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NEW MULTI-ORDER EXACT SOLUTIONS FOR A CLASS OF NONLINEAR EVOLUTION EQUATIONS

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Abstract. In this article multi-order exact solutions of a generalized shallow water wave equation are sought in addition to those belonging to a class of nonlinear systems such as the KdV, modified KdV, Boussinesq, Klein-Gordon and modified Benjamin-Bona-Mahony equation. A modified version of a generalized Lame equation is employed within a perturbative framework and the solutions are identified order by order in terms of Jacobi elliptic functions. The multi-order exact solutions turn out to be new and hold the key feature that they are expressible in terms of an auxiliary function f in a generic way. For appropriate choices of f the previous results reported in the literature are recovered.

Keywords: Travelling waves; Lamé equation; Lamé function; Jacobi elliptic function; Elliptic integral function.

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1 Introduction

Seeking tractable solutions of nonlinear evolution equations has been the focus of intense study for the past several decades (see, *e.g.*, [1]-[11]). In this regard many strategies have been employed that include the Hirota bilinear method [2], homogeneous balance method [3], trigonometric method [4], the hyperbolic method [5], the Jacobi elliptic function method [6] and iterative method [7, 8]. On the other hand, perturbative techniques have also been used to extract multi-order exact periodic solutions based on Lamé equation and Jacobi elliptic functions [9, 10, 11]. In this note we propose a generalized class of Lamé equation in which an auxiliary function $f(\xi)$ is present whose role is to consider correlations between it and the perurbatively reduced nonlinear evolution equation for different orders of the perturbative parameter.

A generalized Lamé equation is expressed as

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2} + \left[\lambda - (n+1)(n+2)a_1 a_2 (f(\xi))^2\right] y = 0, \tag{1.1}$$

where λ stands for an eigenvalue, n is a positive integer and we restrict the auxiliary function $f(\xi)$ to obey an elliptic equation

$$\left(\frac{\mathrm{d}f}{\mathrm{d}\xi}\right)^2 = \left(1 + a_1 f^2\right)\left(a_2 f^2 + a_3\right). \tag{1.2}$$

In (1.1) and (1.2), $a_1,\ a_2,\ a_3$ are real constants. It is well known that equation (1.2) admits of several categories of solutions for different values of $a_1,\ a_2,\ a_3$ all expressible in terms of modulus k of the Jacobi elliptic functions. For later use we summarize a few relevant ones in Table 1.

Table 1: $f(\xi)$ is provided along with the values of the parameters $a_j\,, j = 1, 2, 3$. The complementary modulus is denoted by $k'^2 = 1 - k^2$ [12].

a_1	a_2	a_3	$f(\xi)$
-1	$-k^2$	1	$sn(\xi,k)$
-1	k^2	k'^2	$cn(\xi,k)$
-1	1	$-k'^2$	$dn(\xi,k)$
-1	$-k'^2$	$-k^2$	$nc(\xi,k),\ \textit{i.e.},\ \frac{1}{cn(\xi,k)}$
$-k'^2$	1	-1	$nd(\xi,k),\ \textit{i.e.},\ \frac{1}{dn(\xi,k)}$
1	k'^2	1	$sc(\xi,k),\ \textit{i.e.},\ \frac{sn(\xi,k)}{cn(\xi,k)}$
-1	$-k^2$	1	$cd(\xi,k),\ \textit{i.e.},\ \frac{cn(\xi,k)}{dn(\xi,k)}$
-1	-1	k^2	$dc(\xi,k),\ \textit{i.e.},\ \frac{\mathrm{d}n(\xi,k)}{cn(\xi,k)}$
k^2	$-k'^2$	1	$sd(\xi,k),\ \textit{i.e.},\ \frac{sn(\xi,k)}{dn(\xi,k)}$

It should be mentioned that while we would attend to the enlisted set of $f(\xi)$ in Table 1 to widen the scope of enquiry, some particular cases of $f(\xi)$, namely $sn(\xi,k),\ dn(\xi,k)$ and $cd(\xi,k)$, have been studied in the literature [9, 10, 11] to search for solutions of the relevant PDEs by applying the perturbation techniques on the latter. However, as will be clear from Table 1 there exist other variants of $f(\xi)$ which open up different cases of new solutions not only for the already considered PDEs but other types as well by perturbatively reducing them to an ODE form so as to match with (1.1).

We have selected a physically meaningful model of a generalized shallow water wave equation [13, 14, 15, 16] which is amenable to a perturbative treatment by looking for a travelling wave variable $u(x,t) = u(\xi)$ with $\xi = \gamma(x - ct)$ where γ is the wave number and c is the wave speed. We discuss the procedure of obtaining the solutions in section 2. In section 3 we take up applicability of our scheme to other nonlinear evolution equations namely, the Korteweg de Vries (KdV) equation, modified KdV equation, Boussinesq equation, Klein-Gordon equation and modified Benjamin-Bona-Mahony (mBBM) equation with a view to tracking down new unexplored multi-order solutions. The main point of our analysis is to demonstrate that all such solutions can be written down in a generic form in terms of the function f.

To make equation (1.1) accessible in terms of known Lamé functions, we recast it in terms of a new variable η by applying the transformation $f(\xi) = \sqrt{\eta}$. Using (1.2) this gives

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\eta^2} + \frac{1}{2}\left[\frac{1}{\eta} + \frac{1}{\eta + \frac{1}{a_1}} + \frac{1}{\eta + \mu}\right]\frac{\mathrm{d}y}{\mathrm{d}\eta} - \frac{\nu + (n+1)(n+2)\eta}{4\eta(\eta + \frac{1}{a_1})(\eta + \mu)} y = 0, \tag{1.3}$$

where the parameters μ and ν are defined by $\mu = \frac{a_3}{a_2}$ and $\nu = -\frac{\lambda}{a_1 a_2}$.

Equation (1.3) is readily solvable for the special cases of $n = 1$ and 2 for different choices of μ and ν subject to certain relation between them. The solutions expressible by the corresponding Lamé functions are [9]

$$n = 1 \qquad L_1^I(\xi) = (1 + a_1\eta)^{\frac{1}{2}}(a_3 + a_2\eta)^{\frac{1}{2}} = (1 + a_1 f^2)^{\frac{1}{2}}(a_3 + a_2 f^2)^{\frac{1}{2}}, \tag{1.4}$$
$$\lambda = -(a_2 + a_1 a_3) \quad [\text{provided } \nu = (\mu + \frac{1}{a_1})].$$

$$n = 1 \qquad L_1^{II}(\xi) = \eta^{\frac{1}{2}}(1 + a_1\eta)^{\frac{1}{2}} = f(1 + a_1 f^2)^{\frac{1}{2}}, \tag{1.5}$$
$$\lambda = -(a_2 + 4a_1 a_3) \quad [\text{provided } \nu = (4\mu + \frac{1}{a_1})].$$

$$n = 1 \qquad L_1^{III}(\xi) = \eta^{\frac{1}{2}}(a_3 + a_2\eta)^{\frac{1}{2}} = f(a_3 + a_2 f^2)^{\frac{1}{2}}, \tag{1.6}$$
$$\lambda = -(4a_2 + a_1 a_3) \quad [\text{provided } \nu = (\mu + \frac{4}{a_1})].$$

$$n = 2 \quad L_2(\xi) = \eta^{\frac{1}{2}}(1 + a_1\eta)^{\frac{1}{2}}(a_3 + a_2\eta)^{\frac{1}{2}} = f(1 + a_1 f^2)^{\frac{1}{2}}(a_3 + a_2 f^2)^{\frac{1}{2}}, \tag{1.7}$$
$$\lambda = -4(a_2 + a_1 a_3) \quad [\text{provided } \nu = 4(\mu + \frac{1}{a_1})].$$

Henceforth we will be guided by the solutions (1.4) - (1.7) along with an appropriate f from Table 1 for the various multi-order cases that follow from a nonlinear PDE.

2 A generalized shallow water wave equation

We turn attention to a generalized class of shallow water wave (GSWW) equation given by [13, 14, 15, 16]

$$u_{xxxt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0, \tag{2.1}$$

where $\alpha,\ \beta \in \mathbb{R} - \{0\}$. Equation (2.1) can be derived from a classical study of water waves under the so-called Boussinesq approximation. More interestingly, there also follows various classical and non-classical reductions from GSWW such as the KdV and BBM equations. Investigations of Painlev$\acute{e}$ tests reveal complete integrability for specific values of α and β namely $\alpha = \beta$ or $\alpha = 2\beta$ [14]. Recently we extended [16] the Jacobian elliptic function method to classify new exact travelling wave solutions expressible in terms of quasi-periodic elliptic integral function and doubly-periodic Jacobian elliptic functions.

For the travelling wave solutions $u = u(\xi)$, equation (2.1) can be reduced in terms of the variable $v(\xi) \equiv \frac{\mathrm{d}u}{\mathrm{d}\xi}$ to the ODE form

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} + Pv^2 + Qv + c_1 = 0. \tag{2.2}$$

where $P = \frac{\alpha+\beta}{2\gamma}$, $Q = \frac{1-c}{\gamma^2 c}$ and c_1 is an integrating constant.

To tackle (2.2) perturbatively we set

$$v(\xi) = v_0(\xi) + \epsilon v_1(\xi) + \epsilon^2 v_2(\xi) + \cdots \tag{2.3}$$

where $\epsilon(> 0)$ is a small parameter and $v_0(\xi),\ v_1(\xi),\ v_2(\xi), ...$ represent various multi-order solutions like the zeroth-order, first-order, second-order solutions etc. of equation (2.2). Accordingly we can write $u(\xi)$ as

$$u(\xi) = u_0(\xi) + \epsilon u_1(\xi) + \epsilon^2 u_2(\xi) + \cdots \tag{2.4}$$

where $u_i(\xi) = \int_0^\xi v_i(\tau)\, d\tau$.

Substituting the series (2.3) in equation (2.2) we obtain for each power of ϵ the corresponding equations

$$\epsilon^0 \quad : \quad \frac{\mathrm{d}^2 v_0}{\mathrm{d}\xi^2} + Pv_0^2 + Qv_0 + c_1 = 0\,, \tag{2.5}$$

$$\epsilon^1 \quad : \quad \frac{\mathrm{d}^2 v_1}{\mathrm{d}\xi^2} + (2Pv_0 + Q)v_1 = 0\,, \tag{2.6}$$

$$\epsilon^2 \quad : \quad \frac{\mathrm{d}^2 v_2}{\mathrm{d}\xi^2} + (2Pv_0 + Q)v_2 = -Pv_1^2\,, \tag{2.7}$$

and so on.

We now proceed to solve the above chain of equations by first expanding v_0 namely,

$$v_0 = \sum_{i=0}^{l} A_i f^i, \tag{2.8}$$

where A_i's are constants and then comparing highest order linear and nonlinear terms in (2.5). In this way we obtain $l = 2$ by making use of (1.2). Thus (2.8) gets reduced to the quadratic form

$$v_0 = A_0 + A_1 f + A_2 f^2. \tag{2.9}$$

2.1 Zeroth-order exact solution

With v_0 given by (2.9) we are led to the following system of coupled equations:

$$\begin{aligned}
2a_3A_2 + PA_0^2 + QA_0 + c_1 &= 0, && (2.10)\\
A_1[(a_2 + a_1a_3) + 2PA_0 + Q] &= 0\,, && (2.11)\\
4(a_2 + a_1a_3)A_2 + P(A_1^2 + 2A_0A_2) + QA_2 &= 0\,, && (2.12)\\
A_1[a_1a_2 + PA_2] &= 0\,, && (2.13)\\
A_2[6a_1a_2 + PA_2] &= 0. && (2.14)
\end{aligned}$$

It is readily seen that we generate the following set of consistent solutions $A_0 = -\frac{1}{2P}[Q + 4(a_2 + a_1a_3)]$, $A_1 = 0$ and $A_2 = -\frac{6a_1a_2}{P}$ along with the constraint

$$[16(a_2 + a_1a_3)^2 - 48a_1a_2a_3] - Q^2 + 4Pc_1 = 0. \quad (2.15)$$

From (2.15) we obtain wave speed as

$$c = [1 \pm 4\gamma^2\{(a_2 + a_1a_3)^2 - 3a_1a_2a_3 + \frac{\alpha + \beta}{8\gamma}c_1\}^{\frac{1}{2}}]^{-1} \quad (2.16)$$

where the two signs signal the two directions. Note that in order to have c real we can always adjust the integration constant c_1 to keep $(a_2 + a_1a_3)^2 + \frac{\alpha+\beta}{8\gamma}c_1 > 3a_1a_2a_3$. Further for the finiteness of c we require $(a_2 + a_1a_3)^2 + \frac{\alpha+\beta}{8\gamma}c_1 \neq 3a_1a_2a_3 + \frac{1}{16\gamma^4}$.

From (2.5) and (2.9) we get for the first integral

$$v_0 = \frac{\mathrm{d}u_0}{\mathrm{d}\xi} = -\frac{1}{2P}[Q + 4(a_2 + a_1a_3)] - \frac{6a_1a_2}{P}f^2 \quad (2.17)$$

which in turn gives the zeroth-order solution

$$u_0 = -\frac{1}{2P}[Q + 4(a_2 + a_1a_3)]\xi - \frac{6a_1a_2}{P}\int f^2\, d\xi. \quad (2.18)$$

As is evident, u_0 depends upon the choice of the auxiliary function f. In Table 2 we furnish the various forms for u_0 corresponding to different elliptic functions of Table 1. A sample graph of

Table 2: Wave velocity $c = \frac{1}{1\pm4\gamma^2\sqrt{1-k^2k'^2+\frac{\alpha+\beta}{8\gamma}c_1}}$, $E(\phi, k)$ is the incomplete elliptic integral function of second kind where $sin\phi = sn(\xi, k)$ and $\Lambda = -\frac{1}{2P}[Q - 4(1 + k^2)]$.

$f(\xi)$	Zeroth-order solution u_0
$sn(\xi, k)$ or $cn(\xi, k)$ or $dn(\xi, k)$	$\Lambda\xi + \frac{6}{P}E(\phi, k)$
$nc(\xi, k)$ or $sc(\xi, k)$ or $dc(\xi, k)$	$\Lambda\xi + \frac{6}{P}[E(\phi, k) - sn(\xi, k)dc(\xi, k)]$
$nd(\xi, k)$ or $sd(\xi, k)$ or $cd(\xi, k)$	$\Lambda\xi + \frac{6}{P}[E(\phi, k) - k^2sn(\xi, k)cd(\xi, k)]$

the zeroth-order solution u_0 for $f = sn(\xi, k)$ is depicted in Figure 1 and a plot of wave-speed for variation of the constant c_1 is shown in Figure 2.

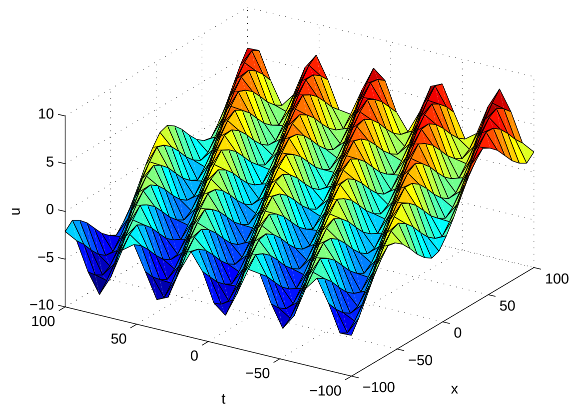

Figure 1: Zeroth-order exact solution corresponding to $f(\xi) = sn(\xi, k)$ for $k = 0.5,\ \alpha = 1,\ \beta = 4,\ \gamma = 1$

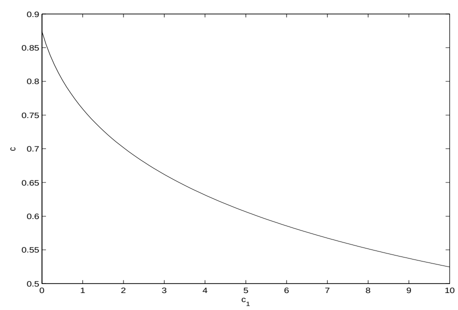

Figure 2: Wave speed c vs constant of integration c_1 for $f(\xi) = sn(\xi, k)$ for $k = 0.5,\ \alpha = 1,\ \beta = 4,\ \gamma = 1$

2.2 First-order exact solution

Knowing v_0 from (2.17), equation (2.6) reads

$$\frac{\mathrm{d}^2 v_1}{\mathrm{d}\xi^2} + [-4(a_2 + a_1 a_3) - 12 a_1 a_2 f^2] v_1 = 0, \tag{2.19}$$

which matches with (1.1) for $\lambda = -4(a_2 + a_1 a_3)$. This gives $n = 2$ implying that the solution of (2.19) can be written as

$$v_1 = \frac{\mathrm{d}u_1}{\mathrm{d}\xi} = c_2 L_2(\xi) = c_2 f (1 + a_1 f^2)^{\frac{1}{2}} (1 + \frac{a_2}{a_3} f^2)^{\frac{1}{2}}, \tag{2.20}$$

where c_2 is an arbitrary constant.

Integration of (2.20) with the use of (1.2) immediately provides the result

$$u_1 = \pm \frac{c_2}{2} f^2 \tag{2.21}$$

which serves as a first order approximation to equation (2.1). Figure 3 gives a plot of u_1 for $f = sn(\xi, k)$.

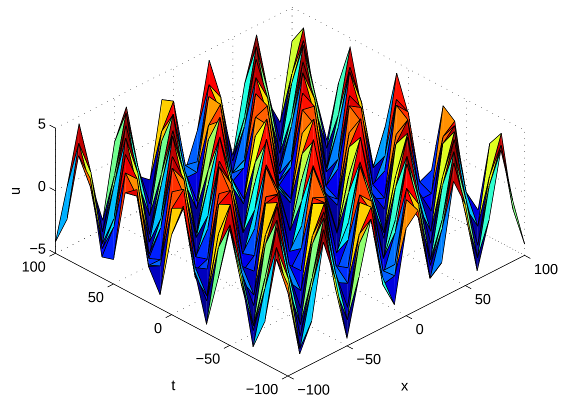

Figure 3: First-order exact solution corresponding to $f(\xi) = sn(\xi, k)$ for $k = 0.5,\ \alpha = 1,\ \beta = 4,\ \gamma = 1,\ c_2 = 1$

2.3 Second-order exact solution

Using (2.17) and (2.20) the second-order equation (2.7) takes the form

$$\frac{\mathrm{d}^2 v_2}{\mathrm{d}\xi^2} + [-4(a_2 + a_1 a_3) - 12 a_1 a_2 f^2] v_2 = -P c_2^2 f^2 (1 + a_1 f^2)(a_3 + a_2 f^2). \tag{2.22}$$

We observe that the coefficients of (2.22) are polynomials in f. This prompts us to take a polynomial ansatz for v_2 namely

$$v_2 = \sum_{i=0}^{l} B_i f^i. \tag{2.23}$$

Substitution of the above into (2.22) we get $l = 4$ on equating the highest power of f from both sides. Thus (2.23) is reduced to the biquadratic form

$$v_2 = B_0 + B_1 f + B_2 f^2 + B_3 f^3 + B_4 f^4. \tag{2.24}$$

Putting (2.24) into (2.22) and equating the coefficients of f^i $(i = 0$ to $6)$ to zero, we have the following system of coupled equations,

$$\begin{aligned} a_3 B_2 - 2(a_2 + a_1 a_3) B_0 &= 0, \\ (a_2 + a_1 a_3) B_1 - 2 a_3 B_3 &= 0, \\ 12 a_3 B_4 - 12 a_1 a_2 B_0 + P c_2^2 a_3 &= 0, \\ (a_2 + a_1 a_3) B_3 - 2 a_1 a_2 B_1 &= 0, \\ -6 a_1 a_2 B_2 + 12(a_2 + a_1 a_3) B_4 + P c_2^2 (a_2 + a_1 a_3) &= 0, \\ a_1 a_2 B_3 &= 0, \\ 8 a_1 a_2 B_4 + P c_2^2 a_1 a_2 &= 0. \end{aligned} \tag{2.25}$$

Solving for $B_0,\ B_1,\ B_2,\ B_3,\ B_4$ we obtain

$$B_0 = -\frac{a_3 c_2^2 P}{24 a_1 a_2},\ B_1 = 0,\ B_2 = -\frac{(a_2 + a_1 a_3) c_2^2 P}{12 a_1 a_2},\ B_3 = 0,\ B_4 = -\frac{c_2^2 P}{8}. \tag{2.26}$$

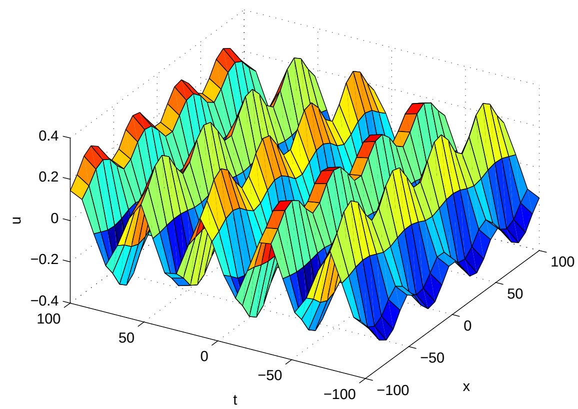

Figure 4: Second-order exact solution corresponding to $f(\xi) = sn(\xi, k)$ for $k = 0.5,\ \alpha = 1,\ \beta = 4,\ \gamma = 1,\ c_2 = 1$

Hence the solution of equation (2.22) can be written as

$$v_2 = \frac{\mathrm{d}u_2}{\mathrm{d}\xi} = -\frac{c_2^2 P}{24a_1a_2}[a_3 + 2(a_2 + a_1a_3)f^2 + 3a_1a_2f^4] \tag{2.27}$$

Using (1.2) we can integrate (2.27) to get

$$u_2 = \pm\frac{c_2^2 P}{24a_1a_2} f(1 + a_1f^2)^{\frac{1}{2}}(a_3 + a_2f^2)^{\frac{1}{2}}. \tag{2.28}$$

u_2 is plotted for $f = sn(\xi, k)$ in Figure 4.

3 Application to other nonlinear evolution equations

In the previous section, we considered a nonlinear second-order ODE containing a functional parameter f and obtained specific solutions in terms of f. It was shown that the use of Jacobi elliptic function method in a perturbative way yielded new multi-order exact solutions that depended upon f in a generic way. In this section we show that our approach can be profitably applied to other nonlinear equations resulting in more general types of solutions that have not been reported to the best of our knowledge. A complete list of our results are presented in Table 3.

(i) **KdV equation**

The concerned equation is of the form

$$u_t + \alpha u u_x + \beta u_{xxx} = 0 \tag{3.1}$$

where α and β are real parameters. Let us choose $f = cd(\xi, k)$ for which $a_1 = -1,\ a_2 = -k^2$ and $a_3 = 1$. Substituting these values into the zeroth-order, first-order and second-order exact solutions listed in Table 3,

Table 3: New general multi-order exact solutions for various non-linear evolution equations

Equation	Multi-order exact solutions $u_n,\ n=0,1,2$
(i) KdV equation $u_t+\alpha uu_x+\beta u_{xxx}=0$	$u_0=\frac{c}{\alpha}-\frac{4\beta\gamma^2}{\alpha}[(a_2+a_1a_3)+3a_1a_2f^2]$, [where $c=\pm\beta\gamma^2\{16(a_2+a_1a_3)^2-48a_1a_2a_3+2\frac{\alpha}{\beta\gamma^2}c_1\}^{\frac{1}{2}}$] $u_1=c_2f(1+a_1f^2)^{\frac{1}{2}}(a_3+a_2f^2)^{\frac{1}{2}}$ $u_2=-\frac{c_2^2\alpha}{48a_1a_2\beta\gamma^2}[a_3+2(a_2+a_1a_3)f^2+3a_1a_2f^4]$
(ii) mKdV equation $u_t+\alpha u^2u_x+\beta u_{xxx}=0$	$u_0=\pm\sqrt{-\frac{6\beta\gamma^2a_1a_2}{\alpha}}\ f$, [where $c=\beta\gamma^2(a_2+a_1a_3)$] $u_1=c_2(1+a_1f^2)^{\frac{1}{2}}(a_3+a_2f^2)^{\frac{1}{2}}$ $u_2=\pm\frac{c_2^2}{2}\sqrt{-\frac{\alpha}{6\beta\gamma^2a_1a_2}}\ f[(a_2+a_1a_3)+2a_1a_2f^2]$
(iii) Boussinesq equation $u_{tt}-c_0^2u_{xx}-\alpha u_{xxxx}-\beta(u^2)_{xx}=0$	$u_0=-\frac{1}{2\beta}[c_0^2-c^2+4\alpha\gamma^2(a_2+a_1a_3)]-\frac{6\alpha\gamma^2}{\beta}a_1a_2f^2$, [where $c=\{c_0^2\pm4\alpha\gamma^2\sqrt{(a_2+a_1a_3)^2-3a_1a_2a_3}\}^{\frac{1}{2}}$] $u_1=c_2f(1+a_1f^2)^{\frac{1}{2}}(a_3+a_2f^2)^{\frac{1}{2}}$ $u_2=-\frac{c_2^2\beta}{24a_1a_2\alpha\gamma^2}[a_3+2(a_2+a_1a_3)f^2+3a_1a_2f^4]$
(iv) Klein-Gordon equation $u_{tt}-u_{xx}+\alpha u+\beta u^3=0$	$u_0=\pm\sqrt{-\frac{2(c^2-1)\gamma^2a_1a_2}{\beta}}\ f$, [where $c=\pm\{1-\frac{\alpha}{\gamma^2(a_2+a_1a_3)}\}^{\frac{1}{2}}$] $u_1=c_2(1+a_1f^2)^{\frac{1}{2}}(a_3+a_2f^2)^{\frac{1}{2}}$ $u_2=\pm\frac{c_2^2}{2}\sqrt{-\frac{\beta}{2(c^2-1)\gamma^2a_1a_2}}\ f[(a_2+a_1a_3)+2a_1a_2f^2]$
(v) mBBM equation $u_t+c_0u_x+u^2u_x+\beta u_{xxt}=0$	$u_0=\pm\sqrt{6\beta\gamma^2ca_1a_2}\ f$, [where $c=\frac{c_0}{1+(a_2+a_1a_3)\beta\gamma^2}$] $u_1=c_2(1+a_1f^2)^{\frac{1}{2}}(a_3+a_2f^2)^{\frac{1}{2}}$ $u_2=\pm\frac{c_2^2}{2}\sqrt{\frac{1}{6\beta\gamma^2ca_1a_2}}\ f[(a_2+a_1a_3)+2a_1a_2f^2]$

we find

$$\begin{aligned}
u_0 &= \frac{c}{\alpha}+\frac{4\beta\gamma^2}{\alpha}(1+k^2)-\frac{12\beta\gamma^2}{\alpha}k^2cd^2(\xi,k)\\
u_1 &= c_2k'^2cd(\xi,k)sd(\xi,k)nd(\xi,k)\\
u_2 &= -\frac{c_2^2\alpha}{48\beta\gamma^2k^2}[1-2(1+k^2)cd^2(\xi,k)+3k^2cd^4(\xi,k)].
\end{aligned} \tag{3.2}$$

These solutions have been obtained in [10] for $\alpha=1$ where the authors used the notations m and k in places of k and γ respectively and their scale is $A=k'^2c_2,\ c_1=0$. The KdV equation has also been studied in [9] and their results can be extracted from ours for $f=sn(\xi,k)$ corresponding to $\alpha=1$.

(ii) **mKdV equation**

$$u_t+\alpha u^2u_x+\beta u_{xxx}=0. \tag{3.3}$$

Our general result listed in Table 3 contains, as special cases, the following results reported in earlier works.

- $f=sn(\xi,k)$ (cf. [9])

$$\begin{aligned} u_0 &= \pm\gamma k\sqrt{-\frac{6\beta}{\alpha}}\, sn(\xi,k) \\ u_1 &= c_2\, cn(\xi,k)dn(\xi,k) \\ u_2 &= \pm\frac{(1+k^2)c_2^2}{12\gamma k}\sqrt{-\frac{6\alpha}{\beta}}\, sn(\xi,k)[\frac{2k^2}{1+k^2}\, sn^2(\xi,k)-1] \end{aligned} \tag{3.4}$$

- $f = cd(\xi,k)$ (cf. [10])

$$\begin{aligned} u_0 &= \pm\gamma k\sqrt{-\frac{6\beta}{\alpha}}\, cd(\xi,k) \\ u_1 &= (1-k^2)c_2\, sd(\xi,k)nd(\xi,k) \\ u_2 &= \pm\frac{(1+k^2)c_2^2}{12\gamma k}\sqrt{-\frac{6\alpha}{\beta}}\, cd(\xi,k)[\frac{2k^2}{1+k^2}\, cd^2(\xi,k)-1] \end{aligned} \tag{3.5}$$

(iii) **Boussinesq equation**

$$u_{tt} - c_0^2 u_{xx} - \alpha u_{xxxx} - \beta(u^2)_{xx} = 0. \tag{3.6}$$

From the general solution given in terms of f (see Table 3) one can deduce the ones obtained in [9] as special case for $f = sn(\xi,k)$.

(iv) **Klein-Gordon equation**

$$u_{tt} - u_{xx} + \alpha u + \beta u^3 = 0. \tag{3.7}$$

The above equation was considered in [11]. Their results correspond to the case $f = dn(\xi,k)$ and we do indeed recover their solutions from Table 3 as given below:

$$\begin{aligned} u_0 &= \pm\sqrt{\frac{2\alpha}{\beta\gamma^2(k^2-2)}}\, dn(\xi,k) \\ u_1 &= k^2 c_2\, sn(\xi,k)cn(\xi,k) \\ u_2 &= \mp\frac{c_2^2}{2}\sqrt{\frac{\beta}{2\gamma^2(c^2-1)}}\, dn(\xi,k)[2-k^2-2dn^2(\xi,k)] \end{aligned} \tag{3.8}$$

(v) **mBBM equation**

$$u_t + c_0 u_x + u^2 u_x + \beta u_{xxt} = 0. \tag{3.9}$$

For $f = sn(\xi,k)$ the multi-order solutions to (3.9) read from Table 3

$$\begin{aligned} u_0 &= \pm\gamma\sqrt{6c\beta}\, sn(\xi,k) \\ u_1 &= c_2\, cn(\xi,k)dn(\xi,k) \\ u_2 &= \mp\frac{c_2^2(1+k^2)}{12\gamma k}\sqrt{\frac{6}{c\beta}}\, sn(\xi,k)[12-\frac{2k^2}{1+k^2}sn^2(\xi,k)]. \end{aligned} \tag{3.10}$$

These results were obtained in the paper [9].

4 Concluding remarks

In this paper we have derived exact multi-order periodic solutions for a generalized shallow water wave equation in a perturbative framework which are based on various types of Jacobi elliptic functions. The presence of an auxiliary function in our scheme, whose particular form makes reference to any particular Jacobi function explicit, can be suitably chosen to connect with other classes of PDE such as the KdV, modified KdV, Boussinesq, Klein-Gordon and modified Benjamin-Bona-Mahony equation. The general character of our results encompasses those special cases which have earlier been studied in the literature.

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