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GRADIENT ESTIMATES FOR HEAT-TYPE EQUATIONS ON EVOLVING MANIFOLDS

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Abstract. In this paper we study gradient estimates for all positive solutions of a class of heat-type equations defined on a Riemannian manifold whose metric is evolving by the generalized geometric flow. In particular, we apply Laplacian comparison theorem and maximum principle to obtain Li–Yau type gradient estimates (local and global). As a corollary, differential Harnack inequalities are derived. The reason for this on the one hand is to show the advantage of using Ricci flow and on the other hand is to prepare ground for applications to other special cases of geometric flow.

Keywords: Gradient estimates, Harnack inequalities, heat-type equations, geometric flows, Ricci flow.

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1 Introduction

Let M be an n-dimensional complete manifold on which a one parameter family of Riemannian metrics $g(t)$, $t \in [0,T]$ is defined. We say that $g_{ij}(x,t)$ is a solution to the generalized geometric flow, if it is evolving by the following equation

$$\frac{\partial}{\partial t} g_{ij}(x,t) = 2h_{ij}(x,t), \qquad (x,t) \in M \times [0,T] \tag{1.1}$$

with $g_{ij}(0) = g_0$, where h_{ij} is a general time-dependent symmetric $(0,2)$-tensor and $T > 0$ is taken to be the maximum time of existence for the flow. The scaling factor 2 in (1.1) is insignificant. By a

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positive solution to the heat-type equation on the manifold we mean a smooth function, at least C^2 in x and C^1 in t, $u(x,t) = u \in C^{2,1}(M \times [0,T])$, which satisfies the following equation

$$\Big(\Delta_g - \frac{\partial}{\partial t} - \mathcal{R}\Big)u(x,t) = 0, \qquad (x,t) \in M \times [0,T], \tag{1.2}$$

where the symbol Δ_g is the Laplace–Beltrami operator acting on functions in space with respect to metric $g(t)$ in time and $\mathcal{R}\colon M \times [0,T] \to \mathbb{R}$ is a C^∞-function on M. We study Li–Yau type gradient estimates for all positive solutions of a class of equations of the form (1.2). A very good example is the conjugate heat equation $\Box^* u = (\Delta_g - \partial_t + g^{ij}h_{ij})u(x,t) = 0$ (*i.e.*, adjoint to the heat operator $\Box = (\Delta_g - \partial_t)$), which the author studied in [1] under both forward and backward in time Ricci flow. The case $h_{ij} = -R_{ij}$ (resp. R_{ij}) is precisely when the manifold is being evolved with respect to forward (resp. backward) Ricci flow. In fact, one of the motivations to study this subject arises from the question: Is there any merit or demerit of flowing Riemannian manifold by the Ricci flow? Here, we show the advantage of using Ricci flow, which is due to the presence of contracted second Bianchi identity. Noticing that Ricci flow theory has led to numerous breakthroughs in the field of Mathematics and Physics, a very much celebrated of which is the complete solution to the Poincaré conjecture (due to G. Perelman's completion of Hamilton's program). On the other hand, the result here gives a general framework to considering special cases of geometric flow, *e.g.* the case of Ricci flow coupled to harmonic map heat flow, where $h_{ij} = -R_{ij} + \gamma\nabla_i u \otimes \nabla_j u$, $\gamma > 0$ and u is a harmonic map between M and another manifold isometrically embedded in Euclidean space as considered in a recent preprint [2] by Băileşteanu.

The importance of gradient estimates as well as those of Harnack inequalities can not be overemphasised in the fields of differential geometry and analysis among their numerous applications. Differential Harnack inequalities are used to study the behaviours of solutions to the heat equation in space-time. Li and Yau's paper [12] can be said to mark the beginning of rigorous applications of these concepts. They derived gradient estimates for positive solutions to the heat operator defined on a complete manifold with static metric, from which they obtained Harnack inequalities. These inequalities were in turn used to establish various lower and upper bounds on the heat kernel. Precisely, Li and Yau's result for static metrics is the following

Theorem A (Li–Yau [12]) *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose there exists some non-negative constant k such that the Ricci curvature $R_{ij}(g) \geq -k$. Let $u \in C^{2,1}(M \times [0,T])$ be any smooth positive solution to the heat equation*

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)u(x,t) = 0 \tag{1.3}$$

in the geodesic ball $\mathcal{B}_{2\rho} \times [0,T]$. Then, the following estimate holds

$$\sup_{x\in\mathcal{B}_\rho}\Big\{\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u}\Big\} \leq \frac{n\alpha^2}{2t} + \frac{C\alpha^2}{\rho^2}\Big(\frac{\alpha^2}{\alpha^2-1} + \sqrt{k}\rho\Big) + \frac{n\alpha^2 k}{2(\alpha-1)} \tag{1.4}$$

for all $(x,t) \in \mathcal{B}_{2\rho,T},\ t > 0$ and some constant C depending only on n and $\alpha > 1$. Moreover, the following estimate

$$\sup_{x\in\mathcal{B}_\rho}\Big\{\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u}\Big\} \leq \frac{n\alpha^2}{2t} + \frac{n\alpha^2 k}{2(\alpha-1)} \tag{1.5}$$

holds for a complete non-compact manifold by letting $\rho \to \infty$.

The above results have been improved by Davies [7, Section 5.3] as follows

$$\sup_{x\in\mathcal{B}_\rho}\left\{\frac{|\nabla u|^2}{u^2}-\alpha\frac{u_t}{u}\right\}\leq\frac{n\alpha^2}{2t}+\frac{n\alpha^2 k}{4(\alpha-1)}. \tag{1.6}$$

If $k=0$, one can choose $\alpha=1$ and the second terms in both (1.5) and (1.6) disappear giving the sharp estimate

$$\frac{|\nabla u|^2}{u^2}-\frac{u_t}{u}\leq\frac{n}{2t}.$$

We remark that α can be chosen as a constant function of time only in such a way that it goes to 1 as $t\to 0$, see for instance Hamilton [9], Huang, Huang and Li [10] and Li and Xu [11]. There have been increasing efforts towards obtaining counterparts of the above estimates on evolving manifolds. In [1], the author studied heat equation and its conjugate, where Laplacian is perturbed with the curvature operator on metrics evolving by either forward or backward Ricci flow. Precisely, assume $u=u(x,t)$ solves the conjugate heat equation

$$(\partial_t-\Delta+R)u(x,t)=0,\qquad (x,t)\in M\times[0,T],\qquad T<\infty, \tag{1.7}$$

where R is the scalar curvature, the metric trace of its Ricci curvature tensor and $T<\infty$ is taking to be the maximum time of existence of the Ricci flow. Define the geodesic cube $\mathcal{Q}_{2\rho}\subset M$ by

$$\mathcal{Q}_{2\rho,T}:=\{(x,t)\in M\times(0,T]:d(x,y,t)\leq 2\rho\},$$

we have

Theorem B ([1]) *Let $(M,g(t)),t\in[0,T]$ be a complete solution to the backward Ricci flow* (i.e., $h_{ij}=R_{ij}$) *with $R\geq -k_1, Rc\geq -k_2$ and $|\nabla R|\leq k_3$, for some constants $k_1,k_2,k_3\geq 0$. Let $u=u(x,t)$ be any positive solution to the heat equation* (1.7) *defined in $\mathcal{Q}_{2\rho,T}\subset(M\times[0,T])$, satisfying $0<u\leq A$. There exist absolute constants C_1,C_2 depending on n such that*

$$\frac{|\nabla u|^2}{u^2}\leq\left(1+\log\left(\frac{A}{u}\right)\right)^2\left(\frac{1}{t}+C_2k_1+4k_2+2k_3+\frac{1}{\rho^2}\Big(\rho C_1\sqrt{k_2}+C_2\Big)\right).$$

Similarly, let $(M,g(t)),t\in[0,T]$ be a complete solution to the forward Ricci flow (i.e., $h_{ij}=-R_{ij}$); *then*

$$\frac{|\nabla u|^2}{u^2}\leq\left(1+\log\left(\frac{A}{u}\right)\right)^2\left(\frac{1}{t}+C_2k_1+2k_3+\frac{C_2}{\rho^2}\right).$$

In [1] we equally proved results for gradient estimates and Harnack inequalities (space-time) for the positive solutions to the heat equation in the geodesic cube $\mathcal{Q}_{2\rho,T}$ of bounded Ricci curvature manifold evolving by the Ricci flow as follows:

Theorem C ([1]) *Let $(M,g(t)),t\in[0,T]$ be a complete solution of the forward Ricci flow such that the Ricci curvature is bounded in $\mathcal{Q}_{2\rho,T}$,* i.e., $-k_1g(x,t)\leq Rc(x,t)\leq k_2g(x,t)$ *for some positive constants k_1 and k_2, with $(x,t)\in\mathcal{Q}_{2\rho,T}\subset(M\times[0,T])$. Suppose a smooth positive function $u\in C^{2,1}(M\times[0,t])$ solves the heat equation* (1.3) *in the cube $\mathcal{Q}_{2\rho,T}$. Then, for any given $\alpha>1$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{\alpha}$ and all $(x,t)\in\mathcal{Q}_{2\rho}\times[0,T]$, the following estimate holds for $f=\log u$*

$$\begin{aligned}\sup_{x\in\mathcal{Q}_{2\rho}}\Big\{|\nabla f|^2-\alpha f_t\Big\}\leq\frac{\alpha np}{4t}+C\alpha^2\left(\frac{\alpha^2p}{\rho^2(\alpha-1)}+\frac{1}{t}+(k_1+k_2)\right)&\\ +\frac{\alpha^2np}{2(\alpha-1)}k_1+\frac{\alpha n}{2}(k_1+k_2)\sqrt{pq},&\end{aligned} \tag{1.8}$$

where C is an arbitrary constant depending only on the dimension of the manifold. Moreover, we have the following Harnack inequalities

$$\frac{u(x_1,t_1)}{u(x_2,t_2)} \leq \left(\frac{t_2}{t_1}\right)^{\frac{np}{4}} \exp\left\{\frac{\alpha\Theta(x_1,t_1;x_2,t_2)}{4(t_2-t_1)} + \frac{(t_2-t_1)}{2(\alpha-1)}A(n,\alpha,p,q,k_1,k_2)\right\}, \tag{1.9}$$

where

$$\Theta(x_1,t_1;x_2,t_2) = \inf \int_0^1 |\dot{\gamma}(s)|^2 \, \mathrm{d}s,$$

the infimum is taken over all the smooth paths $\gamma\colon [t_1,t_2] \to M$ connecting x_1 and x_2, and $A(n,\alpha,p,q,k_1,k_2) = n\alpha p k_1 + n(\alpha-1)(k_1+k_2)\sqrt{pq}$.

We remark that estimates and bounds on parabolic equations behave in a similar way whether the metric is static or moving. This can be justified by the fact that heat diffusion on a bounded geometry with either static or evolving metric behaves like heat diffusion in Euclidean space, many a times, their estimates even coincide. In this paper however, we examine the general framework for the gradient estimates of a class of parabolic equations as in (1.2) under the generalized geometric flow in the form of (1.1). We also derive differential Harnack inequalities and LYH type estimate. The plan is as follows: In Section 2 we highlight the basic tools used in the paper and prove some technical results in a lemma. This lemma further reveals the advantage of contracted second Bianchi identity. In Section 3, we discuss local space-time gradient estimates for the heat-type equation (1.2) on a manifold evolving by the generalized flow (1.1). As an application of these results we obtain Li–Yau type Harnack inequalities.

2 Preliminaries and technical results

Throughout, we work on a C^∞ manifold which is endowed with Riemannian metric $\mathrm{d}s^2 = g = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$, where $\{x^i\}, 1 \leq i \leq n$ is a local coordinate systems and n is the dimension of the manifold. The operator Δ is the Laplace-Beltrami operator on (M,g) which is defined by

$$\Delta_g = \frac{1}{\sqrt{|g|}}\,\partial_i\Big(\sqrt{|g|}\,g^{ij}\partial_j\Big)$$

and $\nabla = g^{ij}\partial_i$ is the gradient operator, where $|g|$ = determinant of g and $g^{ij} = (g_{ij})^{-1}$, inverse metric. A natural function that will be defined on M is the distance function from a given point. Namely, let $p \in M$ and define $d(x,p)$ for all $x \in M$, where $\mathrm{dist}(\cdot,\cdot)$ is the geodesic distance. Note that $d(x,p)$ is only Lipschitz continuous, *i.e.*, everywhere continuous except on the cut locus of p and on the point where x and p coincide. It is then easy to see that

$$|\nabla d| = g^{ij}\partial_i d\,\partial_j d = 1 \qquad \text{on} \qquad M \setminus \big\{\{p\} \cup \mathrm{cut}(p)\big\}.$$

Let $d(x,y,t)$ be the geodesic distance between x and y with respect to the metric $g(t)$, we define a smooth cut-off function $\varphi(x,t)$ with support in the geodesic cube

$$\mathcal{Q}_{2\rho,T} := \{(x,t) \in M \times (0,T] : d(x,y,t) \leq 2\rho\}.$$

For any C^2-function $\psi(s)$ on $[0,+\infty)$ with

$$\psi(s) = \begin{cases} 1, & s \in [0,1], \\ 0, & s \in [2,+\infty) \end{cases}$$

and

$$\psi'(s) \leq 0, \quad \psi''(s) \leq C_1 \quad \text{and} \quad \frac{|\psi'|^2}{\psi} \leq C_2,$$

where C_1, C_2 are absolute constants, such that

$$\varphi(x,t) = \psi\left(\frac{d(x,p,t)}{\rho}\right) \qquad \text{and} \qquad \varphi\big|_{\mathcal{Q}_{2\rho,T}} = 1.$$

We will apply maximum principle and invoke Calabi's trick to assume everywhere smoothness of $\varphi(x)$ since $\psi(s)$ is in general Lipschitz. We need Laplacian comparison theorem to do some calculation on $\varphi(x,t)$. Here is the statement of the theorem: Let M be a complete n-dimensional Riemannian manifold whose Ricci curvature is bounded from below by $Rc \geq (n-1)k$ for some constant $k \in \mathbb{R}$. Then the Laplacian of the distance function satisfies

$$\Delta d(x,p) \leq \begin{cases} (n-1)\sqrt{k}\cot(\sqrt{k}\rho), & k > 0, \\ (n-1)\rho^{-1}, & k = 0, \\ (n-1)\sqrt{|k|}\coth(\sqrt{|k|}\rho), & k < 0. \end{cases}$$

Throughout, we will impose boundedness condition on the Ricci curvature of the metric $g(t)$. We notice that when the metric evolves by the Ricci flow, boundedness and sign assumptions are preserved as long as the flow exists, so it follows that the metrics are uniformly equivalent. Precisely, if $-K_1 g \leq Rc \leq K_2 g$, where $g(t), t \in [0,T]$ is a Ricci flow, then

$$e^{-k_1 T} g(0) \leq g(t) \leq e^{k_2 T} g(0). \tag{2.1}$$

To see the above bounds (2.1) we consider the evolution of a vector form $|X|_g = g(X,X), X \in T_x M$. By the equation of the Ricci flow $\frac{\partial}{\partial t} g(X,X) = -2Rc(X,X),\ 0 \leq t_1 \leq t_2 \leq T$ and by the boundedness of the Ricci curvature we have

$$\left|\frac{\partial}{\partial t} g(X,X)\right| \leq K_2 g(X,X),$$

which implies (by integrating from t_1 to t_2)

$$\left|\log \frac{g(t_2)(X,X)}{g(t_1)(X,X)}\right| \leq K_2 t\Big|_{t_1}^{t_2}.$$

Taking the exponential of this estimate with $t_1 = 0$ and $t_2 = T$ yields $|g(t)| \leq e^{k_2 T} g(0)$ from which the uniform boundedness of the metric follows. See [5] and [6] for details on the theory of the Ricci flow. Similarly, if there holds the boundedness assumption $-cg \leq h \leq Cg$, the metric $g(t)$ is uniformly bounded from below and above for all times $0 \leq t \leq T$ under the geometric flow (1.1). Then, it does not matter what metric we use in the argument that follows.

Most of our calculations are done in local coordinates where $\{x^i\}$ is fixed in a neighbourhood of every point $x \in M$. We switch between the Rc and R_{ij} for the Ricci curvature notation, and any time R_{ij} is used we assume to be on local coordinates at x. We also define some classical identities on function $f \in C^\infty$ at a point x in a local normal coordinate system, namely, Bochner–Weitzenböck identity

$$\Delta |\nabla f|^2 = 2|f_{ij}|^2 + 2 f_j f_{jji} + 2 R_{ij} f_i f_j \tag{2.2}$$

and Ricci identity

$$f_j f_{jii} - f_j f_{jji} = R_{ij} f_i f_j.$$

We make use of the above identities and we try as much as possible to be explicit at any point where they are used.

Lemma 2.1 *Suppose a one-parameter family of smooth metrics $g(t)$ solves the geometric flow* (1.1). *Then, we have the following evolutions*

(1) $\dfrac{\partial}{\partial t} g^{ij} = -2g^{ik} g^{jl} h_{kl} = -2h^{ij}$,

(2) $\dfrac{\partial}{\partial t} \Gamma_{ij}^k = g^{kl}\big(\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij}\big)$,

(3) $\dfrac{\partial}{\partial t} |\nabla f|^2 = -2h_{ij} f_i f_j + 2 f_i f_{ti}$,

(4) $\dfrac{\partial}{\partial t}(\Delta f) = \Delta(f_t) - 2h_{ij} f_{ij} - 2\langle \operatorname{div} h, \nabla f\rangle + \langle \nabla H, \nabla f\rangle$.

Here g^{ij} is the matrix inverse of g_{ij}, Γ_{ij}^k are the Christoffel's symbols with respect to Levi–Civita connection, $\Delta = \Delta_{g(t)}$, $H = g^{ij} h_{ij}$, the metric trace of a symmetric 2-tensor h_{ij} and f is a smooth function defined on M.

Proof. Recall that both g_{ij} and h_{ij} are symmetric tensors and $g^{ij} g_{jl} = \delta_l^i$. Note also that Levi–Civita connection is not a tensor, but the time derivative of a connection is $(2, 1)$ tensor. Then (2) holds as a tensor equation in any coordinate system and at any point. The proofs of (1) and (2) are the same as those of [5, Lemmas 3.1 and 3.2]. We only give the computations in local coordinates which lead to the proofs of (3) and (4):

$$\begin{aligned}
\partial_t(|\nabla f|^2) &= \partial_t(g^{ij}\partial_i f \partial_j f) \\
&= (\partial_t g^{ij})\partial_i f \partial_j f + 2g^{ij}\partial_i f \partial_j f_t \\
&= -2h^{ij}\partial_i f \partial_j f + 2g^{ij}\partial_i f \partial_j f_t \\
&= -2h_{ij} f_i f_j + 2 f_i f_{t,i}.
\end{aligned}$$

Using evolutions in (1) and (2) we have

$$\begin{aligned}
\frac{\partial}{\partial t}(\Delta f) &= \frac{\partial}{\partial t}[g^{ij}(\partial_i\partial_j - \Gamma_{ij}^k \partial_k) f] \\
&= \frac{\partial}{\partial t}(g^{ij})(\partial_i\partial_j - \Gamma_{ij}^k\partial_k) f + g^{ij}(\partial_i\partial_j - \Gamma_{ij}^k\partial_k)\frac{\partial}{\partial t} f - g^{ij}\left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right)\partial_k f \\
&= -2h^{ij}\nabla_i\nabla_j f + \Delta(f_t) - g^{ij} g^{kl}\big(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}\big)\nabla_k f \\
&= \Delta(f_t) - 2h_{ij} f_{ij} - 2g^{kl}\Big(g^{ij}\nabla_i h_{jl} - \frac{1}{2} g^{ij}\nabla_l h_{ij}\Big)\nabla_k f \\
&= \Delta(f_t) - 2h_{ij} f_{ij} - 2\langle \operatorname{div} h, \nabla f\rangle + H_i f_j,
\end{aligned}$$

where div is the divergence operator, *i.e.*, $(\operatorname{div} h)_k = g_{ij}\nabla_i h_{jk}$. □

Remark 2.2 *If it was under the Ricci flow, the contracted second Bianchi identity would make the addition of the last two terms in the RHS of the last equality to vanish. This accounts for the reason we do not need any bound on the gradient of the Ricci curvature when the manifold evolves by the Ricci flow, where as in the next section we need to control the growth of curvature, namely* $|\nabla h|, |\nabla H|$.

3 Gradient estimates and Harnack inequalities

This section extends the results of Sections three and four of [1] (Theorem B above) to a more general situation of geometric flow. This generalization further reveals the advantage of contracted second Bianchi identity we enjoy under the Ricci flow and makes it more obvious that it could be easier to handle the case of Ricci flow under less severe assumption on the curvature. Moreover, this case of generalized flow can be applied to some other special cases such as mean curvature flow, Bernard List extended Ricci flow and Reto Müller Ricci-harmonic map flow. Let $(M, g(t))$, $t \in [0, T]$ be a smooth one parameter family of complete Riemannian metrics evolving by the geometric flow

$$\frac{\partial}{\partial t} g_{ij}(x,t) = 2h_{ij}(x,t), \qquad (x,t) \in M \times [0,T], \tag{3.1}$$

where h_{ij} is a smooth family of symmetric $(0,2)$-tensors which may depend on the metric g, its trace is denoted by $H = g^{ij}h_{ij}$. Let $u = u(x,t)$ be a positive solution to the following heat-type equation

$$\Big(\Delta_g - \frac{\partial}{\partial t} - \mathcal{R}\Big)u(x,t) = 0, \qquad (x,t) \in M \times [0,T], \tag{3.2}$$

where $\mathcal{R}(x,t)$ is a smooth function on $M \times [0,T]$ and Δ is the usual Laplace-Beltrami operator acting on functions with respect to the metric $g(t)$. Here, we use the notation $\mathcal{R}(x,t)$ since we may have to treat it just like scalar curvature of the metric $g(t)$ as in [1] but should not be confused with scalar curvature tensor R. In general, it should be viewed as a potential function, which could even be nonlinear. An interesting case is to make $\mathcal{R} = H$, the metric trace of the symmetric tensor h_{ij}. For some of the cases described above see [8, 12, 16] and [4, Chapters 24 and 25] for related issues. After the completion of our work, the preprint [2] by Bailesteanu was brought to our notice, where he considered a special case of Ricci flow coupled to harmonic map heat flow with $h_{ij} = -R_{ij} + \gamma\nabla_i u \otimes \nabla_j u$, $\gamma > 0$ and u is a harmonic map between M and another manifold isometrically embedded in Euclidean space. This flow was introduced by Muller in [14] and can be traced back to List [13].

In the next, we give the following important lemma which is a generalization of Lemma 3.1 in [1]. It is due to Li and Yau [12]. This is very crucial to derivation of both local and global estimate of Li–Yau type.

Lemma 3.1 *Let* $(M, g(t))$ *be a complete solution to the generalized flow* (3.1) *in some time interval* $[0,T]$. *Suppose there exist some non-negative constants* k_1, k_2, k_3, *and* k_4 *such that* $R_{ij}(g) \geq -K_1 g$, $-k_2 g \leq h \leq k_3 g$ *and* $|\nabla h| \leq k_4$ *for all* $t \in [0,T]$. *For any smooth positive solution*

$u \in C^{2,1}(M \times [0,T])$ to the heat-type equation (3.2) *in the geodesic cube $\mathcal{Q}_{2\rho,T}$, it holds that*

$$
\begin{aligned}
(\Delta - \partial_t)F \geq -2\langle \nabla f, \nabla F\rangle - \big(|\nabla f|^2 - \alpha\partial_t f - \alpha\mathcal{R}\big) + \frac{2\alpha}{np} t\big(|\nabla f|^2 - \partial_t f - \mathcal{R}\big)^2 \\
- 2\big(k_1 + (\alpha-1)k_3\big)t|\nabla f|^2 - 3\alpha n^{\frac{1}{2}} k_4 t|\nabla f| - \frac{\alpha n q}{2} t(k_2+k_3)^2 \\
- 2t(\alpha-1)\langle \nabla f, \nabla R\rangle - \alpha t\Delta\mathcal{R},
\end{aligned}
\tag{3.3}
$$

where $f = \log u$, $F = t\big(|\nabla f|^2 - \alpha\partial_t f - \alpha\mathcal{R}\big)$ and $\alpha \geq 1$ are given such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}$ for any real numbers $p, q > 0$.

Proof. (Still working in local coordinate system). Notice that $f_t = \Delta f + |\nabla f|^2 - \mathcal{R}$. Taking covariant derivative of F we have

$$F_i = t(2f_j f_{ji} - \alpha f_{ti} - \alpha\mathcal{R}_i)$$

and with Bochner's formula we have

$$\Delta F = \sum_{i=1}^n F_{ii} = t\big(2f_{ij}^2 + 2f_j f_{jji} + 2R_{ij}f_{ij} - \alpha\Delta(f_t) - \alpha\Delta\mathcal{R}\big).$$

Using Lemma 2.1, we get

$$
\begin{aligned}
\Delta F &= t\Big[2f_{ij}^2 + 2f_j f_{jji} + 2R_{ij}f_{ij} - \alpha(\Delta f)_t - 2\alpha h_{ij} f_{ij} \\
&\qquad - 2\alpha(\operatorname{div} h)_i f_j + \alpha H_i f_j - \alpha\Delta\mathcal{R}\Big] \\
&= t\big(2f_{ij}^2 - 2\alpha h_{ij} f_{ij}\big) + 2t\big\langle \nabla f, \nabla\big(f_t + \mathcal{R} - |\nabla f|^2\big)\big\rangle - \alpha t\big(f_t + \mathcal{R} - |\nabla f|^2\big)_t \\
&\qquad - 2\alpha t(\operatorname{div} h)_i f_j + \alpha t H_i f_j - \alpha t\Delta\mathcal{R} + 2tR_{ij}f_{ij}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
-\alpha t\big(f_t + \mathcal{R} - |\nabla f|^2\big)_t &= t\big(\alpha|\nabla f|^2 - \alpha f_t - \alpha t\mathcal{R}\big)_t \\
&= t\big\{|\nabla f|^2 - \alpha f_t - \alpha\mathcal{R} + (\alpha-1)|\nabla f|^2\big\}_t \\
&= t\left\{\frac{F}{t} + (\alpha-1)|\nabla f|^2\right\}_t \\
&= F_t - \frac{F}{t} + t(\alpha-1)\big(|\nabla f|^2\big)_t
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
&2t\big\langle \nabla f, \nabla\big(f_t + \mathcal{R} - |\nabla f|^2\big)\big\rangle + t(\alpha-1)\big(|\nabla f|^2\big)_t \\
&\quad = 2t\big\langle \nabla f, \nabla\big(f_t + \mathcal{R} - |\nabla f|^2\big)\big\rangle + 2t(\alpha-1)\langle \nabla f, \nabla(f_t)\rangle - 2t(\alpha-1)h_{ij}f_i f_j \\
&\quad = 2t\big\langle \nabla f, \nabla\big(\alpha f_t + \mathcal{R} - |\nabla f|^2\big)\big\rangle - 2t(\alpha-1)h_{ij}f_i f_j \\
&\quad = -2t\left\langle \nabla f, \nabla\left(\frac{F}{t} + (\alpha-1)\mathcal{R}\right)\right\rangle - 2t(\alpha-1)h_{ij}f_i f_j \\
&\quad = -2\langle \nabla f, \nabla F\rangle - 2t(\alpha-1)\langle \nabla f, \nabla\mathcal{R}\rangle - 2t(\alpha-1)h_{ij}f_i f_j.
\end{aligned}
\tag{3.5}
$$

With (3.4) and (3.5) we get

$$
\begin{aligned}
\Delta F - F_t = t\big(2f_{ij}^2 - 2\alpha h_{ij}f_{ij}\big) - 2\langle \nabla f, \nabla F\rangle - \frac{F}{t} - 2t(\alpha-1)h_{ij}f_if_j \\
- \alpha t\big(2(\operatorname{div} h)_i f_j - H_if_j\big) - 2t(\alpha-1)\langle \nabla f, \nabla \mathcal{R}\rangle - \alpha t\Delta\mathcal{R} + 2tR_{ij}f_{ij}.
\end{aligned}
$$

Now we can choose any two real numbers $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}$, so that we can write

$$
\begin{aligned}
2f_{ij}^2 - 2\alpha h_{ij}f_{ij} &= \frac{2\alpha}{p}f_{ij}^2 + 2\frac{\alpha}{q}\big(f_{ij}^2 - h_{ij}f_{ij}\big) \\
&= \frac{2\alpha}{p}f_{ij}^2 + 2\alpha\left(\frac{1}{\sqrt{q}}f_{ij} - \frac{\sqrt{q}}{2}h_{ij}\right)^2 - \frac{\alpha q}{2}h_{ij}^2 \\
&\geq \frac{2\alpha}{p}f_{ij}^2 - \frac{\alpha q}{2}h_{ij}^2,
\end{aligned}
$$

where we have used completing the square method to arrive at the last inequality. Also by Cauchy–Schwarz inequality

$$(\Delta f)^2 = |g^{ij}\partial_i\partial_j f|^2 \leq n f_{ij}^2$$

holds at an arbitrary point $(x,t) \in \mathcal{Q}_{2\rho,T}$, therefore we have $f_{ij}^2 \geq \frac{1}{n}(\Delta f)^2$. We can also write the boundedness condition on h_{ij} as $-(k_2+k_3)g \leq h_{ij} \leq (k_2+k_3)g$, so that

$$\sup_M |h_{ij}|^2 \leq n(k_2+k_3)^2,$$

since h_{ij} is a symmetric tensor. Therefore, we have

$$t\big(2f_{ij}^2 - 2\alpha h_{ij}f_{ij}\big) \geq \frac{2\alpha t}{np}(\Delta f)^2 - \frac{\alpha n q}{2}t(k_2+k_3)^2. \tag{3.6}$$

Notice also that

$$
\begin{aligned}
\alpha t\Big(2(\operatorname{div} h)_i f_j - H_if_j\Big) &= 2\alpha t\Big(\operatorname{div} h - \frac{1}{2}\nabla H\Big)f_j \\
&= 2\alpha t\Big(g^{ij}\nabla_i h_{jl} - \frac{1}{2}g^{ij}\nabla_i h_{ij}\Big)\nabla_j f \\
&\leq 2\alpha t\left(\frac{3}{2}|g||\nabla h|\right)|\nabla f| \\
&\leq 3\alpha t n^{\frac{1}{2}}k_4|\nabla f|.
\end{aligned}
$$

Using the last inequality and (3.6) with the hypothesis of the theorem, we have

$$
\begin{aligned}
(\Delta - \partial_t)F \geq -2\langle \nabla f, \nabla F\rangle - \frac{F}{t} - 2t(\alpha-1)k_3|\nabla f|^2 - 2tk_1|\nabla f|^2 \\
+ \frac{2\alpha t}{np}\big(|\nabla f|^2 - f_t - \mathcal{R}\big)^2 - 3\alpha t n^{\frac{1}{2}}k_4|\nabla f| - \frac{\alpha n q}{2}t(k_2+k_3)^2 \\
- 2t(\alpha-1)\langle \nabla f, \nabla\mathcal{R}\rangle - \alpha t\Delta\mathcal{R}.
\end{aligned}
$$

Our calculation is valid in the cube $\mathcal{Q}_{2\rho,T}$. Hence the desired claim follows. $\square$

Next we give the general local space-time gradient estimate corresponding to those of (1.8) of Theorem C.

Theorem 3.2 *Let $(M, g(t))$, $t \in [0,T]$ be a complete solution to the geometric flow* (3.1). *Suppose there exist some non-negative constants k_1, k_2, k_3, and k_4 such that $R_{ij}(g) \geq -k_1 g$, $-k_2 g \leq h \leq k_3 g$ and $|\nabla h| \leq k_4$ for all $t \in [0,T]$. Let $u \in C^{2,1}(M \times [0,T])$ be any smooth positive solution to the heat-type equation* (3.2) *in the geodesic cube $\mathcal{Q}_{2\rho,T}$, with $|\nabla \mathcal{R}| \leq \beta$ and $|\Delta \mathcal{R}| \leq \xi$. Then, the following estimate holds*

$$\sup_{x\in\mathcal{Q}_{2\rho}} \big\{|\nabla f|^2 - \alpha f_t - \alpha \mathcal{R}\big\} \leq \frac{\alpha n p}{4t} + \frac{C\alpha}{\rho^2}\left(\frac{\alpha p}{\alpha-1} + \frac{\rho}{t} + \rho\sqrt{k_1} + \rho^2 (k_2+k_3)^2\right)$$
$$+ \frac{\alpha n p}{\alpha - 1}\big(k_1 + (\alpha-1)k_3 + k_4\big) + \frac{(\alpha-1)}{2}\beta t \sqrt{\frac{\alpha n p}{\big(k_1 + (\alpha-1)k_3 + k_4\big)}} \tag{3.7}$$
$$+ \frac{\alpha n}{2}(k_2 + k_3 + 2\sqrt{k_4})\sqrt{pq} + \alpha\sqrt{\frac{np\xi}{2}}$$

for all $(x,t) \in \mathcal{Q}_{2\rho,T}$, $t > 0$ and some constant C depending only on n, where $f = \log u$, $F = t\big(|\nabla f|^2 - \alpha \partial_t f - \alpha\mathcal{R}\big)$ and $\alpha > 1$ are given such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}$ for any real numbers $p, q > 0$.

This can be written in a more compact form

$$\sup_{x\in\mathcal{Q}_{2\rho}} \Big\{|\nabla f|^2 - \alpha f_t - \alpha\mathcal{R}\Big\} \leq \frac{\alpha n p}{4t} + C'\left(k_1 + k_2 + k_3 + k_4 + \sqrt{k_1} + \sqrt{k_4} + \beta + \xi + \frac{1}{t}\right)$$
$$+ \frac{\alpha n p}{\alpha-1}\big(k_1 + (\alpha-1)k_3 + k_4\big),$$

where C' depends on n, ρ and α only.

Proof. We are still using the same notations as in the last lemma; here we write $\widetilde{K} = (k_2 + k_3)^2$. For a fixed $\tau \in (0,T]$ and a smooth cut-off function $\varphi(x,t)$ (also chosen as before), we now estimate the inequality (3.3) at the point $(x_0, t_0) \in \mathcal{Q}_{2\rho,T} \subset (M \times [0,T])$ such that $\operatorname{dist}(x, x_0, t) < \rho$. The argument follows;

$$(\Delta - \partial_t)(\varphi F) = 2\nabla\varphi\nabla F + \varphi(\Delta - \partial_t)F + F(\Delta - \partial_t)\varphi. \tag{3.8}$$

Suppose (φF) attains its maximum value at $(x_0, t_0) \in M \times [0,T]$, for $t_0 > 0$. Since $(\varphi F)(x, 0) = 0$ for all $x \in M$, we have by the maximum principle that

$$\nabla(\varphi F)(x_0,t_0) = 0, \qquad \frac{\partial}{\partial t}(\varphi F)(x_0,t_0) \geq 0, \qquad \Delta(\varphi F)(x_0,t_0) \leq 0, \tag{3.9}$$

where the function (φF) is being considered with support on $\mathcal{Q}_{2\rho} \times [0,T]$ and we have assumed that $(\varphi F)(x_0, t_0) > 0$ for $t_0 > 0$. By (3.9) we notice that

$$(\Delta - \partial_t)(\varphi F)(x_0,t_0) \leq 0.$$

Hence, we have by using the inequality (3.3) and the equation (3.8):

$$
\begin{aligned}
0 \geq (\Delta-\partial_t)(\varphi F) &\geq 2\nabla\varphi\nabla F - 2\varphi|\nabla f|\nabla F - \varphi\frac{F}{t} \\
&\quad + \varphi\Big\{\frac{2\alpha}{np}t\big(|\nabla f|^2-\partial_t f-\mathcal{R}\big)^2 - 2\big(k_1+(\alpha-1)k_3+k_4\big)t|\nabla f|^2 \\
&\quad - 2\alpha^2 nt\Big(\frac{q}{4}\widetilde{K}+k_4\Big)\Big\} - 2t(\alpha-1)|\nabla f||\nabla\mathcal{R}|\varphi - \alpha t\varphi|\Delta\mathcal{R}| + C_3F,
\end{aligned}
\tag{3.10}
$$

where we have used the inequality

$$
3\alpha n^{\frac{1}{2}}k_4t|\nabla f| \leq 2tk_4|\nabla f|^2 + 2\alpha^2 ntk_4
$$

and

$$
(\Delta-\partial_t)\varphi \geq \left(-\frac{C_2}{\rho^2} - \frac{C_1(n-1)}{\rho}\sqrt{k_1}\coth(\sqrt{k_1}\rho) - C_2\widetilde{K}\right) =: C_3,
$$

where C_1, C_2 depend only on n as defined in section three. The calculation of C_3 is as follows:

$$
\frac{|\nabla\varphi|^2}{\varphi} = \frac{|\psi'|^2\cdot|\nabla d|^2}{\rho^2\varphi} \leq \frac{C_2}{\rho^2}
$$

and by the Laplacian comparison theorem we have

$$
\begin{aligned}
\Delta\varphi = \frac{\psi'\Delta d}{\rho} + \frac{\psi''|\nabla d|^2}{\rho} &\geq -\frac{C_1}{\rho}(n-1)\sqrt{k_2}\coth(\sqrt{k_2}\rho) - \frac{C_2}{\rho^2} \\
&\geq -\frac{C_1}{\rho}(n-1)\sqrt{k_2}\coth(\sqrt{k_2}\rho) - \frac{C_2}{\rho^2} \geq -\frac{C_1}{\rho}\sqrt{k_2} - \frac{C_2}{\rho^2}.
\end{aligned}
$$

Next is to estimate time derivative of φ: consider a fixed smooth path $\gamma\colon [a,b]\to M$ whose length at time t is given by $d(\gamma)=\int_a^b|\gamma'(t)|_{g(t)}\,\mathrm{d}r$, where r is the arc length. Differentiating we get

$$
\frac{\partial}{\partial t}(d(\gamma)) = \frac{1}{2}\int_a^b \Big|\gamma'(t)\Big|_{g(t)}^{-1}\frac{\partial g}{\partial t}\big(\gamma'(t),\gamma'(t)\big)\,\mathrm{d}r = \int_\gamma h_{ij}(X,X)\,\mathrm{d}r,
$$

where X is the unit tangent vector to the path γ (for details see [5, Lemma 3.11]). Now

$$
\frac{\partial}{\partial t}\varphi = \psi'\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}t}(d(t)) = \psi'\frac{1}{\rho}\int_\gamma h_{ij}(X,X)\,\mathrm{d}r \leq \frac{\sqrt{C_2}}{\rho}(k_2+k_3)^2\int_\gamma \mathrm{d}r = \sqrt{C_2}\tilde{K}.
$$

Hence

$$
C_3 := -\frac{C_1}{\rho}\sqrt{k_2} - \frac{C_2}{\rho^2} - \sqrt{C_2}\tilde{K}
$$

will be used later.

Notice that since $\nabla(\varphi F)=0$, the product rule tells us that we can always replace $-F\nabla\varphi$ with $\varphi\nabla F$ at the maximum point (x_0,t_0). Indeed, the following identity holds

$$
-\varphi\nabla F\cdot\nabla f = F\nabla\varphi\cdot\nabla f.
$$

The above inequality (3.10) holds in the part of $\mathcal{Q}_{2\rho,T}$, where $\varphi(x,t)$ is strictly positive ($0 < \varphi(x,t) \leq 1$). Multiplying by $(t\varphi)$ after some simple calculation involving the last identity at the maximum point we get

$$\begin{aligned}
0 \geq & -2t\frac{C_2}{\rho^2}\varphi F - \varphi^2 F - 2t\frac{\sqrt{C_2}}{\rho^2}|\nabla f|\varphi^{\frac{3}{2}}F \\
& + \frac{2t^2}{n}\left\{\frac{\alpha}{p}\big(\varphi|\nabla f|^2 - \varphi\partial_t f - \varphi\mathcal{R}\big)^2 - n\big(k_1 + (\alpha-1)k_3 + k_4\big)\varphi^2|\nabla f|^2\right. \\
& \left. -\alpha^2 n^2\left(\frac{q}{4}\widetilde{K} + k_4\right)\varphi^2\right\} - 2t^2(\alpha-1)|\nabla f||\nabla\mathcal{R}|\varphi^2 - \alpha t^2\varphi^2|\Delta\mathcal{R}| + C_3 t\varphi F.
\end{aligned}$$

Using a similar technique as in Li–Yau paper [12] when $t > 0$, let $y = \varphi|\nabla f|^2$ and $z = \varphi(f_t + \mathcal{R})$ to obtain $\varphi^2|\nabla f|^2 \leq \varphi y \leq y$ and $y^{\frac{1}{2}}(y - \alpha z) = \frac{1}{t}|\nabla f|\varphi^{\frac{3}{2}}F$, with $t(y - \alpha z) = \varphi F$. We get

$$\begin{aligned}
0 \geq & -2t\frac{C_2}{\rho^2}\varphi F - \varphi^2 F - 2t\frac{\sqrt{C_2}}{\rho^2}|\nabla f|\varphi^{\frac{3}{2}}F \\
& + \frac{2t^2}{n}\left\{\frac{\alpha}{p}(y-z)^2 - n\big(k_1 + (\alpha-1)k_3 + k_4\big)y - \alpha^2 n^2\left(\frac{q}{4}\widetilde{K} + k_4\right)\varphi^2\right\} \\
& - 2t^2(\alpha-1)y^{\frac{1}{2}}|\nabla\mathcal{R}| - \alpha t^2\varphi^2|\Delta\mathcal{R}| + C_3 t\varphi F.
\end{aligned}$$

Rearranging, using bounds on the gradients of $\mathcal{R}$, namely, $|\nabla\mathcal{R}| \leq \beta$, $|\Delta\mathcal{R}| \leq \xi$, and denoting

$$D = D(\alpha, k_1, k_3, k_4) = \big(k_1 + (\alpha-1)k_3 + k_4\big)$$

we have

$$\begin{aligned}
0 \geq -\varphi^2 F + \frac{2t^2}{n}\Bigg\{ & \frac{\alpha}{p}(y-z)^2 - 2nDy + nDy - np(\alpha-1)\beta y^{\frac{1}{2}} \\
& - \frac{np\sqrt{C_2}}{\rho^2}y(y-\alpha z) - \alpha^2 n^2\left(\frac{q}{4}\widetilde{K} + k_4\right)\varphi^2\Bigg\} \\
& - \alpha t^2\varphi^2\xi + \left(C_3 t - 2t\frac{C_2}{\rho^2}\right)\varphi F.
\end{aligned} \tag{3.11}$$

Notice by direct calculation that

$$\begin{aligned}
(y-z)^2 &= \left[\frac{1}{\alpha}(y-\alpha z) + \frac{\alpha-1}{\alpha}y\right]^2 \\
&= \frac{1}{\alpha^2}(y-\alpha z)^2 + \frac{(\alpha-1)^2}{\alpha^2}y^2 + \frac{2(\alpha-1)}{\alpha^2}y(y-\alpha z).
\end{aligned}$$

Then, the second term in the right-hand side of the last inequality can be simplified as follows:

$$\begin{aligned}
&\frac{2t^2}{n}\bigg\{\frac{\alpha}{p}\bigg[(y-z)^2-\frac{2npD}{\alpha}y+\frac{npD}{\alpha}y-\frac{np(\alpha-1)}{\alpha}\beta y^{\frac12}-\frac{np}{\alpha}\frac{\sqrt{C_2}}{\rho^2}y(y-\alpha z)\bigg]\\
&\qquad-\alpha^2n^2\Big(\frac{q}{4}\widetilde{K}+k_4\Big)\varphi^2\bigg\}\\
&=\frac{2t^2}{n}\bigg\{\frac{\alpha}{p}\bigg[\frac{1}{\alpha^2}(y-\alpha z)^2+\bigg(\frac{(\alpha-1)^2}{\alpha^2}y^2-\frac{2npD}{\alpha}y\bigg)+\bigg(\frac{npD}{\alpha}y-\frac{np}{\alpha}(\alpha-1)\beta y^{\frac12}\bigg)\\
&\qquad+\bigg(\frac{2(\alpha-1)}{\alpha^2}y-\frac{np}{\alpha}\frac{\sqrt{C_2}}{\rho}y^{\frac12}\bigg)(y-\alpha z)\bigg]-\alpha^2n^2\Big(\frac{q}{4}\widetilde{K}+k_4\Big)\varphi^2\bigg\}\\
&\geq\frac{2t^2}{n}\bigg\{\frac{1}{\alpha p}(y-\alpha z)^2-\frac{\alpha n^2pD^2}{(\alpha-1)^2}-\frac{n(\alpha-1)^2}{4D}\beta^2-\frac{C_2\alpha n^2p}{8\rho^2(\alpha-1)}(y-\alpha z)\\
&\qquad-\alpha^2n^2\Big(\frac{q}{4}\widetilde{K}+k_4\Big)\varphi^2\bigg\}\\
&=\frac{2}{\alpha np}(\varphi F)^2-\frac{2\alpha npD^2}{(\alpha-1)^2}t^2-\frac{(\alpha-1)^2}{2D}\beta^2t^2-\frac{C_2\alpha np}{4\rho^2(\alpha-1)}t(\varphi F)\\
&\qquad-2\alpha^2n\Big(\frac{q}{4}\widetilde{K}+k_4\Big)t^2\varphi^2.
\end{aligned}$$

We have used the inequality of the form $ax^2-bx\geq-\frac{b^2}{4a}$, $(a,b>0)$, to compute

$$\begin{aligned}
\frac{(\alpha-1)^2}{\alpha^2}y^2-\frac{2npD}{\alpha}y&\geq-\frac{n^2p^2D^2}{(\alpha-1)^2},\\
\frac{2(\alpha-1)}{\alpha^2}y-\frac{np}{\alpha}\frac{\sqrt{C_2}}{\rho}y^{\frac12}&\geq-C_2\frac{n^2p^2}{8(\alpha-1)\rho^2},\\
\frac{npD}{\alpha}y-\frac{np}{\alpha}(\alpha-1)\beta y^{\frac12}&\geq-\frac{np(\alpha-1)^2}{4\alpha D}\beta^2.
\end{aligned}$$

Therefore, putting all these together into (3.11), we get

$$\begin{aligned}
0\geq&\frac{2}{\alpha np}(\varphi F)^2+\bigg(C_3t-1-2t\frac{C_2}{\rho^2}-\frac{C_2\alpha np}{4\rho^2(\alpha-1)}t\bigg)(\varphi F)\\
&-\bigg(\frac{2\alpha npD^2}{(\alpha-1)^2}t^2+\frac{(\alpha-1)^2}{2D}\beta^2t^2+2\alpha^2n\Big(\frac{q}{4}\widetilde{K}+k_4\Big)t^2\varphi^2+\alpha t^2\varphi^2\xi\bigg).
\end{aligned}$$

Obviously, the last expression is a quadratic polynomial in (φF), then we develop a formula for quadratic inequality of the form $ax^2+bx+c\leq 0$, for $x\in\mathbb{R}$. Note that when $a>0$ and $c<0$, then $b^2-4ac>0$ and we have an upper bound

$$x\leq\frac{-b+\sqrt{b^2-4ac}}{2a}\leq\frac{1}{2a}\Big\{-b+\sqrt{-4ac}\Big\}.$$

The next step is to make the term

$$b=\bigg(C_3t-1-2t\frac{C_2}{\rho^2}-\frac{C_2\alpha np}{4\rho^2(\alpha-1)}t\bigg)$$

more explicit to obtain

$$b = -\left[\frac{C_4}{\rho^2}t\left(\frac{\alpha p}{\alpha-1} + \frac{\rho}{t} + \rho\sqrt{k_1} + \rho^2(k_2+k_3)^2\right) + 1\right],$$

where $C_4 = \max\{C_1, C_2\}$. Hence, we have

$$\begin{aligned}\varphi F \leq\ & \frac{\alpha np}{4} + \frac{\alpha np}{4\rho^2}C_4 t\left(\frac{\alpha p}{\alpha-1} + \frac{\rho}{t} + \rho\sqrt{k_1} + \rho^2(k_2+k_3)^2\right)\\ & + \frac{\alpha np}{\alpha-1}t\big(k_1 + (\alpha-1)k_3 + k_4\big) + \frac{(\alpha-1)}{2}\beta t\sqrt{\frac{\alpha np}{\big(k_1+(\alpha-1)k_3+k_4\big)}}\\ & + \frac{\alpha n}{2}(k_2+k_3+2\sqrt{k_4})t\sqrt{pq} + \alpha t\sqrt{\frac{np\xi}{2}}.\end{aligned}$$

To obtain the required bound on $F(x,\tau)$ for an appropriate range of $x \in M$, we take $\varphi(x,\tau) \equiv 1$ whenever $\operatorname{dist}(x, x_0, \tau) < \rho$ and since (x_0, t_0) is the maximum point for (φF) in $\mathcal{Q}_{2\rho,T}$, we have

$$F(x,\tau) = (\varphi F)(x,\tau) \leq (\varphi F)(x_0, t_0)$$

for all $x \in M$, such that $\operatorname{dist}(x, x_0, \tau) < \rho$ and $\tau \in (0, T]$ was arbitrarily chosen, then we have the conclusion in a more compact way, that

$$\begin{aligned}&\sup_{x\in\mathcal{Q}_{2\rho}}\big\{|\nabla f|^2 - \alpha f_t - \alpha R\big\}\\ &\qquad \leq \frac{\alpha np}{4t} + C'\left(K_1 + k_2 + k_3 + k_4 + \sqrt{k_1} + \sqrt{k_4} + \beta + \xi + \frac{1}{t}\right)\\ &\qquad\quad + \frac{\alpha np}{\alpha-1}\big(k_1 + (\alpha-1)k_3 + k_4\big).\end{aligned}$$

This ends the proof of Theorem 3.2. □

Remark 3.3 *A global estimate follows by letting $\rho \to \infty$ for all $t > 0$. For instance, if we set $p = 2\alpha = q$ and allow ρ to go to infinity, we have the estimate*

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} - \alpha\mathcal{R} \leq \frac{\alpha^2 n}{2t} + C'', \tag{3.12}$$

where C'' is an absolute constant depending on n, τ, α and the upper bounds of $|Rc|$, $|\nabla\mathcal{R}|$, $|\Delta\mathcal{R}|$, $|h|$, $|\nabla h|$.

As an application of the global gradient estimate derived in Theorem 3.2, we obtain the following result for the corresponding Harnack estimates. Here M is complete without (or with empty) boundary. We choose to follow the traditional approach of integrating Harnack quantity along a geodesic path connecting two points and exponentiating the result. Given $x_1, x_2 \in M$ and $t_1, t_2 \in [0, T]$ satisfying $t_1 < t_2$

$$\Theta(x_1, t_1; x_2, t_2) = \inf_{\gamma}\int_{t_1}^{t_2}\left(\left|\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\right|^2 + \frac{4}{A}h\right)\mathrm{d}t,$$

where the infimum is taken over all the smooth paths $\gamma\colon [t_1, t_2] \to M$ connecting x_1 and x_2. The norm $|.|$ depends on t. We now present a lemma whose aim is to give an insight into how the approach goes (see [3] and [15]).

Lemma 3.4 *Let $(M, g(t))$ be a complete solution to the Ricci flow. Let $u\colon M \times [0,T] \to \mathbb{R}$ be a smooth positive solution to the heat equation* (1.1) *and let h be a C^2-function on $M \times [0,T]$. Define $f = -\log u$ and assume that*

$$-\frac{\partial f}{\partial t} \leq \frac{1}{A}\Big(\frac{B}{t} + C - |\nabla f|^2\Big) + h, \qquad (x,t) \in M \times [0,T]$$

for some $A, B, C > 0$. Then, the inequality

$$u(x_1,t_1) \leq u(x_2,t_2)\bigg(\frac{t_2}{t_1}\bigg)^{\frac{B}{A}} \exp\bigg(\frac{A}{4}\Theta(x_1,t_1;x_2,t_2) + \frac{C}{A}(t_2 - t_1)\bigg) \tag{3.13}$$

holds for all (x_1,t_1) and (x_2,t_2) such that $t_1 < t_2$.

Proof. Obtain the time differential of a function f depending on the path γ as follows ($t \in [t_1,t_2]$),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}f(\gamma(t),t) &= \nabla f(\gamma(t),t)\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) - \frac{\partial}{\partial s}f(\gamma(t),s)\Big|_{s=t} \\ &\leq |\nabla f|\Big|\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\Big| + \frac{1}{A}\bigg(\frac{B}{t} + C - |\nabla f|^2\bigg) + h \\ &\leq \frac{A}{4}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\Big|^2 + \frac{B}{At} + \frac{C}{A} + h.\end{aligned}$$

The last inequality was obtained by the application of completing the square method in form of a quadratic inequality satisfying $ax^2 - bx \geq -\frac{b^2}{4a}$, $(a, b > 0)$. Then, integrating over the path from t_1 to t_2, we have

$$\begin{aligned}f(x_2,t_2) - f(x_1,t_1) &= \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}f(\gamma(t),t)\,\mathrm{d}t \\ &\leq \frac{A}{4}\int_{t_1}^{t_2}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\Big|^2\mathrm{d}t + \frac{B}{A}\log t\Big|_{t_1}^{t^2} + \frac{C}{A}(t_2 - t_1) + \int_{t_1}^{t_2} h\,\mathrm{d}t.\end{aligned}$$

The required estimate (3.13) follows immediately after exponentiation.[2] □

Theorem 3.5 *Let $(M, g(t))$, $t \in [0,T]$ be a complete solution to the geometric flow* (3.1)*. Suppose there exist some non-negative constants k_1, k_2, k_3, and k_4 such that $R_{ij}(g) \geq -k_1 g$, $-k_2 g \leq h \leq k_3 g$ and $|\nabla h| \leq k_4$ for all $t \in [0,T]$. Suppose $u(x,t)$ is any positive solution to the heat equation* (3.2)*, where $\mathcal{R} = \mathcal{R}(x,t)$ is a C^2-function with $|\nabla\mathcal{R}| \leq \beta$ and $|\Delta\mathcal{R}| \leq \xi$. Then for any points (x_1,t_1) and (x_2,t_2) in $M \times [0,T]$ such that $0 < t_1 < t_2 \leq T$, and for any $\alpha > 1$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}$, the following estimate*

$$\begin{aligned}\frac{u(x_1,t_1)}{u(x_2,t_2)} \leq \bigg(\frac{t_2}{t_1}\bigg)^{\frac{np}{4}} \exp\bigg\{\int_0^1\bigg(\frac{\alpha|\dot\gamma(s)|^2 + 4(t_2-t_1)^2\mathcal{R}}{4(t_2-t_1)}\bigg)\mathrm{d}s \\ + \frac{(t_2-t_1)}{2(\alpha-1)}A(k_1,k_2,k_3,k_4,\alpha,\beta,\xi)\bigg\}\end{aligned} \tag{3.14}$$

[2] As observed by the referee the inequality (3.13) is obtained by finding the maximum of a quadratic expression in $|\nabla f|^2$.

holds for all $(x,t) \in M \times (0,T]$, *where*

$$A(k_1,k_2,k_3,k_4,\alpha,\beta,\xi) = 2\alpha np(k_1+(\alpha-1)k_3+k_4)+\beta(k_1+(\alpha-1)k_3+k_4)^{\frac{1}{2}}/np \\ +(\alpha-1)(k_2+k_3+2\sqrt{k_4})\sqrt{pq}+2(\alpha-1)\sqrt{np\xi}$$

the infimum is taken over all the smooth paths $\gamma\colon [t_1,t_2] \to M$ *connecting* x_1 *and* x_2.

Proof. After letting $\rho \to \infty$, we have from the estimate (3.7) that

$$-f_t \le \frac{1}{\alpha}\left\{\frac{\alpha np}{4t}+\frac{\alpha}{\alpha-1}A-\frac{|\nabla u|^2}{u^2}\right\}+\mathcal{R}.$$

The desired estimate follows by applying Lemma 3.4 (see also [1, Theorem 4.2] and [15, Section 4.2]). □

We conclude this section with very useful estimates found by Hamilton [9]. He was inspired by the results of Li and Yau [12], hence the estimates are popularly referred to as Li–Yau–Hamilton (LYH) estimates. We state the result for bounded solutions on a closed (*i.e.*, compact without boundary) manifold.

Theorem 3.6 *(LYH gradient estimates). Let* $(M,g(t)), t\in[0,T]$ *be a closed Riemannian manifold evolving by the geometric flow* (3.1) *in some time interval* $[0,T]$. *Let there exist non-negative constants* k_1 *and* k_2 *such that* $R_{ij} \ge -k_1 g_{ij}$, $-k_1 g_{ij} \le h_{ij} \le k_1 g_{ij}$, $|\nabla\mathcal{R}|^2 \le k_2$ *and* $\mathcal{R} \ge -k_2$. *Suppose* u *is a positive solution to the heat-type equation* (3.2) *with* $u \le A < \infty$. *Then*

$$t\frac{|\nabla u|^2}{u^2} \le \big(1+(4k_1+\delta)t\big)\left(k_2+\log\left(\frac{A}{u}\right)\right)+Tk_2 \tag{3.15}$$

for some constant $\delta > 0$.

Proof. Let $f=\log u$ so that $|\nabla f|^2 = |\nabla \log u|^2 = \frac{|\nabla u|^2}{u^2}$ and $\big(\frac{\partial}{\partial t}-\Delta\big)f = |\nabla f|^2-\mathcal{R}$. Define a heat type operator

$$\mathcal{L} := \left(\frac{\partial}{\partial t}-\Delta-2\nabla f\cdot\nabla\right).$$

The idea of the proof is to apply the heat-type operator $\mathcal{L}$ on

$$t\frac{|\nabla u|^2}{u^2}-\big(1+(4k_1+\delta)t\big)\log\left(\frac{A}{u}\right)$$

and then use the weak maximum principle. On compact manifolds the weak maximum principle for parabolic equations asserts that pointwise bounds are preserved ([5, Chapter 4] provides a good survey on this subject). Recall from Lemma 2.1 and the Bochner–Weitzenböck identity (2.2) that

$$\frac{\partial}{\partial t}|\nabla f|^2 = -2h_{ij}f_if_j+2f_if_{ti} \qquad \text{and} \qquad \Delta|\nabla f|^2 = 2|f_{ij}|^2+2f_jf_{jji}+2R_{ij}f_if_j.$$

Hence

$$\left(\frac{\partial}{\partial t}-\Delta\right)|\nabla f|^2 = -2|f_{ij}|^2-2(R_{ij}+h_{ij})f_if_j+2\big\langle\nabla f,\nabla|\nabla f|^2\big\rangle-2\big\langle\nabla f,\nabla\mathcal{R}\big\rangle$$

and

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\big(t|\nabla f|^2\big) &= |\nabla f|^2 - 2t|f_{ij}|^2 - 2t(R_{ij}+h_{ij})f_i f_j \\ &\quad + 2t\big\langle \nabla f, \nabla|\nabla f|^2\big\rangle - 2t\big\langle \nabla f, \nabla\mathcal{R}\big\rangle \\ &\leq (1+4k_1 t)|\nabla f|^2 + 2t\big\langle \nabla f, \nabla|\nabla f|^2\big\rangle - 2t\big\langle \nabla f, \nabla\mathcal{R}\big\rangle.\end{aligned}$$

With the above calculation we obtain

$$\mathcal{L}\big(t|\nabla f|^2\big) \leq (1+4k_1 t)|\nabla f|^2 - 2t\langle \nabla f, \nabla R\rangle.$$

Observe that for any $\delta > 0$

$$-2t|\nabla f||\nabla\mathcal{R}| \leq \delta t|\nabla f|^2 + \delta^{-1}t|\nabla\mathcal{R}|^2.$$

Then

$$\begin{aligned}\mathcal{L}\big(t|\nabla f|^2\big) &\leq \big(1+(4k_1+\delta)t\big)|\nabla f|^2 + \delta^{-1}tk_2 \\ &\leq \big(1+(4k_1+\delta)t\big)|\nabla f|^2 + Tk_2,\end{aligned} \tag{3.16}$$

since $\delta > 0$ and $t \leq T$. Now

$$\begin{aligned}\mathcal{L}\big(1+(4k_1+\delta)t\big)\log\left(\frac{A}{u}\right) &= (4k_1+\delta)\log\left(\frac{A}{u}\right) + \big(1+(4k_1+\delta)t\big)\mathcal{L}\log\left(\frac{A}{u}\right) \\ &= (4k_1+\delta)\log\left(\frac{A}{u}\right) + \big(1+(4k_1+\delta)t\big)\left(\frac{\partial}{\partial t}-\Delta\right)\log\left(\frac{A}{u}\right) \\ &\quad - 2\big(1+(4k_1+\delta)t\big)\left\langle \nabla f, \nabla\log\left(\frac{A}{u}\right)\right\rangle.\end{aligned}$$

Computing

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\log\left(\frac{A}{u}\right) &= \left(\frac{\partial}{\partial t}-\Delta\right)\log A - \left(\frac{\partial}{\partial t}-\Delta\right)\log u \\ &= -|\nabla f|^2 + \mathcal{R} \\ &\geq 2\left\langle \nabla f, \nabla\log\left(\frac{A}{u}\right)\right\rangle + |\nabla f|^2 - k_2.\end{aligned}$$

Then

$$\mathcal{L}\big(1+(4k_1+\delta)t\big)\log\left(\frac{A}{u}\right) \geq \big(1+(4k_1+\delta)t\big)|\nabla f|^2 - \big(1+(4k_1+\delta)t\big)k_2. \tag{3.17}$$

Combining the expressions in (3.16)–(3.17) we obtain

$$\mathcal{L}\left(t|\nabla f|^2 - \big(1+(4k_1+\delta)t\big)\log\left(\frac{A}{u}\right)\right) \leq \Big(T + \big(1+(4k_1+\delta)t\big)\Big)k_2 =: C_4.$$

Note that at $t = 0$,

$$-\log\left(\frac{A}{u}\right) \leq 0.$$

Then by the weak maximum principle we have

$$t|\nabla f|^2 - \big(1+(4k_1+\delta)t\big)\log\left(\frac{A}{u}\right) \leq C_4$$

for all $t \geq 0$, where C_4 depends on t, T, k_1, k_2 and δ. This is precisely

$$t\frac{|\nabla u|^2}{u^2} \leq \big(1+(4k_1+\delta)t\big)\log\left(\frac{A}{u}\right) + \big(1+(4k_1+\delta)t+T\big)k_2,$$

which completes the proof. □

The above result can be extended to the case of complete non-compact manifold, although, a little more effort will be required. The idea here is to use ϵ-regularization method by supposing that the solution $u \geq \epsilon$, replacing u by $u_\epsilon = u + \epsilon$ for a sufficiently small $\epsilon > 0$ and letting ϵ go to zero after the analysis for u_ϵ is completed. An application of this result shows we can bound the maximum of a positive by its integral (see [9]). Furthermore, the estimate yields sharp lower and upper bounds for the fundamental solution, see [4].

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