	0.50	0.306188	0.294317	0.300823	0.300000
	0.75	0.459282	0.441475	0.451234	0.450000
	1.0	0.612376	0.588634	0.601646	0.600000

Table 1: Numerical values when $\alpha = 1$ for equation (4.1)

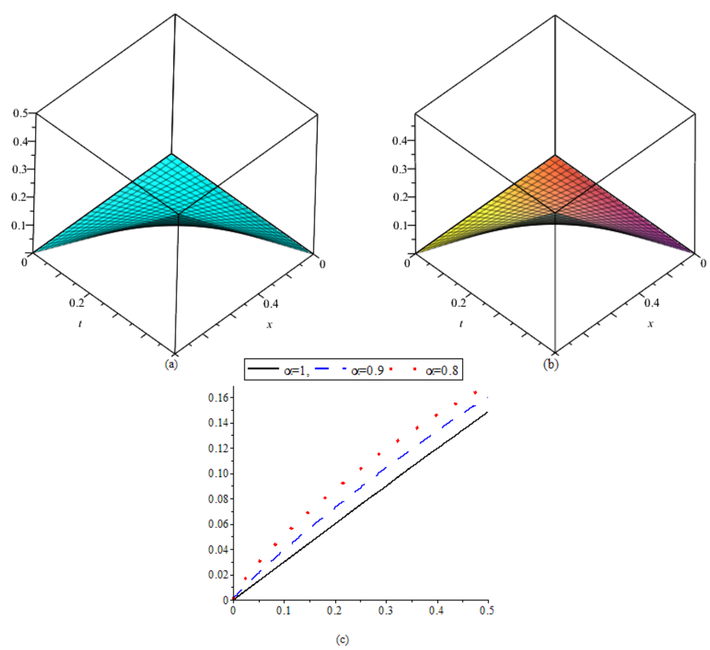

Figure 1: (a) Exact solution (b) The approximate solution in the case $\alpha = 1.0$
(c) The third-order equation (4.1) for different values of α when $x = 0.3$

Example 2 Consider the nonlinear time-fractional hyperbolic equation

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} u(x,t) = \frac{\partial}{\partial x}\big(u(x,t)u_x(x,t)\big), \quad t > 0,\ x \in \mathbb{R},\ 1 < \alpha \leq 2, \tag{4.6}$$

with the initial conditions

$$u(x,0) = x^2, \quad u_t(x,0) = -2x^2. \tag{4.7}$$

Taking the Elzaki transform of equation (4.6) subjected to the initial conditions, we have

$$E\{u(x,t)\} = x^2v^2 - 2x^2v^3 + v^\alpha E\{u_x^2(x,t)u(x,t)u_{xx}(x,t)\}. \tag{4.8}$$

The inverse Elzaki transform implies that

$$u(x,t) = x^2 - 2x^2 t + E^{-1}\{v^\alpha E\{u_x^2(x,t)u(x,t)u_{xx}(x,t)\}\}. \tag{4.9}$$

Now, applying the homotopy perturbation method, we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x^2 - 2x^2 t + p\left\{E^{-1}\left(v^\alpha\left(\sum_{n=0}^{\infty} p^n H_n(u)\right)\right)\right\}, \tag{4.10}$$

where

$$\begin{aligned}
&H_0(u) = u_{0x}^2 + u_0 u_{0xx},\\
&H_1(u) = 2u_{0x}u_{1x} + u_0 u_{1xx} + u_1 u_{0xx},\\
&H_2(u) = 2u_{0x}u_{2x} + u_0 u_{2xx} + u_2 u_{0xx} + u_1 u_{1xx},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .
\end{aligned}$$

Comparing the coefficients of the same powers of p, we get:

$$\begin{aligned}
p^0 : \quad & u_0(x,t) = x^2 - 2tx^2,\\
p^1 : \quad & u_1(x,t) = \frac{6x^2 t^\alpha}{\Gamma(\alpha+1)} - \frac{24x^2 t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{48x^2 t^{\alpha+2}}{\Gamma(\alpha+3)},\\
p^2 : \quad & u_2(x,t) = 72x^2\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{4t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{8t^{2\alpha+2}}{\Gamma(2\alpha+3)}\right)\\
&\qquad - 144x^2\left(\frac{\Gamma(\alpha+2)t^{2\alpha+1}}{\Gamma(\alpha+1)\Gamma(2\alpha+2)} - \frac{4\Gamma(\alpha+3)t^{2\alpha+2}}{\Gamma(\alpha+2)\Gamma(2\alpha+3)}\times\right.\\
&\qquad\qquad \left.\times\frac{8t^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{82\Gamma(\alpha+4)t^{2\alpha+3}}{\Gamma(\alpha+3)\Gamma(2\alpha+4)}\right),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .
\end{aligned}$$

The third-order term approximate solution for (4.6) is given by

$$\begin{aligned}
u(x,t) = x^2\Bigg[&1 - 2t + 6\left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{4t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{8t^{\alpha+2}}{\Gamma(\alpha+3)}\right)\\
&+ 72\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{4t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{8t^{2\alpha+2}}{\Gamma(2\alpha+3)}\right)\\
&- 144\left(\frac{\Gamma(\alpha+2)t^{2\alpha+1}}{\Gamma(\alpha+1)\Gamma(2\alpha+2)} - \frac{4\Gamma(\alpha+3)t^{2\alpha+2}}{\Gamma(\alpha+2)\Gamma(2\alpha+3)}\right.\\
&\left.+ \frac{8t^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{82\Gamma(\alpha+4)t^{2\alpha+3}}{\Gamma(\alpha+3)\Gamma(2\alpha+4)}\right)\Bigg].
\end{aligned}$$

The resulting solution is equivalent to the exact solution in a closed form: $u(x,t) = (x/t+1)^2$, which is the same third-order term approximate solution for (4.6)–(4.7) derived from [16] using VIM. Through applying ADM, the time-fractional hyperbolic differential equation (4.6) can also be solved in [16].

The approximate solutions for $\alpha = 2.0$ obtained for different values of x and t using HPETM, HPM and VIM, are shown in Table 2.

t	x	u_{VIM}	u_{HPM}	u_{HPETM}	u_{Exact}
0.2	0.25	0.043400	0.043400	0.0433951	0.043403
	0.50	0.173600	0.173600	0.173580	0.173611
	0.75	0.390600	0.390600	0.390556	0.390625
	1.0	0.694400	0.694400	0.694321	0.694444
0.4	0.25	0.031779	0.031779	0.031567	0.031888
	0.50	0.127118	0.127118	0.126268	0.127551
	0.75	0.286015	0.286015	0.284103	0.286990
	1.0	0.508471	0.508471	0.505072	0.508471
0.6	0.25	0.023665	0.023665	0.022005	0.024414
	0.50	0.094660	0.094660	0.088018	0.097656
	0.75	0.212984	0.212984	0.198040	0.219727
	1.0	0.378638	0.378638	0.352071	0.390625

Table 2: Numerical values when $\alpha = 2$ for equation (4.6)

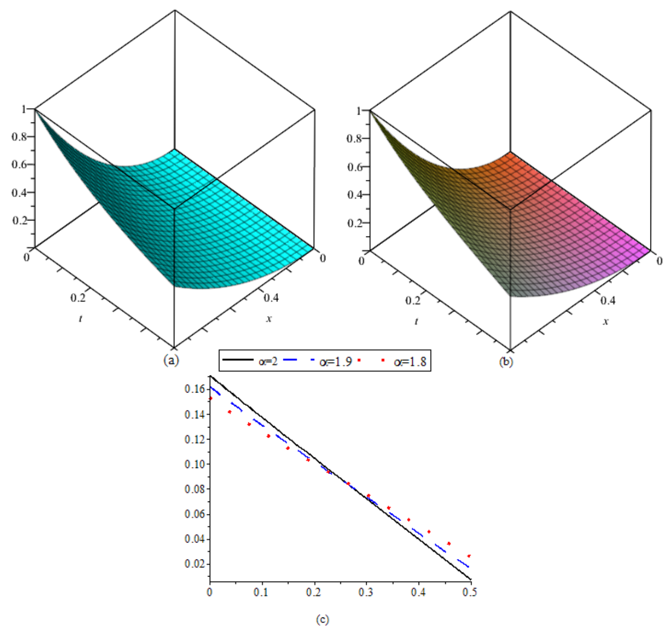

Figure 2: (a) Exact solution (b) The approximate solution in the case $\alpha = 1.0$
(c) The third-order equation (4.6) for different values of α when $x = 0.3$

Example 3 Consider the nonlinear time-fractional Fisher's equation

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}u(x,t) = u_{xx}(x,t) + 6u(x,t)(1-u(x,t)), \quad t>0,\ x\in\mathbb{R},\ 0<\alpha\le 1, \tag{4.11}$$

with the initial condition

$$u(x,0) = \frac{1}{(1+e^x)^2}. \tag{4.12}$$

To find the solution by HPETM, we apply the homotopy perturbation method. After taking the Elzaki and the inverse Elzaki transforms of equation (4.11), we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \frac{1}{(1+e^x)^2} + p\left\{E^{-1}\left(v^\alpha\left(\sum_{n=0}^{\infty} p^n H_n(u)\right)\right)\right\}, \tag{4.13}$$

where:

$$
\begin{aligned}
&H_0(u) = u_{0xx}(x,t) + 6u_0(x,t)(1 - u_0(x,t)),\\
&H_1(u) = u_{1xx}(x,t) + 6u_1(x,t)(1 - 2u_0(x,t)),\\
&H_2(u) = u_{2xx}(x,t) + 6u_2(x,t)(1 - 2u_0(x,t)) - 6u_1^2(x,t),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .
\end{aligned}
$$

Comparing the coefficients of the same powers of p, we get:

$$
\begin{aligned}
p^0: &\quad u_0(x,t) = \frac{1}{(1+e^x)^2},\\
p^1: &\quad u_1(x,t) = \frac{10e^x}{(1+e^x)^3}\frac{t^\alpha}{\Gamma(\alpha+1)},\\
p^2: &\quad u_2(x,t) = \frac{50e^{x(2e^x-1)}}{(1+e^x)^4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .
\end{aligned}
$$

The third-order term approximate solution for (4.11) is given by:

$$
u(x,t) = \frac{1}{(1+e^x)^2} + \frac{10e^x}{(1+e^x)^3}\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{50e^{x(2e^x-1)}}{(1+e^x)^4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.
$$

The solution found here is equivalent to the exact solution in a closed form: $u(x,t) = 1/(1+e^{x-5t})^2$, which results in the same third-order term approximate solution for equations (4.11)–(4.12) obtained from [16] through using VIM. We can further solve the nonlinear time-fractional advection partial differential equation (4.11) in [16] using ADM.

The approximate solutions for $\alpha = 1$ obtained for different values of x and t using HPETM, HPM and VIM, can be seen in Table 3.

t	x	u_{VIM}	u_{HPM}	u_{HPETM}	u_{Exact}
0.1	0.25	0.315940	0.315940	0.317948	0.316042
	0.50	0.249926	0.249926	0.250500	0.250000
	0.75	0.191606	0.191606	0.190964	0.191689
	1.0	0.142411	0.142411	0.140979	0.142537
0.2	0.25	0.459320	0.459320	0.481199	0.461284
	0.50	0.386450	0.386450	0.396941	0.387456
	0.75	0.315478	0.315478	0.315266	0.316042
	1.0	0.249092	0.249092	0.241175	0.250000
0.3	0.25	0.591179	0.591179	0.681440	0.604195
	0.50	0.527635	0.527635	0.581861	0.534447
	0.75	0.459719	0.459719	0.475833	0.461284
	1.0	0.387025	0.387025	0.372917	0.387456

Table 3: Numerical values when $\alpha = 1$ for equation (4.11)

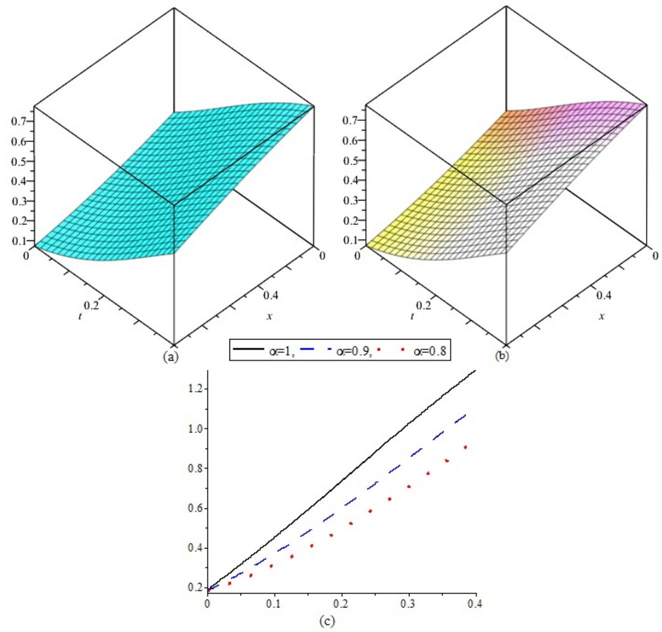

Figure 3: (a) The exact solution (b) The approximate solution in the case $\alpha = 1.0$
(c) The third-order equation (4.11) for different values of α when $x = 0.3$

5 Conclusion

In this paper, we have proposed a mixed version of the Elzaki transform and the homotopy perturbation method in order to solve nonlinear partial differential equations of fractional order. We have shown that the solution of such equations is simple when we use the Adomian decomposition method, but the calculation of Adomian's polynomials is complex in this method. The major advantage of this technique in comparison with the decomposition method is that the developed algorithm can solve nonlinear partial differential equations without the need for Adomian's polynomials.

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