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CAUCHY SYSTEM FOR AN HYPERBOLIC OPERATOR

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Abstract. This paper deals with a control problem of a Cauchy System for an hyperbolic operator. The associate system here which is distributed and singular has in general no solution, and when a solution does exist it is unstable. So instead of considering the control v and the state z separately, we consider the pair control-state (v, z); it suffices then to make sure that the set of admissible pairs (v, z) is non-empty. We establish the existence and the uniqueness of the optimal pair and then we characterize it by using the penalization method.

Keywords: Singular Distributed System, Cauchy System for an hyperbolic operator, Admissible pair, Penalization, Singular Optimal System.

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1 Statement of the problem

Let Ω be an open set with boundary $\partial\Omega = \Gamma = \Gamma_0 \cup \Gamma_1$ of class C^∞, where the boundaries Γ_0, Γ_1 are non-empty and $\Gamma_0 \cap \Gamma_1 = \emptyset$. For $T > 0$ we set $Q = \Omega\times]0, T[,\ \Sigma_1 = \Gamma_1\times]0, T[,\ \ \Sigma_0 = \Gamma_0\times]0, T[$

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and we consider the state z of a system subject to the control $v = (v_0, v_1)$, related by

$$\left\{\begin{array}{rcll} z'' - \Delta z &=& 0 & \text{in } Q \\ z &=& v_0 & \text{on } \Sigma_0 \\ \dfrac{\partial z}{\partial \nu} &=& v_1 & \text{on } \Sigma_0 \\ z(0) &=& 0 & \text{in } \Omega \\ \dfrac{\partial z}{\partial t}(0) &=& 0 & \text{in } \Omega \end{array}\right. \tag{1.1}$$

where $z'' = \dfrac{\partial^2 z}{\partial t^2}$. Let $\mathcal{U}_{ad}^0$ and $\mathcal{U}_{ad}^1$ be two non-empty closed convex subsets of $L^2(\Sigma_0)$. We set

$$\mathcal{A} = \left\{(v_0, v_1, z) \in \mathcal{U}_{ad}^0 \times \mathcal{U}_{ad}^1 \times L^2(Q) \text{ such that (1.1) holds } \right\} \tag{1.2}$$

and we assume that

$$\mathcal{A} \neq \emptyset. \tag{1.3}$$

A triplet (v_0, v_1, z) will be called admissible if it belongs in $\mathcal{A}$.

For $(v = (v_0, v_1), z) \in \mathcal{A}$, we consider the cost function

$$J(v, z) = \frac{1}{2}|z - z_d|^2_{L^2(Q)} + \frac{N_0}{2}|v_0|^2_{L^2(\Sigma_0)} + \frac{N_1}{2}|v_1|^2_{L^2(\Sigma_0)} \tag{1.4}$$

with $N_0, N_1 > 0$ and $z_d \in L^2(Q)$.
We are then interest in the problem:

$$\inf_{(v,z)\in\mathcal{A}} J(v, z). \tag{1.5}$$

Remark 1.1 *The boundary conditions* $(1.1)_2$ *and the initials conditions* $(1.1)_3$ *do have a meaning. Indeed since* $z \in L^2(Q) = L^2(0, T; L^2(\Omega))$ *we have* $z" \in H^{-2}(0, T; L^2(\Omega))$*. As* $z" - \Delta z \in L^2(Q)$*, we deduce that* $\Delta z \in H^{-2}(0, T; L^2(\Omega))$*. Thus* $z(t) \in L^2(\Omega)$ *and* $\Delta z(t) \in L^2(\Omega)$*. Consequently the traces* $z_{|\Sigma}(t)$ *and* $\dfrac{\partial z}{\partial \nu}_{|\Sigma}(t)$ *exist and belong respectively to* $H^{-\frac{1}{2}}(\Gamma)$ *and* $H^{-\frac{3}{2}}(\Gamma)$ *(see [7, p. 79]). On the other hand, since* $z \in L^2(Q)$ *, we have that* $\Delta z \in L^2(0, T; H^{-2}(\Omega))$*. Therefore* $z" \in L^2(0, T; H^{-2}(\Omega))$ *and we deduce that* $z' \in L^2(0, T; H^{-1}(\Omega))$ *and* $(z(0), z(T)) \in [H^{-1}(\Omega)]^2$ *and* $\left(\dfrac{\partial z}{\partial t}(0), \dfrac{\partial z}{\partial t}(T)\right) \in \left[H^{-2}(\Omega)\right]^2$ *(see [11, Theorem 9.2]).*

Here are now some examples in which the set $\mathcal{U}_{ad} \times L^2(Q)$ of admissible pairs (v, z) is non-empty.

Example 1.2 *Let us assume that*

$$\left\{\begin{array}{l} \mathcal{U}_{ad} = L^2(\Sigma_0) \times \mathcal{U}_{ad}^1, \\ \mathcal{U}_{ad}^1 \textit{ is a closed convex set of } L^2(\Sigma_0) \\ \quad \textit{containing at least one function } v_1 \in L^2(\Sigma_0). \end{array}\right. \tag{1.6}$$

We construct the solution ζ of

$$\begin{cases} M\zeta &= 0 & \text{in } Q \\ \dfrac{\partial\zeta}{\partial\nu} &= 0 & \text{on } \Sigma_1 \\ \dfrac{\partial\zeta}{\partial\nu} &= v_1 & \text{on } \Sigma_0 \\ \zeta(0) &= 0 & \text{in } \Omega \\ \dfrac{\partial\zeta}{\partial t}(0) &= 0 & \text{in } \Omega \end{cases} \tag{1.7}$$

where from now on M is given by

$$M = \frac{\partial^2}{\partial t^2} - \Delta. \tag{1.8}$$

Since $v_1 \in L^2(\Sigma_0)$, System (1.7) has a unique $\zeta \in L^2(0,T,H^1(\Omega))$ (see [7, p. 347]). therefore $\zeta_{|\Sigma} \in L^2(0,T,H^{1/2}(\Omega)))$. In particular $\zeta_{|\Sigma_0} \in L^2(\Sigma_0)$. Hence the pair

$$\left((\zeta_{|\Sigma_0}, v_1), \zeta\right)$$

is then admissible, consequently the set of admissible pairs is non-empty.

Example 1.3 *Let us assume*

$$\begin{cases} \mathcal{U}_{ad} = \mathcal{U}_{ad}^0 \times L^2(\Sigma_0), \\ \mathcal{U}_{ad}^0 \text{ is a closed convex set of } L^2(\Sigma_0) \\ \quad \text{containing at least one function } v_0 \in H^1(0,T;H^{\frac{3}{2}}(\Gamma_0)) \cap H_0^2(0,T;H^0(\Gamma_0)). \end{cases} \tag{1.9}$$

We construct ζ solution of

$$\begin{cases} M\zeta &= 0 & \text{in } Q \\ \zeta &= v_0 & \text{on } \Sigma_0 \\ \zeta &= 0 & \text{on } \Sigma_1 \\ \zeta(0) &= 0 & \text{in } \Omega \\ \dfrac{\partial\zeta}{\partial t}(0) &= 0 & \text{in } \Omega \end{cases} \tag{1.10}$$

Since $v_0 \in H^1(0,T;H^{\frac{3}{2}}(\Gamma_0)) \cap H_0^2(0,T;H^0(\Gamma_0))$, we can find (see [12, Theorem 3.1, p. 112]) a function $w \in H^{2,2}(Q)$ such that

$$w|_\Sigma = v_0\, \chi_{\Gamma_0}, \quad w(x,0) = \frac{\partial w}{\partial t}(x,0) = 0 \text{ in } \Omega.$$

Consequently, then $\zeta \in H^{2;1}(Q))$ (see [12, Theorem 3.2, p. 113]). Therefore $\frac{\partial\zeta}{\partial\nu}_{|\Sigma} \in L^2(\Sigma))$. In particular $\frac{\partial\zeta}{\partial\nu}_{|\Sigma_0} \in L^2(\Sigma_0)$. Hence the pair

$$\left(\left(v_0, \frac{\partial\zeta}{\partial\nu}|_{\Sigma_0}\right), \zeta\right)$$

is admissible.

Let us come back now to the general problem (1.5) with $\mathcal{U}_{ad}$ a non-empty closed convex set. It is well known that the problem (1.5) has a unique solution, the optimal pair (u, y) which we are going to characterize.
If (u, y) is the optimal pair the first order Euler-Lagrange conditions give

$$\forall\ (v, z) \in \mathcal{U}_{ad}, \quad (y - z_d, z - y)_{L^2(Q)} + N_0(u_0, v_0 - u_0)_{L^2(\Sigma_0)} + N_1(u_1, v_1 - u_1)_{L^2(\Sigma_0)} \geq 0.$$

We can notice, in this inequality, that the variations of v and z are coupled, then it is important or at least interesting to obtain an optimality system (O.S) in which v and z are not coupled. It is the subject of this paper, precisely we have the following results.

Theorem 1.4 *Assume that (1.3) holds. Let* $\mathcal{U}_{ad} = L^2(\Sigma_0) \times \mathcal{U}^1_{ad}$. *Then the optimal pair* (u, y) *is characterized by the triplet* $(u, y, p) \in \mathcal{U}^1_{ad} \times L^2(Q) \times L^2(0, T; H^1(\Omega))$ *which is a solution of the Singular Optimality System (S.O.S.)*

$$\begin{cases} My = 0 & \text{in } Q \\ y = u_0 & \text{on } \Sigma_0, \\ \dfrac{\partial y}{\partial \nu} = u_1 & \text{on } \Sigma_0, \\ y(0) = 0 & \text{in } \Omega, \\ \dfrac{\partial y}{\partial t}(0) = 0 & \text{in } \Omega, \end{cases} \tag{1.11}$$

$$\begin{cases} Mp = z_d - y & \text{in } Q \\ p = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial p}{\partial \nu} = 0 & \text{on } \Sigma_1, \\ p(T) = 0 & \text{in } \Omega, \\ \dfrac{\partial p}{\partial t}(T) = 0 & \text{in } \Omega, \end{cases} \tag{1.12}$$

$$\frac{\partial p}{\partial \nu} = N_0 u_0 \quad \text{on } \Sigma_0 \tag{1.13}$$

and

$$\forall\ v_1 \in \mathcal{U}^1_{ad}, \quad (-p + N_1 u_1, v_1 - u_1)_{L^2(\Sigma_0)} \geq 0. \tag{1.14}$$

Theorem 1.5 *Assume that (1.3) holds. Let* $\mathcal{U}_{ad} = \mathcal{U}^0_{ad} \times L^2(\Sigma_0)$. *Then the optimal pair* (u, y) *is characterized by the triplet* $(u, y, p) \in \mathcal{U}^1_{ad} \times L^2(Q) \times L^2(Q)$ *which is a solution of the S.O.S.*

$$\begin{cases} My = 0 & \text{in } Q \\ y = u_0 & \text{on } \Sigma_0, \\ \dfrac{\partial y}{\partial \nu} = u_1 & \text{on } \Sigma_0, \\ y(0) = 0 & \text{in } \Omega, \\ \dfrac{\partial y}{\partial t}(0) = 0 & \text{in } \Omega, \end{cases} \tag{1.15}$$

$$\begin{cases} Mp &= z_d - y \quad \textit{in } Q \\ p &= 0 \quad \textit{on } \Sigma_1, \\ \dfrac{\partial p}{\partial \nu} &= 0 \quad \textit{on } \Sigma_1, \\ p(T) &= 0 \quad \textit{in } \Omega, \\ \dfrac{\partial p}{\partial t}(T) &= 0 \quad \textit{in } \Omega, \end{cases} \tag{1.16}$$

$$p = N_1 u_1 \quad \textit{on } \Sigma_0 \tag{1.17}$$

and

$$\begin{aligned} &(p, Mz)_{L^2(Q)} + (y - z_d, z - y)_{L^2(Q)} + \\ &+ N_0(u_0, v_0 - u_0)_{L^2(\Sigma_0)} + N_1(u_1, v_1 - u_1)_{L^2(\Sigma_0)} \geq 0, \quad \forall (v, z) \in \mathcal{A}, \end{aligned} \tag{1.18}$$

where $\mathcal{A}$ is given by (1.2).

Many applications such as the control of enzymatic reactions (cf J. P. Kernevez [5], and the bibliography of this work), the control of the transmission of electrical energy, the control of the form of plasmas, motivate the study of such a problem.
This problem has already been studied in the elliptic case by different authors such for example J. L. Lions [6], O. Nakoulima [14], G. Mophou and O. Nakoulima [13]; M. Barry, O. Nakoulima and G.B. Ndiaye in [15] studied the parabolic case.
The rest of this paper is devoted to the proof of these results.
In section 2 we study the approached problem; section 3 is devoted to the Strong Singular Optimality System. Finally, we give in section 4 the weak Singular Optimality System.

2 Study of the approached problem

Set

$$\mathcal{K} = \begin{cases} v &= (v_0, v_1) \in \mathcal{U}_{ad}^0 \times \mathcal{U}_{ad}^1, \\ z,\, Mz &\in L^2(Q), \\ z &= v_0,\ \dfrac{\partial z}{\partial \nu} = v_1 \text{ on } \Sigma_0 \\ z(0) &= 0,\ \dfrac{\partial z}{\partial t}(0) = 0 \text{ in } \Omega. \end{cases} \tag{2.1}$$

Then $\mathcal{A} \subset \mathcal{K}$ and consequently, $\mathcal{K} \neq \emptyset$. Let $\varepsilon > 0$. For any $(v, z) \in \mathcal{K}$, we can define the functional :

$$J_\varepsilon(v, z) = J(v, z) + \frac{1}{2\varepsilon}|Mz|^2_{L^2(Q)}. \tag{2.2}$$

The optimal control problem is then to find $(u_\varepsilon = (u_{0\varepsilon}, u_{1\varepsilon}), y_\varepsilon)$ such that

$$J(u_\varepsilon, y_\varepsilon) = \inf_{(v,z)\in\mathcal{K}} J_\varepsilon(v, z). \tag{2.3}$$

Proposition 2.1 *Assume that (1.3) holds. Then for any $\varepsilon > 0$, there exists a unique pair $(u_\varepsilon, y_\varepsilon)$ solution to problem* (2.3).

Proof. Since $(u = (u_0, u_1), y) \in A$ is the solution of (1.4), $(u, y) \in \mathcal{K}$ and $J_\varepsilon(v, z) \geq 0$ for all $(v, z) \in \mathcal{K}$, we can define the real number

$$d_\varepsilon = \inf\{J_\varepsilon(v, z), \qquad (v, z) \in \mathcal{K}\}.$$

Let $(v^n = (v_0^n, v_1^n), z^n) \in \mathcal{K}$ be a minimizing sequence such that

$$d_\varepsilon \leq J_\varepsilon(v^n, z^n) < d_\varepsilon + \frac{1}{n} < d_\varepsilon + 1. \tag{2.4}$$

In particular,

$$0 \leq d_\varepsilon \leq J_\varepsilon(u, y) = N_0 \,|u_0|^2_{L^2(\Sigma_0)} + |u_1|^2_{L^2(\Sigma_0)} + |y - z_d|^2_{L(Q)}. \tag{2.5}$$

Therefore, from the form of J_ε, we get

$$|Mz^n|_{L^2(Q)} \leq C\sqrt{\varepsilon}, \tag{2.6a}$$
$$|v_0^n|_{L^2(\Sigma_0)} \leq C, \tag{2.6b}$$
$$|v_1^n|_{L^2(\Sigma_0)} \leq C, \tag{2.6c}$$
$$|z^n|_{L^2(Q)} \leq C, \tag{2.6d}$$

where $C = N_0 \,|u_0|^2_{L^2(\Sigma_0)} + |u_1|^2_{L^2(\Sigma_0)} + |y - z_d|^2_{L(Q)} + 1 > 0$.

Consequently, there exists $y_\varepsilon, \beta \in L^2(Q)$, $u_\varepsilon = (u_{0\varepsilon}, u_{1\varepsilon} \in L^2(\Sigma_0) \times L^2(\Sigma_0)$ and a subsequence extracted from $(v^n = (v_0^n, v_1^n), z^n)$ (still denoted $(v^n = (v_0^n, v_1^n), z^n)$) such that

$$Mz^n \rightharpoonup \beta \quad \text{weakly in } L^2(Q), \tag{2.7a}$$
$$v_0^n \rightharpoonup u_{0\varepsilon} \quad \text{weakly in } L^2(\Sigma_0) \tag{2.7b}$$
$$v_1^n \rightharpoonup u_{1\varepsilon} \quad \text{weakly in } L^2(\Sigma_0) \tag{2.7c}$$
$$z^n \rightharpoonup y_\varepsilon \quad \text{weakly in } L^2(Q). \tag{2.7d}$$

Since $(v_0^n, v_1^n) \in \mathcal{U}_{ad}^0 \times \mathcal{U}_{ad}^1$ which is closed subspace of $L^2(\Sigma) \times L^2(\Sigma)$, we deduce that

$$u = (u_{0\varepsilon}, u_{1\varepsilon}) \in \mathcal{U}_{ad}^0 \times \mathcal{U}_{ad}^1.. \tag{2.8}$$

Using (2.7d), we have

$$z^n \rightharpoonup y_\varepsilon \quad \text{weakly in } \mathcal{D}'(Q)$$

and consequently

$$Mz^n \rightharpoonup My_\varepsilon \quad \text{weakly in } \mathcal{D}'(Q). \tag{2.9}$$

Therefore combining (2.7a) and (2.9) we get

$$My_\varepsilon = \beta$$

we can then write

$$Mz^n \rightharpoonup My_\varepsilon \quad \text{weakly in } L^2(Q). \tag{2.10}$$

Since $y_\varepsilon \in L^2(Q)$, we have $\dfrac{\partial^2 y_\varepsilon}{\partial t^2} \in L^2((0, T; H^{-2}(\Omega))$. Thus, using the same arguments as in Remark1.1 we have that traces $\left(y_\varepsilon|_\Sigma, \dfrac{\partial y_\varepsilon}{\partial \nu}_{|\Sigma}\right)$, $(y_\varepsilon(0), y_\varepsilon(T))$ and $\left(\dfrac{\partial y_\varepsilon}{\partial t}(0), \dfrac{\partial y_\varepsilon}{\partial t}(T)\right)$ exist and belong respectively to $\Big(H^{-2}(0, T; H^{-\frac{1}{2}}(\Gamma)), H^{-2}(0, T; H^{-\frac{3}{2}}(\Gamma))\Big)$, $[H^{-1}(\Omega)]^2$ and to $[H^{-2}(\Omega)]^2$.

Now multiplying My^n by $\varphi \in C^\infty(\overline{Q})$ such that $\varphi(T) = \dfrac{\partial\varphi}{\partial t}(T) = 0$ in Ω, $\varphi = \dfrac{\partial\varphi}{\partial\nu} = 0$ on Σ_1 and integrating by parts over Q, we have

$$(Mz^n, \varphi)_{L^2(Q)} = (z^n, M\varphi)_{L^2(Q)} - (\varphi, v_1^n)_{L^2(\Sigma_0)} + \left(v_0^n, \frac{\partial\varphi}{\partial\nu}\right)_{L^2(\Sigma_0)}$$

because $(v_0^n, v_1^n, z^n) \in \mathcal{K}$.

Passing to the limit in this latter identity when $n \to +\infty$, while using (2.7b), (2.7c), (2.7d) and (2.10), we obtain

$$(My_\varepsilon, \varphi)_{L^2(Q)} = (y_\varepsilon, M\varphi)_{L^2(Q)} - (\varphi, u_{1\varepsilon})_{L^2(\Sigma_0)} + \left(u_{0\varepsilon}, \frac{\partial\varphi}{\partial\nu}\right)_{L^2(\Sigma_0)}$$

which after an integration by parts gives

$$\begin{aligned}(My_\varepsilon, \varphi)_{L^2(Q)} \;=\;& \left\langle \varphi(0), \frac{\partial y_\varepsilon}{\partial t}(0)\right\rangle_{H_0^2(\Omega), H^{-2}(\Omega)} - \left\langle \frac{\partial\varphi}{\partial t}(0), y_\varepsilon(0)\right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\ &+ \left\langle \varphi, \frac{\partial y_\varepsilon}{\partial\nu}\right\rangle_{H_0^2(0,T;H^{3/2}(\Gamma_0)), H^{-2}(0,T;H^{-3/2}(\Gamma_0))} \\ &- \left\langle \frac{\partial\varphi}{\partial\nu}, y_\varepsilon\right\rangle_{H_0^2(0,T;H^{1/2}(\Gamma_0)), H^{-2}(0,T;H^{-1/2}(\Gamma_0))} \\ &+ (My_\varepsilon, \varphi)_{L^2(Q)} - (\varphi, u_{1\varepsilon})_{L^2(\Sigma_0)} + \left(u_{0\varepsilon}, \tfrac{\partial\varphi}{\partial\nu}\right)_{L^2(\Sigma_0)} \\ &\forall\varphi \in C^\infty(\overline{Q}) \text{ such that } \varphi(T) = \frac{\partial\varphi}{\partial t}(T) = 0 \text{ in } \Omega,\ \varphi = \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \Sigma_1.\end{aligned}$$

After simplification, this latter can be rewritten as

$$\begin{aligned}0 \;=\;& \left\langle \varphi(0), \frac{\partial y_\varepsilon}{\partial t}(0)\right\rangle_{H_0^2(\Omega), H^{-2}(\Omega)} - \left\langle \frac{\partial\varphi}{\partial t}(0), y_\varepsilon(0)\right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\ &+ \left\langle \varphi, \frac{\partial y_\varepsilon}{\partial\nu} - u_{1\varepsilon}\right\rangle_{H_0^2(0,T;H^{3/2}(\Gamma_0)), H^{-2}(0,T;H^{-3/2}(\Gamma_0))} \\ &- \left\langle \frac{\partial\varphi}{\partial\nu}, y_\varepsilon - u_{0\varepsilon}\right\rangle_{H_0^2(0,T;H^{1/2}(\Gamma_0)), H^{-2}(0,T;H^{-1/2}(\Gamma_0))} \\ &\forall\varphi \in C^\infty(\overline{Q}) \text{ such that } \varphi(T) = \frac{\partial\varphi}{\partial t}(T) = 0 \text{ in } \Omega,\ \varphi = \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \Sigma_1.\end{aligned} \tag{2.11}$$

Taking in (2.11),

$$\varphi(0) = 0 \text{ in } \Omega,\ \varphi = 0 \quad \text{on } \Sigma_0 \text{ and } \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \Sigma$$

this yields

$$\left\langle \frac{\partial\varphi}{\partial t}(0), y_\varepsilon(0)\right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} = 0$$

and

$$y_\varepsilon(0) = 0 \quad \text{in } \Omega. \tag{2.12}$$

Using the same technique we also obtain

$$\frac{\partial y_\varepsilon}{\partial t}(0) = 0 \ \text{ in } \Omega,\ y_\varepsilon = u_{0\varepsilon} \ \text{ on } \Sigma_0,\ \frac{\partial y_\varepsilon}{\partial \nu} = u_{1\varepsilon} \ \text{ on } \Sigma_0. \tag{2.13}$$

From (2.10), (2.12) and (2.13), we obtain that the pair $(u_\varepsilon, y_\varepsilon) \in \mathcal{K}$.

Finally, by combining (2.10), (2.7b), (2.7c), (2.7d) and the weak lower semi-continuity of J_ε, we obtain

$$J_\varepsilon(u_\varepsilon, y_\varepsilon) \leq \liminf_{n\to\infty} J_\varepsilon(v^n, z^n) = d_\varepsilon.$$

In other words, $(u_\varepsilon, y_\varepsilon)$ is the optimal control. Its uniqueness results from the strict convexity of J_ε. □

Proposition 2.2 *Assume that (1.3) holds. Then $(u_\varepsilon, y_\varepsilon) \in \mathcal{U}_{ad} \times L^2(Q)$ is an optimal solution of problem (2.3) if and only if there exists $p_\varepsilon \in L^2(Q)$ such that the triplet $(u_\varepsilon, y_\varepsilon, p_\varepsilon)$ is solution to the following optimality System*

$$\begin{cases} My_\varepsilon &= \varepsilon p_\varepsilon & \text{in } Q \\ y_\varepsilon(x,0) &= 0 & \text{in } \Omega \\ \dfrac{\partial y_\varepsilon}{\partial t}(0) &= 0 & \text{in } \Omega \\ y_\varepsilon &= u_{0\varepsilon} & \text{on } \Sigma_0, \\ \dfrac{\partial y_\varepsilon}{\partial \nu} &= u_{1\varepsilon} & \text{on } \Sigma_0, \end{cases} \tag{2.14}$$

$$\begin{cases} Mp_\varepsilon &= z_d - y_\varepsilon & \text{in } Q \\ p_\varepsilon(x,T) &= 0 & \text{in } \Omega \\ \dfrac{\partial p_\varepsilon}{\partial t}(T) &= 0 & \text{in } \Omega \\ p_\varepsilon &= 0 & \text{on } \Sigma_1 \\ \dfrac{\partial p_\varepsilon}{\partial \nu} &= 0 & \text{on } \Sigma_1 \end{cases} \tag{2.15}$$

$$\begin{aligned} \forall (v,z) \in \mathcal{K},\ \ (p_\varepsilon, M(z - y_\varepsilon))_{L^2(Q)} - (Mp_\varepsilon, z - y_\varepsilon)_{L^2(Q)} + \\ + N_0(u_{0\varepsilon}, v_0 - u_{0\varepsilon})_{L^2(\Sigma_0)} + N_1(u_{1\varepsilon}, v_1 - u_{1\varepsilon})_{L^2(\Sigma_0)} \ \geq \ 0. \end{aligned} \tag{2.16}$$

Proof. We write the first order Euler-Lagrange condition which characterizes the optimal control $(u_\varepsilon, y_\varepsilon)$:

$$\forall (v,z) \in \mathcal{K},\ \ \frac{d}{d\lambda} J_\varepsilon(u_\varepsilon + \lambda(v - u_\varepsilon), y_\varepsilon + \lambda(z - y_\varepsilon))|_{\lambda=0} \geq 0.$$

After a short calculation we obtain

$$\begin{aligned} \forall (v,z) \in \mathcal{K},\ \ (y_\varepsilon - z_d, z - y_\varepsilon)_{L^2(Q)} + N_0(u_{0\varepsilon}, v_0 - u_{0\varepsilon})_{L^2(\Sigma_0)} + \\ N_1(u_{1\varepsilon}, v_1 - u_{1\varepsilon})_{L^2(\Sigma_0)} + \frac{1}{\varepsilon}(My_\varepsilon, M(z - y_\varepsilon))_{L^2(Q)} \ \geq \ 0. \end{aligned} \tag{2.17}$$

We set

$$p_\varepsilon = \frac{1}{\varepsilon} My_\varepsilon. \tag{2.18}$$

Then $p_\varepsilon \in L^2(Q)$ and (2.17) becomes

$$\begin{array}{ll}\forall (v,z)\in\mathcal{K}, & (p_\varepsilon, M(z-y_\varepsilon))_{L^2(Q)} + (y_\varepsilon - z_d, z-y_\varepsilon)_{L^2(Q)} + \\ & +N_0(u_{0\varepsilon}, v_0-u_{0\varepsilon})_{L^2(\Sigma_0)} + +N_1(u_{1\varepsilon}, v_1-u_{1\varepsilon})_{L^2(\Sigma_0)} \quad \geq \quad 0.\end{array} \tag{2.19}$$

Taking now in (2.19) $z = y_\varepsilon \pm \varphi,\ \varphi \in \mathcal{D}(Q)$ and $v = u_\varepsilon$, we have

$$\forall\ \varphi\in\mathcal{D}(Q),\ \ (p_\varepsilon, M\varphi)_{L^2(Q)} + (y_\varepsilon - z_d, \varphi)_{L^2(Q)} = 0,$$

which after an integration by parts over Q gives

$$\forall\ \varphi\in\mathcal{D}(Q),\ \ (Mp_\varepsilon + y_\varepsilon - z_d, \varphi)_{L^2(Q)} = 0,$$

consequently

$$Mp_\varepsilon = z_d - y_\varepsilon \quad \text{ in } Q. \tag{2.20}$$

Using the same arguments as in Remark1.1 we have that traces $p_\varepsilon|_\Sigma, \dfrac{\partial p_\varepsilon}{\partial\nu}_{|\Sigma}$ exist and belong respectively to $H^{-2}(0,T;H^{-\frac{1}{2}}(\Gamma))$ and $H^{-2}(0,T;H^{-\frac{3}{2}}(\Gamma))$. We can also define $(p_\varepsilon(0), p_\varepsilon(T)) \in [H^{-1}(\Omega)]^2$ and $\left(\dfrac{\partial p_\varepsilon}{\partial t}(0), \dfrac{\partial p_\varepsilon}{\partial t}(T)\right) \in [H^{-2}]^2(\Omega)$.

Choosing now in (2.19) $z = y_\varepsilon \pm \varphi,\ \varphi \in \mathcal{C}^\infty(\overline{Q})$ and $(v_0, v_1) = (u_{0\varepsilon}, u_{1\varepsilon})$, we obtain

$$\begin{array}{l}(p_\varepsilon, M\varphi)_{L^2(Q)} + (y_\varepsilon - z_d, \varphi)_{L^2(Q)} = 0,\\ \text{for all } \varphi \text{ in } \mathcal{C}^\infty(\overline{Q}) \text{ such that } \varphi = \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \Sigma_0,\end{array}$$

which after an integration by parts yields

$$\begin{array}{l}(y_\varepsilon - z_d, \varphi)_{L^2(Q)} + (Mp_\varepsilon, \varphi)_{L^2(Q)} + \left\langle \frac{\partial p_\varepsilon}{\partial\nu}, \varphi \right\rangle_{H^{-2}(0,T;H^{-3/2}(\Gamma_1)), H_0^2(0,T;H^{3/2}(\Gamma_1))} \\ -\left\langle p_\varepsilon, \frac{\partial\varphi}{\partial\nu}\right\rangle_{H^{-2}(0,T;H^{-1/2}(\Gamma_1)), H_0^2(0,T;H^{1/2}(\Gamma_1))} \\ +\left\langle \varphi(T), \dfrac{\partial p_\varepsilon}{\partial t}(T)\right\rangle_{H_0^2(\Omega),H^{-2}(\Omega)} - \left\langle \varphi(0), \dfrac{\partial p_\varepsilon}{\partial t}(0)\right\rangle_{H_0^2(\Omega),H^{-2}(\Omega)} \\ -\left\langle \dfrac{\partial\varphi}{\partial t}(T), p_\varepsilon(T)\right\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} + \left\langle \dfrac{\partial\varphi}{\partial t}(0), p_\varepsilon(0)\right\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} = 0, \\ \text{for all } \varphi \text{ in } \mathcal{C}^\infty(\overline{Q}) \text{ such that } \varphi = \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \Sigma_0.\end{array}$$

Using (2.20) in this latter identity, we deduce that

$$\begin{array}{l}+\left\langle \frac{\partial p_\varepsilon}{\partial\nu}, \varphi \right\rangle_{H^{-2}(0,T;H^{-3/2}(\Gamma_1)), H_0^2(0,T;H^{3/2}(\Gamma_1))} \\ -\left\langle p_\varepsilon, \frac{\partial\varphi}{\partial\nu}\right\rangle_{H^{-2}(0,T;H^{-1/2}(\Gamma_1)), H_0^2(0,T;H^{1/2}(\Gamma_1))} \\ +\left\langle \varphi(T), \dfrac{\partial p_\varepsilon}{\partial t}(T)\right\rangle_{H_0^2(\Omega),H^{-2}(\Omega)} - \left\langle \varphi(0), \dfrac{\partial p_\varepsilon}{\partial t}(0)\right\rangle_{H_0^2(\Omega),H^{-2}(\Omega)} \\ -\left\langle \dfrac{\partial\varphi}{\partial t}(T), p_\varepsilon(T)\right\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} + \left\langle \dfrac{\partial\varphi}{\partial t}(0), p_\varepsilon(0)\right\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} = 0, \\ \text{for all } \varphi \text{ in } \mathcal{C}^\infty(\overline{Q}) \text{ such that } \varphi = \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \Sigma_0.\end{array} \tag{2.21}$$

Choose successively in (2.21), φ such that $\varphi(T) = 0$ in $\Omega, \varphi = 0, \dfrac{\partial \varphi}{\partial \nu} = 0$ on Σ_1; $\varphi = 0, \dfrac{\partial \varphi}{\partial \nu} = 0$ on Σ and $\dfrac{\partial \varphi}{\partial \nu} = 0$ on Σ_1 we then successively deduce that

$$p_\varepsilon(x,T) = 0 \text{ in } \Omega, \tag{2.22}$$

$$\frac{\partial p_\varepsilon}{\partial t}(x,T) = 0 \text{ in } \Omega. \tag{2.23}$$

$$\frac{\partial p_\varepsilon}{\partial \nu} = 0 \text{ on } \Sigma_1 \tag{2.24}$$

and finally

$$p_\varepsilon = 0 \text{ on } \Sigma_1. \tag{2.25}$$

So, (2.20), (2.24), (2.22), (2.23) and (2.25) gives (2.15). From (2.18), (2.12) and (2.13), we obtain (2.14). Replacing $z_d - y_\varepsilon$ by Mp_ε in (2.19), we get (2.16). □

Let us prove that the solution $(u_\varepsilon, y_\varepsilon)$ of the penalized problem converges to our optimal pair.

Proposition 2.3 *Let* $(u_\varepsilon, y_\varepsilon)$ *be the solution to* (2.3). *We then have for* $\varepsilon \to 0$

$$\begin{aligned} u_\varepsilon &\to u \quad \textit{strongly } L^2(\Sigma_0) \times L^2(\Sigma_0), && (2.26)\\ y_\varepsilon &\to y \quad \textit{strongly } L^2(Q), && (2.27)\\ J_\varepsilon &\to J. && (2.28) \end{aligned}$$

where (u, y) *is the optimal pair and* J *the functional defined by (1.4) .*

Proof. We proceed in three steps.

Step 1. We prove the weak convergence of $(u_\varepsilon, y_\varepsilon)$ towards $(\widehat{u} = (\widehat{u}_0, \widehat{u}_1), \widehat{y}) \in \mathcal{U}_{ad}^0 \times \mathcal{U}_{ad}^1 \times L^2(Q)$.

Since (u, y) is solution of (1.5), we have

$$J_\varepsilon(u_\varepsilon, y_\varepsilon) = \inf J_\varepsilon(v, z) \leq J_\varepsilon(u, y) = J(u, y). \tag{2.29}$$

From the structure of J_ε we deduce that

$$\begin{aligned} |y_\varepsilon|^2_{L^2(Q)} &\leq C, && (2.30)\\ |u_\varepsilon|^2_{L^2(\Sigma_0)\times L^2(\Sigma_0)} &\leq C, && (2.31)\\ |My_\varepsilon|_{L^2(Q)} &\leq C\sqrt{\varepsilon} && (2.32) \end{aligned}$$

where the $C's$ are various constants independent of ε. Hence it follows from (2.32) and (2.18) that

$$|p_\varepsilon|_{L^2(Q)} \leq C. \tag{2.33}$$

So, we can therefore pick out from $((u_\varepsilon), (y_\varepsilon))$ a sequence again denoted $((u_\varepsilon), (y_\varepsilon))$, such that

$$u_\varepsilon \rightharpoonup \widehat{u} = (\widehat{u}_0, \widehat{u}_1) \text{ weakly in } L^2(\Sigma_0) \times L^2(\Sigma_0), \tag{2.34}$$

$$y_\varepsilon \rightharpoonup \widehat{y} \text{ weakly in } L^2(Q), \tag{2.35a}$$
$$My_\varepsilon \to 0 \text{ strongly in } L^2(Q). \tag{2.35b}$$

Since $u_\varepsilon \in \mathcal{U}^0_{ad} \times \mathcal{U}^1_{ad}$ which is closed subspace of $L^2(\Sigma_0) \times L^2(\Sigma_0)$, we deduce that

$$\widehat{u} = (\widehat{u}_0, \widehat{u}_1) \in \mathcal{U}^0_{ad} \times \mathcal{U}^1_{ad}. \tag{2.36}$$

From (2.35a), we have

$$y_\varepsilon \rightharpoonup \widehat{y} \quad \text{weakly in } \mathcal{D}'(Q).$$

Consequently

$$My_\varepsilon \rightharpoonup M\widehat{y} \quad \text{weakly in } \mathcal{D}'(Q). \tag{2.37}$$

Therefore combining (2.35b) and (2.37) we get

$$M\widehat{y} = 0. \tag{2.38}$$

Since $\widehat{y} \in L^2(Q)$, we have $\dfrac{\partial^2 \widehat{y}}{\partial t^2} \in L^2((0,T; H^{-2}(\Omega))$. Thus, using the same arguments as in Remark1.1 we have that traces $\left(\widehat{y}|_\Sigma, \dfrac{\partial \widehat{y}}{\partial \nu}_{|\Sigma}\right)$, $(\widehat{y}(0), \widehat{y}(T))$ and $\left(\dfrac{\partial \widehat{y}}{\partial t}(0), \dfrac{\partial \widehat{y}}{\partial t}(T)\right)$ exist and belong respectively to $\left(H^{-2}(0,T; H^{-\frac{1}{2}}(\Gamma)), H^{-2}(0,T; H^{-\frac{3}{2}}(\Gamma))\right)$, $[H^{-1}(\Omega)]^2$ and to $[H^{-2}(\Omega)]^2$.

Now multiplying $(2.14)_1$ by φ in $C^\infty(\overline{Q})$ such that $\varphi(T) = \dfrac{\partial \varphi}{\partial t}(T) = 0$ in Ω, $\varphi = \frac{\partial \varphi}{\partial \nu} = 0$ on Σ_1 and integrating by parts after taking the integral on Q, we have

$$(y_\varepsilon, M\varphi)_{L^2(Q)} - (\varphi, u_{1\varepsilon})_{L^2(\Sigma_0)} + \left(u_{0\varepsilon}, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_0)} = \sqrt{\varepsilon}\,(p_\varepsilon, \varphi)_{L^2(Q)}.$$

Passing to the limit in this latter identity when $\varepsilon \to 0$ while using (2.34) and (2.35a), we obtain

$$(\widehat{y}, M\varphi)_{L^2(Q)} - (\varphi, \widehat{u}_1)_{L^2(\Sigma_0)} + \left(\widehat{u}_0, \tfrac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_0)} = 0,$$
for any φ in $C^\infty(\overline{Q})$ such that $\varphi(T) = \dfrac{\partial \varphi}{\partial t}(T) = 0$ in Ω, $\varphi = \dfrac{\partial \varphi}{\partial \nu} = 0$ on Σ_1,

which after an integration by parts gives

$$\begin{aligned} 0 = & -(\varphi, \widehat{u}_1)_{L^2(\Sigma_0)} + \left(\widehat{u}_0, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_0)} - \left\langle \varphi(0), \frac{\partial \widehat{y}}{\partial t}(0)\right\rangle_{H^2_0(\Omega), H^{-2}(\Omega)} \\ & + \left\langle \frac{\partial \varphi}{\partial t}(0), \widehat{y}(0)\right\rangle_{H^1_0(\Omega), H^{-1}(\Omega)} - \left\langle \varphi, \frac{\partial \widehat{y}}{\partial \nu}\right\rangle_{H^2_0(0,T;H^{3/2}(\Gamma_0)), H^{-2}(0,T;H^{-3/2}(\Gamma_0))} \\ & + \left\langle \frac{\partial \varphi}{\partial \nu}, \widehat{y}\right\rangle_{H^2_0(0,T;H^{1/2}(\Gamma_0)), H^{-2}(0,T;H^{-1/2}(\Gamma_0))}, \end{aligned}$$
for any φ in $C^\infty(\overline{Q})$ such that $\varphi(T) = \dfrac{\partial \varphi}{\partial t}(T) = 0$ in Ω, $\varphi = \dfrac{\partial \varphi}{\partial \nu} = 0$ on Σ_1.

Take successively in this latter equality, $\varphi(0) = 0$ in Ω, $\varphi = \frac{\partial\varphi}{\partial\nu} = 0$ on Σ_0; $\varphi = \frac{\partial\varphi}{\partial\nu} = 0$ on Σ_0 and $\frac{\partial\varphi}{\partial\nu} = 0$ on Σ_0; we successively get

$$\widehat{y}(0) = 0 \quad \text{in } \Omega, \tag{2.39a}$$

$$\frac{\partial\widehat{y}}{\partial t}(0) = 0 \quad \text{in } \Omega, \tag{2.39b}$$

$$\widehat{y} = \widehat{u}_0 \quad \text{on } \Sigma_0, \tag{2.39c}$$

$$\frac{\partial\widehat{y}}{\partial\nu} = \widehat{u}_1 \quad \text{on } \Sigma_0. \tag{2.39d}$$

From (2.38) and (2.39), we obtain that the pair $(\widehat{u}, \widehat{y}) \in \mathcal{A} \subset \mathcal{K}$.

Step 2. We prove that $(\widehat{u}, \widehat{y}) = (u, y)$, where (u, y) is solution of (1.5).

Since $(\widehat{u}, \widehat{y}) \in \mathcal{A}$, we have

$$J(u, y) \leq J((\widehat{u}, \widehat{y}). \tag{2.40}$$

On the other hand, we have

$$J(u_\varepsilon, y_\varepsilon) \leq J_\varepsilon(u_\varepsilon, y_\varepsilon) \leq J_\varepsilon(u, y) = J(u, y).$$

Thus, using (2.35a)and (2.34), we obtain

$$J(\widehat{u}, \widehat{y}) \leq \liminf J_\varepsilon(u_\varepsilon, y_\varepsilon) \leq J(u, y) \tag{2.41}$$

which combining with (2.40) gives

$$J(\widehat{u}, \widehat{y}) = J(u, y).$$

Consequently,

$$(\widehat{u}, \widehat{y}) = (u, y). \tag{2.42}$$

Therefore, it follows from (2.41) that

$$J(u, y) \leq \lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon, y_\varepsilon) \leq J(u, y).$$

That is (2.28), *i.e.*: $J_\varepsilon \to J$.

Step 3. We prove the strong convergence.

As (2.28) can be rewritten as

$$\begin{aligned} &\lim_{\varepsilon\to 0} \left(|y_\varepsilon - z_d|^2_{L^2(Q)} + N_0 \, |u_{o\varepsilon}|^2_{L^2(\Sigma_0)} + N_1 \, |u_{1\varepsilon}|^2_{L^2(\Sigma_0)} \right) \\ &\qquad = |y - z_d|^2_{L^2(Q)} + N_0 \, |u_0|^2_{L^2(\Sigma_0)} + N_1 \, |u_1|^2_{L^2(\Sigma_0)} , \end{aligned} \tag{2.43}$$

by using (2.34), (2.35a) and (2.42), we have

$$\begin{cases} |y - z_d|^2_{L^2(Q)} & \leq \liminf\limits_{\varepsilon\to 0} |y_\varepsilon - z_d|^2_{L^2(Q)} \\ |u_0|^2_{L^2(\Sigma_0)} & \leq \liminf\limits_{\varepsilon\to 0} |u_{o\varepsilon}|^2_{L^2(\Sigma_0)} \\ |u_1|^2_{L^2(\Sigma_0)} & \leq \liminf\limits_{\varepsilon\to 0} |u_{1\varepsilon}|^2_{L^2(\Sigma_0)} \end{cases}$$

which, applied to (2.43), gives

$$\begin{cases} |y - z_d|^2_{L^2(Q)} = \lim_{\varepsilon \to 0} |y_\varepsilon - z_d|^2_{L^2(Q)} \\ |u_0|^2_{L^2(\Sigma_0)} = \lim_{\varepsilon \to 0} |u_{o\varepsilon}|^2_{L^2(\Sigma_0)} \\ |u_1|^2_{L^2(\Sigma_0)} = \lim_{\varepsilon \to 0} |u_{1\varepsilon}|^2_{L^2(\Sigma_0)} . \end{cases} \tag{2.44}$$

Similarly, since

$$\begin{aligned} N_0 \, |u_0 - u_{0\varepsilon}|^2_{L^2(\Sigma_0)} + N_1 \, |u_1 - u_{1\varepsilon}|^2_{L^2(\Sigma_0)} = \; & N_0 \, |u_{0\varepsilon}|^2_{L^2(\Gamma_0)} + N_1 \, |u_{1\varepsilon}|^2_{L^2(\Sigma_0)} \\ & + N_0 \, |u_0|^2_{L^2(\Sigma_0)} + N_1 \, |u_1|^2_{L^2(\Sigma_0)} \\ & - 2 \, \langle N_0 \, u_{0\,\varepsilon}, \, u_0 \rangle_{L^2(\Sigma_0)} \\ & + 2 \, \langle N_1 \, u_{1\,\varepsilon}, \, u_1 \rangle_{L^2(\Sigma_0)} , \end{aligned} \tag{2.45}$$

passing (2.45) to the limit when $\varepsilon \to 0$ while using (2.34), (2.42) $(2.44)_2$ and $(2.44)_3$, we get

$$\lim_{\varepsilon \to 0} \left(N_0 \, |u_0 - u_{0\,\varepsilon}|^2_{L^2(\Sigma_0)} + N_1 \, |u_1 - u_{1\,\varepsilon}|^2_{L^2(\Sigma_0)} \right) = 0$$

which implies (2.26).

It suffices to prove that $y_\varepsilon \to y$ strongly in $L^2(Q)$ to complete the proof of Proposition 2.3.

As we can write

$$|y_\varepsilon - y|^2_{L^2(Q)} = |y_\varepsilon - z_d|^2_{L^2(Q)} + 2 \, \langle y_\varepsilon - z_d, \, z_d - y \rangle_{L^2(Q)} + |z_d - y|^2_{L^2(Q)} , \tag{2.46}$$

passing (2.46) to the limit while using, $(2.44)_1$, (2.35a) and (2.42), we obtain

$$\lim_{\varepsilon \to 0} |y_\varepsilon - y|_{L^2(Q)} = 0.$$

That is (2.27). □

Now we can set up the Singular Optimality System for our optimal pair under some conditions.

3 Strong Singular Optimality System

In this section, we prove Theorem 1.4.

From (2.38), (2.39) and (2.42), we have (1.11).

In view of (2.33), we have that there exists $p \in L^2(Q)$ such that $p_\varepsilon \rightharpoonup p$ in $L^2(Q)$. Therefore using (2.27) and the fact that p_ε is solution to (2.15), we can prove by proceeding as for y_ε in pages 47 and 48 that p satisfies (1.12).

Now, let us take in (2.16) $z = y_\varepsilon \pm \xi$ with $\xi \in C^\infty(\overline{Q})$ such that $\dfrac{\partial \xi}{\partial \nu} = 0$ on Σ_0. Then

$$\xi_{|\Sigma_0} = v_0 - u_{0\varepsilon}, \; \frac{\partial \xi}{\partial \nu}|_{\Sigma_0} = v_1 - u_{1\varepsilon} = 0 \text{ and } \xi(0) = \frac{\partial \xi}{\partial t}(0) = 0 \text{ in } \Omega. \tag{3.1}$$

Therefore, (2.16) becomes

$$(p_\varepsilon, M\xi)_{L^2(Q)} - (Mp_\varepsilon, \xi)_{L^2(Q)} + N_0(u_{0\varepsilon}, \xi)_{L^2(\Sigma_0)} = 0,$$
$$\text{for all } \xi \text{ in } C^\infty(\overline{Q}) \text{such that (3.1) holds.}$$

Integrating by parts the first term of this relation while using on the one hand the fact that p_ε is solution of (2.15), with the traces given in pp. 45, and on the other hand the fact that ξ verifies (3.1), we obtain

$$0 = \left\langle \frac{\partial p_\varepsilon}{\partial \nu} + N_0 u_{0\varepsilon}, \xi \right\rangle_{H^{-2}(0,T;H^{-3/2}(\Gamma_0)),H_0^2(0,T;H^{3/2}(\Gamma_0))} \tag{3.2}$$

from which we deduce that

$$\frac{\partial p_\varepsilon}{\partial \nu} = -N_0 u_{0\varepsilon} \quad \text{on } \Sigma_0. \tag{3.3}$$

Thus we have on the one hand

$$\frac{\partial p_\varepsilon}{\partial \nu} \rightharpoonup \frac{\partial p}{\partial \nu} = -N_0 u_0 \quad \text{weakly in } L^2(\Sigma_0) \tag{3.4}$$

since (2.26) holds, and on the other hand that p_ε is such that

$$\begin{cases} Mp_\varepsilon &= z_d - y_\varepsilon & \text{in } Q \\ \dfrac{\partial p_\varepsilon}{\partial \nu} &= -N_0 u_{0\varepsilon} & \text{on } \Sigma_0 \\ \dfrac{\partial p_\varepsilon}{\partial \nu} &= 0 & \text{on } \Sigma_1. \\ \dfrac{\partial p_\varepsilon}{\partial t}(T) &= 0 & \text{in } \Omega \\ p_\varepsilon(x,T) &= 0 & \text{in } \Omega \end{cases}$$

Therefore, in view of (2.30) and (2.31), we have that there exists $C > 0$ such that $|p_\varepsilon|_{L^2(0,T;H^1(\Omega))} \leq C$ (see [7, p. 347]). This implies that by the continuity of the trace that

$$|p_\varepsilon|_{L^2(\Sigma_0)} \leq C. \tag{3.5}$$

Hence, we have that

$$p_\varepsilon \rightharpoonup p \text{ weakly in } L^2(\Sigma_0). \tag{3.6}$$

Now integrating by parts the first term in (2.16), while using on one hand the fact that p_ε is solution of (2.15), and on the other hand the fact that $z - y_\varepsilon$ verifies

$$z - y_{\varepsilon|\Sigma_0} = v_0 - u_{0\varepsilon},\ \frac{\partial z - y_\varepsilon}{\partial \nu}|_{\Sigma_0} = v_1 - u_{1\varepsilon} \text{ and } z - y_\varepsilon(0) = \frac{\partial z - y_\varepsilon}{\partial t}(0) = 0 \text{ in } \Omega,$$

we obtain

$$\forall (v,z) \in \mathcal{K},\ \left(\frac{\partial p_\varepsilon}{\partial \nu}, v_0 - u_{0\varepsilon}\right)_{L^2(\Sigma_0)} - (p_\varepsilon, v_1 - u_{1\varepsilon})_{L^2(\Sigma_0)}$$
$$+ N_0(u_{0\varepsilon}, z - y_\varepsilon)_{L^2(\Sigma_0)} + N_1(u_{1\varepsilon}, z - y_\varepsilon)_{L^2(\Sigma_0)} \geq 0,$$

which combined with (3.2) gives

$$\forall v_1 \in \mathcal{U}_{ad}^1,\ \ (-p_\varepsilon + N_1 u_{1\varepsilon}, v_1 - u_{1\varepsilon})_{L^2(\Sigma_0)} \geq 0. \tag{3.7}$$

Now let us pass to the limit in (3.7) while using (2.26) and (3.6), we then deduce (1.14). The proof of Theorem 1.4 is then complete.

4 Weak Singular Optimality System

In this section we are going to prove Theorem 1.5.

We have already proved in Theorem 1.4 that (1.15) and (1.16) hold.

Now, let us take in (2.16) $z = y_\varepsilon \pm \xi$ with $\xi \in C^\infty(\overline{Q})$ with $v_0 = u_{0\varepsilon}$. Then ξ is such that

$$\xi_{|\Sigma_0} = 0, \ \frac{\partial \xi}{\partial \nu}|_{\Sigma_0} = v_1 - u_{1\varepsilon} \ \text{ and } \ \xi(0) = \frac{\partial \xi}{\partial t}(0) = 0 \text{ in } \Omega. \tag{4.1}$$

Therefore, (2.16) becomes

$$(p_\varepsilon, M\xi)_{L^2(Q)} - (Mp_\varepsilon, \xi)_{L^2(Q)} + N_1(u_{1\varepsilon}, \xi)_{L^2(\Sigma_0)} = 0,$$
$$\text{for all } \xi \text{ in } C^\infty(\overline{Q}) \text{ such that } (4.1) \text{ holds.}$$

Integrating by parts the first term of this relation while using on the one hand the fact that p_ε is solution of (2.15), with the traces given in p. 45, and on the other hand the fact that ξ verifies (3.1), we obtain

$$0 = \langle p_\varepsilon - N_1 u_{1\varepsilon}, \xi \rangle_{H^{-2}(0,T;H^{-1/2}(\Gamma_0)),H_0^2(0,T;H^{1/2}(\Gamma_0))} \tag{4.2}$$

from which we deduce that

$$p_\varepsilon = N_1 u_{1\varepsilon} \quad \text{ on } \Sigma_0. \tag{4.3}$$

Thus, we have

$$p_\varepsilon \rightharpoonup p = -N_1 u_1 \quad \text{ weakly in } L^2(\Sigma_0) \tag{4.4}$$

since (2.26) holds. Now, from (2.33), we have that there exists $p \in L^2(Q)$ such that

$$p_\varepsilon \rightharpoonup p \text{ weakly in } L^2(Q). \tag{4.5}$$

Finally, passing to the limit in (2.16) while using (2.35b), (2.37), (2.38), (2.26), (2.27) and (4.5) we obtain (1.18). This completes the proof of Theorem 1.5.

References

[1] H. Brezis, *Analyse Fonctionnelle: Théorie et Applications,* Masson, Paris, 1983.

[2] L. Berrahmoune, *Stabilisation d'une classe de systèmes distribués hyperboliques,* Analysis and Optimization of Systems, Lecture notes in Control and Information Science **111** (1988) pp. 484-493.

[3] V. M. Govorov, *Premier problème aux limites pour équations hyperboliques avec données frontières dans L^2,* Dolkady Akad. Nank. (1982), pp. 1044-1047.

[4] T. Kato, *Blow up of solutions of some non linear hyperbolic equations,* Communications on Pure and Applied Mathematics XXXIII (1980) pp. 501-505.

[5] J. P. Kernevez, *Enzyme mathematics,* North-Holland Publishing Company, 1980.

[6] J. L. Lions, *Contrôle des Systèmes Distribués Singuliers,* Gauthiers-Villars, 1983.

[7] J. L. Lions, *Contrôle optimal de systèmes gouvernés par les équations aux dérivées partielles.* Gauthiers-Villars, Paris, 1968.

[8] J. L. Lions, *Controlabilité exacte, perturbation et stabilisation de systèmes distribués,* Hors collection Masson 1988.

[9] J. L. Lions, *Optimal control of unstable hyperbolic systems,* Colloque de Novosibirsk, Juin 1981.

[10] J. L. Lions, *Function spaces and optimal control of distributed systems,* U.F.R.J. Lecture Notes. Rio de Janeiro (1980).

[11] J. L. Lions, E. Magenes, *Problèmes Aux Limites Non Homogènes et Applications,* vol 1. DUNOD, Paris, 1968.

[12] J. L. Lions, E. Magenes, *Problèmes Aux Limites Non Homogènes et Applications,* vol 2. DUNOD, Paris, 1968.

[13] O. Nakoulima, G. M. Mophou, *Control of Cauchy System for an Elliptic Operator,* Acta Mathematica Sinica (English Series)**25**, no.11 (2009) pp. 1819-1834.

[14] O. Nakoulima, *Contrôle de systèmes mal posés de type elliptique,* Journal de Mathématiques Pures et Appliquées **73** (1994) pp. 441-453.

[15] M. Barry, O. Nakoulima, G. B. Ndiaye, *Cauchy System for Parabolic Operator*, International Journal of Evolution Equations **8**, no. 4 (2013) pp. 277-290.

[16] I. Lasiecka, R. Triggiani, *Hyperbolic equations with Dirichlet boundary feedback via position vector: regularity and almost periodic stabilization,*, Part III Applied Mathematics and Optimization **8** (3) (1982) pp. 199-221.