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AN IMPROVED RESULT ON TRIPLE NON-NEGATIVE SOLUTIONS FOR A CLASS OF HIGHER-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This paper investigates the existence of triple non-negative solutions for a kind of higher-order nonlinear fractional differential equations by using Leggett-Williams fixed point theorem, and presents a new existence criterion. The fractional derivative here is the standard Riemann-Liouville one. A new approximation of the Green's function is derived to facilitate the proof of the main results. The study of an illustrative example shows that the new existence criterion obtained in this paper improves the existing results to some extent.

Keywords: Riemann-Liouville fractional derivative; higher-order fractional differential equations; Leggett-Williams fixed point theorem.

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1 Introduction

Fractional differential equations (FDEs) arise in the area of physics [1], chemistry [2], aerodynamics [3], etc. Due to the great importance in both theoretical development and practical applications [32–34], the study of fractional differential equations has drawn a great deal of attention, and a large number of results have been obtained on the existence of positive solutions,

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see [4, 8, 10–22, 24, 28–30] and the references therein. In [22], Zhao *et al*. considered a kind of two-point nonlinear fractional boundary value problem, and obtained some results on the existence of positive solutions by virtue of lower and upper solution method. Xu *et al*. [19] studied the existence of positive solutions for a kind of boundary value problems with Riemann-Liouville's fractional derivative, and established some useful sufficient conditions via the Krasnoselskii-Zabreiko fixed point theorem.

As an important branch of fractional differential equations, higher-order FDEs have been studied recently [1, 11, 28–30]. In [1], higher-order fractional heat-type equations were investigated and some interesting properties on the solution to this type of equations were presented. C. Goodrich [29] considered the following higher-order fractional differential equation:

$$\begin{cases} D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0,\ t \in (0, 1), \\ u^{(i)}(0) = 0,\ 0 \le i \le n-2,\ i \in \mathbb{N};\ [D^{\beta}_{0+}u(t)]_{t=1} = 0,\ 1 \le \beta \le n-2, \end{cases} \tag{1.1}$$

where $n-1 < \alpha \le n, n > 3, n \in \mathbb{N}^+$, and D^{α}_{0+} is the standard Riemann-Liouville fractional derivative of order α. The author derived the Green's function for this problem and presented some results on the existence of one positive solution to FDE (1.1). Later, Wang *et al* [30] studied FDE (1.1) with $f(t, u(t)) = a(t)g(u(\theta(t)))$. They established some sufficient conditions for the existence of multiple positive solutions to this problem by applying fixed point index theory and Leggett-Williams fixed point theorem.

It should be pointed out that the conditions obtained in [30] for the existence of triple positive solutions are strong, and many FDEs which have triple positive solutions do not meet these conditions, see Example 3.6 below. Motivated by this fact, in this paper, we investigate the existence of triple non-negative solutions to FDE (1.1) with general nonlinearity $f(t, u(t))$ by using Leggett-Williams fixed point theorem, and present a new existence criterion. The main contributions of this work are as follows. On one hand, a new approximation of the Green's function is derived, which is crucial to relaxing the conditions. On the other hand, a new criterion is presented for the existence of triple positive solutions to FDE (1.1), which is weaker and improves the existing results to some extent (see Example 3.6).

Throughout this paper, we assume that the nonlinearity $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous. Moreover, let $E = C[0, 1]$ with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$ and $P = \{x \in E : x(t) \ge 0,\ \forall\, t \in [0, 1]\}$. Then, E is a Banach space and P is a normal cone of E. We will consider the existence of non-negative solutions to FDE (1.1) in P.

The rest of this paper is organized as follows. Section 2 contains some necessary preliminaries on the Riemann-Liouville derivative. In Section 3, we investigate the existence of triple non-negative solutions to FDE (1.1), and present a new existence criterion.

2 Preliminaries

In this section, we give some necessary preliminaries on the Riemann-Liouville derivative, which will be used in the sequel.

We first recall some well known results about Riemann-Liouville derivative. For details, please refer to [2–4] and the references therein.

Definition 2.1 *([3]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by*

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,\mathrm{d}s, \tag{2.1}$$

provided that the right side is pointwise defined on $(0, \infty)$, where Γ denotes the Gamma function

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1}\,\mathrm{d}t.$$

Definition 2.2 *([3]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \to \mathbb{R}$ is given by*

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s, \tag{2.2}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α, provided that the right side is pointwise defined on $(0, \infty)$.

One can easily obtain the following properties from the definition of Riemann-Liouville derivative.

Proposition 2.3 *([3]) Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation $D_{0+}^{\alpha} u(t) = 0$ has*

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N},\ C_i \in \mathbb{R},\ i = 1, 2, \cdots, N$$

as unique solutions, where N is the smallest integer greater than or equal to α.

Proposition 2.4 *([3]) Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then,*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N} \tag{2.3}$$

for some $C_i \in \mathbb{R},\ i = 1, 2, \cdots, N$, where N is the smallest integer greater than or equal to α.

Finally, we recall the well-known Leggett-Williams fixed point theorem as follows.

Let $E = (E, \|\cdot\|)$ be a Banach space and $P \subset E$ be a cone on E. A continuous mapping $\omega : P \to [0, +\infty)$ is said to be a concave non-negative continuous functional on P, if ω satisfies $\omega(\lambda x + (1-\lambda)y) \geq \lambda\omega(x) + (1-\lambda)\omega(y)$ for all $x, y \in P$ and $\lambda \in [0,1]$.

Let $a,\ b,\ d > 0$ be constants. Define $P_d = \{x \in P : \|x\| < d\}$, $\overline{P_d} = \{x \in P : \|x\| \leq d\}$ and $P(\omega, a, b) = \{x \in P : \omega(x) \geq a, \|x\| \leq b\}$.

Lemma 2.5 *([31]) Let $E = (E, \|\cdot\|)$ be a Banach space, $P \subset E$ be a cone of E and $c > 0$ be a constant. Suppose there exists a concave non-negative continuous functional ω on P with $\omega(x) \leq \|x\|$ for all $x \in \overline{P_c}$. Let $T : \overline{P_c} \to \overline{P_c}$ be a completely continuous operator. Assume that there are numbers a, b and d with $0 < d < a < b \leq c$, such that*

(i) $\{x \in P(\omega, a, b) : \omega(x) > a\} \neq \emptyset$ *and* $\omega(Tx) > a$ *for all* $x \in P(\omega, a, b)$*;*

(ii) $\|Tx\| < d$ *for all* $x \in \overline{P_d}$*;*

(iii) $\omega(Tx) > a$ *for all* $x \in P(\omega, a, c)$ *with* $\|Tx\| > b$.

Then, T *has at least three fixed points* x_1*,* x_2 *and* x_3 *in* $\overline{P_c}$*. Furthermore,* $x_1 \in P_a$*;* $x_2 \in \{x \in P(\omega, a, c) : \omega(x) > a\}$*;* $x_3 \in \overline{P_c} \setminus (P(\omega, b, c) \cup \overline{P_a})$*.*

3 Main results

In this section, we investigate the existence of triple non-negative solutions to FDE (1.1), and present a new existence criterion. To this end, we need the following results about the Green's function for FDE (1.1).

Lemma 3.1 *([29])* $x(t) \in C[0, 1]$ *is a solution to FDE (1.1), if and only if* $x(t) = Tx(t)$*, where*

$$Tx(t) = \int_0^1 G(t, s) f(s, x(s))\, \mathrm{d}s, \tag{3.1}$$

and

$$G(t, s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, \ 0 \le s \le t \le 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, \ 0 \le t \le s \le 1. \end{cases} \tag{3.2}$$

Lemma 3.2 *([29]) The Green's function* $G(t, s)$ *given in (3.2) has the following property:*

1) $G(t, s)$ *is a continuous function on the unit square* $[0, 1] \times [0, 1]$*;*

2) $G(t, s) \ge 0$ *for each* $(t, s) \in [0, 1] \times [0, 1]$*;*

3) $\max_{t \in [0,1]} G(t, s) = G(1, s)$*, for each* $s \in [0, 1]$.

It is well worth pointing out that Wang et al. [30] studied FDE (1.1) with a special nonlinearity $f(t, x(t)) = a(t)g(x(\theta(t)))$ and obtained an existence criterion for triple positive solutions based on Lemma 3.2. Next, to improve the results in [30], we give a new approximation of the Green's function.

Theorem 3.3 *The Green's function* $G(t, s)$ *has the following property:*

$$t^{\alpha-1} G(1, s) \le G(t, s) \le G(1, s), \ \forall\, t, s \in [0, 1]. \tag{3.3}$$

Proof. On one hand, when $s \le t$, we have

$$\begin{aligned} G(t,s) &= \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - t^{\alpha-1}(1-\frac{s}{t})^{\alpha-1}}{\Gamma(\alpha)} \\ &\geq \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &= \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}(1-(1-s)^{\beta})}{\Gamma(\alpha)} \\ &= t^{\alpha-1}G(1,s). \end{aligned}$$

On the other hand, when $s \geq t$, it is easy to see that

$$\begin{aligned} G(t,s) &= \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} \\ &\geq \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &= t^{\alpha-1}G(1,s). \end{aligned}$$

Thus, $t^{\alpha-1}G(1,s) \leq G(t,s),\ \forall\ t,s \in [0,1]$. From Lemma 3.2, it is easy to see that $G(t,s) \leq G(1,s)$. Hence, (3.3) is true. □

Denote by

$$\Psi = \frac{(\alpha-\beta)\Gamma(\alpha+1)}{\xi^{\alpha-1}(1-\xi)^{\alpha-\beta}(\alpha-(\alpha-\beta)(1-\xi)^{\beta})},$$
$$\overline{f_0} = \overline{\lim_{x\to 0^+}} \sup_{t\in[0,1]} \frac{f(t,x)}{x},\ \overline{f_\infty} = \overline{\lim_{x\to +\infty}} \sup_{t\in[0,1]} \frac{f(t,x)}{x}.$$

To use Lemma 2.5, we choose a constant $\xi \in (0,1)$ and define a functional $\omega : P \to [0,+\infty)$ by

$$\omega(x) = \min_{t\in[\xi,1]} x(t), \tag{3.4}$$

then, one can easily see that ω is a concave non-negative continuous functional on P, and satisfies $\omega(x) \leq \|x\|$ for all $x \in P$.

We have the following result.

Theorem 3.4 *Consider FDE (1.1). Assume that there exist two constants a and b with $0 < a < \xi^{\alpha-1}b$ such that the following conditions hold:*

(H1) *there exists a constant ζ with $0 \leq \zeta < \frac{(\alpha-\beta)\Gamma(\alpha+1)}{\beta}$, such that $\overline{f_\infty} = \zeta$;*

(H2) *there exists a constant η with $0 \leq \eta < \frac{(\alpha-\beta)\Gamma(\alpha+1)}{\beta}$, such that $\overline{f_0} = \eta$;*

(H3) $f(t,x) > \Psi a,\ \forall\ (t,x) \in [\xi,1] \times [a,b]$.

Then, FDE (1.1) has at least three non-negative solutions.

Proof. Let us divide the proof into 4 steps.

Step 1. By (H1), for any $\varepsilon \in (0, \frac{(\alpha-\beta)\Gamma(\alpha+1)}{\beta} - \zeta)$, there exists $\tau > 0$ such that $0 \leq f(t,x) \leq (\zeta+\varepsilon)x$ for all $t \in [0,1]$ and $x > \tau$. Denote $N = \max_{(t,x)\in[0,1]\times[0,\tau]} f(t,x)$. Then

$$0 \leq f(t,x) \leq (\zeta+\varepsilon)x + N, \ \forall\, t \in [0,1], \ x \geq 0.$$

Take $c \geq \max\{b, \frac{\beta N}{(\alpha-\beta)\Gamma(\alpha+1)-\beta(\zeta+\varepsilon)}\}$. Then, for $\|x\| \leq c$, from Theorem 3.3, we have

$$\begin{aligned}
\|Tx\| &= \max_{t\in[0,1]} \int_0^1 G(t,s) f(s,x(s))\,\mathrm{d}s \\
&\leq [N + (\zeta+\varepsilon)\|x\|] \max_{t\in[0,1]} \int_0^1 G(t,s)\,\mathrm{d}s \\
&\leq [N + (\zeta+\varepsilon)\|x\|] \int_0^1 G(1,s)\,\mathrm{d}s \\
&= \frac{\beta}{(\alpha-\beta)\Gamma(\alpha+1)} [N + (\zeta+\varepsilon)\|x\|] \leq c.
\end{aligned}$$

Thus, $T : \overline{P_c} \to \overline{P_c}$.

Next, let us show that $T : \overline{P_c} \to \overline{P_c}$ is completely continuous.

Let $x_n,\ x_0 \in \overline{P_c}$ with $\|x_n - x_0\| \to 0$ as $n \to +\infty$. Then,

$$\begin{aligned}
\|Tx_n - Tx_0\| &= \max_{t\in[0,1]} \left|\int_0^1 G(t,s) f(s,x_n(s))\,\mathrm{d}s - \int_0^1 G(t,s) f(s,x_0(s))\,\mathrm{d}s\right| \\
&\leq \max_{t\in[0,1]} \int_0^1 G(t,s) |f(s,x_n(s)) - f(s,x_0(s))|\,\mathrm{d}s \\
&\leq \int_0^1 G(1,s) |f(s,x_n(s)) - f(s,x_0(s))|\,\mathrm{d}s \to 0, \ n \to +\infty.
\end{aligned}$$

Hence, $T : \overline{P_c} \to \overline{P_c}$ is continuous.

In addition, for any $t_1,\ t_2 \in [0,1]$ and $x \in \overline{P_c}$, we have

$$|(Tx)(t_1) - (Tx)(t_2)| \leq [N + (\zeta+\varepsilon)c] \int_0^1 |G(t_1,s) - G(t_2,s)|\,\mathrm{d}s. \tag{3.5}$$

From Lemma 3.2, $G(t,s)$ is uniformly continuous on $(t,s) \in [0,1]\times[0,1]$, which implies that $|(Tx)(t_1) - (Tx)(t_2)| \to 0$ as $|t_1 - t_2| \to 0$. Moreover, $T(\overline{P_c})$ is bounded. Thus, the Arzela-Ascoli theorem guarantees that $T : \overline{P_c} \to \overline{P_c}$ is compact.

Therefore, $T : \overline{P_c} \to \overline{P_c}$ is completely continuous.

Step 2. Let $x_0(t) = \frac{a+b}{2}$, $\forall\, t \in [0,1]$. Then, $\omega(x_0) = \frac{a+b}{2} > a$ and $\|x_0\| = \frac{a+b}{2} < b$. Thus, $x_0 \in \{x \in P(\omega,a,b) : \omega(x) > a\} \neq \emptyset$.

Now, let us prove that $\omega(Tx) > a$ holds for all $x \in P(\omega,a,b)$. In fact, $x \in P(\omega,a,b)$ implies that $a \le x(t) \le b,\ \forall\, t \in [\xi, 1]$. One can obtain from (H3) and Theorem 3.3 that

$$\begin{aligned}
\omega(Tx) &= \min_{t\in[\xi,1]}(Tx)(t) = \min_{t\in[\xi,1]}\int_0^1 G(t,s)f(s,x(s))\,\mathrm{d}s \\
&> \Psi a \min_{t\in[\xi,1]}\int_\xi^1 G(t,s)\,\mathrm{d}s \\
&\ge \Psi a \min_{t\in[\xi,1]}\int_\xi^1 \frac{t^{\alpha-1}((1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-1})}{\Gamma(\alpha)}\,\mathrm{d}s \\
&\ge \Psi a \frac{\xi^{\alpha-1}}{\Gamma(\alpha)}\int_\xi^1 ((1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-1})\,\mathrm{d}s = a.
\end{aligned}$$

Hence, condition (i) of Lemma 2.5 is true.

Step 3. It is easy to see from (H2) that for all $t \in [0,1]$, $\forall\, 0 < \varepsilon \le \frac{(\alpha-\beta)\Gamma(\alpha+1)}{\beta} - \eta,\ \exists\, \delta > 0,$ such that for $0 \le x < \delta$, we have

$$f(t,x) < (\eta+\varepsilon)x. \tag{3.6}$$

Let $0 < d < \min\{\delta, a\}$. Now, we prove that $\|Tx\| < d$ for all $x \in \overline{P_d} = \{x \in P : \|x\| \le d\}$.

As a matter of fact, $\forall\, x \in \overline{P_d}$, one can see that

$$\begin{aligned}
\|Tx\| &= \max_{t\in[0,1]}\int_0^1 G(t,s)f(s,x(s))\,\mathrm{d}s \\
&< (\eta+\varepsilon)\|x\| \max_{t\in[0,1]}\int_0^1 G(t,s)\,\mathrm{d}s \\
&\le (\eta+\varepsilon)\|x\| \int_0^1 G(1,s)\,\mathrm{d}s \le \|x\| \le d
\end{aligned}$$

Thus, $\|Tx\| < d$, for all $x \in \overline{P_d}$.

Step 4. Let us prove that $\omega(Tx) > a$ holds for all $x \in P(\omega, a, c)$ with $\|Tx\| > b$.
In fact, for $x \in P(\omega, a, c)$ with $\|Tx\| > b$, we have

$$\begin{aligned}
b < \|Tx\| &= \max_{t\in[0,1]}\int_0^1 G(t,s)f(s,x(s))\,\mathrm{d}s \\
&\le \int_0^1 G(1,s)f(s,x(s))\,\mathrm{d}s,
\end{aligned}$$

that is,

$$\int_0^1 G(1,s)f(s,x(s))\,\mathrm{d}s > b.$$

Therefore,

$$\begin{aligned}
\omega(Tx) &= \min_{t\in[\xi,1]}(Tx)(t) = \min_{t\in[\xi,1]}\int_0^1 G(t,s)f(s,x(s))\,\mathrm{d}s \\
&\ge \min_{t\in[\xi,1]}\int_0^1 t^{\alpha-1}G(1,s)f(s,x(s))\,\mathrm{d}s \\
&\ge \xi^{\alpha-1}\int_0^1 G(1,s)f(s,x(s))\,\mathrm{d}s \\
&> \xi^{\alpha-1}b > a.
\end{aligned}$$

To sum up, all conditions of Lemma 2.5 hold. By Lemma 2.5, FDE (1.1) has at least three non-negative solutions. □

Remark 3.5 *From the proof of Theorem 3.4, one can see that at least two of the three non-negative solutions are positive solutions.*

Finally, we give an illustrative example to support our new results.

Example 3.6 *Consider the following higher-order fractional differential equation:*

$$\begin{cases} D_{0+}^{\frac{7}{2}}u(t) + f(t,u(t)) = 0,\ t \in (0,1), \\ u(0) = u'(0) = u''(0) = 0,\ [D_{0+}^{\frac{3}{2}}u(t)]_{t=1} = 0, \end{cases} \tag{3.7}$$

where $f(t,x) = a(t)g(x)$, $a(t) \equiv 1$ *and*

$$g(x) = \begin{cases} 10x,\ 0 \le x \le 0.5, \\ 390x - 190,\ 0.5 < x < 1, \\ 200,\ 1 \le x \le 20, \\ 10x,\ x > 20. \end{cases} \tag{3.8}$$

Choose $\xi = \frac{2}{3}$. A simple calculation shows that $\Psi \approx 185.2136$ and $\frac{(\alpha-\beta)\Gamma(\alpha+1)}{\beta} \approx 15.5090$.

Set $a = 1$ and $b = 10$, then one can see that $\overline{f_\infty} = \overline{f_0} = 10 < \frac{(\alpha-\beta)\Gamma(\alpha+1)}{\beta}$ and

$$f(t,x) = 200 > \Psi a,\ \forall\ (t,x) \in [\frac{2}{3},1] \times [1,10].$$

Thus, (H1)–(H3) hold true. By Theorem 3.4, FDE (3.7) has at least three non-negative solutions.

Remark 3.7 *It is noted that Wang et al. [30] studied FDE (1.1) with* $f(t,x(t)) = a(t)g(x(\theta(t)))$ *and obtained an existence criterion for triple positive solutions. By a simple calculation, it is easy to see that the conditions in [30] are not satisfied for FDE (3.7). Thus, the new existence criterion obtained in this paper improves the existing results in [30] to some extent.*

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