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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO NONLINEAR NONLOCAL FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS

MADHUKANT SHARMA[*], **SHRUTI DUBEY**[†]
Department of Mathematics, Indian Institute of Technology Madras,
Chennai-600 036, India

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Abstract. We investigate a nonlinear nonlocal fractional functional differential equations in a Banach space associated with the family of linear closed operators $\{-A(t) : t \geq 0\}$. The object of our study is to determine the asymptotic behaviour of solutions. Also, we render criteria for stability of the zero solution. We establish our results with the assumption that $-A(t)$ generates a resolvent operator for each $t \geq 0$ and the nonlinear part is continuous in all variables and satisfies certain conditions. We conclude the article with an application of the developed results in which we discuss a nonlocal nonlinear partial fractional functional differential equation.

Keywords: Functional differential equation, fractional calculus, analytic semigroup, resolvent operator.

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1 Introduction

The qualitative theory of infinite dimensional systems has gained a lot of intrinsic interest of many researchers due to its demonstrated applications in biological and ecological systems (see the books of R. Aris [1] and P. Fitzhugh [6]). Nowadays, authors are studying related problems and steadily using an impressive amount of very sophisticated mathematics in their analysis, synthesis, and design of systems (see [2], and the references listed therein).

[*]e-mail address: sharmamk003@gmail.com
[†]e-mail address: sdubey@iitm.ac.in

The prime concern of our paper is to determine the asymptotic behaviour of solutions to the following nonlinear fractional order functional differential equation in a Banach space with a nonlocal condition:

$$\begin{cases} \dfrac{\mathrm{d}^\alpha u(t)}{\mathrm{d}t^\alpha} + A(t)u(t) = f(t, u(t), u_t) \text{ for } t \geq 0, \\ h\big(u_{[-\tau,0]}\big) = \phi. \end{cases} \tag{1.1}$$

$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}$ is understood in the Riemann–Liouville sense with $0 < \alpha \leq 1$ and $\tau > 0$ be a constant. $-A(t)$, $t \geq 0$ is a closed linear operator defined on a dense domain $D(A)$ in X into X such that $D(A)$ is independent of t. $\{-A(t) : t \geq 0\}$ generates a strongly continuous semigroup of evolution operators in a Banach space X. $f\colon [0, \infty) \times X \times C_0 \to X$, $h\colon C_0 \to C_0$ (C_0 will be defined later) are nonlinear maps and for any $t \geq 0$, $u_t \in C_0$ denotes a segment of $u(\cdot)$ at t which is defined by $u_t(s) = u(t + s)$.

Over the past years, a significant development has been done in the direction of functional differential equations due to their applications in various physical problems of science and engineering (see the book of Hale and Verduyn [8]). Most of these works are devoted to the study of functional differential equations of integer order (see for example [4] and the references therein). It is worth mentioning that the differential equations involving differential operators of fractional order describe many physical phenomena such as fluid flows, signal processing, diffusion processes, etc. in a better way than the integer order systems (see [13]). Therefore, in recent years, a considerable attention has been paid to investigate various type of fractional differential equations [9, 10].

On the other hand, it is very important to study the asymptotic behaviour of solutions to distributed parameter systems due to their demonstrated applications in many areas. Many authors have discussed the asymptotic behaviour of solutions and the approach followed by most of the authors rely on proving the existence of asymptotically almost automorphic solutions (see [15] and the references listed therein). The concept of asymptotically almost automorphic functions was firstly introduced by N'Guérékata [11]. Since then these functions have become of great interest to several mathematicians and gained lots of developments and applications. Samuel [14] discussed the asymptotic behaviour of mild solutions of functional differential equations in Banach spaces with quite different approach based on the fractional power of operators, analytic semigroup and evolution operators.

Our aim is to contribute more in this direction. We consider the problem along with a more general nonlocal condition which is the generalization of the nonlocal condition $g(u_0) = x$ with $u_0 \in C_0$ and $x \in X$ (see [5]). We have already established the existence of solutions to (1.1) in [5] under some certain hypotheses which are stated in Section 2. Our main objective in this paper is to discuss the asymptotic behaviour of solutions to (1.1). We establish continuous dependence of solutions on initial data by using Gronwall's inequality and also present the criteria for the zero solution to be stable. To discuss the asymptotic behaviour of solutions, we use the technique similar to [14] with suitable modifications so as to be compatible with fractional order delay systems. Our approach relies mainly on the theory of fractional powers of operators, analytic semigroup, resolvent operators and techniques of nonlinear analysis.

The organization of the paper is as follows. We briefly recall some definitions and preliminary facts about the fractional differential equations in Section 2. The main results concerning the asymptotic behaviour of solutions, continuous dependence of solutions on initial data and the criteria for the stability of the zero solution are given in Section 3. As an application of the developed results, we discuss a nonlocal partial differential equation of fractional order in Section 4.

2 Preliminary results

This section introduces definitions, assumptions, and preliminary facts which are used throughout this paper. The fractional derivative of the function f of order $0 < \alpha < 1$ is defined as [13]

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t (t-s)^{-\alpha} f(s)\,\mathrm{d}s,$$

where f is an abstract continuous function on the interval $[a, b]$.

Suppose E is the Banach space formed from $D(A)$ with the graph norm. Then we have the following definition of resolvent operator generated by $-A(t)$ (see [3]).

Definition 2.1 (Resolvent operator) *Let $0 \le s \le t \le T < \infty$. A resolvent operator for the problem* (1.1) *is a bounded operator-valued function $R(t,s) \in \mathbb{BL}(X)$, having the following properties*:

(i) *$R(t,s)$ is strongly continuous in s and t, $R(s,s) = I$, $0 \le s \le T$, and $\|R(t,s)\| \le Me^{p(t-s)}$ for some constants M, p*;

(ii) *$R(t,s)E \subset E$, $R(t,s)$ is strongly continuous in s and t on E*;

(iii) *for $x \in X$, $R(t,s)x$ is continuously differentiable in $s \in [0,T]$ and*

$$\frac{\partial R(t,s)x}{\partial s} = R(t,s)A(s)x;$$

(iv) *for $x \in X$ and $s \in [0,T]$, $R(t,s)x$ is continuously differentiable in $t \in [s,T]$ and*

$$\frac{\partial R(t,s)x}{\partial t} = -A(t)R(t,s)x,$$

with $\frac{\partial R(t,s)x}{\partial s}$ and $\frac{\partial R(t,s)x}{\partial t}$ strongly continuous on $0 \le s \le t \le T$.

Furthermore, it is assumed that $-A(t)$ satisfies the following assumptions ([7, pp. 108]):

(B1) for each $t \in [0,T]$ the operator $[\lambda I + A(t)]^{-1}$ exists for all λ with $\Re(\lambda) \ge 0$ and

$$\left\| [\lambda I + A(t)]^{-1} \right\| \le \frac{C}{|\lambda| + 1} \quad (\Re(\lambda) \ge 0);$$

(B2) for any t, s, $\zeta \in [0,T]$ we have $\left\| [A(t) - A(\zeta)]A^{-1}(s) \right\| \le C|t-\zeta|^\delta$ $(0 < \delta < 1)$,

where the constants C, δ are independent of t, s, ζ.

Then, under the assumptions (B1) and (B2), we can define the fractional powers of the operator $-A(t)$ (see [12]) as follows:

$$A^{-q}(t) := \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{q-1} e^{-sA(t)}\,\mathrm{d}s,$$

for each $t \in [0,T]$ and $q \in (0,1]$. We note that $A^{-q}(t)$ is a bounded linear operator. Since $0 \in \rho(A(t))$ and $A^{-q}(t)$ is one-to-one, $A^q(t) := (A^{-q}(t))^{-1}$ is a closed linear operator with $D(A^q(t))$ dense in X. Moreover, $\big(D(A^q(t)), \|\cdot\|_{q,t}\big) = X_q(t)$ forms a Banach space with the norm $\|x\|_{q,t} = \|A^q(t)x\|$.

We fix the space $X_q(t_0)$ for some $t_0 \in [0,T]$ and let $C_t := C\big([-\tau,t], X_q(t_0)\big)$ be the space of all continuous functions, endowed with the norm

$$\|\psi\|_t := \sup_{-\tau \le \eta \le t} \|\psi(\eta)\|_{q,t_0}, \quad \psi \in C_t.$$

Then, the space $\big(C_t, \|\cdot\|_t\big)$ forms a Banach space.

On comparing the definitions of resolvent operator and evolution operator, and following the results in Friedman ([7, pp. 160]) and Pazy ([12, Section 2.6]), we can deduce the following lemmas.

Lemma 2.1 *If the conditions* (B1) *and* (B2) *hold, and if* $0 \le \gamma \le 1$, $0 \le \beta \le 1$, *then for any* $0 \le s < t \le T$, $0 \le \xi < T$, *we have*

$$\Big\| A^{\gamma}(\xi)\big[R(t,s) - e^{-(t-s)A(t)}\big]A^{-\beta}(s)\Big\| \le K(\beta,\gamma)|\, t-s\,|^{\delta+\beta-\gamma},$$

where $K(\beta,\gamma)$ *indicates the dependence of constants on* β *and* γ.

Lemma 2.2 *Let* $A(t)$ *be the infinitesimal generator of a resolvent operator* $R(t,s)$. *We denote by* $\rho[A(t)]$ *the resolvent set of* $A(t)$. *If* $0 \in \rho[A(t)]$, *then*

(a) $R(t,s)\colon X \to D(A^q(t))$ *for every* $0 \le s \le t \le T$ *and* $q \ge 0$;

(b) *for every* $u \in D(A^q(t))$ *we have* $R(t,s)A^q(t)u = A^q(t)R(t,s)u$;

(c) *for* $\xi \in [0,T)$ *we have* $\|A^q(\xi)R(t,s)\| \le M_{q,p}|\, t-s\,|^{-q}$.

Employing the Banach contraction principle and resolvent operator theory, in [5] we have established the existence of a mild solution $u \in C_T$ of (1.1) for some $0 < T < \infty$ under the following assumptions.

(H1) The closed linear operator $-A(t)$ generates the resolvent operator $R(t,s)$ with $\|R(t,s)\| \le Me^{p(t-s)}$, $0 \le s \le t \le T$.

(H2) The map $h\colon C_0 \to C_0$ and there exists Lipschitz continuous function $\chi \in C_0$ such that $h(\chi) = \phi$ with $\chi(0) \in D(A)$.

(H3) Let G be an open subset of $[0,\infty) \times X_q(t_0) \times C_0 = \tilde{X}$. For all $(t,x,y) \in G$ there exist constant $L > 0$ and a neighbourhood $F \subseteq G$ of (t,x,y) such that the nonlinear map f is continuous with respect to the first variable on F and it satisfies the following condition on F

$$\|f(t,u,v) - f(t,w,p)\| \le L\big(\|u-w\|_{q,t_0} + \|v-p\|_0\big) \text{ for all } (t,u,v) \text{ and } (t,w,p) \in F.$$

We remark that the mild solution to (1.1) is defined as follows. Let $\chi \in C_0$ be such that $h(\chi) = \phi$. The function $u \in C_T, 0 < T < \infty$, such that

$$u(t) = \begin{cases} \chi(t), & \text{if } t \in [-\tau, 0], \\ R(t,0)\chi(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} R(t,s) f(s, u(s), u_s)\, \mathrm{d}s, & \text{if } t \in [0, T], \end{cases}$$

is called a mild solution of (1.1) on $[-\tau, T]$ (see [3, 5]).

In the subsequent sections, we establish the continuous dependence of solutions on initial data and discuss the asymptotic behaviour of existing solutions.

3 Main results

3.1 Continuous dependence of solutions on initial data

In this subsection, we establish the continuous dependence of solutions on initial conditions by using Gronwall's inequality.

Theorem 3.1 *Let $f\colon [0,T] \times X_q(t_0) \times C_0 \to X$ with $0 < q < \alpha$ and let the hypotheses* (H1)–(H3) *hold. Also, let $0 \in \rho[-A(t)]$. Let $u(t)$ and $\tilde{u}(t)$ solve* (1.1) *for χ, $\tilde{\chi} \in C_0$, respectively. Then, for $t \geq 0$*

$$\|u_t - \tilde{u}_t\|_0 \leq \begin{cases} 2\,\|\chi - \tilde{\chi}\|_0\, e^{(p+M_2)N(t)}, & \text{if } p \geq 0, \\ \|\chi - \tilde{\chi}\|_0\, e^{-p\tau}\Big[1 + e^{\left(p + M_2 e^{-p\tau}\right)N(t)}\Big], & \text{if } p < 0, \end{cases}$$

where $M_2 = \frac{2LM_{q,p}}{(\alpha-q)\Gamma(\alpha)}$ and $N(t) = \max\{t, t^{\alpha-q}\}$.

Proof. For $t \geq 0$, from (H3), Lemma 2.2 and the definition of a mild solution, it follows that

$$\|u(t) - \tilde{u}(t)\|_{q,t_0} \leq e^{pt}\, \|\chi(0) - \tilde{\chi}(0)\|_{q,t_0} + \frac{2LM_{q,p}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-q-1}\, \|u_s - \tilde{u}_s\|_0\, \mathrm{d}s. \tag{3.1}$$

(i) Let $p > 0$. Then from (3.1), for $t \geq 0$, we have

$$\|u_t - \tilde{u}_t\|_0 \leq e^{pt}\, \|\chi - \tilde{\chi}\|_0 + M_2(\alpha - q) \int_0^t (t-s)^{\alpha-q-1}\, \|u_s - \tilde{u}_s\|_0\, \mathrm{d}s.$$

Using Gronwall's inequality we have

$$\begin{aligned} \|u_t - \tilde{u}_t\|_0 &\leq e^{pt}\, \|\chi - \tilde{\chi}\|_0 + M_2\, \|\chi - \tilde{\chi}\|_0\, (\alpha - q) \\ &\qquad \times \int_0^t (t-s)^{\alpha-q-1} \exp\left(ps + M_2(\alpha-q) \int_s^t (t-r)^{\alpha-q-1}\, \mathrm{d}r\right) \mathrm{d}s \\ &\leq e^{pt}\, \|\chi - \tilde{\chi}\|_0 + M_2\, \|\chi - \tilde{\chi}\|_0\, (\alpha-q) e^{pt} \\ &\qquad \times \int_0^t (t-s)^{\alpha-q-1} \exp\left(M_2 (t-s)^{\alpha-q}\right)\, \mathrm{d}s \\ &\leq \left[1 + \exp\left(M_2 t^{\alpha-q}\right)\right] e^{pt}\, \|\chi - \tilde{\chi}\|_0 \\ &\leq 2\, \|\chi - \tilde{\chi}\|_0\, e^{(p+M_2)N(t)}. \end{aligned} \tag{3.2}$$

(ii) Let $p < 0$. Then from (3.1), for $t \geq 0$, we have

$$\|u_t - \tilde{u}_t\|_0 \leq e^{p(t-\tau)} \|\chi - \tilde{\chi}\|_0 + M_2(\alpha - q)e^{-p\tau} \int_0^t (t-s)^{\alpha-q-1} \|u_s - \tilde{u}_s\|_0 \, \mathrm{d}s.$$

Again by Gronwall's inequality, we get

$$\begin{aligned} \|u_t - \tilde{u}_t\|_0 &\leq e^{p(t-\tau)} \|\chi - \tilde{\chi}\|_0 + M_2(\alpha - q)e^{-2p\tau} \|\chi - \tilde{\chi}\|_0 \\ &\quad \times \int_0^t (t-s)^{\alpha-q-1} \exp\left(ps + M_2(\alpha-q)e^{-p\tau} \int_s^t (t-r)^{\alpha-q-1} \, \mathrm{d}r\right) \mathrm{d}s \\ &\leq e^{p(t-\tau)} \|\chi - \tilde{\chi}\|_0 + M_2(\alpha - q)e^{-2p\tau} \|\chi - \tilde{\chi}\|_0 \\ &\quad \times \int_0^t (t-s)^{\alpha-q-1} \exp\left(ps + M_2(t-s)^{\alpha-q}\right) \mathrm{d}s \\ &\leq \|\chi - \tilde{\chi}\|_0 \, e^{-p\tau} \left[1 + e^{\left(p + M_2 e^{-p\tau}\right) N(t)}\right]. \end{aligned} \tag{3.3}$$

Hence, equations (3.2) and (3.3) imply our claim. This completes the proof. □

Remark 3.1 *In addition to hypothesis* (H3)*, if we assume that* $f(t, 0, 0) = 0$ *for* $t \geq 0$ *and* $h(0) = 0$*, then* $u = 0$ *is always a solution of* (1.1) *with* $\chi = 0$*. We say that the solution* $u = 0$ *is stable if for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that the solution* $u(t)$ *of* (1.1) *with* $\|\chi\|_0 < \delta$ *satisfies* $\|u(t)\|_{q,t_0} < \epsilon$ *for all* $t \geq 0$*. Theorem 3.1 gives sufficient condition for the solution* $u = 0$ *of* (1.1) *to be stable, in terms of the growth rate of the resolvent operator* $\{R(t,s)\}_{t \geq s \geq 0}$ *and the Lipschitz constant of* f*. Namely, if* $(p + M_2 e^{-p\tau}) \leq 0$*, then the solution* $u = 0$ *is stable.*

3.2 Asymptotic behaviour of solutions

In this subsection, we discuss the asymptotic behaviour of solutions to (1.1). We require the following hypotheses to prove our main result.

(H4) The assumptions (B1), (B2) hold for all $T < \infty$ with C independent of T and

$$\sup_{0 < t, s < \infty} \left\|A(t)A^{-1}(s)\right\| < \infty.$$

(H5) There exists a closed operator $A(\infty)$ with bounded inverse and domain $D(A)$ such that

$$\left\|(A(t) - A(\infty))A^{-1}(0)\right\| \to 0 \text{ if } t \to 0.$$

If (H4) and (H5) hold, then we can deduce the following bounds ([7, p. 153]):

$$\begin{cases} \|A^q(t)R(t,s)\| \leq K e^{w(t-s)}(t-s)^{-q}, & t - s > 0, w < 0, \\ \|A^q(t_0)R(t,s)\| \leq \tilde{K} e^{w(t-s)}(t-s)^{-q}, & \end{cases} \tag{3.4}$$

where K and $\tilde{K}$ are independent of t, s and $R(t,s)$ is defined for all $0 \leq s \leq t < \infty$.

(H6) $f(\cdot,\cdot,\cdot)$ is continuous in all variables with $0 < q < \alpha$ and satisfies

$$\|f(t,x,y)\| \le L\big(\|x\|_{q,t_0} + \|y\|_0 + \eta(t)\big),$$

where $\eta(t)$ is continuous on $[0,\infty)$ and $L < \frac{e^{w\tau}(-w)^{\alpha-q}\Gamma(\alpha)}{2\tilde{K}\Gamma(\alpha-q)}$.

Theorem 3.2 *Assume that the hypotheses* (H4)–(H6) *hold. Then a solution of* (1.1) *satisfies*

(i) $\|u_t\|_0$ *is bounded as* $t \to \infty$ *if* $\eta(t)$ *is bounded,*

(ii) $\|u_t\|_0 = O(e^{\delta t})$, *where* $w + \left(\frac{2e^{-w\tau}\tilde{K}L\Gamma(\alpha-q)}{\Gamma(\alpha)}\right)^{1/(\alpha-q)} < \delta < 0$ *if* $\eta(t) = O(e^{\delta t})$, *and*

(iii) $\|u_t\|_0 = o(1)$ *if* $\eta(t) = o(1)$.

Proof. Let us choose γ such that $w + \left(\frac{2e^{-w\tau}\tilde{K}L\Gamma(\alpha-q)}{\Gamma(\alpha)}\right)^{1/(\alpha-q)} < \gamma < 0$. Let us define

$$U_t = \sup_{0\le s\le t}\{e^{-\gamma s}\|u_s\|_0\} \quad \text{and} \quad N_t = \sup_{0\le s\le t}\{e^{-\gamma s}\eta(s)\}.$$

For $t > \tau$, the hypothesis (H6) and the inequality (3.4) imply that

$$\begin{aligned}
e^{-\gamma t}\|u_t\|_0 &= e^{-\gamma t}\sup_{-\tau\le\theta\le 0}\|A^q(t_0)u(t+\theta)\| \\
&\le e^{-\gamma t}\sup_{-\tau\le\theta\le 0}\{\|A^q(t_0)R(t+\theta,0)\chi(0)\|\} \\
&\quad + \frac{e^{-\gamma t}}{\Gamma(\alpha)}\sup_{-\tau\le\theta\le 0}\left\{\left\|\int_0^{t+\theta}(t+\theta-s)^{\alpha-1}A^q(t_0)R(t+\theta,s)f(s,u(s),u_s)\,\mathrm{d}s\right\|\right\} \\
&\le e^{-\gamma t}\sup_{-\tau\le\theta\le 0}\left\{\tilde{K}e^{w(t+\theta)}(t+\theta)^{-q}\|\chi(0)\|\right\} \\
&\quad + \frac{e^{-\gamma t}\tilde{K}L}{\Gamma(\alpha)}\sup_{-\tau\le\theta\le 0}\left\{\int_0^{t+\theta}(t+\theta-s)^{\alpha-q-1}e^{w(t+\theta-s)}[\|u(s)\|_{q,t_0} + \|u_s\|_0 + \eta(s)]\,\mathrm{d}s\right\} \\
&\le e^{-\gamma t}\tilde{K}e^{w(t-\tau)}(t-\tau)^{-q}\|\chi(0)\| \\
&\quad + \frac{\tilde{K}L}{\Gamma(\alpha)}\sup_{-\tau\le\theta\le 0}\left\{\int_0^{t+\theta}(t+\theta-s)^{\alpha-q-1}e^{-(\gamma-w)(t+\theta-s)}e^{\gamma\theta}e^{-\gamma s}[2\|u_s\|_0 + \eta(s)]\,\mathrm{d}s\right\} \\
&\le e^{-\gamma t}\tilde{K}e^{w(t-\tau)}(t-\tau)^{-q}\|\chi(0)\| \\
&\quad + \frac{\tilde{K}Le^{-w\tau}}{\Gamma(\alpha)}\sup_{-\tau\le\theta\le 0}\left\{\int_0^{t+\theta}(t+\theta-s)^{\alpha-q-1}e^{-(\gamma-w)(t+\theta-s)}[2U_t + N_t]\,\mathrm{d}s\right\} \\
&\le e^{-\gamma t}\tilde{K}e^{w(t-\tau)}(t-\tau)^{-q}\|\chi(0)\| + \frac{\tilde{K}Le^{-w\tau}\Gamma(\alpha-q)}{\Gamma(\alpha)}(\gamma-w)^{q-\alpha}(2U_t + N_t).
\end{aligned}$$

Since U_t is increasing as a function of t, from the above inequality we have

$$U_t \le e^{-\gamma t}\tilde{K}e^{w(t-\tau)}(t-\tau)^{-q}\|\chi(0)\| + C(\gamma)(2U_t + N_t),$$

where $C(\gamma) = \frac{\tilde{K}Le^{-w\tau}\Gamma(\alpha-q)}{\Gamma(\alpha)}(\gamma - w)^{q-\alpha}$. It is easy to see that $C(\gamma) < \frac{1}{2}$. Since w and τ are fixed, there exists t_2 such that $e^{-w\tau}(t-\tau)^{-q} \leq 1$ for all $t \geq t_2$. Therefore, for all $t \geq t_2$, we have

$$U_t \leq \big(1 - 2C(\gamma)\big)^{-1}\big[\tilde{K}\,\|\chi(0)\| + C(\gamma)N_t\big].$$

Thus, for all $t \geq t_2$, the above inequality implies that

$$\|u_t\|_0 \leq e^{\gamma t}\big(1 - 2C(\gamma)\big)^{-1}\big[\tilde{K}\,\|\chi(0)\| + C(\gamma)N_t\big]. \tag{3.5}$$

(i) First, we assume that $\eta(t)$ is bounded. Then

$$e^{\gamma t}N_t = \sup_{0\leq s\leq t}\{e^{\gamma t}e^{-\gamma s}\eta(s)\} \leq \sup_{0\leq s\leq t}\{\eta(s)\}$$

and the right-hand side of (3.5) is bounded for $t \geq 0$, which implies that $\|u_t\|_0$ is bounded on $[0, \infty)$.

(ii) Next, we assume that $\eta(t) = O(e^{\delta t})$. Also, $w + \left(\frac{2e^{-w\tau}\tilde{K}L\Gamma(\alpha-q)}{\Gamma(\alpha)}\right)^{1/(\alpha-q)} < \delta$, then we choose $\gamma = \delta$. It gives

$$N_t = \sup_{0\leq s\leq t}\{e^{-\delta s}\eta(s)\} \leq \sup_{0\leq s\leq t}\{e^{-\delta s}K' e^{\delta s}\} \leq K',$$

where K' is a constant independent of t. Therefore, the right-hand side of (3.5) is $O(e^{\delta t})$, and so $\|u_t\|_0 = O(e^{\delta t})$.

(iii) Assume that $\eta(t) = o(1)$. Then, from (3.5) we have

$$\begin{aligned}
\|u_t\|_0 &\leq e^{\gamma t}(1 - 2C(\gamma))^{-1}\tilde{K}\,\|\chi(0)\| + e^{\gamma t}(1 - 2C(\gamma))^{-1}C(\gamma)\sup_{0\leq s\leq t_1}\{e^{-\gamma s}\eta(s)\} \\
&\qquad + e^{\gamma t}(1 - 2C(\gamma))^{-1}C(\gamma)\sup_{t_1\leq s\leq t}\{e^{-\gamma s}\eta(s)\} \\
&\leq e^{\gamma t}(1 - 2C(\gamma))^{-1}\tilde{K}\,\|\chi(0)\| + e^{\gamma(t-t_1)}(1 - 2C(\gamma))^{-1}C(\gamma)\sup_{0\leq s\leq t_1}\{\eta(s)\} \\
&\qquad + (1 - 2C(\gamma))^{-1}C(\gamma)\sup_{t_1\leq s\leq t}\{\eta(s)\}.
\end{aligned}$$

The above estimate implies that $\|u_t\|_0 = o(1)$. □

4 Application

Consider the following nonlinear fractional order delay partial differential equation:

$$\begin{cases}
\dfrac{\partial^\alpha z(t,x)}{\partial t^\alpha} + a(t,x)\dfrac{\partial^2}{\partial x^2}z(t,x) = \mathcal{F}(t, z(t,x), z_t), & t \in (0,T], x \in \Omega, \\
z(t,x) = 0, & x \in \partial\Omega, t \in [0,T], \\
g_0(z(t,x)) = \phi_0(x), & t \in [-\tau, 0], x \in \Omega,
\end{cases} \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}$ with sufficiently smooth boundary $\partial\Omega$.

Let $X = L^2(\Omega)$ be the space of functions which are square integrable. Now, it is well known that the operator $A\colon D(A) \subset X \to X$ defined by $Ax = x''$ with domain $D(A) = \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}$ is the infinitesimal generator of the analytic and compact semigroup $(T(t))_{t\geq 0}$ on X which is given by $T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, w_n\rangle w_n$. In particular, we have that $(T(t))_{t\geq 0}$ is a uniformly stable semigroup with $\|T(t)\| \leq e^{-t}$ for all $t \geq 0$.

We define $A(t)\colon D(A(t)) \subset X \to X$ by $-A(t)x(\xi) = a(t,\xi)Ax(\xi)$ for $x \in D(A(t)), t \in [0,T]$, $\xi \in \Omega$, where $D(A(t)) = D(A) = H^2(\Omega) \cap H_0^1(\Omega) = \{H^2(\Omega) : v(x) = 0 \text{ for } x \in \partial\Omega\}$, $t \geq 0$. We require the following assumption for the system (4.1):

(i) $a(t,x)$ is uniformly Hölder continuous in $t \in \mathbb{R}$ with $a(t,x) \leq -\delta_0, (\delta_0 > 0)$ for all $x \in \Omega$.

We can see that the system

$$\begin{cases} u'(t) + A(t)u(t) = 0, & t \geq s, \\ u(s) = x, \end{cases}$$

has an associated evolution family $(U(t,s))_{t\geq s}$ on X, which can be explicitly given by $U(t,s)y = T\left(-a(x,t)(t-s)\right)y$. One can extract the resolvent family $R(t,s)$ from the evolution family of $A(t)$, and then, by using the properties of semigroup and the assumption (i), we have

$$\|A^q(t)R(t,s)\|_X \leq M(t-s)^{-q}e^{-\delta_0(t-s)},$$

where M is independent of t, s and can be calculate by fixing the values of α, q.

We define $u(t)(x) = z(t,x)$ and $u_t = z_t(\theta,\cdot)$, that is, $(u(t+\theta))(x) = z(t+\theta,x)$ for $t \in [0,T]$, $x \in \Omega$, $\theta \in [-\tau,0]$. Also, we define the functions $f\colon [0,T] \times X \times C_0 \to X$ and $g\colon C_0 \to L^2(\Omega)$ by

$$f(t,u(t),u_t)(x) = \mathcal{F}(t,z(t,x),z_t(\theta,x)),$$
$$g(\psi)(x) = g_0(\psi(t,x)).$$

Then we can write (4.1) in an abstract form

$$\begin{cases} \dfrac{\mathrm{d}^\alpha u(t)}{\mathrm{d}t^\alpha} + A(t)u(t) = f(t,u(t),u_t), \\ g(u_0) = \phi_0. \end{cases}$$

Note that the boundary condition is absorbed into the definition of the domain of the operator $A(t)$ and into the requirement that $u(t) \in D(A)$ for all $t \geq 0$. Let g be defined by

$$g(\psi)(x) = \int_{-\tau}^{0} l(s)\psi(s)(x)\,\mathrm{d}s, \qquad l \in L^1([-\tau,0]).$$

Hence, we can write (4.1) as a fractional delay differential equation of the form (1.1), where $h(u_0)(\theta) \equiv g(u_0)$ for $u_0 \in C_0$, $\theta \in [-\tau,0]$ and $\phi(\theta) \equiv \phi_0$ for $\theta \in [-\tau,0]$. Now we can take $\chi(t) = \frac{1}{k}\phi_0$ on $[-\tau,0]$ with $k = \int_{-\tau}^{0} l(s)\,\mathrm{d}s \neq 0$. Let $f(t,u(t),u_t) = c\left(u(t) + \sin(u_t)\right)$, where c is such that $c < \frac{e^{-\delta_0\tau}(\delta_0)^{\alpha-q}\Gamma(\alpha)}{2M\Gamma(\alpha-q)}$. Then it is easy to see that the function f satisfies (H6). Also, under the assumptions (i) and (ii), one can see that the hypotheses (H4) and (H5) hold. Hence, one can apply Theorem 3.2 to see the asymptotic behaviour of solutions to the considered problem.

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