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SOLVABILITY OF NONLINEAR VISCOSITY EQUATION WITH A BOUNDARY INTEGRAL CONDITION

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Abstract. The aim of this paper is to investigate a general nonlinear hyperbolic equation with nonlocal condition. We first recall the theorems concerning the existence, uniqueness and continuous dependence of a strong solution for the linear problem in [7]. Then, by using a priori estimate and applying an iterative process based on results obtained for the linear problem, we prove the existence, uniqueness and continuous dependence of a weak generalized solution of the nonlinear problem.

Keywords: Energy inequality, integral condition, nonlinear viscosity equation, strong solution.

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1 Introduction

We deal with a nonlinear mixed problem having a nonlocal condition, the so-called energy specification. The problem of parameter identification from nonstandard boundary conditions in boundary value problems, originating from various engineering disciplines, is of growing interest. That is, a large number of physical phenomena and many problems in modern physics and technology can be described in terms of nonlocal problems, such as problems in partial differential equations with integral conditions. These nonlocal boundary conditions such as the integral condition $\int_\alpha^\beta u(x,t)\,\mathrm{d}x = E(t)$,

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arise mainly when the data on the boundary cannot be measured directly, but their average values are known.

For linear problems, several authors investigated the initial-boundary value problems in one space variable which involve an integral over the spatial domain of a function of the desired solution may appear in a boundary condition. Along a different line, problems for parabolic equations which combine classical and integral conditions were considered by Batten [1], Ionkin [13], Cannon *et al.* [9, 10, 11, 12], Yurchuk [17], Lin [14], Benouar-Yurchuk [2], Shi [16], Bouziani *et al.* [6, 15]. However, most of these papers consider particular situations like the heat equation in the rectangle $(0, 1) \times (0, T)$. Problems with only boundary integral conditions for a second-order parabolic equation have been treated by Bouziani–Benouar in [8], and for a $2m$-parabolic equation by Bouziani in [5]. Recently, a problem of this type for a second-order pluriparabolic equation has been studied by Bouziani in [4].

In [7], Bouziani deals with the proof of the existence, uniqueness, and continuous dependence of a strong solution upon the data, for an initial-boundary value problem which combine Neumann and integral conditions for a viscosity equation. Motivated by this, we expand the results from [7] by studying a nonlocal nonlinear mixed problem for a nonlinear viscosity equation in the case where the nolinear term $f(x, t, u, u_x)$ is added to the right-hand side of the equation.

2 Formulation of the problem

In this paper, we deal with a class of nonlinear hyperbolic equations with time- and space-variable characteristics, with a nonlocal boundary condition. The precise statement of the problem is as follows.

Let $\beta > 0$, $T > 0$, and $Q = \{(x, t) \in \mathbb{R}^2 : \alpha < x < \beta, 0 < t < T\}$. Find a function $\theta(x, t), (x, t) \in \overline{Q}$, satisfying

$$\mathcal{L}\theta = \frac{\partial^2 \theta}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial \theta}{\partial x}\right) - \frac{\partial^2}{\partial t \partial x}\left(b(x,t)\frac{\partial \theta}{\partial x}\right) + c(x,t)\theta = h\left(x, t, \theta, \frac{\partial \theta}{\partial x}\right), \tag{2.1}$$

together with the initial condition

$$\begin{aligned} \ell_1\theta &= \theta(x, 0) = \Phi(x), \qquad x \in (\alpha, \beta), \\ \ell_2\theta &= \frac{\partial \theta(x, 0)}{\partial t} = \Psi(x), \quad x \in (\alpha, \beta), \end{aligned} \tag{2.2}$$

the Neumann condition

$$\frac{\partial \theta(\alpha, t)}{\partial x} = \mu(t), \qquad t \in (0, T), \tag{2.3}$$

and the integral condition

$$\int_\alpha^\beta \theta(x, t)\, \mathrm{d}x = E(t), \qquad t \in (0, T), \tag{2.4}$$

where $\Phi, \Psi, \mu, E, a, b, c$ and h are known functions.

Condition C(1). For all $(x,t) \in \overline{Q}$, we assume that

$$c_0 \le a(x,t) \le c_1, \quad \frac{\partial a(x,t)}{\partial t} \le c_2, \quad \frac{\partial a(x,t)}{\partial x} \le c_3, \quad c_4 \le b(x,t),$$
$$c_5 \le \frac{\partial b(x,t)}{\partial t} \le c_6, \quad \frac{\partial b(x,t)}{\partial x} \le c_7, \quad \frac{\partial^2 b(x,t)}{\partial t^2} \le c_8, \quad \frac{\partial^2 b(x,t)}{\partial x \partial t} \le c_9, \quad c(x,t) \le c_{10}.$$

Condition C(2). For all $(x,t) \in \overline{Q}$, we assume that

$$\frac{\partial^2 a(x,t)}{\partial x \partial t} \le c_{11}, \quad \frac{\partial^2 b(x,t)}{\partial x^2} \le c_{12}, \quad \frac{\partial^3 b(x,t)}{\partial x \partial t^2} \le c_{13}.$$

In conditions **C(1)**, **C(2)**, and in the rest of the paper, we assume that the constants c_i, $i = 0, \ldots, 17$, are positive.

Condition C(3). The data satisfies the following compatibility conditions

$$\frac{d\Phi(\alpha)}{dx} = \mu(0), \quad \int_\alpha^\beta \Phi(x)\,\mathrm{d}x = E(0), \quad \frac{\mathrm{d}\Psi(\alpha)}{\mathrm{d}x} = \mu'(0), \quad \int_\alpha^\beta \Psi(x)\,\mathrm{d}x = E'(0).$$

To this end, we reduce the inhomogeneous boundary conditions (2.3) and (2.4) to homogeneous conditions, by introducing a new unknown function u defined by: $u(x,t) = \theta(x,t) + K(x,t)$, where

$$K(x,t) = \frac{(x-\alpha)}{2(\beta-\alpha)} \left\{ (3x - \alpha - 2\beta)\mu(t) - \frac{6(x-\alpha)}{(\beta-\alpha)^2} E(t) \right\}.$$

Then, the problem (2.1)–(2.4) becomes:

$$\mathcal{L}u = \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) - \frac{\partial^2}{\partial t \partial x}\left(b(x,t)\frac{\partial u}{\partial x}\right) + c(x,t)u = f\left(x,t,u,\frac{\partial u}{\partial x}\right), \tag{2.5}$$

$$\begin{aligned} \ell_1 u &= u(x,0) = \varphi(x), \\ \ell_2 u &= \frac{\partial u(x,0)}{\partial t} = \varkappa(x), \end{aligned} \tag{2.6}$$

$$\frac{\partial u(\alpha,t)}{\partial x} = 0, \tag{2.7}$$

$$\int_\alpha^\beta u(x,t)\,\mathrm{d}x = 0, \tag{2.8}$$

where

$$f\left(x,t,u,\frac{\partial u}{\partial x}\right) = h\left(x,t,\theta,\frac{\partial \theta}{\partial x}\right) + \mathcal{L}K(x,t),$$

$$\varphi(x) = \Phi(x) + \ell_1 K,$$

$$\varkappa(x) = \Psi(x) + \ell_2 K.$$

Here we assume that the functions φ and $\varkappa$ satisfy conditions of the form (2.7) and (2.8), *i.e.*,

$$\frac{\mathrm{d}\varphi(\alpha)}{\mathrm{d}x} = 0, \quad \int_\alpha^\beta \varphi(x)\,\mathrm{d}x = 0, \quad \frac{\mathrm{d}\varkappa(\alpha)}{\mathrm{d}x} = 0, \quad \int_\alpha^\beta \varkappa(x)\,\mathrm{d}x = 0.$$

Instead of searching for the function θ, we search for the function u. So the solution of problem (2.5)–(2.8) will be given by: $\theta(x,t) = u(x,t) - K(x,t)$.

We shall assume that there exists a positive constant δ such that

$$|f(x,t,u_1,v_1) - f(x,t,u_2,v_2)| \leq \delta\left(|u_1-u_2| + |v_1-v_2|\right) \quad \forall\, u_1,u_2,v_1,v_2 \in L^2(Q), \quad (x,t)\in Q. \tag{$\mathbf{H}_1$}$$

Now, we recall the results on the linear problem which was studied in [7]. We investigate the linear problem; precisely, we consider the equation

$$\mathcal{L}u = \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) - \frac{\partial^2}{\partial t \partial x}\left(b(x,t)\frac{\partial u}{\partial x}\right) + c(x,t)u = f(x,t), \tag{2.9}$$

together with conditions (2.6)–(2.8).

The solution of problem (2.9), (2.6)–(2.8) can be considered as a solution of the operator equation

$$Lu = (f, \varphi, \varkappa),$$

where $L = (\mathcal{L}, \ell_1, \ell_2)$. The operator L maps B into F, where B is the Banach space consisting of functions $\Im_x u \in L^2(Q)$, where $\Im_x u = \int_\alpha^x u(\xi,\cdot)\,\mathrm{d}\xi$, having finite norm

$$\|u\|_B = \left\{ \left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(Q)} + \|u\|^2_{C(0,T;L^2(\alpha,\beta))} + \left\|\Im_x \frac{\partial u}{\partial t}\right\|^2_{C(0,T;L^2(\alpha,\beta))} \right\}^{\frac{1}{2}},$$

and F is the Hilbert space with the finite norm

$$\|Lu\|_F = \left\{ \|\Im_x \mathcal{L}u\|^2_{L^2(Q)} + \|\ell_1 u\|^2_{L^2(\alpha,\beta)} + \|\Im_x \ell_2 u\|^2_{L^2(\alpha,\beta)} \right\}^{\frac{1}{2}}.$$

The domain $D(L)$ of the operator L is the set of all functions u such that $\Im_x u \in L^2(Q)$ for which $\Im_x \frac{\partial u}{\partial t}, \Im_x \frac{\partial^2 u}{\partial t^2}, \Im_x \frac{\partial^2 u}{\partial x^2} \in L^2(Q)$ and satisfying (2.7) and (2.8).

Theorem 1 (see [7]) *Let condition* **C(1)** *be fulfilled. Then the following a priori estimate for the linear problem* (2.9)*,* (2.6)–(2.8)

$$\|u\|_B \leq C\,\|Lu\|_F \tag{2.10}$$

holds for any function $u \in D(L)$, where C is a positive constant independent of u.

Proposition 1 ([7]) *The operator L from B into F has a closure.*

Definition 1 ([7]) *A solution of the equation*

$$\overline{L}u = (f, \varphi, \varkappa),$$

is called a strong solution *of problem* (2.9)*,* (2.6)–(2.8).

Corollary 1 ([7]) *Under the conditions of Theorem* 1*, there is a constant* $C > 0$ *independent of* u *such that*

$$\|u\|_B \leq C \left\|\overline{L}u\right\|_F \qquad \forall u \in D(\overline{L}).$$

Corollary 1 asserts that, if a strong solution exists, it is unique and depends continuously on $(f, \varphi, \varkappa)$, if u is considered in the topology of B and $(f, \varphi, \varkappa)$ is considered in the topology of F.

Corollary 2 ([7]) *The range* $R(\overline{L})$ *of the operator* $\overline{L}$ *equals to the closure* $\overline{R(L)}$ *of* $R(L)$.

Theorem 2 (see [7]) *Let conditions* **C(1)** *and* **C(2)** *be fulfilled. Then for any* $\Im_x f \in L^2(Q)$, $\varphi \in L^2(\alpha, \beta)$ *and* $\Im_x \varkappa \in L^2(\alpha, \beta)$*, problem* (2.9)*,* (2.6)–(2.8) *admits a unique strong solution* $u = \overline{L}^{-1}(f, \varphi, \varkappa) = \overline{L^{-1}}(f, \varphi, \varkappa)$.

3 Solvability of nonlinear problem

This section is devoted to the proof of the existence, uniqueness and continuous dependence of the solution on the data of the problem (2.5)–(2.8). Let us consider the following auxiliary problem with homogeneous equation:

$$\mathcal{L}w = \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial w}{\partial x}\right) - \frac{\partial^2}{\partial t \partial x}\left(b(x,t)\frac{\partial w}{\partial x}\right) + c(x,t)w = 0, \tag{3.1}$$

$$\begin{aligned} \ell_1 w &= w(x,0) = \varphi(x), \\ \ell_2 w &= \frac{\partial w(x,0)}{\partial t} = \varkappa(x), \end{aligned} \tag{3.2}$$

$$\frac{\partial w(\alpha,t)}{\partial x} = 0, \tag{3.3}$$

$$\int_\alpha^\beta w\,(x,t)\;\mathrm{d}x = 0. \tag{3.4}$$

If u is a solution of problem (2.5)–(2.8) and w is a solution of problem (3.1)–(3.4), then $y = u - w$ satisfies

$$\begin{aligned} \mathcal{L}y &= \frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial y}{\partial x}\right) - \frac{\partial^2}{\partial t \partial x}\left(b(x,t)\frac{\partial y}{\partial x}\right) + c(x,t)y \\ &= G\left(x, t, y, \frac{\partial y}{\partial x}\right), \end{aligned} \tag{3.5}$$

$$\begin{aligned} \ell_1 y &= y(x,0) = 0, \\ \ell_2 y &= \frac{\partial y(x,0)}{\partial t} = 0, \end{aligned} \tag{3.6}$$

$$\frac{\partial y(\alpha,t)}{\partial x} = 0, \tag{3.7}$$

$$\int_\alpha^\beta y\,(x,t)\;\mathrm{d}x = 0, \tag{3.8}$$

where $G(x,t,y,\frac{\partial y}{\partial x}) = f(x,t,y+w,\frac{\partial y}{\partial x}+\frac{\partial w}{\partial x})$. As the function f, the function G satisfies the condition ($\mathbf{H}_1$), that is, there exists a positive constant δ such that

$$|G(x,t,u_1,v_1) - G(x,t,u_2,v_2)| \le \delta\left(|u_1-u_2|+|v_1-v_2|\right) \quad \forall\, u_1,u_2,v_1,v_2 \in L^2(Q), \quad (x,t)\in Q. \tag{$\mathbf{H}_2$}$$

According to the results of the previous section, we deduce that problem (3.1)–(3.4) admits a unique solution that depends continuously upon the initial condition (3.2). Therefore it remains to solve the problem (3.5)–(3.8). We shall prove that problem (3.5)–(3.8) has a unique weak solution.

Firstly, we precise the concept of the solution we are considering. Let $v = v(x,t)$ be any function from $\widetilde{C^1}(Q)$, the space of functions v belonging to $C^1(Q)$ having $\frac{\partial^2 v}{\partial x\partial t}$, $\frac{\partial^2 v}{\partial t^2}$ continuous in Q.

We shall compute the integral $\int_Q G\Im_x v\,\mathrm{d}x\,\mathrm{d}t$. For this we assume that $y,v\in\widetilde{C^1}(Q)$, $y(x,0) = 0$, $\frac{\partial y}{\partial t}(x,0) = 0$, $\frac{\partial v}{\partial t}(x,T) = 0$, $v(x,T) = 0$, $\int_\alpha^\beta y(x,t)\,\mathrm{d}x = \int_\alpha^\beta v(x,t)\,\mathrm{d}x = 0$. By using conditions on y and v, we have:

$$\int_Q \frac{\partial^2 y}{\partial t^2}\Im_x v\,\mathrm{d}x\,\mathrm{d}t = -\int_Q v\Im_x\left(\frac{\partial^2 y}{\partial t^2}\right)\mathrm{d}x\,\mathrm{d}t, \tag{3.9}$$

$$-\int_Q \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial y}{\partial x}\right)\Im_x v\,\mathrm{d}x\,\mathrm{d}t = \int_Q va(x,t)\frac{\partial y}{\partial x}\,\mathrm{d}x\,\mathrm{d}t, \tag{3.10}$$

$$-\int_Q \frac{\partial^2}{\partial t\partial x}\left(b(x,t)\frac{\partial y}{\partial x}\right)\Im_x v\,\mathrm{d}x\,\mathrm{d}t = -\int_Q b(x,t)\frac{\partial y}{\partial x}\frac{\partial v}{\partial t}\,\mathrm{d}x\,\mathrm{d}t, \tag{3.11}$$

$$\int_Q c(x,t)y\Im_x v\,\mathrm{d}x\,\mathrm{d}t = -\int_Q v\Im_x\left(c(x,t)y\right)\mathrm{d}x\,\mathrm{d}t, \tag{3.12}$$

$$\int_Q G\Im_x v\,\mathrm{d}x\,\mathrm{d}t = -\int_Q v\Im_x G\,\mathrm{d}x\,\mathrm{d}t. \tag{3.13}$$

It then follows from (3.9)–(3.13) that

$$A(y,v) = -\int_Q v\Im_x G\,\mathrm{d}x\,\mathrm{d}t, \tag{3.14}$$

where

$$A(y,v) = -\int_Q v\Im_x\left(\frac{\partial^2 y}{\partial t^2}\right)\mathrm{d}x\,\mathrm{d}t + \int_Q va(x,t)\frac{\partial y}{\partial x}\,\mathrm{d}x\,\mathrm{d}t - \int_Q b(x,t)\frac{\partial y}{\partial x}\frac{\partial v}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_Q v\Im_x\left(c(x,t)y\right)\mathrm{d}x\,\mathrm{d}t.$$

Definition 2 *For every $v\in\widetilde{C^1}(Q)$, a function $y\in L^2(0,T;H^1(\alpha,\beta))$ is called a* weak solution *of problem* (3.5)–(3.8) *if* (3.7) *and* (3.14) *holds.*

Let us construct an iteration sequence in the following way. Starting with $y^{(0)} = 0$, the sequence

$\left\{y^{(n)}\right\}_{n\in N}$ is defined as follows: given the element $y^{(n-1)}$, then for $n = 1, 2, \ldots$ solve the problem:

$$\begin{aligned}&\frac{\partial^2 y^{(n)}}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial y^{(n)}}{\partial x}\right) - \frac{\partial^2}{\partial t \partial x}\left(b(x,t)\frac{\partial y^{(n)}}{\partial x}\right) + c(x,t)y^{(n)} \\ &= G\left(x, t, y^{(n-1)}, \frac{\partial y^{(n-1)}}{\partial x}\right),\end{aligned} \tag{3.15}$$

$$\begin{aligned}y^{(n)}(x,0) &= 0, \\ \frac{\partial y^{(n)}(x,0)}{\partial t} &= 0,\end{aligned} \tag{3.16}$$

$$\frac{\partial y^{(n)}(\alpha,t)}{\partial x} = 0, \tag{3.17}$$

$$\int_\alpha^\beta y^{(n)}(x,t)\ \mathrm{d}x = 0. \tag{3.18}$$

Theorem 2 asserts that for a fixed n, each problem (3.15)–(3.18) has a unique solution $y^{(n)}(x,t)$. If we set $Z^{(n)}(x,t) = y^{(n+1)}(x,t) - y^{(n)}(x,t)$, then we have the new problem

$$\begin{aligned}&\frac{\partial^2 Z^{(n)}}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial Z^{(n)}}{\partial x}\right) - \frac{\partial^2}{\partial t \partial x}\left(b(x,t)\frac{\partial Z^{(n)}}{\partial x}\right) + c(x,t)Z^{(n)} \\ &= P^{(n-1)}(x,t),\end{aligned} \tag{3.19}$$

$$\begin{aligned}Z^{(n)}(x,0) &= 0, \\ \frac{\partial Z^{(n)}(x,0)}{\partial t} &= 0,\end{aligned} \tag{3.20}$$

$$\frac{\partial Z^{(n)}(\alpha,t)}{\partial x} = 0, \tag{3.21}$$

$$\int_\alpha^\beta Z^{(n)}(x,t)\ \mathrm{d}x = 0, \tag{3.22}$$

where

$$P^{(n-1)}(x,t) = G\left(x, t, y^{(n)}, \frac{\partial y^{(n)}}{\partial x}\right) - G\left(x, t, y^{(n-1)}, \frac{\partial y^{(n-1)}}{\partial x}\right).$$

Lemma 1 *Assume that condition* ($\mathbf{H}_2$) *holds. Then for the linearized problem* (3.19)–(3.22)*, we have the a priori estimate*

$$\left\|Z^{(n)}\right\|_{L^2(0,T;\,H^1(\alpha,\beta))} \leq Tg^* \left\|Z^{(n-1)}\right\|_{L^2(0,T;\,H^1(\alpha,\beta))}, \tag{3.23}$$

where g^* *is a positive constant given by*

$$g^* = \exp\left(\max\left(\frac{c_2 + c_8}{2}, \frac{(\beta-\alpha)^2 c_{10}^2}{4c_4}\right)T\right)\frac{\left(8c_{16}(\beta-\alpha)^2 + 1\right)\delta^2}{\min\left(1, \frac{c_0+g}{2}\right)}.$$

Proof. Applying the operator $\Im_x$ to equation (3.19) by taking into account conditions (3.20) and (3.22), multiplying the obtained equality by $2\Im_x \frac{\partial Z^{(n)}}{\partial t}$ and integrating over $Q^\tau := (\alpha, \beta) \times (0, \tau)$, where $0 \leq \tau \leq T$, we get

$$\begin{aligned} &2\int_{Q^\tau} \Im_x \frac{\partial^2 Z^{(n)}}{\partial t^2} \Im_x \frac{\partial Z^{(n)}}{\partial t} \,\mathrm{d}x\,\mathrm{d}t - 2\int_{Q^\tau} a(x,t)\frac{\partial Z^{(n)}}{\partial x} \Im_x \frac{\partial Z^{(n)}}{\partial t} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad - 2\int_{Q^\tau} \frac{\partial}{\partial t}\left(b(x,t)\frac{\partial Z^{(n)}}{\partial x}\right) \Im_x \frac{\partial Z^{(n)}}{\partial t} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + 2\int_{Q^\tau} \Im_x \left(c(x,t)Z^{(n)}\right) \Im_x \frac{\partial Z^{(n)}}{\partial t} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = 2\int_{Q^\tau} \Im_x P^{(n-1)} \Im_x \frac{\partial Z^{(n)}}{\partial t} \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.24}$$

Integrating by parts the first three integrals on the left-hand side of (3.24), we obtain

$$2\int_{Q^\tau} \Im_x \frac{\partial^2 Z^{(n)}}{\partial t^2} \Im_x \frac{\partial Z^{(n)}}{\partial t} \,\mathrm{d}x\,\mathrm{d}t = \int_\alpha^\beta \left(\Im_x \frac{\partial Z^{(n)}(x,\tau)}{\partial t}\right)^2 \mathrm{d}x, \tag{3.25}$$

$$\begin{aligned} &-2\int_{Q^\tau} a\,\frac{\partial Z^{(n)}}{\partial x} \Im_x \frac{\partial Z^{(n)}}{\partial t} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_\alpha^\beta a(x,\tau)\left(Z^{(n)}\right)^2(x,\tau)\,\mathrm{d}x - \int_{Q^\tau} \frac{\partial a}{\partial t}\left(Z^{(n)}\right)^2 \mathrm{d}x\,\mathrm{d}t \\ &\quad + 2\int_{Q^\tau} \frac{\partial a}{\partial x} Z^{(n)} \Im_x \frac{\partial Z^{(n)}}{\partial t} \,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{3.26}$$

$$\begin{aligned} &-2\int_{Q^\tau} \frac{\partial}{\partial t}\left(b\frac{\partial Z^{(n)}}{\partial x}\right) \Im_x \frac{\partial Z^{(n)}}{\partial t} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = 2\int_{Q^\tau} b\left(\frac{\partial Z^{(n)}}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t + \int_\alpha^\beta \frac{\partial b(x,\tau)}{\partial t}\left(Z^{(n)}\right)^2(x,\tau)\,\mathrm{d}x \\ &\quad - \int_{Q^\tau} \frac{\partial^2 b}{\partial t^2}\left(Z^{(n)}\right)^2 \mathrm{d}x\,\mathrm{d}t + 2\int_{Q^\tau} \frac{\partial b}{\partial x}\frac{\partial Z^{(n)}}{\partial t} \Im_x \frac{\partial Z^{(n)}}{\partial t} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + 2\int_{Q^\tau} \frac{\partial^2 b}{\partial x \partial t} Z^{(n)} \Im_x \frac{\partial Z^{(n)}}{\partial t} \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.27}$$

Substituting (3.25)–(3.27) into (3.24), we get

$$\begin{aligned}
&2\int_{Q^\tau} b\left(\frac{\partial Z^{(n)}}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad+\int_\alpha^\beta \left\{\left(a+\frac{\partial b}{\partial t}\right)\left(Z^{(n)}(x,\tau)\right)^2+\left(\Im_x\frac{\partial Z^{(n)}(x,\tau)}{\partial t}\right)^2\right\}\mathrm{d}x \\
&\quad= 2\int_{Q^\tau}\Im_x P^{(n-1)}\Im_x\frac{\partial Z^{(n)}}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad+\int_{Q^\tau}\left(\frac{\partial a}{\partial t}+\frac{\partial^2 b}{\partial t^2}\right)\left(Z^{(n)}\right)^2\mathrm{d}x\,\mathrm{d}t-2\int_{Q^\tau}\left(\frac{\partial a}{\partial x}+\frac{\partial^2 b}{\partial x\partial t}\right)Z^{(n)}\Im_x\frac{\partial Z^{(n)}}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad-2\int_{Q^\tau}\frac{\partial b}{\partial x}\frac{\partial Z^{(n)}}{\partial t}\Im_x\frac{\partial Z^{(n)}}{\partial t}\,\mathrm{d}x\,\mathrm{d}t-2\int_{Q^\tau}\Im_x\left(c(x,t)Z^{(n)}\right)\Im_x\frac{\partial Z^{(n)}}{\partial t}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{3.28}$$

Estimating the first and the three last integrals on the right-hand side of (3.28), by applying elementary inequalities, we get

$$\begin{aligned}
&2\int_{Q^\tau}\Im_x P^{(n-1)}\,\Im_x\frac{\partial Z^{(n)}}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le\int_{Q^\tau}\left(\Im_x P^{(n-1)}\right)^2\mathrm{d}x\,\mathrm{d}t+\int_{Q^\tau}\left(\Im_x\frac{\partial Z^{(n)}}{\partial t}\right)^2\mathrm{d}x\,\mathrm{d}t,
\end{aligned}\tag{3.29}$$

$$\begin{aligned}
&-2\int_{Q^\tau}\left(\frac{\partial a}{\partial x}+\frac{\partial^2 b}{\partial x\partial t}\right)Z^{(n)}\Im_x\frac{\partial Z^{(n)}}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le 2\int_{Q^\tau}\left\{\left(\frac{\partial a}{\partial x}\right)^2+\left(\frac{\partial^2 b}{\partial x\partial t}\right)^2\right\}\left(Z^{(n)}\right)^2\mathrm{d}x\,\mathrm{d}t \\
&\qquad+\int_{Q^\tau}\left(\Im_x\frac{\partial Z^{(n)}}{\partial t}\right)^2\mathrm{d}x\,\mathrm{d}t,
\end{aligned}\tag{3.30}$$

$$\begin{aligned}
&-2\int_{Q^\tau}\frac{\partial b}{\partial x}\frac{\partial Z^{(n)}}{\partial t}\Im_x\frac{\partial Z^{(n)}}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le c_4\int_{Q^\tau}\left(\frac{\partial Z^{(n)}}{\partial t}\right)^2\mathrm{d}x\,\mathrm{d}t+\frac{1}{c_4}\int_{Q^\tau}\left(\frac{\partial b}{\partial x}\right)^2\left(\Im_x\frac{\partial Z^{(n)}}{\partial t}\right)^2\mathrm{d}x\,\mathrm{d}t,
\end{aligned}\tag{3.31}$$

$$\begin{aligned}
&-2\int_{Q^\tau}\Im_x\left(cZ^{(n)}\right)\Im_x\frac{\partial Z^{(n)}}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le\frac{(\beta-\alpha)^2}{2}\int_{Q^\tau}c^2\left(Z^{(n)}\right)^2\mathrm{d}x\,\mathrm{d}t+\int_{Q^\tau}\left(\Im_x\frac{\partial Z^{(n)}}{\partial t}\right)^2\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{3.32}$$

Therefore, by formulas (3.29)–(3.32) and condition **C(1)**, we obtain

$$\begin{aligned}\int_0^\tau \left\| \frac{\partial Z^{(n)}}{\partial t} \right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t + \left\| Z^{(n)}(x,\tau) \right\|^2_{L^2(\alpha,\beta)} + \left\| \Im_x \frac{\partial Z^{(n)}(x,\tau)}{\partial t} \right\|^2_{L^2(\alpha,\beta)} \\ \leq \left\{ \int_0^\tau \left\| \Im_x P^{(n-1)} \right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t \right\} \\ + c_{14} \int_0^\tau \left\{ \left\| Z^{(n)} \right\|^2_{L^2(\alpha,\beta)} + \left\| \Im_x \frac{\partial Z^{(n)}}{\partial t} \right\|^2_{L^2(\alpha,\beta)} \right\} \mathrm{d}t,\end{aligned} \tag{3.33}$$

where

$$c_{14} = \frac{\max\left(c_2 + c_8 + c_3^2 + c_9^2 + \frac{(\beta-\alpha)^2}{2} c_{10}^2, 3 + c_7^2/c_4 \right)}{\min\left(c_4, c_0 + c_5, 1\right)}.$$

Estimating the last integral on the right hand side of inequality (3.33). To this end, using Gronwall's Lemma, it follows

$$\begin{aligned}\int_0^\tau \left\| \frac{\partial Z^{(n)}}{\partial t} \right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t + \left\| Z^{(n)}(x,\tau) \right\|^2_{L^2(\alpha,\beta)} + \left\| \Im_x \frac{\partial Z^{(n)}(x,\tau)}{\partial t} \right\|^2_{L^2(\alpha,\beta)} \\ \leq c_{15} \left\{ \int_0^T \left\| \Im_x P^{(n-1)} \right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t \right\},\end{aligned}$$

where $c_{15} = \exp\left(c_{14} T\right)$.

The right-hand side here is independent of τ. Hence we replace the left-hand side by the upper bound with respect to τ. Then we get

$$\begin{aligned}\int_0^\tau \left\| \frac{\partial Z^{(n)}}{\partial t} \right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t + \left\| \Im_x \frac{\partial Z^{(n)}(x,\tau)}{\partial \tau} \right\|^2_{L^2(\alpha,\beta)} + \left\| Z^{(n)}(x,\tau) \right\|^2_{L^2(\alpha,\beta)} \\ \leq c_{15} \int_0^\tau \left\| \Im_x P^{(n-1)} \right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t,\end{aligned} \tag{3.34}$$

in light of the last inequality, we have

$$\left\| \Im_x P^{(n-1)} \right\|^2_{L^2(\alpha,\beta)} \leq 4\left(\beta-\alpha\right)^2 \left\| P^{(n-1)} \right\|^2_{L^2(\alpha,\beta)}.$$

So, by virtue of condition ($\mathbf{H}_2$), we obtain

$$\begin{aligned}&4\left(\beta-\alpha\right)^2 \left(\int_{Q^\tau} \left(P^{(n-1)} \right)^2 \mathrm{d}x\, \mathrm{d}t \right) \\ &\leq 4\left(\beta-\alpha\right)^2 \delta^2 \int_{Q^\tau} \left(\left| Z^{(n-1)}(x,t) \right| + \left| \frac{\partial Z^{(n-1)}(x,t)}{\partial x} \right| \right)^2 \mathrm{d}x\, \mathrm{d}t \\ &\leq 8\left(\beta-\alpha\right)^2 \delta^2 \int_0^\tau \left(\left\| Z^{(n-1)}(\cdot,t) \right\|^2_{L^2(\alpha,\beta)} + \left\| \frac{\partial Z^{(n-1)}(\cdot,t)}{\partial x} \right\|^2_{L^2(\alpha,\beta)} \right) \mathrm{d}t.\end{aligned} \tag{3.35}$$

Substituting (3.35) into (3.34), we get

$$\begin{aligned}&\int_0^\tau \left\|\frac{\partial Z^{(n)}}{\partial t}\right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t + \left\|\frac{\partial Z^{(n)}}{\partial t}\right\|^2_{B_2^1(\alpha,\beta)} + \left\|Z^{(n)}\right\|^2_{L^2(\alpha,\beta)} \\ &\qquad \leq 8c_{15}\left(\beta-\alpha\right)^2\delta^2 \int_0^\tau \left(\left\|Z^{(n-1)}\left(\cdot,t\right)\right\|^2_{L^2(\alpha,\beta)} + \left\|\frac{\partial Z^{(n-1)}\left(\cdot,t\right)}{\partial x}\right\|^2_{L^2(\alpha,\beta)}\right) \mathrm{d}t,\end{aligned} \tag{3.36}$$

On the other hand, applying the operator $\Im_x$ to equation (3.19), we get

$$\Im_x \frac{\partial^2 Z^{(n)}}{\partial t^2} - \left(a(x,t)\frac{\partial Z^{(n)}}{\partial x}\right) - \frac{\partial}{\partial t}\left(b(x,t)\frac{\partial Z^{(n)}}{\partial x}\right) + \Im_x\left(c(x,t)Z^{(n)}\right) = \Im_x\left(P^{(n-1)}\right).$$

Multiplying the obtained equality by $-\frac{\partial^2 Z^{(n)}}{\partial x \partial t}$ and integrating over $Q^\tau := (\alpha,\beta)\times(0,\tau)$, where $0 \leq \tau \leq T$, and using conditions (3.20), (3.21) and (3.22), we obtain

$$\begin{aligned}&\frac{1}{2}\left\|\frac{\partial Z^{(n)}}{\partial t}\right\|^2_{L^2(\alpha,\beta)} + \frac{c_0+c_5}{2}\left\|\frac{\partial Z^{(n)}}{\partial x}\right\|^2_{L^2(\alpha,\beta)} + c_4\int_0^\tau \left\|\frac{\partial^2 Z^{(n)}}{\partial x \partial t}\right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t \\ &\quad \leq \frac{c_2+c_8}{2}\int_0^\tau \left\|\frac{\partial Z^{(n)}}{\partial x}\right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t + \frac{c_{10}^2}{2\varepsilon}\int_0^\tau \left\|Z^{(n)}\right\|^2_{B_2^1(\alpha,\beta)} \mathrm{d}t \\ &\quad\quad + \frac{\varepsilon}{2}\int_0^\tau \left\|\frac{\partial^2 Z^{(n)}}{\partial x \partial t}\right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t + \frac{1}{2}\int_0^\tau \left\|\frac{\partial Z^{(n)}}{\partial t}\right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t + \frac{1}{2}\int_0^\tau \left\|P^{(n-1)}\right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t\end{aligned} \tag{3.37}$$

where we have

$$\left\|Z^{(n)}\right\|^2_{B_2^1(\alpha,\beta)} \leq \frac{(\beta-\alpha)^2}{2}\left\|Z^{(n)}\right\|^2_{L^2(\alpha,\beta)}. \tag{3.38}$$

By using (3.35) and (3.38), the inequality (3.37) becomes

$$\begin{aligned}&\frac{1}{2}\left\|\frac{\partial Z^{(n)}}{\partial t}\right\|^2_{L^2(\alpha,\beta)} + \frac{c_0+c_5}{2}\left\|\frac{\partial Z^{(n)}}{\partial x}\right\|^2_{L^2(\alpha,\beta)} + c_4\int_0^\tau \left\|\frac{\partial^2 Z^{(n)}}{\partial x \partial t}\right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t \\ &\quad \leq \frac{c_2+c_8}{2}\int_0^\tau \left\|\frac{\partial Z^{(n)}}{\partial x}\right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t + \frac{(\beta-\alpha)^2 c_{10}^2}{4\varepsilon}\int_0^\tau \left\|Z^{(n)}\right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t \\ &\quad\quad + \frac{\varepsilon}{2}\int_0^\tau \left\|\frac{\partial^2 Z^{(n)}}{\partial x \partial t}\right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t + \frac{1}{2}\int_0^\tau \left\|\frac{\partial Z^{(n)}}{\partial t}\right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t \\ &\quad\quad + \delta^2\int_0^\tau \left(\left\|Z^{(n-1)}\left(\cdot,t\right)\right\|^2_{L^2(\alpha,\beta)} + \left\|\frac{\partial Z^{(n-1)}\left(\cdot,t\right)}{\partial x}\right\|^2_{L^2(\alpha,\beta)}\right) \mathrm{d}t.\end{aligned} \tag{3.39}$$

Combining (3.36) and (3.39), and by putting $\varepsilon = c_4$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^\tau \left\| \frac{\partial Z^{(n)}}{\partial t} \right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t + \left\| \frac{\partial Z^{(n)}}{\partial t} \right\|^2_{B_2^1(\alpha,\beta)} + \frac{c_4}{2}\int_0^\tau \left\| \frac{\partial^2 Z^{(n)}}{\partial x \partial t} \right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t \\
&\quad + \left\| Z^{(n)} \right\|^2_{L^2(\alpha,\beta)} + \frac{c_0 + c_5}{2} \left\| \frac{\partial Z^{(n)}}{\partial x} \right\|^2_{L^2(\alpha,\beta)} \\
&\qquad \le \frac{c_2 + c_8}{2} \int_0^\tau \left\| \frac{\partial Z^{(n)}}{\partial x} \right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t + \frac{(\beta-\alpha)^2 c_{10}^2}{4c_4} \int_0^\tau \left\| Z^{(n)} \right\|^2_{L^2(\alpha,\beta)} \mathrm{d}t \\
&\qquad\quad + \left(8c_{16}(\beta-\alpha)^2 + 1\right)\delta^2 \int_0^\tau \left(\left\| Z^{(n-1)}(\cdot,t) \right\|^2_{H^1(\alpha,\beta)} \right) \mathrm{d}t.
\end{aligned} \tag{3.40}$$

After discarding the first three terms on the left-hand side of (3.40), by Gronwall's Lemma, it follows that

$$\left\| Z^{(n)} \right\|^2_{L^2(\alpha,\beta)} + \left\| \frac{\partial Z^{(n)}}{\partial x} \right\|^2_{L^2(\alpha,\beta)} \le g^* \left\| Z^{(n-1)} \right\|_{L^2(0,T;\, H^1(\alpha,\beta))}, \tag{3.41}$$

where

$$g^* = \exp\left(\max\left(\frac{c_2 + c_8}{2}, \frac{(\beta-\alpha)^2 c_{10}^2}{4c_4} \right) T \right) \frac{\left(8c_{16}(\beta-\alpha)^2 + 1\right)\delta^2}{\min\left(1, \frac{c_0+g}{2}\right)}.$$

Integrating the resulted inequality over the interval $(0,T)$, we obtain

$$\left\| Z^{(n)} \right\|^2_{L^2(0,T;\, H^1(\alpha,\beta))} \le T g^* \left\| Z^{(n-1)} \right\|_{L^2(0,T;\, H^1(\alpha,\beta))}. \tag{3.42}$$

From the criteria of convergence of series, we see that the series $\sum_{n=1}^{\infty} Z^{(n)}$ converges if $Tg^* < 1$, that is, if

$$\delta < \sqrt{\frac{\min\left(1, \frac{c_0+g}{2}\right)}{T\exp\left(\max\left(\frac{c_2+c_8}{2}, \frac{(\beta-\alpha)^2 c_{10}^2}{4c_4}\right)T\right)\left(8c_{16}(\beta-\alpha)^2+1\right)}}.$$

Since $Z^{(n)}(x,t) = y^{(n+1)}(x,t) - y^{(n)}(x,t)$, then it follows that the sequence $\{y^{(n)}\}_{n\in N}$ defined by

$$y^{(n)}(x,t) = \sum_{i=0}^{n-1} Z^{(i)} + y^{(0)}(x,t),$$

converges to an element $y \in L^2\left(0,T;\ H^1(\alpha,\beta)\right)$. $\square$

We must show that the limit function y is a solution of the problem under study. To do this, we will show that y satisfies the conditions (3.7) and (3.14) as mentioned in Definition 2. So, we consider the weak formulation of problem (3.15)–(3.18),

$$A(y,v) = -\int_Q v \Im_x G\left(x,t,y^{(n-1)}, \frac{\partial y^{(n-1)}}{\partial x}\right) \mathrm{d}x\,\mathrm{d}t. \tag{3.43}$$

From (3.43), we have

$$\begin{aligned} &A\left(y^{(n)}-y,v\right)+A\left(y,v\right)\\ &\quad=-\int_Q v\left(\Im_x G\left(x,t,y^{(n-1)},\frac{\partial y^{(n-1)}}{\partial x}\right)-\Im_x G\left(x,t,y,\frac{\partial y}{\partial x}\right)\right)\mathrm{d}x\,\mathrm{d}t\\ &\qquad-\int_Q v\Im_x G\left(x,t,y,\frac{\partial y}{\partial x}\right)\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.44}$$

However, by applying Cauchy–Schwarz inequality, we get

$$\begin{aligned} &A\left(y^{(n)}-y,v\right)\\ &=-\int_Q v\Im_x\left(\frac{\partial^2 y^{(n)}}{\partial t^2}-\frac{\partial^2 y}{\partial t^2}\right)\mathrm{d}x\,\mathrm{d}t+\int_Q va(x,t)\left(\frac{\partial y^{(n)}}{\partial x}-\frac{\partial y}{\partial x}\right)\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_Q v\frac{\partial}{\partial t}\left[b(x,t)\left(\frac{\partial y^{(n)}}{\partial x}-\frac{\partial y}{\partial x}\right)\right]\mathrm{d}x\,\mathrm{d}t-\int_Q v\Im_x\left[c(x,t)\left(y^{(n)}-y\right)\right]\mathrm{d}x\,\mathrm{d}t\\ &\qquad\le g_2\left\|v\right\|_{L^2(0,T;\,L_2(\alpha,\beta))}\left[\begin{array}{c}\left\|y^{(n)}-y\right\|_{L^2(0,T;\,H^1(\alpha,\beta))}\\ +\left\|\frac{\partial^2 y^{(n)}}{\partial t^2}-\frac{\partial^2 y}{\partial t^2}\right\|_{L^2(0,T;\,L_2(\alpha,\beta))}\\ +\left\|\frac{\partial^2 y^{(n)}}{\partial x\partial t}-\frac{\partial^2 y}{\partial x\partial t}\right\|_{L^2(0,T;\,L_2(\alpha,\beta))}\end{array}\right], \end{aligned} \tag{3.45}$$

where

$$g_2=\max\left(c_5,\left(c_1+c_6\right),\frac{\left(\beta-\alpha\right)c_{10}}{\sqrt{2}}\right),$$

and we have

$$\begin{aligned} &-\int_Q v\left(\Im_x G\left(x,t,y^{(n-1)},\frac{\partial y^{(n-1)}}{\partial x}\right)-\Im_x G\left(x,t,y,\frac{\partial y}{\partial x}\right)\right)\mathrm{d}x\,\mathrm{d}t\\ &\qquad\le\frac{\left(\beta-\alpha\right)c_{10}}{\sqrt{2}}\delta\left(\left\|y^{(n)}-y\right\|_{L^2(0,T;\,H^1(\alpha,\beta))}\left\|v\right\|_{L^2(0,T;\,L^2(\alpha,\beta))}\right). \end{aligned} \tag{3.46}$$

Taking into account (3.45) and (3.46), and passing to the limit in (3.44) as $n\to\infty$, we obtain

$$A\left(y,v\right)=-\int_\Omega v\Im_x G\left(x,t,y,\frac{\partial y}{\partial x}\right)\mathrm{d}x\,\mathrm{d}t.$$

Therefore, we have established the following result:

Theorem 3 *Assume that condition* ($\mathbf{H}_2$) *holds and*

$$\delta<\sqrt{\frac{\min\left(1,\frac{c_0+g}{2}\right)}{T\exp\left(\max\left(\frac{c_2+c_8}{2},\frac{(\beta-\alpha)^2c_{10}^2}{4c_4}\right)T\right)\left(8c_{16}\left(\beta-\alpha\right)^2+1\right)}}.$$

Then problem (3.5)–(3.8) *admits a weak solution in* $L^2\left(0,T;\;H^1(0,b)\right)$.

It remains to prove that problem (3.5)–(3.8) admits a unique solution.

Theorem 4 *Under the condition* ($\mathbf{H}_2$)*, the solution of the problem* (3.5)–(3.8) *is unique.*

Proof. Suppose that y_1 and y_2 in $L^2\left(0,T;\ H^1(\alpha,\beta)\right)$ are two solutions of (3.5)–(3.8). Then $h = y_1 - y_2$ satisfies $h \in L^2\left(0,T;\ H^1(\alpha,\beta)\right)$ and

$$\frac{\partial^2 h}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial h}{\partial x}\right) - \frac{\partial^2}{\partial t \partial x}\left(b(x,t)\frac{\partial h}{\partial x}\right) + c(x,t)h = \psi\left(x,t\right), \quad (x,t) \in \overline{\Omega}, \tag{3.47}$$

$$\begin{aligned} h(x,0) &= 0, \\ \frac{\partial h(x,0)}{\partial t} &= 0, \end{aligned} \tag{3.48}$$

$$\frac{\partial h(\alpha,t)}{\partial x} = 0, \tag{3.49}$$

$$\int_\alpha^\beta h\left(x,t\right)\, \mathrm{d}x = 0, \tag{3.50}$$

$$\psi\left(x,t\right) = G\left(x,t,y_1,\frac{\partial y_1}{\partial x}\right) - G\left(x,t,y_2,\frac{\partial y_2}{\partial x}\right).$$

Following the same procedure as in the proof of Lemma 1, for problem (3.47)–(3.50) we get

$$\|h\|_{L^2(0,T;\,H^1(\alpha,\beta))} \leq Tg^* \,\|h\|_{L^2(0,T;\,H^1(\alpha,\beta))}. \tag{3.51}$$

Since $Tg^* < 1$, then from (3.51) it follows that

$$\left(1 - Tg^*\right) \|h\|_{L^2(0,T;\,H^1(\alpha,\beta)} \leq 0,$$

from which we conclude that $y_1 = y_2$ in $L^2\left(0,T;\ H^1(\alpha,\beta)\right)$. □

4 Conclusion

In this article, by using a priori estimate and applying an iterative process based on the results obtained for the linear problem, we prove the existence, uniqueness and continuous dependence of the weak generalized solution of the nonlinear mixed problem for nonlinear hyperbolic equation with integral condition. This new study is a generalisation of the work of Bouziani in [7] where he studied the linear problem. Also, this problem is very interesting in the modelling of the dynamics of groundwater, population models, complex dynamics and also it seems to be very important in the study of the theory of partial differential nonlinear equations in functional Sobolev spaces.

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