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STABILITY OF A QUADRATIC FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN $\mathcal{L}$–FUZZY n–NORMED SPACES

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Abstract. In this paper, we prove the Hyers–Ulam–Rassias stability of the following version of a quadratic functional equation in non-Archimedean $\mathcal{L}$–fuzzy n–normed spaces

$$f(l\,x+y)+f(l\,x-y)=2\,l^2\,f(x)+2\,f(y)\ .$$

Keywords: Non-Archimedean fuzzy n–normed space; quadratic functional equation; stability.

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1 Introduction

Fuzzy notion as first introduced by Zadeh [20] has been widely involved in different subjects of mathematics. In Zadeh's definition, a fuzzy set is characterized by a function from a nonempty set X to $[0,1]$. Goguen generalized the notion of a fuzzy subset of X to an $\mathcal{L}$–fuzzy subset, namely a function from X to a lattice L [6].

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Fuzzy set theory is a powerful tool for modelling uncertainty and vagueness in various problems arising in the field of science and engineering.

Gähler [5] introduced an attractive theory of 2-norm and n–norm on a linear space. A systematic development of an n–normed linear space has been extensively made by Kim and Cho [11], A. Misiak [13] and Gunawan [7]. Gunawan and Mashadi [7] gave a simple way to derive an $(n-1)$–norm from the n–norm and realized that any n–normed space is an $(n-1)$–normed space. A detailed theory of fuzzy normed linear space can be found in [2, 3, 4]. Narayanan and Vijayabalaji [14] have extended the notion of n–normed linear space to fuzzy n–normed linear space.

Stability problem of a functional equation was first posed in [19] which was answered in [8] and then generalized in [1, 15] for additive mappings and linear mappings respectively. Since then, several stability problems for various functional equations have been investigated in [16, 9, 10, 12].

Several results for the Hyers–Ulam–Rassias stability of many functional equations have been proved by several researchers. Our goal is to determine some stability results concerning the functional equation

$$f(l\,x+y)+f(l\,x-y)=2\,l^2\,f(x)+2\,f(y)$$

in non-Archmedian $\mathcal{L}$–fuzzy n–normed spaces.

2 Preliminaries

In this section, we provide a collection of definitions and related results which are essential and used in the next discussions.

Definition 2.1 *([5]). Let X be a real vector space of dimension greater than 1 and let $\|\bullet,\bullet\|$ be a real-valued function on $X\times X$ satisfying the following conditions:*

(1) $\|x,y\|=0$ if any only if x and y are linearly dependent,

(2) $\|x,y\|=\|y,x\|$,

(3) $\|\alpha x,y\|=|\alpha|\,\|x,y\|$, where $\alpha\in\mathbb{R}$,

(4) $\|x,y+z\|\le\|x,y\|+\|x,z\|$.

Then, $\|\bullet,\bullet\|$ is called a 2-norm on X and the pair $(X,\|\bullet,\bullet\|)$ is called a 2-normed linear space.

Definition 2.2 *([7]). Let $n\in\mathbb{N}$ and X be a real linear space of dimension $d\ge n$. (Here we allow d to be infinite.) A real valued function $\|\bullet,...,\bullet\|$ on $\underbrace{X\times\cdots\times X}_{n}=X^n$ satisfying the following four properties:*

(1) $\|x_1,x_2,\dots,x_n\|=0$ if any only if $x_1,x_2,\dots x_n$ are linearly dependent,

(2) $\|x_1,x_2,\dots,x_n\|$ is invariant under any permutation of the arguments,

(3) $\|x_1,x_2,\dots,\alpha\,x_n\|=|\alpha|\,\|x_1,x_2,\dots,x_n\|$, for any $\alpha\in\mathbb{R}$,

(4) $\|x_1,x_2,\dots x_{n-1},y+z\|\le\|x_1,x_2,\dots,x_{n-1},y\|+\|x_1,x_2,\dots,x_{n-1},z\|$

is called an n–norm on X and the pair $(X, \|\bullet, \ldots, \bullet\|)$ is called an n–normed linear space.

Definition 2.3 *([7]). A sequence (x_k) in an n–normed linear space $(X, \|\bullet, \ldots, \bullet\|)$ is said to be convergent to $x \in X$ (in the n–norm) if for any $a_1, .., a_{n-1} \in X$,*

$$\lim_{k\to\infty} \|a_1, a_2, \ldots, a_{n-1}, x_k - x\| = 0 \,.$$

Definition 2.4 *([7]). A sequence (x_k) in an n–normed linear space $(X, \|\bullet, \ldots, \bullet\|)$ is called a Cauchy sequence if for any $a_1, .., a_{n-1} \in X$,*

$$\lim_{k,m\to\infty} \|a_1, a_2, \ldots, a_{n-1}, x_k - x_m\| = 0 \,.$$

Definition 2.5 *([7]). An n–normed linear space is said to be complete if every Cauchy sequence is convergent.*

Definition 2.6 *([14]). Let X be a linear space over a field $\mathbb{F}$. A fuzzy subset N of $X^n \times \mathbb{R}$ is called a fuzzy n–norm on X if and only if:*

(N1) For all $t \in \mathbb{R}$ with $t \leq 0$, $N(x_1, x_2, \ldots, x_n, t) = 0$,

(N2) For all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, \ldots, x_n, t) = 1$ if and only $x_1, x_2, \ldots x_n$ are linearly dependent,

(N3) $N(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$,

(N4) For all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$, if $c \neq 0$, $c \in \mathbb{F}$,

(N5) For all $s, t \in \mathbb{R}$,

$$N(x_1, x_2, \ldots, x_n + x'_n, s + t) \geq \min\{N(x_1, x_2, \ldots, x_n, s), N(x_1, x_2, \ldots, x'_n, t)\} \,,$$

(N6) $N(x_1, x_2, \ldots, x_n, t)$ is a non-decreasing function of $t \in \mathbb{R}$ and

$$\lim_{t\to\infty} N(x_1, x_2, \ldots, x_n, t) = 1 \,.$$

Then (X, N) is called a fuzzy n–normed linear space or in short f-n-NLS.

Example 2.7 *Let $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ be an n–normed space. Define*

$$\forall t \in \mathbb{R},\ (x_1, x_2, \ldots, x_n) \in X^n :\ N(x_1, x_2, \ldots, x_n, t) = \begin{cases} \frac{t}{t + \|x_1, x_2, \ldots, x_n\|} & t > 0, \\ 0 & t < 0 \end{cases}$$

Then (X, N) is an f-n-NLS.

Definition 2.8 *Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice and let U be a non-empty set called the universe. An $\mathcal{L}$–fuzzy set in U is defined as a mapping $\mathcal{A} : U \to L$. For each u in U, $\mathcal{A}(u)$ represents the degree (in L) to which u is an element of $\mathcal{A}$.*

Definition 2.9 *A t–norm on $\mathcal{L}$ is a mapping $*_L : L^2 \to L$ satisfying the following conditions:*

(i) $\forall x \in L \ : \ x *_L 1_{\mathcal{L}} = x$ *(boundary condition),*

(ii) $\forall (x,y) \in L^2 \ : \ x *_L y = y *_L x$ *(commutativity),*

(iii) $\forall (x,y,z) \in L^3 \ : \ x *_L (y *_L z) = (x *_L y) *_L z$ *(associativity),*

(iv) $\forall (x,y,x',y') \in L^4 \ : \ x \leq_L x'$ *and* $y \leq_L y' \Rightarrow x *_L y \leq_L x' *_L y')$ *(monotonicity).*

Definition 2.10 *A t–norm $*_L$ on $\mathcal{L}$ is said to be continuous if, for any $x, y \in \mathcal{L}$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y, respectively, $\lim_{n\to\infty}(x_n *_L y_n) = x *_L y$.*

Definition 2.11 *The triple $(V, \mathcal{P}, *_L)$ is said to be an $\mathcal{L}$–fuzzy n–normed space if V is vector space, $*_L$ is a continuous t–norm on $\mathcal{L}$ and $\mathcal{P}$ is an $\mathcal{L}$–fuzzy set on $V^n \times (0,\infty)$ satisfying the following conditions:*

For all $x_1, x_2, \dots, x_n, x'_n \in V$ and $t, s \in (0,\infty)$:

(a) $\mathcal{P}(x_1, x_2, \dots x_n, t) >_L 0_{\mathcal{L}}$,

(b) $\mathcal{P}(x_1, x_2, \dots x_n, t) = 1_{\mathcal{L}}$ *if and only if* $x_1, x_2, \dots x_n$ *are linearly dependent,*

(c) $\mathcal{P}(x_1, x_2, \dots x_n, t)$ *is invariant under any permutation of* $x_1, x_2, \dots x_n$,

(d) $\mathcal{P}(x_1, x_2, \dots, \alpha\, x_n, t) = \mathcal{P}(x_1, x_2, \dots, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$,

(e) $\mathcal{P}(x_1, x_2, \dots x_n, t) *_L \mathcal{P}(x_1, x_2, \dots x'_n, s) \leq_L \mathcal{P}(x_1, x_2, \dots, x_n + x'_n, t+s)$,

(f) $\mathcal{P}(x_1, x_2, \dots x_n, \bullet) : (0,\infty) \to L$ *is continuous and* $\lim_{t\to\infty}\mathcal{P}(x,t) = 1_{\mathcal{L}}$.

In this case, $\mathcal{P}$ is called an $\mathcal{L}$–fuzzy n–norm.

Definition 2.12 *[17]. A negator on $\mathcal{L}$ is any decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$.*

Definition 2.13 *If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L$, then $\mathcal{N}$ is called an involutive negator.*

In this paper, the involutive negator $\mathcal{N}$ is fixed.

Definition 2.14 *A sequence (x_k) in an $\mathcal{L}$–fuzzy n–normed space $(V, \mathcal{P}, *_L)$ is called a Cauchy sequence if, for each $\varepsilon \in L - \{0_{\mathcal{L}}\}$ and $t > 0$ and any $a_1, .., a_{n-1} \in X$, there exists $n_0 \in \mathbb{N}$ such that, for all $k, m \geq n_0$, $\mathcal{P}(a_1, a_2, \dots, a_{n-1}, x_k - x_m, t) >_L \mathcal{N}(\varepsilon)$, where $\mathcal{N}$ is a negator on $\mathcal{L}$.*

A sequence (x_k) is said to be convergent to $x \in V$ in the $\mathcal{L}$–fuzzy n–normed space $(V, \mathcal{P}, *_L)$ if $\mathcal{P}(a_1, a_2, \dots a_{n-1}, x_k - x, t) \to 1_{\mathcal{L}}$, whenever $n \to +\infty$ for all $t > 0$ and $a_1, .., a_{n-1} \in X$.

An $\mathcal{L}$–fuzzy n–normed space $(V, \mathcal{P}, *_L)$ is said to be complete if and only if every Cauchy sequence in V is convergent.

Definition 2.15 *[17]. Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|.| : \mathbb{K} \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have:*

(1) $|a| \geq 0$*, and equality holds if and only if* $a = 0$*,*

(2) $|ab| = |a|\,|b|$*,*

(3) $|a + b| \leq \max\{|a|, |b|\}$*.*

Note that $|n| \leq 1$ for each integer n. We always assume, in addition that $|.|$ is non-tivial, i.e., there exists an $a_0 \in \mathbb{K}$ such that $|a_0| \neq 0, 1$.

3 Stability of quadratic functional equation in $\mathcal{L}$–fuzzy n–normed spaces

Let $\mathbb{K}$ be a non-Archimedean field, X a vector space over $\mathbb{K}$ and $(Y, \mathcal{P}, *_L)$ a non-Archimedean $\mathcal{L}$–fuzzy n-Banach space over $\mathbb{K}$.

In this section we investigate the quadratic functional equation. We define an $\mathcal{L}$–fuzzy approximately quadratic mapping.

Definition 3.1 *Let* Ψ *be an* $\mathcal{L}$*–fuzzy set on* $\underbrace{X \times \ldots \times X}_{n} \times [0, \infty)$ *such that* $\Psi(x_1, x_2, \ldots x_n, \bullet)$ *is nondecreasing,*

$$\Psi(cx_1, cx_2, \ldots cx_n, t) \geq_L \Psi(x_1, x_2, \ldots, x_n, \frac{t}{|c|}), \ \ \forall x_1, x_2, \ldots, x_n \in X, \ c \neq 0$$

and

$$\lim_{t\to\infty} \Psi(x_1, x_2, \ldots, x_n, t) = 1_{\mathcal{L}}, \ \ \forall x_1, x_2, \ldots, x_n \in X, \ \ t > 0.$$

A mapping $f : X \to Y$ *is said to be* Ψ*–approximately quadratic in non-Archimedean* $\mathcal{L}$*–fuzzy* n*–normed space if*

$$\mathcal{P}(f(l\,x + y) + f(l\,x - y) - 2\,l^2 f(x) - 2\,f(x\,y), x_2, \ldots x_n, t) \geq_L \Psi(x, x_2, \ldots x_n, t) \tag{3.1}$$

$$\forall x, y, x_2, \ldots x_n \in X, t > 0$$

Throughout this paper, we denote $a_1 *_L a_2 *_L \ldots *_L a_n$ by $\prod_{j=1}^{n} a_j$.

Theorem 3.2 *Let* $\mathbb{K}$ *be a non-Archimedean field,* X *a vector space over* $\mathbb{K}$ *and* $(Y, \mathcal{P}, *_L)$ *a non-Archimedean* $\mathcal{L}$*–fuzzy* n*–normed space over* $\mathbb{K}$*. Let* $f : X \to Y$ *be a* Ψ*–approximately quadratic mapping in non-Archimedean* $\mathcal{L}$*–fuzzy* n*–normed space. If there exist an* $\alpha \in \mathbb{R}$ $(\alpha > 0)$ *and an integer* k*,* $k \geq 2$ *with* $|l^k| < \alpha$ *and* $|l| \neq 1$ *and* $l \neq 0$ *such that*

$$\Psi(l^{-k}x, l^{-k}x_2, \ldots, l^{-k}x_n, t) \geq_L \Psi(x, x_2, \ldots x_n, \alpha t), \tag{3.2}$$

and

$$\lim_{n\to\infty} \prod_{j=n}^{\infty} \mathcal{M}(x, x_2, \ldots, x_n, \frac{\alpha^j t}{|l|^{kj}}) = 1_{\mathcal{L}},$$

then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mathcal{P}(f(x) - Q(x), x_2, \dots x_n, t) \geq_L \prod_{i=1}^{\infty} \mathcal{M}(x, x_2, \dots x_n, \frac{\alpha^{i+1} t}{|l|^{ki}}), \quad \forall x_i \in X,\ t > 0, \tag{3.3}$$

where

$$\mathcal{M}(x, x_2, \dots x_n, t) = \mathcal{M}_k(x, x_2, \dots x_n, t) := \Psi(x, x_2, x_3, \dots x_n, t) *_L \Psi(lx, x_2, x_3, \dots x_n, t) *_L \dots *_L \Psi(l^{k-1}x, x_2, x_3, \dots, x_n, t)\,.$$

Proof. First we show by induction on j that for $x, x_2, ..., x_n \in X, t > 0$ and $j \geq 1$,

$$\mathcal{P}(f(l^j x) - l^{2j} f(x), x_2, \dots, x_n, t) \geq_L \mathcal{M}_j(x, x_2, \dots, x_n, t) \tag{3.4}$$

Putting $y = 0$ in (3.1), we obtain

$$\mathcal{P}(2\, f(l\, x) - 2\, l^2\, f(x), x_2, \dots, x_n, t) \geq_L \Psi(x, x_2, x_3, \dots, x_n, t)\,,$$

and

$$\mathcal{P}(f(lx) - l^2 f(x), x_2, \dots, x_n, t) \geq_L \Psi(x, x_2, x_3, \dots, x_n, 2t) \geq_L \Psi(x, x_2, x_3, \dots, x_n, t)\,.$$

This proves (3.4) for $j = 1$. Let (3.4) hold for some $j > 1$. Replacing y by 0 and x by $l^j x$ in (3.1), we get

$$\mathcal{P}(f(l^{j+1} x) - l^2 f(l^j x), x_2, \dots, x_n, t) \geq_L \Psi(l^j x, x_2, x_3, \dots, x_n, t)\,,$$

Since $|l| \leq 1$, it follows that

$$\begin{aligned}
&\mathcal{P}(f(l^{j+1}\, x) - l^{2\,(j+1)}\, f(x), x_2, \dots, x_n, t) \\
&\geq_L \mathcal{P}(f(l^{j+1}\, x) - l^2\, f(l^j\, x), x_2, \dots, x_n, t) *_L \mathcal{P}(l^2 f(l^j\, x) - l^{2\,(j+1)}\, f(x), x_2, \dots, x_n, t) \\
&= \mathcal{P}(f(l^{j+1} x) - l^2 f(l^j x), x_2, \dots, x_n, t) *_L \mathcal{P}(f(l^j x) - l^{2j} f(x), x_2, \dots, x_n, \frac{t}{|l^2|}) \\
&\geq_L \mathcal{P}(f(l^{j+1} x) - l^2 f(l^j x), x_2, \dots, x_n, t) *_L \mathcal{P}(f(l^j x) - l^{2j} f(x), x_2, \dots, x_n, t) \\
&\geq_L \Psi(l^j x, x_2, x_3, \dots, x_n, t) *_L \mathcal{M}_j(x, x_2, \dots, x_n, t) \ = \ \mathcal{M}_{j+1}(x, x_2, \dots, x_n, t),
\end{aligned}$$

and thus (3.4) holds for all $j \geq 1$. In particular, we have

$$\mathcal{P}(f(l^k x) - l^{2k} f(x), x_2, \dots, x_n, t) \geq_L \mathcal{M}(x, x_2, \dots, x_n, t). \tag{3.5}$$

Replacing x by $l^{-(kn+k)} x$ in (3.5) and using inequality (3.1), we obtain

$$\begin{aligned}
\mathcal{P}(f(\frac{x}{l^{kn}}) - l^{2k} f(\frac{x}{l^{kn+k}}), x_2, \dots, x_n, t) &\geq_L \mathcal{M}(\frac{x}{l^{kn+k}}, x_2, \dots, x_n, t) \\
&\geq_L \mathcal{M}(x, x_2, \dots, x_n, \alpha^{n+1} t),
\end{aligned}$$

and so

$$\begin{aligned}
\mathcal{P}((l^{2\,k})^n\, f(\frac{x}{(l^k)^n}) - (l^{2\,k})^{n+1}\, f(\frac{x}{(l^k)^{n+1}}), x_2, \dots, x_n, t) &\geq_L \mathcal{M}(x, x_2, \dots, x_n, \frac{\alpha^{n+1}}{|(l^{2\,k})^n|}\, t) \\
&\geq_L \mathcal{M}(x, x_2, \dots, x_n, \frac{\alpha^{n+1}}{|(l^k)^n|} t)\,.
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
&\mathcal{P}((l^{2k})^n f(\frac{x}{(l^k)^n}) - (l^{2k})^{n+p} f(\frac{x}{(l^k)^{n+p}}), x_2, \ldots, x_n, t) \\
&\geq_L \prod_{j=n}^{n+p} (\mathcal{P}((l^{2k})^j f(\frac{x}{(l^k)^j}) - (l^{2k})^{j+p} f(\frac{x}{(l^k)^{j+p}}), x_2, \ldots, x_n, t)) \\
&\geq_L \prod_{j=n}^{n+p} \mathcal{M}(x, x_2, \ldots, x_n, \frac{\alpha^{j+1}}{|(l^k)^j|} t)\,.
\end{aligned}$$

Since $\lim_{n\to\infty} \prod_{j=n}^{\infty} \mathcal{M}(x, x_2, \ldots, x_n, \frac{\alpha^{j+1}}{|(l^k)^j|} t) = 1_{\mathcal{L}}$, the sequence

$$\left\{ (l^{2k})^n f(\frac{x}{(l^k)^n}) \right\}_{n\in\mathbb{N}}$$

is a Cauchy sequence in the non-Archmedean $\mathcal{L}$–fuzzy n–normed space $(Y, \mathcal{P}, *_L)$. Hence we can define a mapping $Q : X \to Y$ such that

$$\lim_{n\to\infty} \mathcal{P}((l^{2\,k})^n f(\frac{x}{(l^k)^n}) - Q(x), x_2, \ldots, x_n, t) = 1_{\mathcal{L}}. \tag{3.6}$$

Next, for all $n \geq 1$, we have

$$\begin{aligned}
&\mathcal{P}(f(x) - (l^{2k})^n f(\frac{x}{(l^k)^n}), x_2, \ldots, x_n, t) \\
&= \mathcal{P}\Big(\sum_{i=0}^{n-1} (l^{2k})^i f(\frac{x}{(l^k)^i}) - (l^{2k})^{i+1} f(\frac{x}{(l^k)^{i+1}}), x_2, \ldots, x_n, t\Big) \\
&\geq_L \prod_{i=0}^{n-1} \mathcal{P}\Big((l^{2k})^i f(\frac{x}{(l^k)^i}) - (l^{2k})^{i+1} f(\frac{x}{(l^k)^{i+1}}), x_2, \ldots, x_n, t\Big) \\
&\geq_L \prod_{i=0}^{n-1} \mathcal{M}(x, x_2, \ldots, x_n, \frac{\alpha^{i+1}}{|(l^k)^i|} t)\,,
\end{aligned}$$

and so

$$\begin{aligned}\mathcal{P}(f(x)-Q(x),x_2,\ldots,x_n,t) \geq_L\ & \mathcal{P}\Big(f(x)-(l^{2k})^n f(\frac{x}{(l^k)^n}),x_2,\ldots,x_n,t\Big)\\ &\quad *_L \mathcal{P}\Big((l^{2k})^n f(\frac{x}{(l^k)^n})-Q(x),x_2,\ldots,x_n,t\Big)\\ \geq_L\ & \prod_{i=0}^{n-1}\mathcal{M}(x,x_2,\ldots,x_n,f(\frac{\alpha^{j+1}}{|(l^k)^i|})\Big)\\ &\quad *_L \mathcal{P}\Big((l^{2k})^n f(\frac{x}{(l^k)^n})-Q(x),x_2,\ldots,x_n,t\Big). \qquad (3.7)\end{aligned}$$

Taking the limit as $n\to\infty$ in (3.7), we obtain

$$\mathcal{P}(f(x)-Q(x),x_2,\ldots,x_n,t) \geq_L \prod_{i=1}^{\infty}\mathcal{M}(x,x_2,\ldots,x_n,\frac{\alpha^{i+1}}{|(l^k)^i|}t)$$

which proves (3.4). As $*_L$ is continuous, from a well-known result in $\mathcal{L}$–fuzzy normed space (see [18, Chapter 12]), it follows that

$$\begin{aligned}&\lim_{n\to\infty}\mathcal{P}\big((l^{2k})^n f(l^{-kn}(l\,x+y))+(l^{2k})^n f(l^{-kn}(l\,x-y))\\ &\qquad -2\,(l^{2k})^n f(l^{-kn}(x))-2\,(l^{2k})^n f(l^{-kn}(y)),x_2,\ldots,x_n,t\big)\\ &\quad = \mathcal{P}\big(Q(l\,x+y)+Q(l\,x-y)-2\,l^2\,Q(x)-2\,Q(y),x_2,\ldots,x_n,t\big).\end{aligned}$$

for almost all $t>0$.

On the other hand, replacing x_i by $l^{-kn}x_i$ in (3.2) and (3.1), we get

$$\begin{aligned}&\lim_{n\to\infty}\mathcal{P}\big((l^{2k})^n f(l^{-kn}(l\,x+y))+(l^{2k})^n f(l^{-kn}(l\,x-y))\\ &\qquad -2\,(l^{2k})^n f(l^{-kn}(x))-2(l^{2k})^n f(l^{-kn}(y)),l^{-kn}\,x_2,\ldots,l^{-kn}\,x_n,t\big)\\ &\quad \geq_L \Psi(l^{-kn}x,l^{-kn}\,x_2,\ldots,l^{-kn}\,x_n,\frac{t}{|l^{2k}|^n})\\ &\quad \geq_L \Psi(x,x_2,\ldots,x_n,\frac{\alpha^n\,t}{|l^k|^n}).\end{aligned}$$

Since $\lim_{n\to\infty}\Psi(x_1,x_2,\ldots,x_n,\frac{\alpha^n t}{|l^k|^n})=1_{\mathcal{L}}$, we infer that Q is a quadratic mapping.

For the uniqueness of Q, let $Q':X\to Y$ be another quadratic mapping such that

$$\mathcal{P}(Q'(x)-f(x),x_2,\ldots,x_n,t) \geq_L \mathcal{M}(x,x_2,\ldots,x_n,t).$$

Then we have,

$$\begin{aligned}&\mathcal{P}\big(Q(x)-Q'(x),x_2,\ldots,x_n,t\big)\\ &\geq_L \mathcal{P}\big(Q(x)-(l^{2k})^n f(\frac{x}{(l^k)^n}),x_2,\ldots,x_n,t\big) *_L \mathcal{P}\big((l^{2k})^n f(\frac{x}{(l^k)^n})-Q'(x),x_2,\ldots,x_n,t\big).\end{aligned}$$

Therefore from (3.6), we conclude that $Q=Q'$. This completes the proof. □

4 Stability of pexiderized quadratic functional equation in $\mathcal{L}$–fuzzy n–normed spaces

In this chapter we consider the stability problem for the pexiderized functional equation.

Definition 4.1 *Let Ψ be as in Definition 3.1 and $g, h : X \to Y$ be functions. A mapping $f : X \to Y$ is said to be Ψ–approximately pexiderized quadratic in non-Archimedean $\mathcal{L}$–fuzzy n–normed space if*

$$\mathcal{P}(f(x+y) + f(x-y) - 2g(x) - 2h(y), x_2, \ldots, x_n, t) \geq_L \Psi(x, x_2, \ldots, x_n, t), \qquad (4.1)$$

for all $x, y, x_2, ..., x_n \in X$ and all $t > 0$.

Proposition 4.2 *Let $\mathbb{K}$ be a non-Archimedean field, X a vector space over $\mathbb{K}$ and $(Y, \mathcal{P}, *_L)$ a non-Archimedean $\mathcal{L}$–fuzzy Banach space over $\mathbb{K}$. Let $f : X \to Y$ be a Ψ–approximately pexiderized quadratic mapping in non-Archimedean $\mathcal{L}$–fuzzy n–normed space. Suppose that f, g and h are odd. If there exist $\alpha \in \mathbb{R}$ $(\alpha > 0)$ and an integer $k \geq 2$ with $|2^k| < \alpha$ and $|2| \neq 0$ such that*

$$\Psi(2^{-k}x, 2^{-k}x_2, \ldots, 2^{-k}x_n, t) \geq_L \Psi(x, x_2, \ldots, x_n, \alpha t), \quad \forall x, x_2, ..., x_n \in X,\ t > 0, \qquad (4.2)$$

and

$$\lim_{m\to\infty} \prod_{j=m}^{\infty} \mathcal{M}(x, x_2, \ldots, x_n, \frac{\alpha^j t}{|2|^{kj}}) = 1_{\mathcal{L}}, \quad \forall x, x_2, ..., x_n \in X,\ t > 0,$$

then there exists an additive mapping $T : X \to Y$ such that

$$\mathcal{P}(f(x) - T(x), x_2, \ldots, x_n, t) \geq_L \prod_{i=1}^{\infty} \mathcal{M}(x, x_2, \ldots, x_n, \frac{\alpha^{i+1}t}{|2|^{ki}}), \quad \forall x, x_2, ..., x_n \in X,\ t > 0, \qquad (4.3)$$

and

$$\mathcal{P}\big(g(x) + h(x) - T(x), x_2, \ldots, x_n, t\big) \geq_L \prod_{i=1}^{\infty} \mathcal{M}\big(x, x_2, \ldots, x_n, \frac{\alpha^{i+1}t}{|2|^{ki}}\big)$$

where

$$\mathcal{M}(x, x_2, \ldots, x_n, t) = \Psi(x, x_2, \ldots, x_n, t) *_L \Psi(2x, x_2, \ldots, x_n, t), ..., \\ *_L \Psi(2^{k-1}x, x_2, \ldots, x_n, t) *_L \Psi(0, x_2, \ldots, x_n, t) \quad \forall x, x_2, ..., x_n \in X, t > 0\ .$$

Proof. By changing the roles of x and y in (4.1) we get,

$$\mathcal{P}\big(f(x+y) - f(x-y) - 2g(y) - 2h(x), x_2, \ldots, x_n, t\big) \geq_L \Psi(y, x_2, \ldots, x_n, t). \qquad (4.4)$$

It follows from (4.1) and (4.4) that

$$\begin{aligned} &\mathcal{P}(f(x+y) - g(x) - h(y) - g(y) - h(x), x_2, \ldots, x_n, t) \\ &\geq_L \mathcal{P}(f(x+y) - g(x) - h(y) - g(y) - h(x), x_2, \ldots, x_n, \frac{t}{|2|}) \\ &\geq_L \mathcal{P}(f(x+y) + f(x-y) - 2g(x) - 2h(y), x_2, \ldots, x_n, t) \\ &\quad *_L \mathcal{P}(f(x+y) - f(x-y) - 2g(y) - 2h(x), x_2, \ldots, x_n, t) \\ &\geq_L \Psi(x, x_2, \ldots, x_n, t) *_L \Psi(y, x_2, \ldots, x_n, t). \end{aligned} \qquad (4.5)$$

If we put $y = 0$ in (4.5), then we get

$$\mathcal{P}(f(x) - g(x) - h(x), x_2, \ldots, x_n, t) \ \geq_L \ \Psi(x, x_2, \ldots, x_n, t) *_L \Psi(0, x_2, \ldots, x_n, t). \tag{4.6}$$

Similarly by putting $x = 0$ in (4.5), we have

$$\mathcal{P}(f(y) - g(y) - h(y), x_2, \ldots, x_n, t) \ \geq_L \ \Psi(0, x_2, \ldots, x_n, t) *_L \Psi(y, x_2, \ldots, x_n, t). \tag{4.7}$$

From (4.5), (4.6) and (4.7) we conclude that

$$\begin{aligned}
&\mathcal{P}(f(x+y) - f(x) - f(y), x_2, \ldots, x_n, t) \\
&\geq_L \ \mathcal{P}(f(x+y) - g(x) - h(y) - g(y) - h(x), x_2, \ldots, x_n, t) \\
&\qquad *_L \mathcal{P}(f(x) - g(x) - h(x), x_2, \ldots, x_n, t) *_L \mathcal{P}(f(y) - g(y) - h(y), x_2, \ldots, x_n, t) \\
&\geq_L \ \Psi(x, x_2, \ldots, x_n, t) *_L \Psi(y, x_2, \ldots, x_n, t) *_L \Psi(0, x_2, \ldots, x_n, t).
\end{aligned} \tag{4.8}$$

We show by induction on j, that for all $x, x_2, ..., x_n \in X, t > 0$ and $j \geq 1$,

$$\mathcal{P}(f(2^j x) - 2^j f(x), x_2, \ldots, x_n, t) \ \geq_L \ \mathcal{M}_j(x, x_2, \ldots, x_n, t). \tag{4.9}$$

If we put $x = y$ in (4.8), then we get

$$\begin{aligned}
&\mathcal{P}(f(2\,x) - 2\,f(x), x_2, \ldots, x_n, t) \\
&\ \geq_L \ \Psi(x, x_2, \ldots, x_n, t) *_L \Psi(0, x_2, \ldots, x_n, t).
\end{aligned} \tag{4.10}$$

This proves (4.9) for $j = 1$. Let (4.9) holds for some $j > 1$. Replacing x by $2^j x$ in (4.10), we get

$$\begin{aligned}
&\mathcal{P}(f(2^{j+1}x) - 2\,f(2^j x), x_2, \ldots, x_n, t) \\
&\geq_L \ \Psi(2^j x, x_2, \ldots, x_n, t) *_L \Psi(0, x_2, \ldots, x_n, t).
\end{aligned}$$

Since $|2| \leq 1$, it follows that

$$\begin{aligned}
&\mathcal{P}(f(2^{j+1}x) - 2^{j+1} f(x), x_2, \ldots, x_n, t) \\
&\geq_L \ \mathcal{P}(f(2^{j+1}x) - 2\,f(2^j x), x_2, \ldots, x_n, t) *_L \mathcal{P}(2\,f(2^j x) - 2^{j+1} f(x), x_2, \ldots, x_n, t) \\
&= \mathcal{P}(f(2^{j+1}x) - 2\,f(2^j x), x_2, \ldots, x_n, t) *_L \mathcal{P}(f(2^j x) - 2^j f(x), x_2, \ldots, x_n, \frac{t}{|2|}) \\
&\geq_L \ \mathcal{P}(f(2^{j+1}x) - 2\,f(2^j x), x_2, \ldots, x_n, t) *_L \mathcal{P}(f(2^j x) - 2^j f(x), x_2, \ldots, x_n, t) \\
&\geq_L \ \Psi(2^j x, x_2, \ldots, x_n, t) *_L \Psi(0, x_2, \ldots, x_n, t) *_L \mathcal{M}_j(x, x_2, \ldots, x_n, t) \\
&\qquad = \ \mathcal{M}_{j+1}(x, x_2, \ldots, x_n, t).
\end{aligned}$$

Thus (4.9) holds for all $j \geq 1$. In particular, we have

$$\mathcal{P}(f(2^k x) - 2^k f(x), x_2, \ldots, x_n, t) \ \geq_L \ \mathcal{M}(x, x_2, \ldots, x_n, t). \tag{4.11}$$

Replacing x by $2^{-(kn+k)}x$ in (4.11) and using inequality (4.1), we obtain

$$\begin{aligned}
\mathcal{P}(f(\frac{x}{2^{kn}}) - 2^k f(\frac{x}{2^{kn+k}}), x_2, \ldots, x_n, t) \ &\geq_L \ \mathcal{M}(\frac{x}{2^{kn+k}}, x_2, \ldots, x_n, t) \\
&\geq_L \mathcal{M}(x, x_2, \ldots, x_n, \alpha^{n+1} t).
\end{aligned}$$

and so

$$\mathcal{P}(2^{kn} f(\frac{x}{2^{kn}}) - 2^{k(n+1)} f(\frac{x}{2^{k(n+1)}}), x_2, \dots, x_n, t) \geq_L \mathcal{M}(x, x_2, \dots, x_n, \frac{\alpha^{n+1} t}{|2^{kn}|}).$$

Hence it follows that

$$\begin{aligned}
&\mathcal{P}(2^{kn} f(\frac{x}{2^{kn}}) - 2^{k(n+p)} f(\frac{x}{2^{k(n+p)}}), x_2, \dots, x_n, t) \\
&\geq_L \prod_{j=n}^{n+p} \mathcal{P}(2^{kj} f(\frac{x}{2^{kj}}) - 2^{k(j+1)} f(\frac{x}{2^{k(j+1)}}), x_2, \dots, x_n, t) \geq_L \prod_{j=n}^{n+p} \mathcal{M}(x, x_2, \dots, x_n, \frac{\alpha^{j+1} t}{|2^{kj}|}).
\end{aligned}$$

Since $\lim_{n\to\infty} \prod_{j=n}^{\infty} \mathcal{M}(x, x_2, \dots, x_n, \frac{\alpha^{j+1}t}{|2^{kj}|}) = 1_{\mathcal{L}}$ for $x \in X$ and $t > 0$, $\{2^{kn} f(\frac{x}{2^{kn}})\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the non-Archmedean $\mathcal{L}$–fuzzy Banach space $(Y, \mathcal{P}, *_L)$. Hence we can define a mapping $T : X \to Y$ such that

$$\lim_{n\to\infty} \mathcal{P}(2^{kn} f(\frac{x}{2^{kn}}) - T(x), x_2, \dots, x_n, t) = 1_{\mathcal{L}}. \tag{4.12}$$

Next, for all $n \geq 1, x \in X$ and $t > 0$, we have

$$\begin{aligned}
\mathcal{P}\Big(f(x) - 2^{kn} f(\frac{x}{2^{kn}}), x_2, \dots, x_n, t\Big) &= \mathcal{P}\Big(\sum_{i=0}^{n-1} 2^{ki} f(\frac{x}{2^{ki}}) - 2^{k(i+1)} f(\frac{x}{2^{k(i+1)}}), x_2, \dots, x_n, t\Big) \\
&\geq_L \prod_{i=0}^{n-1} \mathcal{P}\Big(2^{ki} f(\frac{x}{2^{ki}}) - 2^{k(i+1)} f(\frac{x}{2^{k(i+1)}}), x_2, \dots, x_n, t\Big) \\
&\geq_L \prod_{i=0}^{n-1} \mathcal{M}\Big(x, x_2, \dots, x_n, \frac{\alpha^{i+1} t}{|2^{ki}|}\Big)
\end{aligned}$$

and so

$$\begin{aligned}
&\mathcal{P}(f(x) - T(x), x_2, \dots, x_n, t) \\
&\qquad \geq_L \mathcal{P}\big(f(x) - 2^{kn} f(\frac{x}{2^{kn}}), x_2, \dots, x_n, t\big) *_L \mathcal{P}\big(2^{kn} f(\frac{x}{2^{kn}}) - T(x), x_2, \dots, x_n, t\big) \\
&\qquad \geq_L \prod_{i=0}^{n-1} \mathcal{M}\big(x, x_2, \dots, x_n, \frac{\alpha^{i+1} t}{|2^{ki}|}\big) *_L \mathcal{P}\big(2^{kn} f(\frac{x}{2^{kn}}) - T(x), x_2, \dots, x_n, t\big).
\end{aligned} \tag{4.13}$$

Taking the limit as $n \to \infty$ in (4.13), we obtain

$$\mathcal{P}(f(x) - T(x), x_2, \dots, x_n, t) \geq_L \prod_{i=1}^{\infty} \mathcal{M}(x, x_2, \dots, x_n, \frac{\alpha^{i+1} t}{|2^{ki}|}), \tag{4.14}$$

which proves (4.3). As $*_L$ is continuous, from a well-known result in $\mathcal{L}$–fuzzy n–normed space (see [18, Chapter 12]), it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \mathcal{P}\big(2^{kn} f(2^{-kn}(x+y)) - 2^{kn} f(2^{-kn} x) - 2^{kn} f(2^{-kn} y), x_2, \dots, x_n, t\big) \\
&\qquad = \mathcal{P}\big(T(x+y) - T(x) - T(y), x_2, \dots, x_n, t\big),
\end{aligned}$$

for almost all $t > 0$.

On the other hand, replacing x by $2^{-kn}x$ in (4.8), we get

$$\begin{aligned}
&\mathcal{P}(2^{kn}f(2^{-kn}(x+y)) - 2^{kn}f(2^{-kn}x) - 2^{kn}f(2^{-kn}y), x_2, \dots, x_n, t)\\
&\geq_L \Psi(2^{-kn}x, x_2, \dots, x_n, \frac{t}{|2^{kn}|}) *_L \Psi(2^{-kn}y, x_2, \dots, x_n, \frac{t}{|2^{kn}|}) *_L\\
&\quad *_L \Psi(0, x_2, \dots, x_n, \frac{t}{|2^{kn}|})\\
&\geq_L \Psi(x, x_2, \dots, x_n, \frac{\alpha^n t}{|2^{kn}|}) *_L \Psi(y, x_2, \dots, x_n, \frac{\alpha^n t}{|2^{kn}|}) *_L\\
&\quad *_L \Psi(0, x_2, \dots, x_n, \frac{\alpha^n t}{|2^{kn}|})
\end{aligned}$$

Since each items of the right hand side of above inequality tends to 1 as $n \to \infty$, we infer that T is an additive mapping.

It follows from (4.6) and (4.14) that

$$\begin{aligned}
&\mathcal{P}\big(g(x) + h(x) - T(x), x_2, \dots, x_n, t\big)\\
&\geq_L \mathcal{P}\big(f(x) - T(x), x_2, \dots, x_n, t\big) *_L \mathcal{P}\big(g(x) + h(x) - f(x), x_2, \dots, x_n, t\big)\\
&\geq_L \prod_{i=1}^{\infty} \mathcal{M}(x, x_2, \dots, x_n, \frac{\alpha^{i+1} t}{|2^{ki}|}) *_L \Psi(x, x_2, \dots, x_n, t) *_L \Psi(0, x_2, \dots, x_n, t)\\
&\geq_L \prod_{i=1}^{\infty} \mathcal{M}(x, x_2, \dots, x_n, \frac{\alpha^{i+1} t}{|2^{ki}|}).
\end{aligned}$$

For the uniqueness of T, let $T' : X \to Y$ be another additive mapping such that

$$\mathcal{P}(T'(x) - f(x), x_2, \dots, x_n, t) \geq_L \mathcal{M}(x, x_2, \dots, x_n, t).$$

Then we have for all $x, x_2, \dots, x_n \in X$ and $t > 0$,

$$\begin{aligned}
&\mathcal{P}(T(x) - T'(x), x_2, \dots, x_n, t)\\
&\geq_L \mathcal{P}(T(x) - 2^{kn}f(\frac{x}{2^{kn}}), x_2, \dots, x_n, t) *_L \mathcal{P}(2^{kn}f(\frac{x}{2^{kn}}) - T'(x), x_2, \dots, x_n, t).
\end{aligned}$$

Therefore we conclude from (4.12), that $T = T'$. This completes the proof. □

Proposition 4.3 *Let $\mathbb{K}$ be a non-Archimedean field, X a vector space over $\mathbb{K}$ and $(Y, \mathcal{P}, *_L)$ a non-Archimedean $\mathcal{L}$–fuzzy Banach space over $\mathbb{K}$. Let $f : X \to Y$ be a Ψ–approximately pexiderized quadratic mapping in non-Archimedean $\mathcal{L}$–fuzzy n–normed space. Suppose that f,g and h are even and $f(0) = g(0) = h(0) = 0$. If there exist an $\alpha \in \mathbb{R}$ ($\alpha > 0$) and an integer k, $k \geq 2$ with $|2^k| < \alpha$ and $|2| \neq 0$ such that*

$$\Psi(2^{-k}x, 2^{-k}x_2, \dots, 2^{-k}x_n, t) \geq_L \Psi(x, x_2, \dots, x_n, \alpha t), \quad \forall x, x_2, ..., x_n \in X,\ t > 0,$$

and

$$\lim_{m\to\infty} \prod_{j=m}^{\infty} \mathcal{M}(x, x_2, \dots, x_n, \frac{\alpha^j t}{|2|^{kj}}) = 1_{\mathcal{L}}, \quad \forall x, x_2, ..., x_n \in X,\ t > 0\ ,$$

then there exists a quadratic mapping $Q : X \to Y$ such that

$$\mathcal{P}(f(x) - Q(x), x_2, \dots, x_n, t) \geq_L \prod_{i=1}^{\infty} \mathcal{M}(x, x_2, \dots, x_n, \frac{\alpha^{i+1} t}{|2|^{ki}}), \quad \forall x, x_2, ..., x_n \in X,\ t > 0, \tag{4.15}$$

and

$$\mathcal{P}(Q(x) - g(x), x_2, \dots, x_n, t) \geq_L \prod_{i=1}^{\infty} \mathcal{M}(x, x_2, \dots, x_n, \frac{\alpha^{i+1} t}{|2|^{ki}}),$$

and

$$\mathcal{P}(Q(x) - h(x), x_2, \dots, x_n, t) \geq_L \prod_{i=1}^{\infty} \mathcal{M}(x, x_2, \dots, x_n, \frac{\alpha^{i+1} t}{|2|^{ki}}),$$

where $\forall x, x_2, ..., x_n \in X,\ t > 0,$

$$\mathcal{M}(x, x_2, \dots, x_n, t) = \Psi(x, x_2, \dots, x_n, t) *_L \Psi(2x, x_2, \dots, x_n, t), ..., \\ *_L \Psi(2^{k-1} x, x_2, \dots, x_n, t) *_L \Psi(0, x_2, \dots, x_n, t) \quad \forall x, x_2, ..., x_n \in X, t > 0\ .$$

Proof. Put $x = y$ in (4.1). Then for all $x, x_2, ..., x_n \in X$ and $t > 0$,

$$\mathcal{P}(f(2x) - 2g(x) - 2h(x), x_2, \dots, x_n, t) \geq_L \Psi(x, x_2, \dots, x_n, t).$$

Put $x = 0$ in (4.1), we get

$$\mathcal{P}(2\,f(y) - 2h(y), x_2, \dots, x_n, t) \geq_L \Psi(0, x_2, \dots, x_n, t), \tag{4.16}$$

for all $x, x_2, ..., x_n \in X$ and $t > 0$. For $y = 0$, (4.1) becomes

$$\mathcal{P}(2\,f(x) - 2g(x), x_2, \dots, x_n, t) \geq_L \Psi(x, x_2, \dots, x_n, t). \tag{4.17}$$

Combining (4.1), (4.16) and (4.17) we get

$$\begin{aligned} &\mathcal{P}(f(x+y) + f(x-y) - 2\,f(x) - 2\,f(y), x_2, \dots, x_n, t) \\ &\quad \geq_L \mathcal{P}(f(x+y) + f(x-y) - 2g(x) - 2h(y), x_2, \dots, x_n, t) *_L \\ &\qquad *_L \mathcal{P}(2\,f(y) - 2h(y), x_2, \dots, x_n, t) *_L \mathcal{P}(f(2x) - 2g(x), x_2, \dots, x_n, t) \\ &\quad \geq_L \Psi(x, x_2, \dots, x_n, t) *_L \Psi(0, x_2, \dots, x_n, t). \end{aligned} \tag{4.18}$$

We show by induction on j, that for $x \in X, t > 0$ and $j \geq 1$,

$$\mathcal{P}(f(2^j x) - 4^j f(x), x_2, \dots, x_n) \geq_L \mathcal{M}_j(x, x_2, \dots, x_n, t). \tag{4.19}$$

Similarly to the proof of Proposition 4.2, we can obtain the results. Here, by (4.15) and (4.17) we get

$$\begin{aligned} &\mathcal{P}(Q(x) - g(x), x_2, \dots, x_n, t) \\ &\quad \geq_L \mathcal{P}(Q(x) - f(x), x_2, \dots, x_n, t) *_L \mathcal{P}(f(x) - g(x), x_2, \dots, x_n, t) \\ &\quad \geq_L \prod_{i=1}^{\infty} \mathcal{M}(x, x_2, \dots, x_n, \frac{\alpha^{i+1} t}{|2|^{ki}}) *_L \Psi(x, x_2, \dots, x_n, t) \\ &\quad = \prod_{i=1}^{\infty} \mathcal{M}(x, x_2, \dots, x_n, \frac{\alpha^{i+1} t}{|2|^{ki}}). \end{aligned}$$

A similar inequality holds for h. □

Theorem 4.4 *Let $\mathbb{K}$ be a non-Archimedean field, X a vector space over $\mathbb{K}$ and $(Y, \mathcal{P}, *_L)$ a non-Archimedean $\mathcal{L}$–fuzzy Banach space over $\mathbb{K}$. Let $f : X \to Y$ be a Ψ–approximately quadratic mapping in non-Archimedean $\mathcal{L}$–fuzzy n–normed space. Suppose that $f(0) = 0$. If there exist an $\alpha \in \mathbb{R}$ $(\alpha > 0)$ and an integer k, $k \geq 2$ with $|2^k| < \alpha$ and $|2| \neq 0$ such that*

$$\forall x, x_2, ..., x_n \in X, t > 0 \;:\; \Psi(2^{-k}x, 2^{-k}x_2, \ldots, 2^{-k}x_n, t) \geq_L \Psi(x, x_2, \ldots, x_n, \alpha t),$$

and

$$\lim_{m\to\infty} \prod_{j=n}^{\infty} \mathcal{M}(x, x_2, \ldots, x_n, \frac{\alpha^j t}{|2|^{kj}}) = 1_{\mathcal{L}},$$

then there are unique mappings T and Q from X to Y such that T is additive, Q is quadratic and

$$\mathcal{P}(f(x) - T(x) - Q(x), x_2, \ldots, x_n, t) \geq_L \prod_{i=1}^{\infty} \mathcal{M}(x, x_2, \ldots, x_n, \frac{\alpha^{i+1} t}{|2|^{ki}}), \tag{4.20}$$

for all $x, x_2, ..., x_n \in X$ and all $t > 0$, where

$$\mathcal{M}(x, x_2, \ldots, x_n, t) = \Psi(x, x_2, \ldots, x_n, t) *_L \Psi(2x, x_2, \ldots, x_n, t), ..., \\ *_L \Psi(2^{k-1}x, x_2, \ldots, x_n, t) *_L \Psi(0, x_2, \ldots, x_n, t) \quad \forall x, x_2, ..., x_n \in X, t > 0\,.$$

Proof. Passing to the odd part f^o and even part f^e of f we deduce from (4.1) that

$$\mathcal{P}\big(f^o(x+y) + f^o(x-y) - 2\, f^o(x) - 2\, f^o(x), x_2, \ldots, x_n, t\big) \geq_L \Psi(x, x_2, \ldots, x_n)\,.$$

And

$$\mathcal{P}\big(f^e(x+y) + f^e(x-y) - 2\, f^e(x) - 2\, f^e(y), x_2, \ldots, x_n, t\big) \geq_L \Psi(x, x_2, \ldots, x_n)\,.$$

Using the proof of Proposition 4.2 and 4.3, we get a unique additive mapping T and a unique quadratic mapping Q satisfying

$$\forall x, x_2, ..., x_n \in X, t > 0 : \mathcal{P}\big(f^o(x) - T(x), x_2, \ldots, x_n, t\big) \geq_L \prod_{i=1}^{\infty} \mathcal{M}\big(x, x_2, \ldots, x_n, \frac{\alpha^{i+1} t}{|2|^{ki}}\big)\,,$$

also

$$\forall x, x_2, ..., x_n \in X, t > 0 : \mathcal{P}\big(f^e(x) - Q(x), x_2, \ldots, x_n, t\big) \geq_L \prod_{i=1}^{\infty} \mathcal{M}\big(x, x_2, \ldots, x_n, \frac{\alpha^{i+1} t}{|2|^{ki}}\big)\,,$$

therefore

$$\begin{aligned} &\mathcal{P}\big(f(x) - T(x) - Q(x), x_2, \ldots, x_n, t\big) \\ &\geq_L \mathcal{P}\big(f^o(x) - T(x), x_2, \ldots, x_n, t\big) *_L \mathcal{P}\big(f^e(x) - Q(x), x_2, \ldots, x_n, t\big) \\ &\geq_L \prod_{i=1}^{\infty} \mathcal{M}\big(x, x_2, \ldots, x_n, \frac{\alpha^{i+1} t}{|2|^{ki}}\big)\,. \end{aligned}$$

□

Remark 4.5 *By replacing $\Psi(x, x_2, ..., x_n, t)$ with $\Psi(x, x_2, ..., x_n, t) *_L \Psi(y, x_2, ..., x_n, t)$ in the right hand of inequalities (3.1) and (4.1) in Definitions 3.1 and 4.1 we can have similar results as in Theorem 3.2, Propositions 4.2, 4.3 and Theorem 4.4.*

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