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UNIQUE SOLVABILITY OF INITIAL-BOUNDARY-VALUE PROBLEMS FOR ANISOTROPIC ELLIPTIC-PARABOLIC EQUATIONS WITH VARIABLE EXPONENTS OF NONLINEARITY

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Abstract. Existence and uniqueness of weak solutions of initial-boundary-value problems for second order elliptic-parabolic equations are proved. These equations have the exponents of nonlinearity depending on the points of domain and the direction of differentiation. The weak solutions belong to some generalized Sobolev spaces.

Keywords: nonlinear equation, anisotropic equation, elliptic-parabolic equation, degenerate parabolic equation, initial-boundary-value problem, variable exponents of nonlinearity, generalized Lebesgue space, generalized Sobolev space.

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1 Introduction

The object of our investigations is the degenerate equations of the following type

$$\frac{\partial}{\partial t}B(x,t,u)+A(u)=F(x,t),\quad (x,t)\in Q, \tag{1.1}$$

where $Q:=\Omega\times(0,T)$, Ω is a domain in $\mathbb{R}^n$ ($n\in\mathbb{N}$), A is a differential expression of the elliptic type, B, F are given functions. The function B may be such that $B(x,t,u)=u$ for every $u\in\mathbb{R}$ and for a.e. $(x,t)\in Q_1$, and $B(x,t,u)=0$ for every $u\in\mathbb{R}$ and for a.e. $(x,t)\in Q\setminus Q_1$, where Q_1 is an arbitrary measurable subset of Q. These equations are the *elliptic-parabolic equations* (see [32]). The various problems for equation (1.1) are investigated in [2], [3], [5], [18], [19], [20], [31], [32] etc. For the linear B, A corresponding problems are considered in [31], [32]. If the nonlinear function B independs of the spatial variable x, then the solvability of the problems for systems of equations (1.1) is proved in [2]. The case of the dependence of the nonlinear function B only on x and u is considered in [20].

The mentioned above papers are devoted the equations with the constant exponent of nonlinearity, for example, equations (1.1) with $A(u)=-\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, where $p=const$. If the exponent of nonlinearity p is a function of the variable x, then (1.1) are the *equations with variable exponents of nonlinearity* (see [27]). Nowadays, nonlinear differential equations with variable exponents of nonlinearity actively being studied (see, for example, [1], [4], [6], [7], [8], [9], [10], [11], [12], [13], [25], [26], [27] and references given there). These equations describe many physical processes (see [25], [30]) such us electromagnetic fields, electrorheological fluids, image reconstruction processes, current flow in variable temperature field. The solutions of these problems belong to some generalized Lebesgue and Sobolev spaces. The mentioned spaces were firstly introduced in [29]. Properties of these spaces were studied in [16], [22], [24], [28], [29], [33] and others.

In this paper we consider the initial-boundary-value problems for the elliptic-parabolic equations with variable exponents of nonlinearity. A typical example of the equations being studied here are

$$\frac{\partial}{\partial t}(b(x)u)-\sum_{i=1}^{n}\Big(\widehat{a}_i(x,t)|u_{x_i}|^{p_i(x)-2}u_{x_i}\Big)_{x_i}+\widehat{a}_0(x,t)|u|^{p_0(x)-2}u=f(x,t), \tag{1.2}$$

$(x,t)\in Q$, where b is a measurable bounded function such that $b(x)\geq 0$ for a.e. $x\in\Omega$, $\widehat{a}_0,\dots,\widehat{a}_n$ are measurable positive functions, p_0, …, p_n (the exponents of nonlinearity) are measurable bounded functions such that $p_0(x)>1$, …, $p_n(x)>1$ for a.e. $x\in\Omega$, f is an integrable function.

Notice that the unique solvability of the initial-boundary-value problems for the equations similar to (1.2) with $b(x)\equiv const>0$ are investigated in [1], [4], [17], [21], [34]. In case $p_1(x)\equiv const$, …, $p_n(x)\equiv const$ the problems for the equations similar to (1.2) are studied in [3], [18], [20], [31], [32]. In this paper we consider the mentioned cases together. Such situation was investigated only in [5], but here we use the weaker conditions for initial date as compared to [5].

2 Notation and auxiliary facts

Let $n \in \mathbb{N}, T > 0$ be some numbers, $|\cdot|$ be norm of the space $\mathbb{R}^n$, i.e. $|x| := (|x_1|^2 + \ldots + |x_n|^2)^{1/2}$ if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with the piecewise smooth boundary $\partial\Omega$, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 is the closure of an open set on $\partial\Omega$, in particular, either $\Gamma_0 = \emptyset$ or $\Gamma_0 = \partial\Omega$, $\Gamma_1 := \partial\Omega \setminus \Gamma_0$, $\nu = (\nu_1, \ldots, \nu_n)$ is a unit outward pointing normal vector on the $\partial\Omega$. Put $Q := \Omega \times (0,T)$, $\Sigma_0 := \Gamma_0 \times (0,T)$, $\Sigma_1 := \Gamma_1 \times (0,T)$.

Let us introduce some functional spaces. Let either $G = \Omega$ or $G = Q$. Suppose that $r \in L_\infty(\Omega)$, $r(x) \geq 1$ for a.e. $x \in \Omega$. Consider a linear subspace $L_{r(\cdot)}(G)$ of the space $L_1(G)$ which consists of the measurable functions v such that $\rho_{G,r}(v) < \infty$, where $\rho_{G,r}(v) := \int_\Omega |v(x)|^{r(x)}\, \mathrm{d}x$ if $G = \Omega$, and $\rho_{G,r}(v) := \int_Q |v(x,t)|^{r(x)}\, \mathrm{d}x\, \mathrm{d}t$ if $G = Q$. This is a Banach space with respect to the norm $||v||_{L_{r(\cdot)}(G)} := \inf\{\lambda > 0 \mid \rho_{G,r}(v/\lambda) \leq 1\}$ (see [22, p. 599]) and it is called a *generalized Lebesgue space*. Note that if $r(x) = r_0 = \text{const} \geq 1$ for a.e. $x \in \Omega$ then $||\cdot||_{L_{r(\cdot)}(G)}$ equals to the standard norm $||\cdot||_{L_{r_0}(G)}$ of the Lebesgue space $L_{r_0}(G)$. Note also that the set $C(\overline{G})$ is dense in $L_{r(\cdot)}(G)$ (see [22, p. 603]). According to [22, p. 599], if $\operatorname*{ess\,inf}_{x\in\Omega} r(x) > 1$, then the space $L_{r(\cdot)}(G)$ is reflexive and the dual space $[L_{r(\cdot)}(G)]^*$ equals $L_{r'(\cdot)}(G)$, where the function r' is defined by the equality $\frac{1}{r(x)} + \frac{1}{r'(x)} = 1$ for a.e. $x \in \Omega$. But there exist properties of standard Legesgue spaces that is not valid for generalized Lebesgue spaces (see [33] for more details). Thus the properties of the generalized Lebesgue spaces are not simple corollaries of the correspondent properties of the standard Lebesgue spaces (see, for example, the mentioned papers, and Lemmas 1, 2 below).

Consider the functions $p = (p_0, \ldots, p_n) : \Omega \to \mathbb{R}^{n+1}$, and $b : \Omega \to \mathbb{R}$ such that following conditions are satisfied:

($\mathcal{P}$) for every $i \in \{0, 1, \ldots, n\}$ $p_i : \Omega \to \mathbb{R}$ is a measurable function such that $p_i^- := \operatorname*{ess\,inf}_{x\in\Omega} p_i(x) > 1$, $p_i^+ := \operatorname*{ess\,sup}_{x\in\Omega} p_i(x) < +\infty$,

($\mathcal{B}$) $b \in L_\infty(\Omega)$, $b(x) \geq 0$ for a.e. $x \in \Omega$.

Denote by $p' = (p_0', \ldots, p_n') : \Omega \to \mathbb{R}^{n+1}$ the vector-function such that $\frac{1}{p_i(x)} + \frac{1}{p_i'(x)} = 1$ for a.e. $x \in \Omega$ ($i = \overline{0,n}$).

Now let us give the definitions of the following functional spaces. First, denote by $W^1_{p(\cdot)}(\Omega)$ a generalized Sobolev space of functions $v \in L_{p_0(\cdot)}(\Omega)$ such that $v_{x_1} \in L_{p_1(\cdot)}(\Omega)$, $\ldots$, $v_{x_n} \in L_{p_n(\cdot)}(\Omega)$. This is a Banach space with respect to the norm

$$||v||_{W^1_{p(\cdot)}(\Omega)} := ||v||_{L_{p_0(\cdot)}(\Omega)} + \sum_{i=1}^n ||v_{x_i}||_{L_{p_i(\cdot)}(\Omega)}.$$

Let $\widetilde{W}^1_{p(\cdot)}(\Omega)$ be the subspace of $W^1_{p(\cdot)}(\Omega)$ that equals to the closure of the space $\widetilde{C}^1(\overline{\Omega}) := \{v \in C^1(\overline{\Omega}) \mid v|_{\Gamma_0} = 0\}$ with respect to the norm $||\cdot||_{W^1_{p(\cdot)}(\Omega)}$.

Put by definition $\mathbb{V}^b_p := \{v \in \widetilde{W}^1_{p(\cdot)}(\Omega) \mid b^{1/2}v \in L_2(\Omega)\}$. It is easy to verify that $\mathbb{V}^b_p$ is a Banach space with respect to the norm

$$||v||_{\mathbb{V}^b_p} := ||v||_{W^1_{p(\cdot)}(\Omega)} + ||b^{1/2}v||_{L_2(\Omega)}.$$

Further, denote by $W^{1,0}_{p(\cdot)}(Q)$ a space of functions $w \in L_{p_0(\cdot)}(Q)$ such that $w_{x_1} \in L_{p_1(\cdot)}(Q), \ldots,$ $w_{x_n} \in L_{p_n(\cdot)}(Q)$. Consider this space with the norm

$$||w||_{W^{1,0}_{p(\cdot)}(Q)} := ||w||_{L_{p_0(\cdot)}(Q)} + \sum_{i=1}^{n} ||w_{x_i}||_{L_{p_i(\cdot)}(Q)}.$$

Define $\widetilde{W}^{1,0}_{p(\cdot)}(Q)$ be the subspace of the space $W^{1,0}_{p(\cdot)}(Q)$ that equals the closure of

$$\widetilde{C}^{1,0}(\overline{Q}) := \big\{ w \in C(\overline{Q}) \;\big|\; w_{x_i} \in C(\overline{Q}) \ (i = \overline{1,n}), \ \ w|_{\Sigma_0} = 0 \big\}$$

with respect to the norm $|| \cdot ||_{W^{1,0}_{p(\cdot)}(Q)}$.

By definition, put $\mathbb{U}^b_p := \{ w \in \widetilde{W}^{1,0}_{p(\cdot)}(Q) \mid b^{1/2} w \in C([0,T]; L_2(\Omega)) \}$. It is easy to verify that this is Banach space with respect to the norm

$$||w||_{\mathbb{U}^b_p} := ||w||_{W^{1,0}_{p(\cdot)}(Q)} + \max_{t \in [0,T]} ||b^{1/2}(\cdot) w(\cdot,t)||_{L_2(\Omega)}\,.$$

Clearly, for every $w \in \mathbb{U}^b_p$ we have $w(\cdot,t) \in \mathbb{V}^b_p$ for a.e. $t \in [0,T]$.

Finally, denote by $\mathbb{F}_{p'}$ a space of vector-functions $(f_0, f_1, \ldots, f_n)$ such that $f_i \in L_{p'_i(\cdot)}(Q)$, and $f_i = 0$ a.e. in some neighborhood of the surface Σ_1 $(i = \overline{0,n})$. Denote by $\mathbb{H}^b$ the closure of the space $\{ b^{1/2} v \,\big|\, v \in C(\overline{\Omega}) \}$ with respect to the norm $|| \cdot ||_{L_2(\Omega)}$.

Note that if $b(x) \geq b_0 = \text{const} > 0$ for a.e. $x \in \Omega$, then $\mathbb{V}^b_p$ continuously embeds in $L_2(\Omega)$, $\mathbb{U}^b_p$ continuously embeds in $L_2(Q)$, and $\mathbb{H}^b = L_2(\Omega)$.

3 Statement of the problem and main result

In this paper we consider the problem of the finding the function $u : \overline{Q} \to \mathbb{R}$ satisfying (in some sense) the equation

$$\frac{\partial}{\partial t}(b(x)u) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x,t,u,\nabla u) + a_0(x,t,u,\nabla u)$$
$$= - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(x,t) + f_0(x,t), \quad (x,t) \in Q, \tag{3.1}$$

the boundary conditions

$$u(x,t)\Big|_{(x,t)\in\Sigma_0} = 0, \qquad \sum_{i=1}^{n} a_i(x,t,u,\nabla u)\, \nu_i(x)\Big|_{(x,t)\in\Sigma_1} = 0, \tag{3.2}$$

and the initial condition

$$\big(b^{1/2}(x) u(x,t)\big)|_{t=0} = u_0(x), \quad x \in \Omega. \tag{3.3}$$

Here $b : \Omega \to [0, +\infty)$, $a_i : Q \times \mathbb{R}^{1+n} \to \mathbb{R}$, $f_i : Q \to \mathbb{R}$ $(i = \overline{0,n})$, $u_0 : \Omega \to \mathbb{R}$ are given real-valued functions. Notice that the equality $b = 0$ may be studied on any subset of Ω and the spatial part of the differential expression in the left-hand side of equation (3.1) is elliptic.

We consider a weak solutions of problem (3.1)–(3.3), and thus we introduce following classes of the initial data. Define $\mathbb{A}_p(1-3)$ to be the set of the functions $(a_0, a_1, \ldots, a_n)$ satisfying the following assumptions:

($\mathcal{A}1$) for every $i \in \{0, 1, \ldots, n\}$

$$Q \times \mathbb{R}^{1+n} \ni (x, t, s, \xi) \mapsto a_i(x, t, s, \xi) \in \mathbb{R}$$

is the Caratheodory function, i.e. $a_i(x, t, \cdot, \cdot) : \mathbb{R}^{1+n} \to \mathbb{R}$ is the continuous function for a.e. $(x, t) \in Q$, and $a_i(\cdot, \cdot, s, \xi) : Q \to \mathbb{R}$ is the measurable function for every $(s, \xi) \in \mathbb{R}^{1+n}$;

($\mathcal{A}2$) for every $i \in \{0, 1, \ldots, n\}$, for every $(s, \xi) \in \mathbb{R}^{1+n}$, and for a.e. $(x, t) \in Q$ the following estimate is valid

$$|a_i(x, t, s, \xi)| \leq C_1 \big(|s|^{p_0(x)/p_i'(x)} + \sum_{j=1}^{n} |\xi_j|^{p_j(x)/p_i'(x)}\big) + h_i(x, t),$$

where $C_1 = \mathit{const} > 0$, $h_i \in L_{p_i'(\cdot)}(Q)$;

($\mathcal{A}3$) for every $(s_1, \xi^1), (s_2, \xi^2) \in \mathbb{R}^{1+n}$ and for a.e. $(x, t) \in Q$ the inequality

$$\sum_{i=1}^{n} \big(a_i(x, t, s_1, \xi^1) - a_i(x, t, s_2, \xi^2)\big)(\xi_i^1 - \xi_i^2) + \big(a_0(x, t, s_1, \xi^1) - a_0(x, t, s_2, \xi^2)\big)(s_1 - s_2) \geq 0 \tag{3.4}$$

holds.

Now we can give a definition of the weak solution to (3.1)–(3.3).

Definition. If p, b satisfy conditions ($\mathcal{P}$), ($\mathcal{B}$) respectively, $u_0 \in \mathbb{H}^b$, $(f_0, f_1, \ldots, f_n) \in \mathbb{F}_{p'}$, $(a_0, a_1, \ldots, a_n) \in \mathbb{A}_p(1-3)$. The function $u \in \mathbb{U}_p^b$ is called *weak solution* of problem (3.1)–(3.3) if u satisfies the initial condition (3.3), that is

$$\lim_{t \to +0} \int_{\Omega} |b^{1/2}(x) u(x, t) - u_0(x)|^2 \, \mathrm{d}x = 0, \tag{3.5}$$

and the integral equality

$$\iint_{Q} \Big\{ \sum_{i=1}^{n} a_i(x, t, u, \nabla u)\, v_{x_i}\, \varphi + a_0(x, t, u, \nabla u)\, v\, \varphi - b(x)\, u\, v\, \varphi' \Big\} \mathrm{d}x\, \mathrm{d}t = \iint_{Q} \Big\{ \sum_{i=1}^{n} f_i\, v_{x_i}\, \varphi + f_0\, v\, \varphi \Big\} \mathrm{d}x\, \mathrm{d}t \tag{3.6}$$

holds for every $v \in \mathbb{V}_p^b$ and $\varphi \in C_0^1(0, T)$.

Denote by $\mathbb{A}_p(1-3,3^*)$ a subset of the functions $(a_0,a_1,\dots,a_n)\in\mathbb{A}_p(1-3)$ satisfying the following condition:

($\mathcal{A}3^*$) if $s_1\neq s_2$ then the sign " $\geq$ " in the inequality (3.4) must be

replaced by the sign " $>$ " for a.e. $x\in\Omega$ such that $b(x)=0$.

Theorem 1. If p,b satisfy conditions ($\mathcal{P}$), ($\mathcal{B}$) respectively, $u_0\in\mathbb{H}^b$, $(f_0,f_1,\dots,f_n)\in\mathbb{F}_{p'}$, $(a_0,a_1,\dots,a_n)\in\mathbb{A}_p(1-3,3^*)$, then the weak solution of problem (3.1)–(3.3) is unique.

Denote by $\mathbb{A}_p(1-4)$ the subset of the functions $(a_0,a_1,\dots,a_n)\in\mathbb{A}_p(1-3)$ satisfying the following property:

($\mathcal{A}4$) for every $(s,\xi)\in\mathbb{R}^{1+n}$ and for a.e. $(x,t)\in Q$ we have

$$\sum_{i=1}^{n} a_i(x,t,s,\xi)\xi_i + a_0(x,t,s,\xi)s \geq K_1\Big(\sum_{i=1}^{n}|\xi_i|^{p_i(x)}+|s|^{p_0(x)}\Big)-g(x,t),$$

where $g\in L_1(Q)$, and $K_1=\text{const}>0$.

Note that the function g from above satisfies $g(x,t)\geq 0$ for a.e. $(x,t)\in Q$. This follows from inequality in condition ($\mathcal{A}4$) if $\xi_1=\ldots=\xi_n=0$, $s=0$.

Theorem 2. If p,b satisfy conditions ($\mathcal{P}$), ($\mathcal{B}$) respectively, $u_0\in\mathbb{H}^b$, $(f_0,f_1,\dots,f_n)\in\mathbb{F}_{p'}$, $(a_0,a_1,\dots,a_n)\in\mathbb{A}_p(1\text{–}4)$ then problem (3.1)–(3.3) has a weak solution u. Moreover, an arbitrary weak solution u of this problem satisfies following estimate

$$\max_{t\in[0,T]}\int_{\Omega} b(x)|u(x,t)|^2\,\mathrm{d}x+\iint_{Q}\Big\{\sum_{i=1}^{n}\big|u_{x_i}(x,t)\big|^{p_i(x)}+|u(x,t)|^{p_0(x)}\Big\}\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq C_2\iint_{Q}\Big\{\sum_{i=1}^{n}\big|f_i(x,t)\big|^{p_i'(x)}+g(x,t)\Big\}\,\mathrm{d}x\,\mathrm{d}t+C_3\int_{\Omega}|u_0(x)|^2\,\mathrm{d}x, \qquad (3.7)$$

where C_2,C_3 are positive constants depending only on K_1 and p_i^- $(i=\overline{0,n})$.

Let $\mathbb{A}_p(1-3,3^*,4)$ be the subset of the functions $(a_0,a_1,\dots,a_n)\in\mathbb{A}_p(1-3)$ satisfying the conditions ($\mathcal{A}3^*$) and ($\mathcal{A}4$).

Collorary 1. If p,b satisfy conditions ($\mathcal{P}$), ($\mathcal{B}$) respectively, $u_0\in\mathbb{H}^b$, $(f_0,f_1,\dots,f_n)\in\mathbb{F}_{p'}$, $(a_0,a_1,\dots,a_n)\in\mathbb{A}_p(1-3,3^*,4)$ then problem (3.1)–(3.3) has a unique weak solution and estimate (3.7) holds.

4 Auxiliary statements

We need some technical statements that play an important role in obtaining the main results.

Consider a family of functions $\{\omega_\rho:\mathbb{R}\to\mathbb{R}\mid\rho>0\}$ such that for every $\rho>0$ we have $\omega_\rho(z):=(1/\rho)\,\omega_1(z/\rho)$, $z\in\mathbb{R}$, where $\omega_1\in C_0^\infty(\mathbb{R})$ is a standard mollifier (see [15, p. 629]), i.e. $\operatorname{supp}\omega_1\subset[-1,1]$, $\omega_1(z)\geq 0$, $\omega_1(-z)=\omega_1(z)$ if $z\in\mathbb{R}$, $\int_{\mathbb{R}}\omega_1(z)\,\mathrm{d}z=1$.

For every $\rho > 0$ we define the mollification of any $\psi \in L_1(Q)$ by the rule

$$\psi_\rho(x,t) := \int_{\mathbb{R}} \widehat{\psi}(x,\tau)\omega_\rho(\tau - t)\ \mathrm{d}\tau \quad \text{for a.e.} \quad (x,t) \in Q,$$

where $\widehat{\psi}(x,t) := \begin{cases} \psi(x,t), & x \in \Omega,\ t \in (0,T), \\ 0, & x \in \Omega,\ t \notin (0,T). \end{cases}$

The following statement is well-known for standard Lebesgue spaces (see [15]). For the generalized Lebesgue spaces you can find this statement in [11] (see also the proof of Lemma 1 in [7]) but we will prove Lemma 1 in other way for convenience.

Lemma 1. If $r \in L_\infty(\Omega)$, $r(x) \geq 1$ for a.e. $x \in \Omega$, then for every function $f \in L_{r(\cdot)}(Q)$ we have

$$f_\rho \underset{\rho \to 0}{\longrightarrow} f \quad \text{strongly in} \quad L_{r(\cdot)}(Q).$$

Proof. It is enough to show that for every $\varepsilon > 0$ there exists constant $\delta > 0$ such that for every $\rho \in (0,\delta)$ we have

$$\iint_Q |f_\rho(x,t) - f(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t < \varepsilon. \tag{4.1}$$

Let $\{\widetilde{h}_l\}_{l=1}^\infty \subset C(\overline{Q})$ be a sequence of functions such that $\widetilde{h}_l \underset{l\to\infty}{\longrightarrow} f$ strongly in $L_{r(\cdot)}(Q)$, i.e.

$$\iint_Q |f(x,t) - \widetilde{h}_l(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t \underset{l\to\infty}{\longrightarrow} 0. \tag{4.2}$$

For every $l \in \mathbb{N}$ we put $h_l(x,t) = \begin{cases} \widetilde{h}_l(x,t), & (x,t) \in \overline{Q}, \\ \widetilde{h}_l(x,0), & x \in \overline{\Omega},\ t \leq 0, \\ \widetilde{h}_l(x,T), & x \in \overline{\Omega},\ t \geq T. \end{cases}$

Thus $h_l \in C(\overline{\Omega} \times \mathbb{R})$. Let

$$h_{l,\rho}(x,t) := \int_{\mathbb{R}} h_l(x,\tau)\omega_\rho(\tau - t)\,\mathrm{d}\tau, \quad (x,t) \in \overline{\Omega} \times \mathbb{R}, \quad \rho > 0.$$

Take an arbitrary $l \in \mathbb{N}$ and $\rho > 0$. Using the well-known inequality
$(a+b+c)^q \leq 3^{q-1}(a^q + b^q + c^q)$, $a,b,c \geq 0$, $q \geq 1$, we get

$$\begin{aligned} \iint_Q |f_\rho(x,t) - f(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t \leq 3^{r^+-1}\Big[\iint_Q |f_\rho(x,t) - h_{l,\rho}(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t \\ + \iint_Q |h_{l,\rho}(x,t) - h_l(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t + \iint_Q |h_l(x,t) - f(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t\Big], \end{aligned} \tag{4.3}$$

where $r^+ := \operatorname*{ess\,sup}_{x\in\Omega} r(x)$.

Consider the first integral of the right-hand side of inequality (4.3). Using the Hölder inequality, for a.e. $x \in \Omega$ we get

$$|f_\rho(x,t) - h_{l,\rho}(x,t)| = \int\limits_{|\tau - t| \le \rho} (\widehat{f}(x,\tau) - h_l(x,\tau))\omega_\rho(\tau - t)\,\mathrm{d}\tau$$
$$\le C_4\Big(\int\limits_{|\tau-t|\le\rho} |\widehat{f}(x,\tau) - h_l(x,\tau)|^{r(x)} d\tau\Big)^{\frac{1}{r(x)}}\ 2^{\frac{1}{r'(x)}}\rho^{-\frac{1}{r(x)}},$$

where $C_4 > 0$ is some constant independent of l, ρ. From this after simple transformations we have

$$\iint\limits_Q |f_\rho(x,t) - h_{l,\rho}(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t \le C_5\Big[\int\limits_{-\rho}^{0}\int\limits_\Omega |h_l(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int\limits_T^{T+\rho}\int\limits_\Omega |h_l(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t + \iint\limits_Q |f(x,t) - h_l(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t\Big], \tag{4.4}$$

where the constant $C_5 > 0$ does not depends on ρ and l.

Combining (4.4) with (4.3), we obtain

$$\iint\limits_Q |f_\rho(x,t) - f(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C_6\Big[\iint\limits_Q |f(x,t) - h_l(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t + \iint\limits_Q |h_{l,p}(x,t) - h_l(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int\limits_{-\rho}^{0}\int\limits_\Omega |h_l(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t + \int\limits_T^{T+\rho}\int\limits_\Omega |h_l(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t\Big], \tag{4.5}$$

where $C_6 > 0$ is independent of ρ, l.

Fix $\varepsilon > 0$. Taking into account (4.2), we choose the number $l_\varepsilon \in \mathbb{N}$ such that

$$C_6 \iint\limits_Q |f(x,t) - h_{l_\varepsilon}(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t < \frac{\varepsilon}{3}. \tag{4.6}$$

We can prove that

$$\max_{(x,t)\in\overline{Q}} |h_{l,p}(x,t) - h_l(x,t)| \underset{\rho\to 0}{\longrightarrow} 0 \tag{4.7}$$

for every fixed $l \in \mathbb{N}$ in the standard way. Since (4.7) holds, there exists $\delta_1 > 0$ such that

$$\max_{(x,t)\in\overline{Q}} |h_{l_\varepsilon,\rho}(x,t) - h_{l_\varepsilon}(x,t)| \le 1$$

if $\rho \in (0, \delta_1)$. Hence, for every $\rho \in (0, \delta_1)$ we get

$$\iint_Q |h_{l_\varepsilon,\rho}(x,t) - h_{l_\varepsilon}(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t \le \mathrm{mes}_{n+1}Q \max_{(x,t)\in\overline{Q}} |h_{l_\varepsilon,\rho}(x,t) - h_{l_\varepsilon}(x,t)|,$$

where $\mathrm{mes}_{n+1}Q$ is the Lebesgue measure of the set Q. Then there exists $\delta_2 \in (0, \delta_1)$ such that

$$C_6 \iint_Q |h_{l,p}(x,t) - h_l(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t < \frac{\varepsilon}{3} \tag{4.8}$$

for all $\rho \in (0, \delta_2)$.

Since $\mathrm{mes}_{n+1}\{\Omega \times (-\rho, 0)\} = \rho\,\mathrm{mes}_n\Omega \underset{\rho\to+0}{\longrightarrow} 0$, and $\mathrm{mes}_{n+1}\{\Omega \times (T, T+\rho)\} = \rho\,\mathrm{mes}_n\Omega \underset{\rho\to+0}{\longrightarrow} 0$, using the absolute continuity of the Lebesgue integral, we choose a number $\delta \in (0, \delta_2)$ such that for every $\rho \in (0, \delta)$ the inequality

$$C_6\Big[\int_{-\rho}^{0}\int_\Omega |h_{l_\varepsilon}(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t + \int_T^{T+\rho}\int_\Omega |h_{l_\varepsilon}(x,t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t\Big] < \frac{\varepsilon}{3} \tag{4.9}$$

holds. Therefore, from (4.5) in view of (4.6), (4.8), (4.9) it follows (4.1). □

Lemma 2. Suppose that b satisfies condition ($\mathcal{B}$), and $w \in \widetilde{W}^{1,0}_{p(\cdot)}(Q)$ such that $b^{1/2}w \in L_2(Q)$, and the identity

$$\iint_Q \Big\{\sum_{i=1}^n g_i v_{x_i}\varphi + g_0 v\varphi - bwv\varphi'\Big\}\,\mathrm{d}x\,\mathrm{d}t = 0,\ v \in \mathbb{V}_p^b,\ \varphi \in C_0^1(0,T), \tag{4.10}$$

holds for some functions $g_j \in L_{p'_j(\cdot)}(Q)$ ($j = \overline{0,n}$). Then $b^{1/2}w \in C([0,T]; L_2(\Omega))$ and for every $\theta \in C^1([0,T])$, $v \in \mathbb{V}_p^b$, and $t_1, t_2 \in [0,T]$, $t_1 < t_2$, we have

$$\theta(t_2)\int_\Omega b(x)w(x,t_2)v(x)\,\mathrm{d}x - \theta(t_1)\int_\Omega b(x)w(x,t_1)v(x)\,\mathrm{d}x + \int_{t_1}^{t_2}\int_\Omega \Big\{\sum_{i=1}^n g_i v_{x_i}\theta + g_0 v\theta - bwv\theta'\Big\}\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{4.11}$$

$$\frac{1}{2}\theta(t_2)\int_\Omega b(x)|w(x,t_2)|^2\,\mathrm{d}x - \frac{1}{2}\theta(t_1)\int_\Omega b(x)|w(x,t_1)|^2\,\mathrm{d}x - \frac{1}{2}\int_{t_1}^{t_2}\int_\Omega b|w|^2\theta'\,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\int_\Omega \Big\{\sum_{i=1}^n g_i w_{x_i} + g_0 w\Big\}\theta\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{4.12}$$

Proof. We use some ideas of the proof from [32, Proposition 1.2, p. 106]. Let us construct functions

$$\widehat{w}(x,t) := \begin{cases} w(x,-t), & -T < t < 0, \\ w(x,t), & 0 \le t \le T, \\ w(x,2T-t), & T < t < 2T, \end{cases}$$

$$\widehat{g}_i(x,t) := \begin{cases} -g_i(x,-t), & -T < t < 0, \\ g_i(x,t), & 0 \le t \le T, \\ -g_i(x,2T-t), & T < t < 2T \end{cases} \qquad (i=\overline{0,n}).$$

Firstly we prove the equality

$$\int\limits_{-T}^{2T}\int\limits_{\Omega}\Big\{\sum_{i=1}^{n}\widehat{g}_i\, v_{x_i}\varphi + \widehat{g}_0\, v\,\varphi - b\,\widehat{w}\,v\,\varphi'\Big\}\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{4.13}$$

for every $\varphi \in C_0^1(-T,2T)$, $v \in \mathbb{V}_p^b$. It is easy to show that equality (4.13) holds for every $v \in \mathbb{V}_p^b$, and arbitrary $\varphi \in C_0^1(-T,2T)$ such that $\operatorname{supp}\varphi \subset (-T,0)\cup(0,T)\cup(T,2T)$ (notice that it is enough to make the corresponding substitution of variable t into identity (4.10)).

Suppose that $\operatorname{supp}\varphi \cap \{0,T\} \ne \emptyset$. It can be assumed without loss of generality that $\operatorname{supp}\varphi \subset (-T,T)$, $0 \in \operatorname{supp}\varphi$. Then for every $m \in \mathbb{N}$ choose a function $\chi_m \in C^1(\mathbb{R})$ such that: 1) $|\chi_m(t)| \le 1$, $|\chi_m'(t)| \le 2m$, and $\chi_m(-t) = \chi_m(t)$ if $t \in \mathbb{R}$; 2) $\chi_m(t) = 1$ if $t \in (-\infty,-2/m)\cup(2/m,+\infty)$; 3) $\chi_m(t) = 0$ if $t \in (-1/m,1/m)$ (for example, we may put $\chi_m(t) = \frac{1}{2}\sin(\pi m(t-\frac{3}{2m}))+\frac{1}{2}$ if $t \in (\frac{1}{m},\frac{2}{m})$). It is understood that for every $t \in \mathbb{R}\backslash\{0\}$ we have $\chi_m(t) \underset{m\to+\infty}{\longrightarrow} 1$. Therefore equality (4.13) is fulfilled for $v \in \mathbb{V}_p^b$ and with φ instead of $\chi_m\varphi$, where $m \in \mathbb{N}$. The simple transformations yield

$$\int\limits_{-T}^{T}\int\limits_{\Omega}\Big\{\sum_{i=1}^{n}\widehat{g}_i\, v_{x_i}\,\varphi + \widehat{g}_0\, v\,\varphi - b\widehat{w}\,v\,\varphi'\Big\}\chi_m\,\mathrm{d}x\,\mathrm{d}t - \int\limits_{-2/m}^{2/m}\int\limits_{\Omega} b\widehat{w}\,v\,\varphi\,\chi_m'\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{4.14}$$

where $m \in \mathbb{N}$, $v \in \mathbb{V}_p^b$, $\varphi \in C_0^1(-T,2T)$, $\operatorname{supp}\varphi \subset (-T,T)$.

Consider the second term of the left-hand side of (4.14). Notice that $\varphi(t)-\varphi(-t) = 2\varphi'(\xi(t))\,t$, where $t > 0$ and $\xi(t)$ is some number between $-t$ and t. Then after simple transformations we obtain

$$\int\limits_{-2/m}^{2/m}\int\limits_{\Omega} b\widehat{w}\,v\,\varphi\,\chi_m'\,\mathrm{d}x\,\mathrm{d}t = 2\int\limits_{1/m}^{2/m}\int\limits_{\Omega} t\,\chi_m'(t)\,\varphi'(\xi(t))b(x)w(x,t)\,v(x)\,\mathrm{d}x\,\mathrm{d}t. \tag{4.15}$$

Note that

$$|t\,\chi_m'(t)\,\varphi'(\xi(t))\,b(x)w(x,t)\,v(x)\,| \le C_7|b(x)w(x,t)\,v(x)|, \tag{4.16}$$

$(x,t) \in \Omega \times (0,T)$, where $C_7 > 0$ is some independent of m constant, and the right-hand side of the inequality (4.16) belongs to $L_1(Q)$. From (4.15) by (4.16) we have

$$\int\limits_{-2/m}^{2/m} \int\limits_{\Omega} b\widehat{w}\, v\, \varphi\, \chi'_m \,\mathrm{d}x\,\mathrm{d}t \underset{m\to+\infty}{\longrightarrow} 0. \tag{4.17}$$

Letting $m \to +\infty$ in (4.14) and taking into account (4.17) and the Dominated Convergence Theorem (see [15, p. 648]) we obtain (4.13).

Now let $\{\omega_\rho \mid \rho > 0\}$ be the functions from the beginning of this subsection. Choose a number $k_0 \in \mathbb{N}$ such that $1/k_0 < T/2$. By definition, for each $k \geq k_0$ put

$$\widehat{w}_k(x,\tau) := \int\limits_{\mathbb{R}} \widehat{w}(x,t)\, \omega_{1/k}(t-\tau)\,\mathrm{d}t,$$

$$\widehat{g}_{i,k}(x,\tau) := \int\limits_{\mathbb{R}} \widehat{g}_i(x,t)\, \omega_{1/k}(t-\tau)\ \mathrm{d}t, \quad i \in \{0,\dots,n\},$$

for a.e. $x \in \Omega$ and for every $\tau \in [-T/2, T]$.

According to Lemma 3.1, we have

$$\widehat{w}_k \underset{k\to\infty}{\longrightarrow} \widehat{w} \quad \text{in} \quad L_{p_0(\cdot)}\big(\Omega \times (-T/2,T)\big), \tag{4.18}$$

$$\widehat{w}_{k,x_i} \underset{k\to\infty}{\longrightarrow} \widehat{w}_{x_i} \quad \text{in} \quad L_{p_i(\cdot)}\big(\Omega \times (-T/2,T)\big), \quad i = \overline{1,n}, \tag{4.19}$$

$$b^{1/2}\widehat{w}_k \underset{k\to\infty}{\longrightarrow} b^{1/2}\widehat{w} \quad \text{in} \quad L_2\big(\Omega \times (-T/2,T)\big), \tag{4.20}$$

$$\widehat{g}_{i,k} \underset{k\to\infty}{\longrightarrow} \widehat{g}_i \quad \text{in} \quad L_{p'_i(\cdot)}\big(\Omega \times (-T/2,T)\big), \quad i = \overline{0,n}. \tag{4.21}$$

Note that $b^{1/2}\widehat{w}_k \in C([-T/2,T]; L^2(\Omega))$, $k \geq k_0$.

For each $\tau \in [T/2, T]$, $k \geq k_0$, substituting $\omega_{1/k}(\cdot - \tau)$ instead of $\varphi(\cdot)$ in (4.13), and using the simple transformations, we get

$$\int\limits_{\Omega} \bigg\{ b(x)\frac{\partial}{\partial\tau}\widehat{w}_k(x,\tau)\, v(x) + \sum_{i=1}^{n} \widehat{g}_{i,k}(x,\tau)\, v_{x_i}(x) + \widehat{g}_{0,k}(x,\tau)\, v(x) \bigg\}\,\mathrm{d}x = 0. \tag{4.22}$$

Let $k, l \in \mathbb{N}$ be arbitrary numbers such that $k, l \geq k_0$. Put $\widehat{w}_{kl} := \widehat{w}_k - \widehat{w}_l$, $\widehat{g}_{i,kl} := \widehat{g}_{i,k} - \widehat{g}_{i,l}$ $(i = \overline{0,n})$. The difference between (4.22) and the same equality with $k = l$ equals

$$\int\limits_{\Omega} \big\{ b(x)\frac{\partial}{\partial\tau}\widehat{w}_{kl}(x,\tau)\, v(x) + \sum_{i=1}^{n} \widehat{g}_{i,kl}(x,\tau)\, v_{x_i}(x) + \widehat{g}_{0,kl}(x,\tau)\, v(x) \big\}\,\mathrm{d}x = 0, \tag{4.23}$$

where $v \in \mathbb{V}_p^b$, $\tau \in [-T/2, T]$.

Take a function $\theta \in C^1(\mathbb{R})$. For every $\tau \in [-T/2, T]$ the functions $\widehat{w}_{kl}(\cdot,\tau)\,\theta(\tau)$ belongs to $v \in \mathbb{V}_p^b$. Substituting $\widehat{w}_{kl}(\cdot,\tau)\,\theta(\tau)$ instead of $v(\cdot)$ in (4.23), integrating the obtained equality over $\tau \in (t_1, t_2)$ $(-T/2 \le t_1 < t_2 \le T)$, we have

$$\frac{1}{2}\int\limits_{\Omega} b(x)|\widehat{w}_{kl}(x,\tau)|^2\theta(\tau)\Big|_{\tau=t_1}^{\tau=t_2}\,\mathrm{d}x - \frac{1}{2}\int\limits_{t_1}^{t_2}\int\limits_{\Omega} b(x)|\widehat{w}_{kl}(x,\tau)|^2\theta'(\tau)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+\int\limits_{t_1}^{t_2}\int\limits_{\Omega}\Big\{\sum_{i=1}^{n}\widehat{g}_{i,kl}(x,\tau)(\widehat{w}_{kl}(x,\tau))_{x_i}+\widehat{g}_{0,kl}(x,\tau)\widehat{w}_{kl}(x,\tau)\Big\}\,\theta(\tau)\,\mathrm{d}x\,\mathrm{d}\tau=0. \qquad (4.24)$$

Now suppose

$$0 \le \theta(\tau) \le 1 \quad \text{if} \quad \tau \in \mathbb{R}, \qquad \theta(\tau) = 0 \quad \text{if} \quad \tau \le -T/2,$$
$$\theta(\tau) = 1 \quad \text{if} \quad \tau \ge 0, \qquad |\theta'(\tau)| \le 4/T \quad \text{if} \quad \tau \in [-T/2,\, 0].$$

Taking $t_1 = -T/2$ and $t_2 = t \in [0, T]$ in (4.24) we obtain

$$\max_{t\in[0,T]}\int\limits_{\Omega} b(x)|\widehat{w}_{kl}(x,t)|^2\,\mathrm{d}x \le \frac{4}{T}\int\limits_{-T/2}^{0}\int\limits_{\Omega} b(x)|\widehat{w}_{kl}(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau$$

$$+2\int\limits_{-T/2}^{T}\int\limits_{\Omega}\Big\{\sum_{i=1}^{n}|\widehat{g}_{i,kl}(x,\tau)|\,|(\widehat{w}_{kl}(x,\tau))_{x_i}|+|\widehat{g}_{0,kl}(x,\tau)|\,|\widehat{w}_{kl}(x,\tau)|\Big\}\,\mathrm{d}x\,\mathrm{d}\tau. \qquad (4.25)$$

From (4.25), taking into account (4.18)–(4.21), we get

$$b^{1/2}\,\widehat{w}_{k,l} \underset{k,\,l\to+\infty}{\longrightarrow} 0 \quad \text{in} \quad C([0,T]; L_2(\Omega)).$$

This yields that $\{b^{1/2}\widehat{w}_k\}_{k=1}^{\infty}$ is a Cauchy sequence in the space $C([0,T];\, L_2(\Omega))$ and

$$b^{1/2}\widehat{w}_k \underset{k\to+\infty}{\longrightarrow} b^{1/2}\widehat{w} \quad \text{in} \quad C([0,T]; L_2(\Omega)). \qquad (4.26)$$

Hence $b^{1/2}w \in C([0,T]; L_2(\Omega))$.

Take an arbitrary function $\theta \in C^1([0,T])$, and take any points $t_1, t_2 \in [0,T]$ such that $t_1 < t_2$. Multiplying (4.22) by $\theta(\tau)$ and integrating the obtained equality over $\tau \in [t_1, t_2]$ we get

$$\int\limits_{t_1}^{t_2}\int\limits_{\Omega} b(x)\left[\frac{\partial}{\partial\tau}\,\widehat{w}_k(x,\tau)\right] v(x)\,\theta(\tau)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+\int\limits_{t_1}^{t_2}\int\limits_{\Omega}\Big\{\sum_{i=1}^{n}\widehat{g}_{i,k}(x,\tau)v_{x_i}(x)\theta(\tau)+\widehat{g}_{0,k}(x,\tau)v(x)\,\theta(\tau)\Big\}\,\mathrm{d}x\,\mathrm{d}\tau = 0. \qquad (4.27)$$

Using the formula of integration by parts for first term in the left-hand side of (4.27), and letting $k \to +\infty$ in obtained identity, in view of (4.21), (4.26), we get (4.11).

For each $\tau \in [T/2, T]$, $k \geq k_0$, substituting in (4.22) $\widehat{w}_k(\cdot,\tau)\theta(\tau)$ instead of $v(\cdot)$, we integrate this equality over $\tau \in (t_1, t_2)$. Similarly to (4.24) we get

$$\frac{1}{2}\int\limits_{\Omega} b(x)|\widehat{w}_k(x,\tau)|^2\theta(\tau)\Big|_{\tau=t_1}^{\tau=t_2} \mathrm{d}x - \frac{1}{2}\int\limits_{t_1}^{t_2}\int\limits_{\Omega} b(x)|\widehat{w}_k(x,\tau)|^2\theta'(\tau)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+\int\limits_{t_1}^{t_2}\int\limits_{\Omega}\Big\{\sum_{i=1}^{n}\widehat{g}_{i,k}(x,\tau)(\widehat{w}_k(x,\tau))_{x_i} + \widehat{g}_{0,k}(x,\tau)\widehat{w}_k(x,\tau)\Big\}\,\theta(\tau)\,\mathrm{d}x\,\mathrm{d}\tau = 0. \tag{4.28}$$

Letting $k \to +\infty$ in (4.28), and using (4.18)–(4.21), (4.26) we get (4.12). □

5 Proof of the main results

For every functions $w \in L_1(Q)$ such that $w_{x_1}, \ldots, w_{x_n} \in L_1(Q)$ we denote

$$a_j(w)(x,t) := a_j(x,t,w(x,t),\nabla w(x,t)), \quad (x,t) \in Q, \quad j = \overline{0,n}.$$

Proof of Theorem 2.1. Suppose that problem (3.1)–(3.3) has two weak solutions u_1 and u_2. Consider the difference between (3.6) with $u = u_2$ and (3.6) with $u = u_1$. By Lemma 3.2 with $w = u_1 - u_2$, $\theta \equiv 1$, $t_1 = 0$, $t_2 = \tau \in (0,T]$, we get (see (4.12))

$$\frac{1}{2}\int\limits_{\Omega} b(x)w^2(x,\tau)\,\mathrm{d}x + \int\limits_{0}^{\tau}\int\limits_{\Omega}\Big\{\sum_{i=1}^{n}(a_i(u_1) - a_i(u_2))(u_{1,x_i} - u_{2,x_i})$$
$$+ (a_0(u_1) - a_0(u_2))(u_1 - u_2)\Big\}\,\mathrm{d}x\,\mathrm{d}t = 0, \quad \tau \in (0,T].$$

This equality and $(\mathcal{A}3)$ yield $b(x)w^2(x,t) = 0$ and

$$\sum_{i=1}^{n}(a_i(u_1) - a_i(u_2))(u_{1,x_i} - u_{2,x_i}) + (a_0(u_1) - a_0(u_2))(u_1 - u_2) = 0$$

for a.e. $(x,t) \in Q$. The first equality implies that $w(x,t) = 0$ for a.e. $(x,t) \in G$, where $G := \big\{(x,t) \mid b(x) > 0,\ t \in (0,T)\big\}$. The second equality and condition $(\mathcal{A}3^*)$ imply that $w(x,t) = 0$ for a.e. $(x,t) \in Q\backslash G$. Therefore, $w(x,t) = 0$ for a.e. $(x,t) \in Q$. □

Proof of Theorem 2.2. We use Galerkin's method. Let $\{w_j \mid j \in \mathbb{N}\}$ be a linear independent set of the functions from $\widetilde{C}^1(\overline{\Omega})$ whose finite linear combinations are dense in $\mathbb{V}_p^b$, and, additionally, the finite linear combinations of the functions $\{b^{1/2}w_j \mid j \in \mathbb{N}\}$ are dense in $\mathbb{H}^b$. Then we take a sequence $\{u_{0,m} = \sum_{k=1}^{m}\alpha_k^m w_k\}_{m=1}^{\infty}$ of the finite linear combinations of the functions $\{w_j \mid j \in \mathbb{N}\}$ such that

$$||u_0 - b^{1/2}u_{0,m}||_{L_2(\Omega)} \underset{m\to\infty}{\longrightarrow} 0. \tag{5.1}$$

Notice that for every $\eta \in [0,1]$ and for a.e. $x \in \Omega$ we have

$$|b^{1/2}(x) - (b(x)+\eta)^{1/2}|^2 |u_{0,m}(x)|^2 \le 4(b(x)+1)|u_{0,m}(x)|^2.$$

Then, taking into account the Dominated Convergence Theorem (see [15, p. 648]), for every $m \in \mathbb{N}$ we get

$$||b^{1/2}u_{0,m} - (b+\eta)^{1/2}u_{0,m}||_{L_2(\Omega)} \underset{\eta\to+0}{\longrightarrow} 0.$$

Therefore there exists a sequence of positive numbers $\{\eta_m\}_{m=1}^{\infty}$ such that $\eta_m \underset{m\to\infty}{\longrightarrow} 0$, and

$$||b^{1/2}u_{0,m} - (b+\eta_m)^{1/2}u_{0,m}||_{L_2(\Omega)} \underset{m\to\infty}{\longrightarrow} 0. \tag{5.2}$$

Put by definition

$$b_m(x) := b(x) + \eta_m, \quad m \in \mathbb{N}, \quad x \in \Omega. \tag{5.3}$$

Therefore, taking into account (5.1) and (5.2), we have

$$||u_0 - b_m^{1/2}u_{0,m}||_{L_2(\Omega)} \underset{m\to\infty}{\longrightarrow} 0. \tag{5.4}$$

According to Galerkin's method, for every $m \in \mathbb{N}$ we put

$$u_m(x,t) = \sum_{k=1}^{m} c_{m,k}(t) w_k(x), \quad (x,t) \in Q,$$

where $c_{m,1},\dots,c_{m,m}$ are solutions of the Cauchy problem for the system of ordinary differential equations

$$\int_\Omega b_m u_{m,t} w_j \,\mathrm{d}x + \int_\Omega \Big\{ \sum_{i=1}^{n} (a_i(u_m) - f_i) w_{j,x_i} + (a_0(u_0) - f_0) w_j \Big\} \,\mathrm{d}x = 0, \quad j = \overline{1,m}, \tag{5.5}$$

$$u_m|_{t=0} = u_{0,m}. \tag{5.6}$$

The linear independence of functions $w_1,\dots,w_m$ yields that the matrix $(a^m_{k,j})^m_{k,j=1}$ is positive-definite, where $a^m_{k,j} = \int_\Omega b_m w_k w_j \,\mathrm{d}x$ $(k,j=\overline{1,m})$. Thus the system of ordinary differential equations (5.5) can be transformed to the normal form. Hence, according to the theorems of existence and extension of the solution to this problem (see [14]), we get the global solution $c_{1,m}, \dots.., c_{m,m}$ of problem (5.5), (5.6). This solution is defined on the interval $[0, T_m\rangle$, where $T_m \le T$. Here the braces "$\rangle$" means either ")" or "]". Further we will get the estimates that imply the equality $[0,T_m\rangle = [0,T]$.

Multiply the equation of system (5.5) with number $j \in \{1,\dots,m\}$ by $c_{m,j}$ and sum over $j \in \{1,\dots,m\}$. Integrating the obtained equality over $t \in [0,\tau] \subset [0,T_m\rangle$, and using the integration-by-parts formula, we have

$$\frac{1}{2}\int_\Omega b_m(x)|u_m(x,\tau)|^2\,\mathrm{d}x-\frac{1}{2}\int_\Omega b_m(x)|u_{0,m}(x)|^2\,\mathrm{d}x$$
$$+\int_0^\tau\int_\Omega\Big\{\sum_{i=1}^n a_i(u_m)u_{m,x_i}+a_0(u_m)u_m\Big\}\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_0^\tau\int_\Omega\Big\{\sum_{i=1}^n f_iu_{m,x_i}+f_0u_m\Big\}\,\mathrm{d}x\,\mathrm{d}t,\quad \tau\in(0,T_m\rangle. \tag{5.7}$$

Further we need Young's inequality in the form

$$ab\le\varepsilon|a|^{r(x)}+\varepsilon^{-\frac{1}{r^--1}}|b|^{r'(x)},\quad a,b\in\mathbb{R},\quad q>1,\quad 0<\varepsilon<1, \tag{5.8}$$

for a.e. $x\in\Omega$, where $r\in L^\infty(\Omega)$, $r(x)>1$, $r'(x):=r(x)/(r(x)-1)$ for a.e. $x\in\Omega$, $r^-:=\operatorname*{ess\,inf}_{x\in\Omega}r(x)$.

Take an arbitrary value $\varepsilon\in(0,1)$. From (5.7), using condition $(\mathcal{A}4)$ and inequality (5.8) with small enough $\varepsilon\in(0,1)$ (for example, $\varepsilon=\frac{1}{2}\min\{1,K_1\}>0$), we get

$$\int_\Omega b_m(x)|u_m(x,\tau)|^2\,\mathrm{d}x+K_1\int_0^\tau\int_\Omega\Big\{\sum_{i=1}^n|u_{m,x_i}(x,t)|^{p_i(x)}$$
$$+|u_m(x,t)|^{p_0(x)}\Big\}\,\mathrm{d}x\,\mathrm{d}t\le C_8\int_0^\tau\int_\Omega\sum_{i=0}^n|f_i(x,t)|^{p_i'(x)}\,\mathrm{d}x\,\mathrm{d}t$$

$$+2\int_0^\tau\int_\Omega g(x,t)\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega b_m(x)|u_{0,m}(x)|^2\,\mathrm{d}x,\quad\tau\in(0,T_m\rangle. \tag{5.9}$$

From (5.4) it follows that the sequence $\{\int_\Omega b_m(x)u_{0,m}^2(x)\,\mathrm{d}x\}_{m=1}^\infty$ is bounded. Hence from (5.9) we get the following estimates

$$\int_\Omega b_m(x)u_m^2(x,\tau)\ \mathrm{d}x\le C_9, \tag{5.10}$$

$$\int_0^\tau\int_\Omega\Big\{\sum_{i=1}^n|u_{m,x_i}(x,t)|^{p_i(x)}+|u_m(x,t)|^{p_0(x)}\Big\}\ \mathrm{d}x\,\mathrm{d}t\le C_{10}, \tag{5.11}$$

where $C_9,\ C_{10}>0$ are independent of m,T_m. Estimate (5.10) implies that there exists an independent of T_m constant that bounds the functions $c_{m,1},\dots,c_{m,m}$ on $[0,T_m\rangle$. Thus $[0,T_m\rangle=[0,T]$.

Condition $(\mathcal{A}2)$ and estimates (5.11) yield

$$\iint_Q|a_i(u_m)(x,t)|^{p_i'(x)}\,\mathrm{d}x\,\mathrm{d}t\le C_{11},\quad i=\overline{0,n}, \tag{5.12}$$

where $C_{11} > 0$ is independent of m.

Since the spaces $L_{p_i(\cdot)}(Q)$, $L_{p_i'(\cdot)}(Q)$ $(i = \overline{0,n})$ are reflexive (see [22, p. 600]), and estimates (5.10)-(5.12) hold, we obtain the existence of subsequence (we call it $\{u_m\}_{m\in\mathbb{N}}$ again), the functions $v_* \in L_2(\Omega)$, $\widetilde{u} \in L_\infty(0,T;L_2(\Omega))$, $u \in L_{p_0(\cdot)}(Q)$, and $\chi_i \in L_{p_i'(\cdot)}(Q)$ $(i = \overline{0,n})$ such that $u_{x_i} \in L_{p_i(\cdot)}(Q)$ $(i = \overline{1,n})$, and

$$b_m^{1/2}(\cdot)u_m(\cdot,T) \underset{m\to\infty}{\longrightarrow} v_*(\cdot) \quad \text{weakly in} \quad L_2(\Omega), \tag{5.13}$$

$$b_m^{1/2}u_m \underset{m\to\infty}{\longrightarrow} \widetilde{u} \quad *-\text{weakly in} \quad L_\infty(0,T;L_2(\Omega)), \tag{5.14}$$

$$u_m \underset{m\to\infty}{\longrightarrow} u \quad \text{weakly in} \quad \widetilde{W}^{1,0}_{p(\cdot)}(Q), \tag{5.15}$$

$$a_i(u_m) \underset{m\to\infty}{\longrightarrow} \chi_i \;\; \text{weakly in} \;\; L_{p_i'(\cdot)}(Q) \quad (i = \overline{1,n}). \tag{5.16}$$

Let us prove that u is a weak solution of problem (3.1)–(3.3). First note that

$$b_m^{1/2} \underset{m\to\infty}{\longrightarrow} b^{1/2} \quad \text{strongly in} \quad L_2(\Omega) \quad \text{and almost everywhere on} \quad \Omega. \tag{5.17}$$

Now let us prove that

$$\widetilde{u}(x,t) = b^{1/2}(x)u(x,t) \quad \text{for a.e.} \quad (x,t) \in Q. \tag{5.18}$$

Indeed, take a function $\psi \in C(\overline{Q})$. Then (5.14) yields that

$$\iint\limits_Q b_m^{1/2}u_m\psi \,\mathrm{d}x\,\mathrm{d}t \underset{m\to\infty}{\longrightarrow} \iint\limits_Q \widetilde{u}\psi \,\mathrm{d}x\,\mathrm{d}t. \tag{5.19}$$

Taking into account (5.17) and the Dominated Convergence Theorem (see [15, p. 648]) it is easy to show that $b_m^{1/2}\psi \underset{m\to\infty}{\longrightarrow} b^{1/2}\psi$ in $L_{p_0'(\cdot)}(Q)$. Hence, by (5.15) we obtain

$$\iint\limits_Q u_m b_m^{1/2}\psi \,\mathrm{d}x\,\mathrm{d}t \underset{m\to\infty}{\longrightarrow} \iint\limits_Q u b^{1/2}\psi \,\mathrm{d}x\,\mathrm{d}t. \tag{5.20}$$

Relations (5.19), (5.20) imply that for every $\psi \in C(\overline{Q})$ the equality

$$\iint\limits_Q \widetilde{u}\psi \,\mathrm{d}x\,\mathrm{d}t = \iint\limits_Q b^{1/2}u\psi \,\mathrm{d}x\,\mathrm{d}t$$

holds, i.e. equality (5.18) is true.

Fix the numbers $j, m \in \mathbb{N}$ such that $m \ge j$. Multiplying the equation of system (5.5) with number j by the function $\theta \in C^1([0,T])$ we integrate the obtained equality in $t \in [0,T]$. Letting $m \to \infty$, and taking into account (5.4), (5.6), (5.13)–(5.18), we get

$$\theta(T)\int\limits_Q b^{1/2}(x)v_*(x)w_j(x)\,\mathrm{d}x - \theta(0)\int\limits_Q b^{1/2}(x)u_0(x)w_j(x)\,\mathrm{d}x$$
$$-\iint\limits_Q buw_j\theta' \,\mathrm{d}x\,\mathrm{d}t + \iint\limits_Q \Big\{\sum_{i=1}^n (\chi_i - f_i)w_{j,x_i} + (\chi_0 - f_0)w_j\Big\}\theta \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{5.21}$$

This equality yields that for every $v \in \mathbb{V}_p^b$ and $\theta \in C^1([0,T])$ the equality

$$\theta(T)\int\limits_{\Omega} b^{1/2}(x)v_*(x)v(x)\ \mathrm{d}x - \theta(0)\int\limits_{\Omega} b^{1/2}(x)u_0(x)v(x)\ \mathrm{d}x$$
$$-\iint\limits_{Q} buv\theta'\,\mathrm{d}x\,\mathrm{d}t + \iint\limits_{Q}\Big\{\sum_{i=1}^{n}(\chi_i - f_i)v_{x_i} + (\chi_0 - f_0)v\Big\}\theta\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{5.22}$$

holds.

Notice that if we take $\theta = \varphi \in C_0^1(0,T)$ in (5.22) then for every $v \in \mathbb{V}_p^b$ and $\varphi \in C_0^1(0,T)$ we have the equality

$$\iint\limits_{Q}\Big\{\sum_{i=1}^{n}(\chi_i - f_i)v_{x_i}\varphi + (\chi_0 - f_0)v\varphi - buv\varphi'\Big\}\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{5.23}$$

According to Lemma 3.2, (5.23) implies that

$$b^{1/2}u \in C([0,T];L_2(\Omega)) \tag{5.24}$$

and for every $v \in \mathbb{V}_p^b$ and $\theta \in C^1([0,T])$ the equality

$$\theta(T)\int\limits_{\Omega} b(x)u(x,T)v(x)\ \mathrm{d}x - \theta(0)\int\limits_{\Omega} b(x)u(x,0)v(x)\ \mathrm{d}x$$
$$-\iint\limits_{Q} buv\theta'\,\mathrm{d}x\,\mathrm{d}t + \iint\limits_{Q}\Big\{\sum_{i=1}^{n}(\chi_i - f_i)v_{x_i} + (\chi_0 - f_0)v\Big\}\theta\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{5.25}$$

holds.

From (5.22) and (5.25) we get

$$b^{1/2}(x)u(x,0) = u_0(x), \quad b^{1/2}(x)u(x,T) = v_*(x) \quad \text{for a.e.} \quad x \in \Omega. \tag{5.26}$$

In view of (5.15) and (5.24) we conclude that $u \in \mathbb{U}_p^b$. First equality from (5.26) implies (3.5). According to (5.23) to prove (3.6) it is enough to show that the equality

$$\iint\limits_{Q}\Big\{\sum_{i=1}^{n}\chi_i v_{x_i}\varphi + \chi_0 v\varphi\Big\}\,\mathrm{d}x\,\mathrm{d}t = \iint\limits_{Q}\Big\{\sum_{i=1}^{n}a_i(u)v_{x_i}\varphi + a_0(u)v\varphi\Big\}\,\mathrm{d}x\,\mathrm{d}t \tag{5.27}$$

is valid for every $v \in \mathbb{V}_p^b$ and $\varphi \in C_0^1(0,T)$. For this we use the monotonicity method (see [23]). Take an arbitrary function $w \in \widetilde{W}_{p(\cdot)}^{1,0}(Q)$. Using condition $(\mathcal{A}3)$ for every $m \in \mathbb{N}$ we have

$$W_m := \iint\limits_{Q}\Big\{\sum_{i=1}^{n}(a_i(u_m) - a_i(w))(u_{m,x_i} - w_{x_i})$$
$$+ (a_0(u_m) - a_0(w))(u_m - w)\Big\}\,\mathrm{d}x\,\mathrm{d}t \geq 0.$$

Hence,

$$W_m = \iint\limits_Q \Big\{ \sum_{i=1}^n a_i(u_m)u_{m,x_i} + a_0(u_m)u_m \Big\} \,\mathrm{d}x\,\mathrm{d}t - \iint\limits_Q \Big\{ \sum_{i=1}^n \big[a_i(u_m)w_{x_i} + a_i(w)(u_{m,x_i} - w_{x_i})\big] + a_0(u_m)w + a_0(w)(u_m - w)\Big\} \,\mathrm{d}x\,\mathrm{d}t \geq 0, \quad m \in \mathbb{N}. \tag{5.28}$$

From (5.28), using (5.7) with $\tau = T$, we obtain

$$W_m = \iint\limits_Q \Big\{ \sum_{i=1}^n f_i u_{m,x_i} + f_0 u_m \Big\} \,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int\limits_\Omega b_m(x)|u_m(x,T)|^2 \,\mathrm{d}x + \frac{1}{2}\int\limits_\Omega b_m(x)|u_{0,m}(x)|^2 \,\mathrm{d}x - \iint\limits_Q \Big\{ \sum_{i=1}^n \big[a_i(u_m)w_{x_i} + a_i(w)(u_{m,x_i} - w_{x_i})\big] + a_0(u_m)w + a_0(w)(u_m - w)\Big\} \,\mathrm{d}x\,\mathrm{d}t \geq 0, \tag{5.29}$$

where $m \in \mathbb{N}$.

Taking into account (5.13) and the second equality of (5.26) we have

$$\liminf_{m\to\infty} ||b_m^{1/2}(\cdot)u_m(\cdot,T)||_{L_2(\Omega)} \geq ||b^{1/2}(\cdot)u(\cdot,T)||_{L_2(\Omega)}. \tag{5.30}$$

By (5.4), (5.15), (5.16), (5.30), from (5.29) we get

$$0 \leq \limsup_{m\to\infty} W_m \leq \iint\limits_Q \Big\{ \sum_{i=1}^n f_i u_{x_i} + f_0 u \Big\} \,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int\limits_\Omega b(x)|u(x,T)|^2 \,\mathrm{d}x + \frac{1}{2}\int\limits_\Omega |u_0(x)|^2 \,\mathrm{d}x - \iint\limits_Q \Big\{ \sum_{i=1}^n \big[\chi_i w_{x_i} + a_i(w)(u_{x_i} - w_{x_i})\big] + \chi_0 w + a_0(w)(u - w)\Big\} \,\mathrm{d}x\,\mathrm{d}t. \tag{5.31}$$

From (5.23), using Lemma 3.2 with $\theta \equiv 1$ and first equality of (5.26), we obtain

$$\iint\limits_Q \Big\{ \sum_{i=1}^n \chi_i u_{x_i} + \chi_0 u \Big\} \,\mathrm{d}x\,\mathrm{d}t = \iint\limits_Q \Big\{ \sum_{i=1}^n f_i u_{x_i} + f_0 u \Big\} \,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int\limits_\Omega b(x)|u(x,T)|^2 \,\mathrm{d}x + \frac{1}{2}\int\limits_\Omega |u_0(x)|^2 \,\mathrm{d}x. \tag{5.32}$$

Thus, (5.31) and (5.32) imply that

$$\iint\limits_Q \Big\{ \sum_{i=1}^n (\chi_i - a_i(w))(u_{x_i} - w_{x_i}) + (\chi_0 - a_0(w))(u - w) \Big\} \,\mathrm{d}x\,\mathrm{d}t \geq 0. \tag{5.33}$$

Using the standard monotonous method, we get (5.27). Therefore u is a weak solution of problem (3.1)–(3.3).

Finally let us prove estimate (3.7). Take arbitrary weak solution u to problem (3.1)–(3.3). From (3.6), using Lemma 2 with $\theta \equiv 1$, $t_1 = 0$, $t_2 = \tau \in (0, T]$ (see (4.12)), we get

$$\frac{1}{2}\int\limits_{\Omega} b(x)|u(x,\tau)|^2 \, \mathrm{d}x + \int\limits_0^{\tau}\int\limits_{\Omega} \Big\{ \sum_{i=1}^{n} a_i(u)u_{x_i} + a_0(u)u \Big\} \, \mathrm{d}x\,\mathrm{d}t$$
$$= \int\limits_0^{\tau}\int\limits_{\Omega} \Big\{ \sum_{i=1}^{n} f_i u_{x_i} + f_0 u \Big\} \, \mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega} b(x)|u_0(x)|^2 \, \mathrm{d}x.$$

From this similar as to show inequality (5.9), taking into account ($\mathcal{A}4$), (5.8), we obtain (3.7). □

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