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NONLINEAR PARABOLIC INEQUALITIES IN SOBOLEV SPACE WITH VARIABLE EXPONENT

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Abstract. We prove the existence of an entropy solution for the obstacle parabolic problem associated to the equation:

$$\begin{cases} u \geq \psi & \text{a.e. in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f & \text{in } Q = \Omega \times (0,T), \end{cases}$$

where the second term f belongs to $L^1(Q)$ and $u_0 \in L^1(\Omega)$. The critical growth condition on H is with respect to ∇u; no growth with respect to u and no sign conditions and the coercivity conditions are assumed. The main methods are the so-called 'penalization methods'.

Keywords: Entropy solution, nonlinear parabolic problem, penalized equation.

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1 Introduction

In the present paper we establish an existence result of an entropy solution for a class of nonlinear parabolic problems of the type:

$$(\mathcal{P})\begin{cases} u \geq \psi & \text{a.e. in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f & \text{in } Q = \Omega \times (0,T), \\ u(x,0) = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T). \end{cases}$$

In the problem $(\mathcal{P})$, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, T is a positive real number, while the data $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. The operator $-\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray–Lions operator which is coercive; H is a nonlinear lower order term.

More precisely, this paper deals with the existence of a solution to the obstacle parabolic problems associated to $(\mathcal{P})$ in the sense of an entropy solution (see Definition 3.1).

The existence of solutions to nonlinear parabolic inequalities with L^1-data in Orlicz spaces was studied by R. Aboulaich, B. Achchab, D. Meskine and A. Souissi in [1], where the case $H(x,t,u,\nabla u) = 0$ was considered. In the case where $a(x,t,u,\nabla u) = |\nabla u|^{p(x)-2}\nabla u$ and $H(x,t,u,\nabla u) = 0$, M. Bendahmane, P. Wittbold and A. Zimmermann [3] proved the existence and uniqueness of renormalized solutions to nonlinear parabolic equations with L^1-data. Recently, M. Sanchón and J. M. Urbano [15] have studied a Dirichlet problem $(\mathcal{P})$ of $p(x)$-Laplace equation and have obtained the existence and uniqueness of an entropy solution for L^1-data where $H(x,t,u,\nabla u) = 0$ and $Au = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. In the case where $H(x,t,u,\nabla u) = -g(u)M(|\nabla u|)$, the existence of solutions of some unilateral problems in the framework of Orlicz spaces has been established by M. Kbiri Alaoui, D. Meskine and A. Souissi [11] with the penalization methods.

The plan of the paper is as follows. In Section 2 we collect some important propositions and results of variable exponent Lebesgue–Sobolev spaces that will be used thoroughout the paper. In Section 3 we give the basic assumptions and the definition of an entropy solution of $(\mathcal{P})$. In Section 4 we establish the existence of such a solution in Theorem 4.2. Section 5 is devoted to an example which illustrates the abstract result.

2 Mathematical preliminaries on variable exponent Sobolev spaces

2-1. Sobolev space with exponent variable. In this section, we recall some definitions and basic properties of the generalised Lebesgue–Sobolev spaces with variable exponent: $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. We refer to Fan and Zhao [9] for further properties of variable exponent Lebesgue–Sobolev spaces.

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$. We say that a continuous real-valued function p is log-Hölder continuous in Ω if there is a constant C such that

$$|p(x) - p(y)| \leq \frac{C}{|\log|x-y||} \quad \forall x, y \in \bar{\Omega} \text{ such that } |x-y| < \frac{1}{2}. \tag{2.1}$$

Moreover, let us set $C_+(\bar{\Omega})$=$\{p \in C(\bar{\Omega}) : \min_{x\in\bar{\Omega}} p(x) > 1\}$. For any $p \in C_+(\bar{\Omega})$ we define

$$p^- = \inf_{x\in\Omega} p(x) \quad \text{and} \quad p^+ = \sup_{x\in\Omega} p(x).$$

We define the variable exponent Lebesgue space for $p \in C_+(\bar{\Omega})$ by:

$$L^{p(x)}(\Omega) = \left\{ u\colon \Omega \to \mathbb{R} : u \text{ is measurable with } \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty \right\}.$$

This space endowed with the (Luxembourg) norm defined by the formula

$$||u||_{L^{p(x)}(\Omega)} = ||u||_{p(x)} = \inf\left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \le 1 \right\}$$

is a separable and reflexive Banach space. The dual space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ (see [12]). If p is a constant function, then the variable exponent Lebesgue space coincides with the classical Lebesgue space.

Proposition 2.1 ([9, 12], Generalised Hölder inequality)

(i) *For any functions $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ we have*

$$\left| \int_\Omega uv\,\mathrm{d}x \right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) ||u||_{p(x)} ||v||_{p'(x)} \le 2||u||_{p(x)} ||v||_{p'(x)}.$$

(ii) *For all $p, q \in C_+(\bar{\Omega})$ such that $p(x) \le q(x)$ a.e. in Ω, we have $L^{q(x)} \hookrightarrow L^{p(x)}$ and the embedding is continuous.*

The modular of the space $L^{p(x)}(\Omega)$, that is, the mapping $\rho\colon L^{p(x)}(\Omega) \to \mathbb{R}$, is defined by the formula

$$\rho(u) = \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x \quad \text{for all } u \in L^{p(x)}(\Omega).$$

Lemma 2.1 ([9]) *If $u \in L^{p(x)}(\Omega)$, then*

$$\min\left\{ ||u||^{p^-}_{p(x)}, ||u||^{p^+}_{p(x)} \right\} \le \rho(u) \le \max\left\{ ||u||^{p^-}_{p(x)}, ||u||^{p^+}_{p(x)} \right\}.$$

The next proposition shows that there is a gap between the modular and the norm in $L^{p(x)}(\Omega)$.

Proposition 2.2 ([10, 17]) *For $u \in L^{p(x)}(\Omega)$ and $\{u_k\}_{k\in\mathbb{N}} \subset L^{p(x)}(\Omega)$ the following assertions hold*:

(i) $u \neq 0 \;\Rightarrow\; \left[||u||_{p(x)} = \lambda \;\Leftrightarrow\; \rho\left(\frac{u}{\lambda}\right) = 1 \right]$;

(ii) $||u||_{p(x)} > 1 \;\Rightarrow\; ||u||^{p^-}_{p(x)} \le \rho(u) \le ||u||^{p^+}_{p(x)}$;

(iii) $||u||_{p(x)} < 1 \;\Rightarrow\; ||u||^{p^+}_{p(x)} \le \rho(u) \le ||u||^{p^-}_{p(x)}$;

(iv) $\lim_{k\to\infty} ||u_k||_{p(x)} = 0 \;\Leftrightarrow\; \lim_{k\to\infty} \rho(u_k) = 0;$

(v) $\lim_{k\to\infty} ||u_k||_{L^{p(x)}(\Omega)} = \infty \;\Leftrightarrow\; \lim_{k\to\infty} \rho(u_k) = \infty.$

Similarly to the definition of $L^{p(x)}(\Omega)$, we define the variable Sobolev space by

$$W^{1,p(x)}(\Omega) = \Big\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \Big\}.$$

The space $W^{1,p(x)}(\Omega)$ is endowed with the norm:

$$||u||_{1,p(x)} = ||u||_{p(x)} + ||\nabla u||_{p(x)} \qquad \forall u \in W^{1,p(x)}(\Omega).$$

We do not assume that the continuous function $p(x) \in C(\bar{\Omega})$ is log-Hölder continuous. If the log-Hölder continuity condition (2.1) holds, then we denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, *i.e.*,

$$W_0^{1,p(x)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,p(x)}(\Omega)}$$

and

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{for } p(x) < N, \\ \infty & \text{for } p(x) \geq N. \end{cases}$$

Proposition 2.3 ([10])

(i) *Assuming* $1 < p^- \leq p^+ < \infty$ *and* $p \in C(\bar{\Omega})$, *the spaces* $W^{1,p(x)}(\Omega)$ *and* $W_0^{1,p(x)}(\Omega)$ *are separable and reflexive Banach spaces.*

(ii) *If* $q \in C_+(\bar{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \bar{\Omega}$, *then* $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ *is continuous and compact. In particular, we have that* $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ *is continuous and compact* (*for more details see [8, Theorem 8.4.2]*).

(iii) Poincaré inequality: *For* $u \in W_0^{1,p(x)}(\Omega)$ *with* $p \in C(\bar{\Omega})$ *and* $p^- > 1$ *there exists a constant* $C > 0$ *such that* $||u||_{p(x)} \leq C||\nabla u||_{p(x)}$ *holds, where the positive constant* C *depends on* $p(x)$ *and* Ω.

Remark 2.1 *By* (iii) *of Proposition 2.3 we know that* $||\nabla u||_{p(x)}$ *and* $||u||_{1,p(x)}$ *are equivalent norms on* $W_0^{1,p(x)}(\Omega)$.

We will also use the standard notations for Bochner spaces, *i.e.*, if $q \geq 1$ and X is a Banach space, then $L^q(0,T;X)$ denotes the space of strongly measurable functions $u\colon (0,T) \to X$ for which $t \mapsto \|u(t)\|_X \in L^q(0,T)$. Moreover, $C([0,T];X)$ denotes the space of continuous functions $u\colon [0,T] \to X$, endowed with the norm $\|u\|_{C([0,T];X)} = \max_{t\in[0,T]} \|u(t)\|_X$. Set

$$L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) = \Big\{ u\colon (0,T) \to W_0^{1,p(x)}(\Omega) \ : u \text{ is measurable and } \Big(\int_0^T \|u(t)\|^{p^-}_{W_0^{1,p(x)}(\Omega)} \,\mathrm{d}t \Big)^{\frac{1}{p^-}} < \infty \Big\}.$$

Similarly to the definition of $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$, we define also the space

$$L^\infty(0,T;X)=\big\{u\colon(0,T)\to X: u \text{ is measurable and } \exists\, C>0\ \|u(t)\|_X\le C \text{ a.e.}\big\}.$$

Let us recall that the space $L^\infty(0,T;X)$ is endowed with the following norm

$$\|u\|_{L^\infty(0,T;X)}=\inf\big\{C>0:\|u(t)\|_X\le C \text{ a.e.}\big\}.$$

We introduce the functional space (see [3])

$$V=\Big\{u\in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)):|\nabla u|\in L^{p(x)}(Q)\Big\}, \tag{2.2}$$

which endowed with the norm:

$$\|u\|_V=\|\nabla u\|_{L^{p(x)}(Q)}$$

or, the equivalent norm:

$$|||u|||_V=\|u\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))}+\|\nabla u\|_{L^{p(x)}(Q)},$$

is a separable and reflexive Banach space. The equivalence of the two norms is an easy consequence of the continuous embedding $L^{p(x)}(Q)\hookrightarrow L^{p^-}(0,T;L^{p(x)}(\Omega))$ and the Poincaré inequality. We state some further properties of V in the following lemma.

Lemma 2.2 *Let V be defined as in* (2.2) *and let its dual space be denoted by V^*. Then*

(i) *we have the following continuous dense embeddings*:

$$L^{p^+}(0,T;W_0^{1,p(x)}(\Omega))\hookrightarrow V\hookrightarrow L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)).$$

In particular, since $D(Q)$ is dense in $L^{p^+}(0,T;W_0^{1,p(x)}(\Omega))$, it is dense in V and for the corresponding dual spaces we have

$$L^{(p^-)'}(0,T;(W_0^{1,p(x)}(\Omega))^*)\hookrightarrow V^*\hookrightarrow L^{(p^+)'}(0,T;(W_0^{1,p(x)}(\Omega))^*).$$

Note that we have the following continuous dense embeddings

$$L^{p^+}(0,T;L^{p(x)}(\Omega))\hookrightarrow L^{p(x)}(Q)\hookrightarrow L^{p^-}(0,T;L^{p(x)}(\Omega)).$$

(ii) *one can represent the elements of V^* as follows*: *if $T\in V^*$, then there exists $F=(f_1,\dots,f_N)\in(L^{p'(x)}(Q))^N$ such that $T=\operatorname{div}F$ and*

$$\langle T,\xi\rangle_{V^*,V}=\int_0^T\int_\Omega F.\nabla\xi\,\mathrm{d}x\,\mathrm{d}t$$

for any $\xi\in V$. Moreover, we have

$$\|T\|_{V^*}=\max\big\{\|f_i\|_{L^{p'(x)}(Q)}:i=1,\dots,N\big\}.$$

Remark 2.2 *The space $V \cap L^\infty(Q)$, endowed with the norm defined by the formula*:

$$\|v\|_{V\cap L^\infty(Q)} := \max\big\{\|v\|_V, \|v\|_{L^\infty(Q)}\big\}, \quad v \in V \cap L^\infty(Q),$$

is a Banach space. In fact, it is the dual space of the Banach space $V^* + L^1(Q)$ endowed with the norm:

$$\|v\|_{V^*+L^1(Q)} := \inf\big\{\|v_1\|_{V^*} + \|v_2\|_{L^1(Q)} : v = v_1 + v_2, v_1 \in V^*, v_2 \in L^1(Q)\big\}.$$

2-2. Some technical results. The aim of this section is to state several technical results, which will be needed in the sequel.

Let u' stand for the generalized derivative of u, *i.e.*,

$$\int_0^T u'(t)\varphi(t)\,\mathrm{d}t = -\int_0^T u(t)\varphi'(t)\,\mathrm{d}t \quad \text{for all} \quad \varphi \in C_0^\infty(0,T).$$

Lemma 2.3 ([14]) $W := \big\{u \in V : u_t \in V^* + L^1(Q)\big\} \hookrightarrow C([0,T]; L^1(\Omega))$ *and*

$$W \cap L^\infty(Q) \hookrightarrow C([0,T]; L^2(\Omega)).$$

3 Assumption on data and the definition of an entropy solution

Throughout the paper, we assume that the following assumptions hold true.

Assumption (H1)
Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$, $T > 0$ be given and let us set $Q = \Omega \times (0,T)$. Moreover, let $p \in C_+(\bar{\Omega})$ and assume that $p(x)$ satisfies the log-Hölder condition (2.1) with $1 < p^- \leq p(x) \leq p^+ < \infty$. We consider a Leray–Lions operator defined by the formula:

$$Au = -\operatorname{div}\big(a(x,t,u,\nabla u)\big),$$

where $a\colon \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function (*i.e.*, measurable with respect to x in Ω for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and continuous with respect to $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω), which satisfies the following conditions: there exists $k(x,t) \in L^{p'(x)}(Q)$ such that for almost every $(x,t) \in Q$, all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and $\beta > 0$

$$\big|a(x,t,s,\xi)\big| \leq \beta\Big[k(x,t) + |s|^{p(x)-1} + |\xi|^{p(x)-1}\Big], \tag{3.1}$$

$$\big[a(x,t,s,\xi) - a(x,t,s,\eta)\big](\xi - \eta) > 0 \quad \forall\, \xi \neq \eta \in \mathbb{R}^N, \tag{3.2}$$

$$a(x,t,s,\xi)\cdot\xi \geq \alpha|\xi|^{p(x)}, \tag{3.3}$$

where α is a strictly positive constant.

Assumption (H2)
Furthermore, let $H\colon \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for all $(x,t) \in Q$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the growth condition

$$|H(x,t,s,\xi)| \leq \gamma(x,t) + g(s)|\xi|^{p(x)} \tag{3.4}$$

is satisfied, where $g\colon \mathbb{R} \to \mathbb{R}^+$ is a continuous positive function that belongs to $L^1(\mathbb{R})$ and $\gamma(x,t) \in L^1(Q)$.

Let ψ be a measurable function with values in $\bar{\mathbb{R}}$ such that $\psi \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) \cap L^\infty(Q)$ and let

$$K_\psi = \Big\{ u \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) : u \geq \psi \text{ a.e. in } \Omega \times (0,T) \Big\}.$$

We recall that for $k > 0$ and $s \in \mathbb{R}$ *the truncation function* T_k is defined by

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k\frac{s}{|s|}, & \text{if } |s| > k, \end{cases}$$

and we define $\phi_k(s) = \frac{1}{k}T_k(s)$.

Definition 3.1 *Let $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. A real-valued function u defined on Q is an entropy solution of the problem $(\mathcal{P})$ if*

$$u \geq \psi \text{ a.e. in } \Omega \times (0,T), \tag{3.5}$$

$$\begin{aligned} &T_k(u) \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) \quad \text{for all } k \geq 0, \\ &u \in C(0,T;L^1(\Omega)), \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\int_\Omega S_k\big(u(T)-v(T)\big)\,\mathrm{d}x - \int_\Omega S_k\big(u_0 - v(0)\big)\,\mathrm{d}x \\ &\quad + \int_Q \frac{\partial v}{\partial t} T_k(u-v)\,\mathrm{d}x\,\mathrm{d}t + \int_Q H(x,t,u,\nabla u)T_k(u-v)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_Q a(x,t,u,\nabla u)\nabla T_k(u-v)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq \int_Q f T_k(u-v) dx\,\mathrm{d}t \quad \forall v \in K_\psi \cap L^\infty(Q), \end{aligned} \tag{3.7}$$

where $S_k(s) = \int_0^s T_k(r)\,\mathrm{d}r$ and $\frac{\partial v}{\partial t} \in L^{p'^-}(0,T;W^{-1,p'(x)}(\Omega))$.

4 Existence results

In this section, we establish the following existence theorem. We begin with the following

Lemma 4.1 ([3]) *Assume* (3.1)–(3.3) *and let $(u_n)_n$ be a sequence in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ such that $u_n \rightharpoonup u$ weakly in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ and*

$$\int_Q \big[a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)\big]\nabla(u_n - u)\,\mathrm{d}x \to 0.$$

Then $u_n \to u$ strongly in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$.

Theorem 4.2 *Let $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$ and $p \in C_+(\bar{\Omega})$. Assume that* (H1) *and* (H2) *hold true. Then there exists at least one entropy solution u of the problem $(\mathcal{P})$ (in the sense of Definition 3.1).*

Proof. The above theorem is proven in the following five steps.

Step 1. Approximate problem. For $n > 0$ let us define the following respective approximation of H, f and u_0:

$$H_n(x,t,s,\xi) \quad = \quad \frac{H(x,t,s,\xi)}{1+\frac{1}{n}|H(x,t,s,\xi)|},$$

and select f_n and u_{0n} so that

$$f_n \in L^{p'(x)}(Q),\ f_n \to f \text{ a.e. in } Q \text{ and strongly in } L^1(Q) \text{ as } n \to \infty, \tag{4.1}$$

$$u_{0n} \in D(\Omega),\ u_{0n} \to u_0 \text{ a.e. in } \Omega \text{ and strongly in } L^1(\Omega) \text{ as } n \to \infty. \tag{4.2}$$

Note that $H_n(x,t,s,\xi)$ satisfies the following conditions

$$|H_n(x,t,s,\xi)| \le |H(x,t,s,\xi)|$$

and

$$|H_n(x,t,s,\xi)| \le n \quad \text{for all} \quad (s,\xi) \in \mathbb{R} \times \mathbb{R}^N.$$

Let us now consider the approximate problem

$$(\mathcal{P}_n) \begin{cases} \dfrac{\partial u_n}{\partial t} - \operatorname{div}(a(x,t,u_n,\nabla u_n)) + H_n(x,t,u_n,\nabla u_n) & \\ \qquad\qquad + nT_n(u_n-\psi)^- \phi_{1/n}(u_n) = f_n & \text{in } D'(Q), \\ u_n(t=0) = u_{0n} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega \times (0,T). \end{cases}$$

Moreover, since $f_n \in L^{p'^-}(0,T;W^{-1,p'(x)}(\Omega))$, proving the existence of a weak solution $u_n \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ of $(\mathcal{P}_n)$ is an easy task (see [13]).

Step 2. A priori estimates. Let us begin with the following

Proposition 4.1 *Let u_n be a solution of the approximate problem $(\mathcal{P}_n)$. Then there exists a constant C (which does not depend on n and k) such that:*

$$\|T_k(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))} \le Ck \quad \forall\, k > 0.$$

Proof. Let $v = T_k(u_n)^+ \chi_{(0,\tau)} \exp(G(u_n))$, where $G(s) = \int_0^s \frac{g(r)}{\alpha}\,\mathrm{d}r$ (the function g appears in (3.4)). Choosing v as a test function in the approximate problem $(\mathcal{P}_n)$ with $\tau \in (0,T)$, by (3.4) and (3.3), we get

$$\begin{aligned} &\int_{Q^\tau} \frac{\partial u_n}{\partial t} \exp(G(u_n)) T_k(u_n)^+ \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{Q^\tau} a(x,t,u_n,\nabla u_n) \exp(G(u_n)) \nabla T_k(u_n)^+ \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_{Q^\tau} nT_n(u_n-\psi)^- \exp(G(u_n)) \phi_{1/n}(u_n) T_k(u_n)^+ \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad\quad \le \int_{Q^\tau} [\gamma(x,t) + f_n] \exp(G(u_n)) T_k(u_n)^+ \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.3}$$

On the other hand, taking $v = T_k(u_n)^-\chi_{(0,\tau)}\exp(-G(u_n))$ as a test function in the problem $(\mathcal{P}_n)$, we deduce as in (4.3) that

$$\begin{aligned}\int_{Q^\tau}\frac{\partial u_n}{\partial t}&\exp(-G(u_n))T_k(u_n)^-\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_{Q^\tau}a(x,t,u_n,\nabla u_n)\exp(-G(u_n))\nabla T_k(u_n)^-\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_{Q^\tau}\gamma(x,t)\exp(-G(u_n))T_k(u_n)^-\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad+\int_{Q^\tau}nT_n(u_n-\psi)^-\exp(-G(u_n))T_k(u_n)^-\phi_{1/n}(u_n)\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\quad\geq\int_{Q^\tau}f_n\exp(-G(u_n))T_k(u_n)^-\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}\tag{4.4}$$

which, by using (4.3), gives

$$\begin{aligned}\int_0^\tau\!\!\int_\Omega\frac{\partial u_n}{\partial t}&\exp(G(u_n))T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_{Q^\tau}a\big(x,t,u_n,\nabla T_k(u_n)^+\big)\nabla T_k(u_n)^+\exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t\\ &\quad-\int_{Q^\tau}nT_n(u_n-\psi)^-\exp(G(u_n))\phi_{1/n}(u_n)T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\leq\int_{Q^\tau}\big[\gamma(x,t)+f_n\big]\exp(G(u_n))T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Since $G(u_n)\leq\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}$, we have $|\varphi_k(r)|\leq k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)|r|$, where $\varphi_k(r)=\int_0^r T_k(s)^+\exp(G(r))\,\mathrm{d}s$. Then

$$\int_{Q^\tau}\frac{\partial u_n}{\partial t}\exp(G(u_n))T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t\geq\int_\Omega\varphi_k(u_n(\tau))\,\mathrm{d}x-k\exp\bigg(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\bigg)\|u_{0n}\|_{L^1(\Omega)}.$$

Then, we deduce that

$$\begin{aligned}\int_\Omega&\varphi_k(u_n(\tau))\,\mathrm{d}x\\ &+\int_{Q^\tau}a(x,t,u_n,\nabla T_k(u_n)^+)\nabla T_k(u_n)^+\exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_{Q^\tau}nT_n(u_n-\psi)^-\exp(G(u_n))\phi_{1/n}(u_n)T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\leq k\exp\bigg(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\bigg)\big[\|\gamma\|_{L^1(Q)}+\|f_n\|_{L^1(Q)}+\|u_{0n}\|_{L^1(\Omega)}\big]\leq C_1k.\end{aligned}\tag{4.5}$$

Since a satisfies (3.3), by the fact that $\varphi_k(u_n(\tau))\geq 0$, for every $n>0$ we get

$$\begin{aligned}\alpha\int_{Q_\tau}&|\nabla T_k(u_n)^+|^{p(x)}\exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_{Q^\tau}nT_n(u_n-\psi)^-\exp(G(u_n))\phi_{1/n}(u_n)T_k(u_n)^+\,\mathrm{d}x\,\mathrm{d}t\leq C_1k,\end{aligned}\tag{4.6}$$

where C_1 is a constant which varies from line to line and which depends only on the data. It follows that

$$0 \leq \int_{Q_\tau} nT_n(u_n-\psi)^- \exp(G(u_n))\phi_{1/n}(u_n)\frac{T_k(u_n)^+}{k}\,\mathrm{d}x\,\mathrm{d}t \leq C_1,$$

and as $k \to 0$ by Fatou's lemma we deduce that

$$\int_{\{u_n\geq 0\}} nT_n(u_n-\psi)^- \exp(G(u_n))\phi_{1/n}(u_n)\,\mathrm{d}x\,\mathrm{d}t \leq C_1. \tag{4.7}$$

Thanks to (4.6), we have

$$\alpha\int_{Q_\tau} |\nabla T_k(u_n)^+|^{p(x)} \exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t \leq C_1 k, \tag{4.8}$$

and we deduce that

$$\alpha\int_{Q} |\nabla T_k(u_n)^+|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t \leq C_1 k. \tag{4.9}$$

Now, using $v = T_k(u_n)^-\chi_{(0,\tau)}\exp(-G(u_n))$ as a test function in (4.4), with $k > 0$, for every $\tau \in [0,T]$ we get

$$\begin{aligned}\int_{Q^\tau} & a(x,t,u_n,\nabla u_n)\exp(-G(u_n))\nabla u_n\chi_{\{-k\leq u_n\leq 0\}}\,\mathrm{d}x\,\mathrm{d}t\\ &- \int_{Q^\tau} nT_n(u_n-\psi)^-\exp(-G(u_n))\phi_{1/n}(u_n)T_k(u_n)^-\,\mathrm{d}x\,\mathrm{d}t\\ &\leq k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\big[\|\gamma\|_{L^1(Q)}+\|f_n\|_{L^1(Q)}+\|u_{0n}\|_{L^1(\Omega)}\big]\\ &- \int_\Omega \varphi_k^1(u_n(\tau))\,\mathrm{d}x,\end{aligned} \tag{4.10}$$

where $\varphi_k^1 = \int_0^r T_k(s)^-\exp(-G(s))\,\mathrm{d}s$. Then

$$\begin{aligned}\int_\Omega & \varphi_k^1(u_n(\tau))\,\mathrm{d}x\\ &+ \int_{Q^\tau} a(x,t,u_n,\nabla u_n)\exp(-G(u_n))\nabla u_n\chi_{\{-k\leq u_n\leq 0\}}\,\mathrm{d}x\,\mathrm{d}t\\ &- \int_{Q^\tau} nT_n(u_n-\psi)^-\exp(-G(u_n))\phi_{1/n}(u_n)T_k(u_n)^-\,\mathrm{d}x\,\mathrm{d}t\\ &\leq k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\big[\|\gamma\|_{L^1(Q)}+\|f_n\|_{L^1(Q)}+\|u_{0n}\|_{L^1(\Omega)}\big] \leq C_2 k.\end{aligned} \tag{4.11}$$

Since a satisfies (3.3) and due to the fact that $\varphi_k^1(u_n(\tau)) \geq 0$, for every $n > 0$ we get

$$\begin{aligned}\alpha\int_{Q_\tau} & |\nabla T_k(u_n)^-|^{p(x)}\exp(-G(u_n))\,\mathrm{d}x\,\mathrm{d}t\\ &- n\int_{Q_\tau} T_n(u_n-\psi)^-T_k(u_n)^-\exp(-G(u_n))\phi_{1/n}(u_n)\,\mathrm{d}x\,\mathrm{d}t \leq C_2 k,\end{aligned} \tag{4.12}$$

where C_2 is a positive constant, and we conclude that

$$0 \leq -\int_{\{u_n\leq 0\}} nT_n(u_n-\psi)^-\exp(-G(u_n))\phi_{1/n}(u_n) \leq C_2, \tag{4.13}$$

and

$$\alpha \int_Q |\nabla T_k(u_n)^-|^{p(x)} \,\mathrm{d}x\,\mathrm{d}t \leq C_2 k. \tag{4.14}$$

Combining (4.9), (4.14) and Lemma 2.1, we deduce that

$$\begin{aligned} \int_0^T \min\left\{ \|\nabla T_k(u_n)\|_{p(x)}^{p^+}, \|\nabla T_k(u_n)\|_{p(.)}|^{p^-} \right\} \mathrm{d}t \leq \rho(\nabla T_k(u_n)) \leq C_3 k \\ \Rightarrow \|T_k(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))} \leq k C_3. \end{aligned} \tag{4.15}$$

Then, $T_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ independently of n for any $k > 0$. □

Now we turn to proving the almost everywhere convergence of u_n. Consider a non-decreasing function $g_k \in C^2(\mathbb{R})$ such that

$$g_k(s) = \begin{cases} s, & \text{if } |s| \leq \frac{k}{2}, \\ k, & \text{if } |s| \geq k. \end{cases}$$

Multiplying the approximate equation by $g_k'(u_n)$, we get

$$\begin{aligned} \frac{\partial g_k(u_n)}{\partial t} - \operatorname{div}\big(a(x,t,u_n,\nabla u_n) g_k'(u_n)\big) + a(x,t,u_n,\nabla u_n) g_k''(u_n)\nabla u_n \\ + nT_n(u_n - \psi)^- g_k'(u_n)\phi_{1/n}(u_n) + H_n(x,t,u_n,\nabla u_n) g_k'(u_n) = f_n g_k'(u_n) \end{aligned} \tag{4.16}$$

in the sense of distributions. This, thanks to the fact that g_k' has compact support, implies that $g_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$, while its time derivative $\frac{\partial g_k(u_n)}{\partial t}$ is bounded in $L^1(Q) + V^*$. Due to the choice of g_k, we conclude that for each k the sequence $T_k(u_n)$ converges almost everywhere in Q, which implies that the sequence u_n converges almost everywhere to some measurable function v in Q. Thus, by using the same argument as in [4, 5, 6], we can show the following lemma.

Lemma 4.3 *Let u_n be a solution of the approximate problem $(\mathcal{P}_n)$. Then $u_n \to u$ a.e. in Q.*

We can deduce from (4.15) that

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{in} \quad L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)),$$

which by (3.3) implies that for every $k > 0$ there exists a function $h_k \in (L^{p'(x)}(Q))^N$ such that

$$a(x,u,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup h_k \quad \text{in} \quad (L^{p'(x)}(Q))^N. \tag{4.17}$$

Lemma 4.4 ([2]) *Let u_n be a solution of the approximate problem $(\mathcal{P}_n)$. Then,*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x,t,u_n,\nabla(u_n))\nabla u_n \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{4.18}$$

Lemma 4.5 ([3]) *Let $g \in L^{p(x)}(Q)$ and $g_n \in L^{p(x)}(Q)$ with $\|g_n\|_{p(x)} \leq C$ for $1 < p(x) < \infty$, if $g_n(x) \to g(x)$ a.e. on Q. Then $g_n \rightharpoonup g$ in $L^{p(x)}(Q)$.*

Step 3. Almost everywhere convergence of the gradients. This step is devoted to introducing for a fixed $k \geq 0$ a time regularization of the function $T_k(u)$ in order to perform the monotonicity method. This specific time regularization of $T_k(u)$ (for fixed $k \geq 0$) is defined as follows. Let $(\upsilon_0^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$\upsilon_0^\mu \in L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega) \text{ for all } \mu > 0, \tag{4.19}$$

$$\|\upsilon_0^\mu\|_{L^\infty(\Omega)} \leq k \text{ for all } \mu > 0, \tag{4.20}$$

$$\upsilon_0^\mu \to T_k(u_0) \text{ a.e. in } \Omega \text{ and } \frac{1}{\mu}\|\upsilon_0^\mu\|_{L^{p(x)}(\Omega)} \to 0 \text{ as } \mu \to \infty. \tag{4.21}$$

For fixed k, $\mu > 0$ let us consider the unique solution $(T_k(u))_\mu \in L^\infty(Q) \cap L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ of the monotone problem:

$$\frac{\partial (T_k(u))_\mu}{\partial t} + \mu\big((T_k(u))_\mu - T_k(u)\big) = 0 \quad \text{in} \quad D'(Q), \tag{4.22}$$

$$(T_k(u))_\mu(t=0) = \upsilon_0^\mu \quad \text{in} \quad \Omega. \tag{4.23}$$

Note that due to (4.22), for $\mu > 0$ and $k \geq 0$, we have

$$\frac{\partial (T_k(u))_\mu}{\partial t} \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)). \tag{4.24}$$

We just recall here that (4.22)–(4.23) imply that

$$(T_k(u))_\mu \to T_k(u) \quad \text{a.e. in} \quad Q \tag{4.25}$$

as well as weakly in $L^\infty(Q)$ and strongly in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ as $\mu \to \infty$. Note that for any μ and any $k \geq 0$ we have

$$\|(T_k(u))_\mu\|_{L^\infty(Q)} \leq \max\big(\|T_k(u)\|_{L^\infty(Q)}; \|\upsilon_0^\mu\|_{L^\infty(\Omega)}\big) \leq k. \tag{4.26}$$

We introduce a sequence of increasing $C^\infty(\mathbb{R})$-functions S_m such that

$$S_m(r) = r \text{ for } |r| \leq m, \qquad \text{supp}(S'_m) \subset [-(m+1), m+1], \qquad \|S''_m\|_{L^\infty(\mathbb{R})} \leq 1,$$

for any $m \geq 1$, and we denote by $\omega(n,\mu,\eta,m)$ the quantities such that

$$\lim_{m\to\infty}\lim_{\eta\to 0}\lim_{\mu\to\infty}\lim_{n\to\infty} \omega(n,\mu,\eta,m) = 0.$$

The main estimate is

Lemma 4.6 ([2, 6]) *We have*

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, T_\eta\big(u_n - (T_k(u))_\mu\big)^+ \exp(G(u_n)) S'_m(u_n) \right\rangle \mathrm{d}t \geq \omega(n,\mu,\eta) \quad \forall m \geq 1. \tag{4.27}$$

Taking now $v = T_\eta\big(u_n - (T_k(u))_\mu\big)^+ S'_m(u_n)\exp(G(u_n))$ in $(\mathcal{P}_n)$ and using (3.3) and (3.4), we get

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial u_n}{\partial t}, T_\eta\big(u_n - (T_k(u))_\mu\big)^+ \exp(G(u_n)) S'_m(u_n) \right\rangle \mathrm{d}t \\
&+ \int_Q a\big(x,t,u_n,\nabla u_n\big)\nabla\big(T_\eta\big(u_n - (T_k(u))_\mu\big)^+\big)\exp(G(u_n))S'_m(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{\{m\le |u_n|\le m+1\}} a\big(x,t,u_n,\nabla u_n\big) T_\eta\big(u_n - (T_k(u))_\mu\big)^+ \exp(G(u_n)) S''_m(u_n)\nabla u_n\,\mathrm{d}x\,\mathrm{d}t \\
&+ n\int_Q T_n(u_n-\psi)^- T_\eta\big(u_n - (T_k(u))_\mu\big)^+ \exp(G(u_n))S'_m(u_n)\phi_{\frac{1}{n}}(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\le C\eta.
\end{aligned} \tag{4.28}$$

From (4.7), (4.18), (4.27) and (4.28) it follows that

$$\begin{aligned}
&\int_Q a\big(x,t,u_n,\nabla u_n\big)\nabla\big(T_\eta\big(u_n - (T_k(u))_\mu\big)^+\big)\exp(G(u_n))S'_m(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le C\eta + \omega(n,\mu,\eta,m),
\end{aligned} \tag{4.29}$$

where C is a constant independent of n and m. On the other hand, let

$$A = \{0 \le T_k(u_n) - (T_k(u))_\mu < \eta\} \quad \text{and} \quad B = \{0 \le u_n - (T_k(u))_\mu < \eta\}.$$

Then, we have

$$\begin{aligned}
&\int_Q a\big(x,t,u_n,\nabla u_n\big)\nabla\big(T_\eta\big(u_n - (T_k(u))_\mu\big)^+\big)\exp(G(u_n))S'_m(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_B a\big(x,t,u_n,\nabla u_n\big)\big(\nabla u_n - \nabla(T_k(u))_\mu\big)\exp(G(u_n))S'_m(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_A a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\big(\nabla T_k(u_n) - \nabla(T_k(u))_\mu\big)\exp(G(u_n))S'_m(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{\{|u_n|>k\}\cap B} a\big(x,t,u_n,\nabla u_n\big)\big(\nabla u_n - \nabla(T_k(u))_\mu\big)\exp(G(u_n))S'_m(u_n)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.30}$$

By the coercivity condition (3.3) and the definition of S'_m ($S'_m(u_n) = 1$ a.e. in $\{|u_n| \le k\}$ if $k \le m$), in view of (4.29) and (4.30), we get

$$\begin{aligned}
&\int_A a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\big(\nabla T_k(u_n) - \nabla(T_k(u))_\mu\big)\exp(G(u_n)S'_m(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le \int_{\{|u_n|>k\}\cap B} a\big(x,t,u_n,\nabla u_n\big)\nabla(T_k(u))_\mu \exp(G(u_n))S'_m(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad + C\eta + \omega(n,\mu,\eta,m).
\end{aligned} \tag{4.31}$$

Since $a\big(x,t,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n)\big)$ is bounded in $(L^{p'(x)}(Q))^N$, there exists some $h_{k+\eta} \in (L^{p'(x)}(Q))^N$ such that $a\big(x,t,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n)\big) \rightharpoonup h_{k+\eta}$ weakly in $(L^{p'(x)}(Q))^N$. Consequently

$$\int_{\{|u_n|>k\}\cap B} a\big(x,t,u_n,\nabla u_n\big)|\nabla(T_k(u))_\mu| \exp(G(u_n))S'_m(u_n)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\{|u|>k\}\cap\{0\le u-(T_k(u))_\mu<\eta\}} h_{k+\eta}\nabla(T_k(u))_\mu \exp(G(u))S'_m(u)\,\mathrm{d}x\,\mathrm{d}t + \omega(n).$$

Thanks to (4.25) one easily has

$$\int_{\{|u|>k\}\cap\{0\le u-(T_k(u))_\mu<\eta\}} h_{k+\eta}\nabla(T_k(u))_\mu \exp(G(u))S'_m(u)\,\mathrm{d}x\,\mathrm{d}t = \omega(\mu).$$

Hence

$$\begin{aligned}\int_A a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\big(\nabla T_k(u_n)-\nabla(T_k(u))_\mu\big)\exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t\\ \le C\eta+\omega(n,\mu,\eta,m).\end{aligned} \tag{4.32}$$

On the other hand, note that

$$\begin{aligned}&\int_A a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\big(\nabla T_k(u_n)-\nabla(T_k(u))_\mu\big)\exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t\\ &\quad=\int_A a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\big(\nabla T_k(u_n)-\nabla T_k(u)\big)\exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad+\int_A a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\big(\nabla T_k(u)-\nabla(T_k(u))_\mu\big)\exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{4.33}$$

and the last integral tends to 0 as $n\to\infty$ and $\mu\to\infty$. Indeed, we have that

$$\int_A a\big(T_k(u_n),\nabla T_k(u_n)\big)\big(\nabla T_k(u)-\nabla(T_k(u))_\mu\big)\exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t\to$$
$$\to\int_{\{0\le T_k(u)-(T_k(u))_\mu<\eta\}} h_k\big(\nabla T_k(u)-\nabla(T_k(u))_\mu\big)\exp(G(u))\,\mathrm{d}x\,\mathrm{d}t$$

as $n\to\infty$. It is obvious that

$$\int_{\{0\le T_k(u)-(T_k(u))_\mu<\eta\}} h_k\big(\nabla T_k(u)-\nabla(T_k(u))_\mu\big)\exp(G(u))\,\mathrm{d}x\,\mathrm{d}t\to 0 \quad\text{as}\quad \mu\to\infty.$$

We deduce then that

$$\begin{aligned}\int_A a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\big(\nabla T_k(u_n)-\nabla T_k(u)\big)\exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t\\ \le C\eta+\epsilon(n,\mu,\eta,m).\end{aligned} \tag{4.34}$$

Let

$$\begin{aligned}M_n=\big(\big[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\big]\big[\nabla T_k(u_n)-\nabla T_k(u)\big]\big)\times\\ \times\big(\exp(G(u_n))\big).\end{aligned}$$

Then for any $0 < \theta < 1$ we write

$$\begin{aligned} I_n &= \int_{\{|u_n-(T_k(u))_\mu| \geq 0} M_n^\theta \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_{\{|T_k(u_n)-(T_k(u))_\mu| \leq \eta, \; u_n - T_k(u)_\mu \geq 0\}} M_n^\theta \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_{\{|T_k(u_n)-(T_k(u))_\mu| > \eta, \; u_n-(T_k(u))_\mu \geq 0\}} M_n^\theta \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

Since $a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)$ is bounded in $(L^{p'(x)}(Q))^N$, while $\nabla T_k(u_n)$ is bounded in $(L^{p(x)}(Q))^N$, by applying Hölder's inequality, we obtain

$$\begin{aligned} I_n \leq C_1 &\left(\int_{\{0 \leq T_k(u_n)-(T_k(u))_\mu < \eta\}} M_n \, \mathrm{d}x \, \mathrm{d}t \right)^\theta \\ &+ C_2 \operatorname{meas}\Big\{ (x,t) \in Q : \big|T_k(u_n) - (T_k(u))_\mu\big| > \eta \, , \; u_n - (T_k(u))_\mu \geq 0 \Big\}^{1-\theta}. \end{aligned} \tag{4.35}$$

On the other hand, we have

$$\begin{aligned} &\int_{\{0 \leq T_k(u_n)-(T_k(u))_\mu) < \eta\}} M_n \, \mathrm{d}x \, \mathrm{d}t \\ &\quad = \int_{\{0 \leq T_k(u_n)-(T_k(u))_\mu) < \eta\}} a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) \times \\ &\qquad\qquad \times \big(\nabla T_k(u_n) - \nabla T_k(u)\big) \exp(G(u_n)) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad - \int_{\{0 \leq T_k(u_n)-(T_k(u))_\mu) < \eta\}} a\big(x,t,T_k(u_n),\nabla T_k(u)\big) \times \\ &\qquad\qquad \times \big(\nabla T_k(u_n) - \nabla T_k(u)\big) \exp(G(u_n)) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad = I_n^1 + I_n^2. \end{aligned} \tag{4.36}$$

Using (4.34), we have

$$I_n^1 \leq C \, \eta + w(n,\mu,\eta,m). \tag{4.37}$$

Concerning I_n^2, that is, the second term on the right-hand side of the (4.36), it is easy to see that

$$I_n^2 = w(n,\mu). \tag{4.38}$$

Because $a_i\big(x,t,T_k(u_n),\nabla T_k(u)\big) \to a_i\big(x,t,T_k(u),\nabla T_k(u)\big)$ strongly in $L^{p'(x)}(Q)$ for all $i = 1,\dots,N$, and $\frac{\partial T_k(u_n)}{\partial x_i} \rightharpoonup \frac{\partial T_k(u)}{\partial x_i}$ in $L^{p(x)}(Q)$, combining (4.35)–(4.38), yields

$$I_n \leq C_1\big(C\eta + w(n,\mu,\eta,m)\big)^\theta + C_2\big(w(n,\mu)\big)^{1-\theta},$$

and by passing to the limit sup over n, μ and η

$$\begin{aligned} \int_{\{u_n-(T_k(u))_\mu \geq 0\}} &\Big(\big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))\big] \times \\ &\qquad \times \big[\nabla T_k(u_n) - \nabla T_k(u)\big]\Big)^\theta \, \mathrm{d}x \, \mathrm{d}t = w(n). \end{aligned} \tag{4.39}$$

On the other hand, if we choose $v = T_\eta\big(u_n - (T_k(u))_\mu\big)^- \exp(-G(u_n))$ in $(\mathcal{P}_n)$, we obtain

$$\int_{\{u_n - T_k(u)_\mu \le 0\}} \Big(\big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))\big] \times \times \big[\nabla T_k(u_n) - \nabla T_k(u)\big]\Big)^\theta \,\mathrm{d}x\,\mathrm{d}t = w(n). \tag{4.40}$$

Moreover, (4.39) and (4.40) imply that

$$\int_Q \Big(\big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))\big] \times \times \big[\nabla T_k(u_n) - \nabla T_k(u)\big]\Big)^\theta \,\mathrm{d}x\,\mathrm{d}t = w(n), \tag{4.41}$$

which implies that

$$T_k(u_n) \to T_k(u) \quad \text{in} \quad L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) \quad \forall\, k \ge 0. \tag{4.42}$$

By [7, Theorem 3.3] (see also [4, 5]), there exists a subsequence also denoted by u_n such that

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \quad Q, \tag{4.43}$$

which implies that

$$a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) \rightharpoonup a\big(x,t,T_k(u),\nabla T_k(u)\big) \quad \text{in} \quad (L^{p'(x)}(Q))^N. \tag{4.44}$$

Step 4. Equi-integrability of the nonlinearity sequence. Since $H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u)$ a.e. in Q, using Vitali's theorem we shall now prove that $H_n(x,t,u_n,\nabla u_n)$ converges to $H(x,t,u,\nabla u)$ strongly in $L^1(Q)$.

Choosing $\varphi = \rho_h(u_n) = \int_0^{u_n} g(s)\chi_{\{s>h\}}\,\mathrm{d}s \exp(G(u_n))$ as a test function in the approximate problem $(\mathcal{P}_n)$, by (4.3) and (3.3), we obtain

$$\begin{aligned}
&\Big[\int_\Omega \theta_h(u_n)\,\mathrm{d}x\Big]_0^T + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n g(u_n)\chi_{\{u_n>h\}} \exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_Q nT_n(u_n-\psi)^- \exp(G(u_n))\phi_{\frac{1}{n}}(u_n) \int_0^{u_n} g(s)\chi_{\{s>h\}}\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\quad \le \Big(\int_h^\infty g(s)\chi_{\{s>h\}}\,\mathrm{d}s\Big) \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)}\Big],
\end{aligned}$$

where $\theta_h(r) = \int_0^r \rho_h(\tau)\,\mathrm{d}\tau$, which implies, since $\theta_h \ge 0$, that

$$\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n g(u_n)\chi_{\{u_n>h\}}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \Big(\int_h^\infty g(s)\,\mathrm{d}s\Big) \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \int_\Omega \theta_h(u_{0n})\,\mathrm{d}x + C\Big].
\end{aligned}$$

Using (3.3) we have

$$\alpha\int_{\{u_n>h\}}|\nabla u_n|^{p(x)}g(u_n)\,\mathrm{d}x\,\mathrm{d}t \le \Big(\int_h^\infty g(s)\,\mathrm{d}s\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)}+\|\gamma\|_{L^1(Q)}+\int_\Omega \theta_h(u_{0n})\,\mathrm{d}x + C\Big],$$

and then

$$\int_{\{u_n>h\}}|\nabla u_n|^{p(x)}g(u_n)\,\mathrm{d}x\,\mathrm{d}t \le C_1\Big(\int_h^\infty g(s)\,\mathrm{d}s\Big).$$

Since $g\in L^1(\mathbb{R})$, we deduce that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}}|\nabla u_n|^{p(x)}g(u_n)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Similarly, taking $\varphi=\rho_h(u_n)=\int_{u_n}^0 g(s)\chi_{\{s<-h\}}\exp(-G(u_n))\,\mathrm{d}s$ as a test function in $(\mathcal{P}_n)$, we conclude that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}}|\nabla u_n|^{p(x)}g(u_n)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Consequently

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}}|\nabla u_n|^{p(x)}g(u_n)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

which, for h large enough and for a subset E of Q, implies that

$$\lim_{\operatorname{meas} E\to 0}\int_E|\nabla u_n|^{p(x)}g(u_n)\,\mathrm{d}x\,\mathrm{d}t \le \max_{|u_n|\le h}(g(s))\lim_{\operatorname{meas} E\to 0}\int_E|\nabla T_h(u_n)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t + \int_{\{|u_n|>h\}}|\nabla u_n|^{p(x)}g(u_n)\,\mathrm{d}x\,\mathrm{d}t.$$

So we conclude that $g(u_n)|\nabla u_n|^{p(x)}$ is equi-integrable, which implies that

$$g(u_n)|\nabla u_n|^{p(x)}\to g(u)|\nabla u|^{p(x)} \quad \text{in} \quad L^1(Q).$$

Consequently, by using (3.4) we conclude that

$$H_n(x,t,u_n,\nabla u_n)\to H(x,t,u,\nabla u) \quad \text{in} \quad L^1(Q). \tag{4.45}$$

Step 5. Passing to the limit. Let us consider the following three substeps.

5-1. We show that u satisfies (3.5).

Proposition 4.2 *Let u_n be a solution of the approximate problem $(\mathcal{P}_n)$. Then $u\ge\psi$ a.e. in Q.*

Proof. Thanks to (4.7) and (4.13), we get $\int_Q nT_n(u_n-\psi)^-\exp(G(u_n))\,\mathrm{d}x\,\mathrm{d}t \le C$. So, by Fatou's Lemma, we infer that $\int_Q(u-\psi)^-\,\mathrm{d}x\,\mathrm{d}t=0$, which implies that $(u-\psi)^-=0$ a.e. in Q. Consequently, we conclude that $u\ge\psi$ a.e. in Q. □

5-2. We claim that $u \in C(0,T;L^1(\Omega))$. We will show that

$$u_n \to u \quad \text{in} \quad C(0,T;L^1(\Omega)).$$

Since $T_k(u) \in K_\psi$, for every $k \geq \|\psi\|_{L^\infty}$ there exists a sequence $v_j \in K_\psi \cap D(\bar{Q})$ such that

$$v_j \to T_k(u) \quad \text{in} \quad L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$$

for the modular convergence.

Let $\omega_{j,\mu}^{i,l} = (T_l(v_j))_\mu + e^{-\mu t}T_l(\eta_i)$ with $\eta_i \geq 0$ converge to u_0 in $L^1(\Omega)$, where $(T_l(v_j))_\mu$ is the mollification of $T_l(v_j)$ with respect to time. Note that $\omega_{j,\mu}^{i,l}$ is a smooth function having the following properties:

$$\frac{\partial \omega_{j,\mu}^{i,l}}{\partial t} = \mu(T_l(v_j) - \omega_{j,\mu}^{i,l}), \qquad \omega_{j,\mu}^{i,l}(0) = T_l(\eta_i), \qquad |\omega_{j,\mu}^{i,l}| \leq l, \tag{4.46}$$

$$\omega_{j,\mu}^{i,l} \to T_l(v_j) \quad \text{in} \quad L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \quad \text{as} \quad \mu \to \infty. \tag{4.47}$$

Choosing now $v = T_k(u_n - \omega_{j,\mu}^{i,l})\chi_{(0,\tau)}$ as a test function in $(\mathcal{P}_n)$, yields

$$\begin{aligned} &\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_{j,\mu}^{i,l}) \right\rangle_{Q^\tau} + \int_{Q^\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - \omega_{j,\mu}^{i,l})\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{Q^\tau} nT_n(u_n-\psi)^- \phi_{1/n}(u_n) T_k(u_n - \omega_{j,\mu}^{i,l})\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_{Q^\tau} H_n(x,t,u_n,\nabla u_n) T_k(u_n - \omega_{j,\mu}^{i,l})\,\mathrm{d}x\,\mathrm{d}t = \int_{Q^\tau} f_n T_k(u_n - \omega_{j,\mu}^{i,l})\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.48}$$

We have (see [1])

$$\left\langle \frac{\partial \omega_{j,\mu}^{i,l}}{\partial t}, T_k(u_n - \omega_{j,\mu}^{i,l}) \right\rangle_{Q^\tau} = \mu \int_{Q^\tau} (T_k(v_j) - \omega_{j,\mu}^{i,l})) T_k(u_n - \omega_{j,\mu}^{i,l}) \geq \epsilon(n,j,\mu,l). \tag{4.49}$$

And by using (3.4) and the fact that

$$\int_{Q^\tau} nT_n(u_n-\psi)^- \phi_{1/n}(u_n) T_k(u_n - \omega_{j,\mu}^{i,l})\,\mathrm{d}x\,\mathrm{d}t \geq 0,$$

we deduce that

$$\begin{aligned} &\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_{j,\mu}^{i,l}) \right\rangle_{Q^\tau} + \int_{Q^\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - \omega_{j,\mu}^{i,l})\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq \int_{Q^\tau} [f_n + \gamma] T_k(u_n - \omega_{j,\mu}^{i,l})\,\mathrm{d}x\,\mathrm{d}t + \int_{Q^\tau} g(u_n)|\nabla u_n|^{p(x)} T_k(u_n - \omega_{j,\mu}^{i,l})\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.50}$$

On the one hand, we have

$$\begin{aligned} I &= \int_{Q^\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - \omega_{j,\mu}^{i,l})\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\{|T_k(u_n) - \omega_{j,\mu}^{i,l}| \leq k\}} a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla \omega_{j,\mu}^{i,l}]\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.51}$$

In the following, we pass to the limit in (4.51): By letting n and μ go to infinity, since $a(x,t,u_n,\nabla u_n) \rightharpoonup a(x,t,u,\nabla u)$ in $L^{p'(x)}(Q)$, in view of Lebesgue's theorem, we have

$$I = \int_{\{|T_k(u)-T_l(v_j)|\le k\}} a(x,t,u,\nabla u)[\nabla T_k(u) - \nabla T_l(v_j)]\,\mathrm{d}x\,\mathrm{d}t + \epsilon(n,\mu).$$

Consequently, by taking the limit as $j \to \infty$, we deduce that

$$I = \epsilon(n,\mu,j,l).$$

On the other hand, we have

$$J = \int_{Q^\tau} g(u_n)|\nabla u_n|^{p(x)} T_k(u_n - \omega^{i,l}_{j,\mu})\,\mathrm{d}x\,\mathrm{d}t. \tag{4.52}$$

In the following, we pass to the limit in (4.52): Taking the limit as $n \to \infty$ in (4.52), since $g(u_n)|\nabla u_n|^{p(x)} \to g(u)|\nabla u|^{p(x)}$ in $L^1(Q)$, in view of Lebesgue's theorem, we obtain

$$J = \int_{Q^\tau} g(u)|\nabla u|^{p(x)} T_k(u - \omega^{i,l}_{j,\mu})\,\mathrm{d}x\,\mathrm{d}t + \epsilon(n).$$

Consequently, by letting μ and j go to infinity, we have

$$J = \epsilon(n,\mu,j,l).$$

Similarly to (4.52) and by using (4.1), we have

$$\int_{Q^\tau} [f_n + \gamma] T_k(u_n - \omega^{i,l}_{j,\mu})\,\mathrm{d}x\,\mathrm{d}t = \epsilon(n,\mu,j,l),$$

and by using Vitali's theorem, we get

$$\limsup_{k\to\infty}\limsup_{i\to 0}\limsup_{j\to\infty}\limsup_{\mu\to\infty}\lim_{n\to\infty}\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega^{i,l}_{j,\mu})\right\rangle_{Q^\tau} \le 0 \tag{4.53}$$

uniformly on τ. Therefore, by writing

$$\begin{aligned}\int_\Omega S_k\big(u_n(\tau) - \omega^{i,l}_{j,\mu}(\tau)\big)\,\mathrm{d}x &= \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega^{i,l}_{j,\mu})\right\rangle_{Q^\tau} - \left\langle \frac{\partial \omega^{i,l}_{j,\mu}}{\partial t}, T_k(u_n - \omega^{i,l}_{j,\mu})\right\rangle_{Q^\tau} \\ &\quad + \int_\Omega S_k\big(u_n(0) - T_l(\eta_i)\big)\,\mathrm{d}x,\end{aligned} \tag{4.54}$$

and using (4.49), (4.53) and (4.54), we see that

$$\int_\Omega S_k\big(u_n(\tau) - \omega^{i,l}_{j,\mu}(\tau)\big)\,\mathrm{d}x \le \epsilon(n,j,\mu,l), \tag{4.55}$$

which implies, by writing

$$\begin{aligned}\int_\Omega S_k\Big(\frac{u_n(\tau) - u_m(\tau)}{2}\Big)\,\mathrm{d}x \le \frac{1}{2}\Big(&\int_\Omega S_k\big(u_n(\tau) - \omega^{i,l}_{j,\mu}(\tau)\big)\,\mathrm{d}x \\ &+ \int_\Omega S_k\big(u_m(\tau) - \omega^{i,l}_{j,\mu}(\tau)\big)\,\mathrm{d}x\Big),\end{aligned} \tag{4.56}$$

that

$$\int_\Omega S_k\left(\frac{u_n(\tau)-u_m(\tau)}{2}\right)\mathrm{d}x \le \epsilon_1(n,m).$$

We deduce then that

$$\int_\Omega |u_n(\tau)-u_m(\tau)|\,\mathrm{d}x \le \epsilon_2(n,m) \quad \text{independently of } \tau, \tag{4.57}$$

and thus (u_n) is a Cauchy sequence in $C(0,T;L^1(\Omega))$, and since $u_n \to u$ a.e. in Q, we deduce that

$$u_n \to u \quad \text{in} \quad C(0,T;L^1(\Omega)). \tag{4.58}$$

5-3. We show that u satisfies (3.7)**.** Let $v \in K_\psi \cap L^\infty(Q)$, $\frac{\partial v}{\partial t} \in L^{p'^-}(0,T;W^{-1,p(x)}(\Omega))$. By pointwise multiplication of the approximate problem $(\mathcal{P}_n)$ by $T_k(u_n-v)$, we get

$$\begin{aligned}
&\int_\Omega S_k\big(u_n(T)-v(T)\big)\,\mathrm{d}x - \int_\Omega S_k\big(u_{0n}-v(0)\big)\,\mathrm{d}x \\
&\quad + \int_Q \frac{\partial v}{\partial t}T_k(u_n-v)\,\mathrm{d}x\,\mathrm{d}t + \int_Q a(x,t,u_n,\nabla u_n)\nabla T_k(u_n-v)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_Q H_n(x,t,u_n,\nabla u_n)T_k(u_n-v)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_Q nT_n(u_n-\psi)^-\phi_{\frac{1}{n}}T_k(u_n-v)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_Q fT_k(u_n-v)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where $S_k(s)=\int_0^s T_k(r)\,\mathrm{d}r$. Since $\int_Q nT_n(u_n-\psi)^-\phi_{\frac{1}{n}}T_k(u_n-v)\,\mathrm{d}x\,\mathrm{d}t \ge 0$, we deduce then that

$$\begin{aligned}
&\int_\Omega S_k\big(u_n(T)-v(T)\big)\,\mathrm{d}x - \int_\Omega S_k\big(u_{0n}-v(0)\big)\,\mathrm{d}x \\
&\quad + \int_Q \frac{\partial v}{\partial t}T_k(u_n-v)\,\mathrm{d}x\,\mathrm{d}t + \int_Q a(x,t,u_n,\nabla u_n)\nabla T_k(u_n-v)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_Q H_n(x,t,u_n,\nabla u_n)T_k(u_n-v)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \int_Q fT_k(u_n-v)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.59}$$

Let us pass to the limit with $n \to \infty$ in each term in (4.59). We saw that $u_n \to u$ in $C(0,T;L^1(\Omega))$. Therefore, $u_n(t) \to u(t)$ in $L^1(\Omega)$ for all $t \le T$.

As S_k is Lipschitz continuous with constant k, when $n \to \infty$ we have

$$\int_\Omega S_k(u_n-v)(T)\,\mathrm{d}x \to \int_\Omega S_k(u-v)(T)\,\mathrm{d}x$$

and

$$\int_\Omega S_k(u_n-v)(0)\,\mathrm{d}x = \int_\Omega S_k(u_{0n}-v(0))\,\mathrm{d}x \to \int_\Omega S_k(u_0-v(0))\,\mathrm{d}x.$$

Let $v \in K_\psi \cap L^\infty(Q)$. Then $\frac{\partial v}{\partial t} \in L^{p'^-}(0,T;W^{-1,p'(x)}(\Omega))$ and by Lebesgue's theorem, we have

$$\int_0^T \left\langle \frac{\partial v}{\partial t}, T_k(u_n - v) \right\rangle \mathrm{d}t \to \int_0^T \left\langle \frac{\partial v}{\partial t} T_k(u - v) \right\rangle \mathrm{d}t.$$

On the other hand, we note that $M = \|v\|_\infty$. Then we get

$$\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - v)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T\int_\Omega a(x,t,T_{k+M}(u_n),\nabla T_{k+M}(u_n))\nabla T_k(T_{k+M}(u_n) - v)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T\int_\Omega a(x,t,T_{k+M}(u_n),\nabla T_{k+M}(u_n))\nabla T_{k+M}(u_n))\mathbf{1}_{\{|T_{k+M}(u_n)-v|\le k\}}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T\int_\Omega a(x,t,T_{k+M}(u_n),\nabla T_{k+M}(u_n)\nabla v\mathbf{1}_{\{|T_{k+M}(u_n)-v|\le k\}}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

As $T_{k+M}(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$, $\nabla u_n \to \nabla u$ a.e. in Q, by Lebesgue's theorem, we deduce that

$$\begin{aligned}
&\int_0^T\int_\Omega a(x,t,T_{k+M}(u_n),\nabla T_{k+M}(u_n)\nabla v\mathbf{1}_{\{|T_{k+M}(u_n)-v|\le k\}}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \to \int_0^T\int_\Omega a(x,t,T_{k+M}(u),\nabla T_{k+M}(u)\nabla v\mathbf{1}_{\{|T_{k+M}(u-v)|\le k\}}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Then

$$\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n T_k(u_n - v)\,\mathrm{d}x\,\mathrm{d}t \to \int_Q a(x,t,u,\nabla u)\nabla u T_k(u-v)\,\mathrm{d}x\,\mathrm{d}t.$$

Since $H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u)$ in $L^1(\Omega)$, as $|T_k(u_n - v)| \le k$ and $T_k(u_n - v) \rightharpoonup T_k(u - v)$ weakly in $L^\infty(Q)$, by Lebesgue's theorem, we have

$$\int_0^T\int_\Omega H_n(x,t,u_n,\nabla u_n)T_k(u_n - v)\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega H(x,t,u,\nabla u)T_k(u-v)\,\mathrm{d}x\,\mathrm{d}t.$$

Due to (4.1) and the fact that $u_n \to u$ a.e. in Q, we have

$$\int_0^T\int_\Omega f_n T_k(u_n - v)\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega f T_k(u-v)\,\mathrm{d}x\,\mathrm{d}t.$$

Due to (4.59), we have

$$\begin{aligned}
&\int_\Omega S_k\big(u(T) - v(T)\big)\,\mathrm{d}x - \int_\Omega S_k\big(u_0 - v(0)\big)\,\mathrm{d}x \\
&\qquad + \int_Q \frac{\partial v}{\partial t} T_k(u-v)\,\mathrm{d}x\,\mathrm{d}t + \int_Q H(x,t,u,\nabla u)T_k(u-v)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\quad + \int_Q a(x,t,u,\nabla u)\nabla T_k(u-v)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad \le \int_Q f T_k(u-v)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

As a conclusion of Steps 1–5, the proof of Theorem 4.2 is complete. □

5 Example

Let us consider the following special case. Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $p(x) = \sin|x| + 3$, $p \in C_+(\bar{\Omega})$ and

$$H(x,t,s,\xi) = \frac{-4s}{1+s^4}|\xi|^{p(x)}.$$

The Carathéodory function $H(x,t,s,\xi)$ satisfies the condition (3.4). Indeed,

$$|H(x,t,s,\xi)| \leq \frac{4|s|}{1+s^4}|\xi|^{p(x)} = g(s)|\xi|^{p(x)},$$

where $g(s) = \frac{4|s|}{1+s^4}$ is a continuous and positive function which belongs to $L^1(\mathbb{R})$. Note that $H(x,t,s,\xi)$ does not satisfy the sign condition and the coercivity condition. Set

$$Au = -\triangle_{p(x)} = -\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big). \tag{5.1}$$

We have $\big(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\big)(u-v) > 0$ for almost all $x \in \Omega$, $u, v \in \mathbb{R}^N$ and $u \neq v$, and so the monotonicity condition is satisfied.

The operator $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is a Carathéodory function satisfying the growth condition (3.1) and the coercivity (3.3).

Define the obstacle function

$$\psi(x,t) = [t\chi_{(0,\tau)}(t) + c(1-\chi_{(0,\tau)}(t))]w(x),$$

where $\tau \in (0,T)$ is fixed, c is a real constant and $w \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$.

Remark 5.1 *Finally, the hypotheses of Theorem 4.2 are satisfied. Therefore, the following problem*

$$u \geq \psi \text{ a.e. in } \Omega \times (0,T),$$

$$T_k(u) \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) \quad \text{for all } k \geq 0,$$
$$u \in C(0,T;L^1(\Omega)),$$

$$\begin{aligned}
&\int_\Omega S_k(u-v)(T)\,\mathrm{d}x - \int_\Omega S_k(u-v)(0)\,\mathrm{d}x \\
&\quad + \int_Q \frac{\partial v}{\partial t}T_k(u-v)\,\mathrm{d}x\,\mathrm{d}t + \int_Q |\nabla u|^{p(x)-2}\nabla u \nabla T_k(u-v)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_Q \frac{4u}{1+u^4}|\nabla u|^{p(x)}T_k(u-v)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad \leq \int_Q fT_k(u-v)\,\mathrm{d}x\,\mathrm{d}t \quad \forall v \in K_\psi \cap L^\infty(Q),
\end{aligned}$$

where $S_k(s) = \int_0^s T_k(r)\exp(r)\,\mathrm{d}r$ *and* $\frac{\partial v}{\partial t} \in L^{p'^-}(0,T;W^{-1,p'(x)}(\Omega))$*, has at least one unilateral an entropy solution.*

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