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CONTROLLABILITY OF IMPULSIVE FRACTIONAL ORDER SEMILINEAR EVOLUTION EQUATIONS WITH NONLOCAL CONDITIONS

NUTAN KUMAR TOMAR *
School of Basic Sciences, Indian Institute of Technology Patna,
Patna-800 013, Bihar, India.

JAYDEV DABAS †
Department of Applied Science and Engineering, Indian Institute of Technology,
Roorkee, Saharanpur Campus, Saharanpur-247001,UP, India.

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Abstract. The controllability problem for a class of impulsive fractional-order semilinear control system with nonlocal initial condition has been considered. Sufficient condition for the controllability is established by means of solution operator and application of Banach contraction theorem.

Keywords: Controllability, Fractional order differential equation, Nonlocal conditions, Contractions, Mild solution, Impulsive conditions.

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1 Introduction

It is a well known fact that the problem of controllability of semilinear systems in infinite-dimensional spaces can be converted into solvability problem of a functional operator equation in appropriate Banach spaces, and fixed-point theory has been widely used in the literature to establish this solvability [10, 15, 28, 29]. Nonlocal conditions are generally more practical for the physical measurements as compared to the classical conditions. The importance of nonlocal conditions has

*e-mail address: chinidma@gmail.com
†e-mail address: jay.dabas@gmail.com

been discussed in the pioneering work [7]. Nonlocal conditions were used in [12] to describe, for instance, the diffusion phenomenon of a small amount of gas in a transparent tube. The concept of fixed point theory has been extended to infinite-dimensional semilinear control systems with nonlocal initial conditions, among others, we refer to the papers [4, 6, 14].

Impulsive differential equations have become important in recent years as mathematical models of phenomena in both physical and social sciences (see for instance [13, 17, 19, 27] and references in these papers). In [16, 8] authors studied the controllability of impulsive functional differential systems in Banach spaces by Schaefer's fixed point theorem. In [21] authors considered the controllability of impulsive neutral integro-differential infinite-dimensional systems with infinite delay. For more recent development in the field of controllability of impulsive differential systems in infinite dimensional spaces, we refer to the work of Sakthivel and his co-workers [23, 24, 25] and references therein.

No doubt, enough literature is available for integral-order infinite dimensional systems, but as far as controllability is concerned there is a wide gap between integral-order and fractional-order infinite-dimensional systems. Recently, fractional differential equations attracted many authors as these equations can represent many engineering, physics, continuum mechanics, signal processing, electromagnetic, and economics problems in more efficient way, see for instance [22] and subsequently the papers [19, 2, 5, 11, 18, 20, 31, 32] and references therein.

In this paper, we are concerned with the controllability of the fractional order differential equation in a Banach space X

$$\begin{aligned} \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}x(t) &= f(t, x(t), x(a_1(t)), \ldots, x(a_m(t))) \\ &\quad + Ax(t) + Bu(t),\ t \in J = [0, T],\ t \neq t_i, \qquad (1.1) \\ x(0) + g(x) &= x_0, \qquad (1.2) \\ \Delta x(t_i) &= I_i(x(t_i^-)), \qquad (1.3) \end{aligned}$$

In [27], authors have investigated the existence of mild solutions of the system

$$\left\{ \begin{aligned} D^\alpha x(t) &= Ax(t) + f(t, x(t)),\ t \in [0, T],\ t \neq t_k, \\ x(0) &= x_0 \in X, \\ \Delta x|_{t=t_k} &= I_k(x(t_k^-)),\ k = 1, \ldots, m, \end{aligned} \right. \qquad (1.4)$$

and corrected the errors in [19], and generalized some existing results. Recently in [9] authors extended the definition of mild solutions given in [27] to the systems of the form (1.1)–(1.3) when $B \equiv 0$.

Although there are few recent papers on the controllability of fractional-order semilinear systems, see [3, 30, 26] and references therein, but it should be noted down here that all these papers, when impulsive effect is considered, are based on the mild solutions given in [19]. In this paper we corrected the errors of mild solutions in previous papers on the controllability of fractional-order semilinear systems.

The organization of this paper is as follows. In Section 2, we present some necessary definitions and preliminary results that will be used to prove our main result. The proof of our main result is given in Section 3. To apply the results of this paper one example is presented in the last section.

2 Preliminaries

In this paper, we shall establish the sufficient conditions for the controllability of system (1.1)–(1.3), where the state variable $x(\cdot)$ takes values in Banach Space $(X, \|\cdot\|_X)$ and the control function $u(\cdot)$ belongs to $Y = L^2([0, T]; U)$, the Banach space of admissible control functions with a Banach space U. $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}$ is Caputo's fractional derivative of order $0 < \alpha < 1$, $i = 1, 2, \cdots, p$, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$. The linear operator A, defined from the domain $D(A) \subset X$ into X, is such that A generates α–resolvent family $\{S_\alpha(t) : t \geq 0\}$ of bounded linear operators in X, for the theory of resolvent operator we refer [22, 1]. $B : U \to X$ is a bounded linear operator. The nonlinear map f is defined from $J \times X^{m+1}$ into X, for each of j the map a_j is defined on $J := [0, T]$ into J and $\Delta x(t_i) = x(t_i^+) - x(t_i^-), x(t_i^+), x(t_i^-)$ denotes the right and the left limit of x at t_i, respectively. In general the derivatives $x'(t_i)$ do not exist, we assume that $x'(t_i) = x'(t_i - 0)$ at the point of discontinuity t_i of the solution $t \to x(t)$.

Before proceeding further, we recall some basic definitions and properties from the fractional calculus and operators theory.

The Mittag–Leffler function is an important function that finds widespread use in the world of fractional calculus. Just as the exponential naturally arises out of the solution to integer order differential equations, the Mittag–Leffler function plays an important role in the solution of non-integer order differential equations. The standard definition of the Mittag–Leffler function (see [22]) is given as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

It is also common to represent the Mittag–Leffler function in two arguments, α and β, such that

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_{H_a} e^\mu \frac{\mu^{\alpha-\beta}}{\mu^\alpha - z} \,\mathrm{d}\mu,\ \alpha, \beta > 0,\ z \in \mathbb{C},$$

where H_a is a Hankel path, that is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^{\frac{1}{\alpha}}$ counter clockwise. The Laplace transform of the Mittag–Leffler function is given as:

$$L(t^{\beta-1} E_{\alpha,\beta}(-\rho^\alpha t^\alpha)) = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha + \rho^\alpha},\ \Re e\, \lambda > \rho^{\frac{1}{\alpha}},\ \rho > 0.$$

Definition 2.1 *Caputo's derivative of order α for a function $f : [0, \infty) \to \mathbb{R}$ is defined as*

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} f^{(n)}(s) \,\mathrm{d}s,$$

for $n - 1 < \alpha < n,\ n \in N$. If $0 < \alpha \leq 1$, then

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} f^{(1)}(s) \,\mathrm{d}s.$$

The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$L\{D_t^\alpha f(t); \lambda\} = \lambda^\alpha \widehat{f}(\lambda) - \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} f^{(k)}(0); \quad n - 1 < \alpha \leq n.$$

Definition 2.2 *([2, Definition 2.3]) Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. Let $\rho(A)$ be the resolvent set of A. We call A the generator of an α–resolvent family if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha : R_+ \to L(X)$ such that $\{\lambda^\alpha : \Re e\,\lambda > \omega\} \subset \rho(A)$ and*

$$(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x\,\mathrm{d}t,\ \Re e\,\lambda > \omega,\ x \in X. \tag{2.1}$$

In this case, $S_\alpha(t)$ is called the $\alpha-$resolvent family generated by A.

Definition 2.3 *([1, Definition 2.1]) Let A be a closed linear operator with the domain $D(A)$ defined in a Banach space X and $\alpha > 0$. We say that A is the generator of a solution operator if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(X)$ such that $\{\lambda^\alpha : \Re e\,\lambda > \omega\} \subset \rho(A)$ and*

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x\,\mathrm{d}t,\ \Re e\,\lambda > \omega,\ x \in X, \tag{2.2}$$

in this case, $S_\alpha(t)$ is called the solution operator generated by A.

Let us consider the set of functions

$$\begin{aligned} PC(J,X) &= \{x : J \to X : x \in C((t_k, t_{k+1}], X), k = 0, 1, \ldots p \text{ and there exist} \\ &\quad\ x(t_k^-) \text{ and } x(t_k^+) \text{ for } k = 1, \ldots, p \text{ with } x(t_k^-) = x(t_k)\}, \end{aligned}$$

endowed with the norm $\|x\|_{PC} = \sup_{t\in J} \|x(t)\|_X$. Then $(PC(J,X), \|.\|_{PC})$ is a Banach space.

Lemma 2.4 *([27]) Consider the following Cauchy problem*

$$\begin{cases} D_t^\alpha x(t) + Ax(t) = Bu(t) \\ +f(t, x(t), x(a_1(t)), \ldots, x(a_m(t))),\ t > t_0,\ t_0 \geq 0,\ 0 < \alpha < 1, \\ x(t_0) = x_0 \in X. \end{cases} \tag{2.3}$$

If f satisfies the uniform Hölder condition with exponent $\beta \in (0, 1]$ and A is a sectorial operator, then the unique solution of this Cauchy problem is given by

$$\begin{aligned} x(t) &= T_\alpha(t - t_0)x(t_0^+) + \int_0^t S_\alpha(t-s)Bu(s)\,\mathrm{d}s \\ &\quad + \int_{t_0}^t S_\alpha(t-\theta)f(\theta, x(\theta), x(a_1(\theta)), \ldots, x(a_m(\theta)))\,\mathrm{d}\theta, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} T_\alpha(t) &= E_{\alpha,1}(-At^\alpha) = \frac{1}{2\pi i}\int_{\widehat{B}_r} e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha + A}\,\mathrm{d}\lambda, \\ S_\alpha(t) &= t^{\alpha-1}E_{\alpha,\alpha}(-At^\alpha) = \frac{1}{2\pi i}\int_{\widehat{B}_r} e^{\lambda t}\frac{1}{\lambda^\alpha + A}\,\mathrm{d}\lambda, \end{aligned}$$

where $\widehat{B}_r$ denotes the Bromwich path. $S_\alpha(t)$ is called the $\alpha-$resolvent family and $T_\alpha(t)$ is the solution operator, generated by $-A$.

Proof. The proof is similar as Lemma 3.1 in [9]. □

Definition 2.5 *A solution $x \in PC(J, X)$ of the integral equation*

$$x(t) = \begin{cases} T_\alpha(t)(x_0 - g(x)) + \int_0^t S_\alpha(t-s)Bu(s)\,\mathrm{d}s \\ + \int_0^t S_\alpha(t-s)f(s, x(s), x(a_1(s)), \ldots, x(a_m(s)))\,\mathrm{d}s, & t \in [0, t_1]; \\ T_\alpha(t-t_1)[x(t_1^-) + I_1(x(t_1^-))] + \int_{t_1}^t S_\alpha(t-s)Bu(s)\,\mathrm{d}s \\ + \int_{t_1}^t S_\alpha(t-s)f(s, x(s), x(a_1(s)), \ldots, x(a_m(s)))\,\mathrm{d}s, & t \in (t_1, t_2]; \\ \quad \vdots \\ T_\alpha(t-t_p)[x(t_p^-) + I_p(x(t_p^-))] + \int_{t_p}^t S_\alpha(t-s)Bu(s)\,\mathrm{d}s \\ + \int_{t_p}^t S_\alpha(t-s)f(s, x(s), x(a_1(s)), \ldots, x(a_m(s)))\,\mathrm{d}s, & t \in (t_p, T]. \end{cases} \tag{2.5}$$

is called a mild solution of the problem (1.1).

Note that, mild solution (2.5) depends on control functions $u(\cdot)$. The solution of (1.1)–(1.3) under a control $u(\cdot)$, denoted by $x(\cdot; u)$, is called the trajectory (state) function of (1.1) under $u(\cdot)$. The set of all possible terminal states, denoted by

$$K_T(f) := \{x(T; u) \in X : u \in L^2([0, T]; U)\}, \tag{2.6}$$

is called the reachable set of system (1.1) at terminal time T.

Definition 2.6 *System* (1.1)–(1.3) *is said to be controllable on J if $K_T(f) = X$.*

3 The main results

If $\alpha \in (0, 1)$ and $A \in A^\alpha(\theta_0, \omega_0)$, then for any $x \in X$ and $t > 0$, (see [27] for details), we have

$$\|T_\alpha(t)\|_{L(X)} \le M_1 e^{\omega t} \text{ and } \|S_\alpha(t)\|_{L(X)} \le Ce^{\omega t}(1 + t^{\alpha-1}),\ t > 0,\ \omega > \omega_0.$$

Let

$$\widetilde{M}_T = \sup_{0 \le t \le T} \|T_\alpha(t)\|_{L(X)} \text{ and } \widetilde{M}_S = \sup_{0 \le t \le T} Ce^{\omega t}(1 + t^{1-\alpha}),$$

where $L(X)$ is the Banach space of bounded linear operators from X into X equipped with its natural topology. So we have

$$\|T_\alpha(t)\|_{L(X)} \le \widetilde{M}_T \text{ and } \|S_\alpha(t)\|_{L(X)} \le t^{\alpha-1}\widetilde{M}_S.$$

Now we introduce the following assumptions:

(H1) There exists a constant $L_g > 0$ such that $\|g(x) - g(y)\|_X \le L_g\|x - y\|_X$.

(H2) The nonlinear map $f : [0, T] \times X^{m+1} \to X$ is continuous and there exists a constant L_f such that

$$\|f(t, x_1, x_2, \ldots, x_{m+1}) - f(s, y_1, y_2, \ldots, y_{m+1})\|_X \le L_f[|t - s| + \sum_{i=1}^{m+1} \|x_i - y_i\|_X],$$

for all $(x_1, \ldots, x_{m+1})$ and $(y_1, \ldots, y_{m+1})$ in X^{m+1} and $t \in [0, T]$.

(H3) The function $I_k : X \to X$ are continuous and there exists $L_k > 0$ such that

$$\|I_k(x) - I_k(y)\|_X \leq L_k \|x - y\|_X, \quad x, y \in X, k = 1, 2, \ldots, p,$$

where $L = \max\{L_k\} > L_g$.

(H4) For $k = 1,\, 2,\, 3, \cdots, p+1$, the linear operators $W_k : L^2([t_{k-1}, t_k]; U) \to X$ defined by

$$W_k u = \int_{t_{k-1}}^{t_k} S_\alpha(t-s) B u(s)\, \mathrm{d}s,$$

has an invertible operator W_k^{-1} taking values in $L^2([t_{k-1}, t_k]; U) \backslash \ker(W_k)$ and there exists a positive constant M_k such that $\|BW_k^{-1}\| \leq M_k$, and $M = \max\{M_k\}$

Theorem 3.1 *Suppose that assumptions (H1)–(H4) are satisfied and*

$$\Theta = \big(1 + M\widetilde{M}_S \frac{T^\alpha}{\alpha}\big)\, \big(\widetilde{M}_T(1+L) + \widetilde{M}_S L_f (m+1) \frac{T^\alpha}{\alpha}\big) < 1\,.$$

Then system (1.1)–(1.3) *is controllable on* J.

Proof. Let $z \in PC(J, X)$ be any arbitrary function, now to transfer the system (1.1) from initial state to $z(T)$, consider the control

$$u(t) = \begin{cases} W_1^{-1}\big[z(t_1) - T_\alpha(t_1)(x_0 - g(x)) \\ \qquad - \int_0^{t_1} S_\alpha(t_1 - s) f(s, x(s), x(a_1(s)), \ldots, x(a_m(s)))\, \mathrm{d}s\big](t), & t \in [0, t_1]; \\ W_2^{-1}\big[z(t_2) - T_\alpha(t_2 - t_1)[x(t_1^-) + I_1(x(t_1^-))] \\ \qquad - \int_{t_1}^{t_2} S_\alpha(t_2 - s) f(s, x(s), x(a_1(s)), \ldots, x(a_m(s)))\, \mathrm{d}s](t), & t \in (t_1, t_2]; \\ \quad \vdots \\ W_{p+1}^{-1}\big[z(T) - T_\alpha(T - t_p)[x(t_p^-) + I_p(x(t_p^-))] \\ \qquad + \int_{t_p}^{T} S_\alpha(T - s) f(s, x(s), x(a_1(s)), \ldots, x(a_m(s)))\, \mathrm{d}s\big](t), & t \in (t_p, T]. \end{cases} \tag{3.1}$$

Define a mapping N from $PC(J, X)$ into itself by

$$(Nx)(t) = \begin{cases} T_\alpha(t)(x_0 - g(x)) + \int_0^t S_\alpha(t-s) B W_1^{-1}\big[z(t_1) - T_\alpha(t_1)(x_0 - g(x)) \\ \qquad - \int_0^{t_1} S_\alpha(t_1 - s)\, f(s, x(s), x(a_1(s)), \ldots, x(a_m(s)))\, \mathrm{d}s\big](s)\, \mathrm{d}s \\ \quad + \int_0^t S_\alpha(t-s) f(s, x(s), x(a_1(s)), \ldots, x(a_m(s)))\, \mathrm{d}s, & t \in [0, t_1]; \\ T_\alpha(t - t_1)[x(t_1^-) + I_1(x(t_1^-))] \\ \quad + \int_{t_1}^t S_\alpha(t-s) B W_2^{-1}\big[z(t_2) - T_\alpha(t_2 - t_1)[x(t_1^-) + I_1(x(t_1^-))] \\ \qquad - \int_{t_1}^{t_2} S_\alpha(t_2 - s) f(s, x(s), x(a_1(s)), \ldots, x(a_m(s)))\ \mathrm{d}s\big](s)\, \mathrm{d}s \\ \quad + \int_{t_1}^t S_\alpha(t-s) f(s, x(s), x(a_1(s)), \ldots, x(a_m(s)))\, \mathrm{d}s, & t \in (t_1, t_2]; \\ \qquad \vdots \\ T_\alpha(t - t_p)[x(t_p^-) + I_p(x(t_p^-))] \\ \quad + \int_{t_p}^t S_\alpha(t-s) B W_{p+1}^{-1}\big[z(T) - T_\alpha(T - t_p)[x(t_p^-) + I_p(x(t_p^-))] \\ \qquad - \int_{t_p}^{T} S_\alpha(T - s) f(s, x(s), x(a_1(s)), \ldots, x(a_m(s)))\ \mathrm{d}s\big](s)\, \mathrm{d}s \\ \quad + \int_{t_p}^t S_\alpha(t-s) f(s, x(s), x(a_1(s)), \ldots, x(a_m(s)))\, \mathrm{d}s, & t \in (t_p, T]. \end{cases}$$

For convenience, let us take

$$\begin{aligned} C(s,x) &= BW_1^{-1}\big[z(t_1) - T_\alpha(t_1)(x_0 - g(x)) \\ &\quad - \int_0^{t_1} S_\alpha(t_1 - s) f(s, x(s), x(a_1(s)), \dots, x(a_m(s)))\, \mathrm{d}s\big](s), \end{aligned} \tag{3.2}$$

and for $i = 2,\, 3, \dots, p+1$,

$$\begin{aligned} D_i(s,\,x) &= BW_i^{-1}\big[z(t_i) - T_\alpha(t_i - t_{i-1})[x(t_{i-1}^-) + I_{i-1}(x(t_{i-1}^-))] \\ &\quad - \int_{t_{i-1}}^{t_i} S_\alpha(t_i - s) f(s, x(s), x(a_1(s)), \dots, x(a_m(s)))\, \mathrm{d}s\big](s). \end{aligned} \tag{3.3}$$

From our assumptions, we have

$$\|C(s,x) - C(s,y)\| \le M_1\,\big[\widetilde{M}_T L_g + \widetilde{M}_S L_f (m+1)\frac{T^\alpha}{\alpha}\big]\|x - y\|_{PC}, \tag{3.4}$$

and

$$\|D_i(s,\,x) - D_i(s,\,y)\| \le M_i\,\big[\widetilde{M}_T(1 + L_{i-1}) + \widetilde{M}_S L_f (m+1)\frac{T^\alpha}{\alpha}\big]\|x - y\|_{PC}. \tag{3.5}$$

Now we show that N is a contraction mapping on $PC(J,X)$. We have

$$\begin{aligned} &\|Nx(t) - Ny(t)\|_X \\ &\le \begin{cases} \|T_\alpha(t)\|(\|g(x) - g(y)\|_X) + \int_0^t \|S_\alpha(t-s)\|\,\|C(s,x) - C(s,y)\|\,\mathrm{d}s \\ \quad + \int_0^t \|S_\alpha(t-s)\|\,\|f(s,x(s),x(a_1(s)),\dots,x(a_m(s))) \\ \qquad - f(s,y(s),y(a_1(s)),\dots,y(a_m(s)))\|\,\mathrm{d}s, & t \in [0,t_1]; \\ \|T_\alpha(t-t_1)\|(\|x(t_1^-) - y(t_1^-)\| + \|I_1(x(t_1^-)) - I_1(y(t_1^-))\|) \\ \quad + \int_{t_1}^t \|S_\alpha(t-s)\|\,\|D_2(s,x) - D_2(s,y)\|\,\mathrm{d}s \\ \quad + \int_{t_1}^t \|S_\alpha(t-s)\|\,\|f(s,x(s),x(a_1(s)),\dots,x(a_m(s))) \\ \qquad - f(s,y(s),y(a_1(s)),\dots,y(a_m(s)))\|\,\mathrm{d}s, & t \in (t_1,t_2]; \\ \quad \vdots \\ \|T_\alpha(t-t_p)\|_{L(X)}(\|x(t_p^-) - y(t_p^-)\|_X + \|I_p(x(t_p^-)) - I_p(y(t_p^-))\|_X) \\ \quad + \int_{t_p}^t \|S_\alpha(t_p-s)\|\,\|D_{p+1}(s,x) - D_{p+1}(s,y)\|\,\mathrm{d}s \\ \quad + \int_{t_p}^t \|S_\alpha(t-s)\|_{L(X)}\,\|f(s,x(s),x(a_1(s)),\dots,x(a_m(s))) \\ \qquad - f(s,y(s),y(a_1(s)),\dots,y(a_m(s)))\|_X\,\mathrm{d}s, & t \in (t_p,T]. \end{cases} \end{aligned}$$

Applying the assumptions (H1)–(H3), and using the estimates (3.4)–(3.5), we obtain

$$\begin{aligned} &\|Nx(t) - Ny(t)\|_X \\ &\le \begin{cases} (1 + M\widetilde{M}_S\frac{T^\alpha}{\alpha})(\widetilde{M}_T L_g + \widetilde{M}_S L_f(m+1)\frac{T^\alpha}{\alpha})\|x-y\|_{PC}, & t \in [0,t_1]; \\ (1 + M\widetilde{M}_S\frac{T^\alpha}{\alpha})(\widetilde{M}_T(1+L_1) + \widetilde{M}_S L_f(m+1)\frac{T^\alpha}{\alpha})\|x-y\|_{PC}, & t \in (t_1,t_2]; \\ \quad \vdots \\ (1 + M\widetilde{M}_S\frac{T^\alpha}{\alpha})(\widetilde{M}_T(1+L_p) + \widetilde{M}_S L_f(m+1)\frac{T^\alpha}{\alpha})\|x-y\|_{PC}, & t \in (t_p,T]. \end{cases} \end{aligned}$$

Which implies that for $t \in [0,T]$,

$$\|Nx - Ny\|_{PC} \le \Theta\,\|x - y\|_{PC}.$$

Since $\Theta < 1$, hence N is a contraction map. Therefore, by Banach contraction principle N has a unique fixed point x such that $(Nx)(t) = x(t)$. This fixed point is then a solution of the system (1.1)–(1.3), and clearly $x(T) = z(T)$, which implies that the system is controllable on J. This completes the proof of the theorem. □

4 Example

To illustrate the application of the theory we consider the the following partial differential equation with fractional derivative of the form:

$$\begin{aligned} D_t^\alpha y(t,x) + \frac{\partial^2 y(t,x)}{\partial x^2} &= Bu(t) + f(t, y(t,x), y(a_1(t),x), \ldots, y(a_m(t),x)), \\ &\quad (t,x) \in [0,T] \times (0,\pi),\ t \neq \frac{T}{2}, \\ y(t,0) &= y(t,\pi),\ t \in [0,T], \\ y(0,x) + g(y) &= y_0, \\ \Delta y|_{t=\frac{T}{2}} &= I_1(\frac{T^-}{2}), \end{aligned}$$

where $T > 0$, $0 < \alpha < 1$; $y(t,.)$, $y_0 \in L^2([0,\pi]$. Then the above example resembles the control system (1.1)–(1.3), if we take

(1) $X = L^2([0,\pi])$ as the state space and $y(t,\cdot) = \{y(t,x) : 0 \leq x \leq \pi\}$ as the state.

(2) input trajectory $u(t,.) \in U$ as the control, where U is any Banach space.

(3) $A : D(A) \subset X \to X$ defined by $D(A) = \{z \in X : \frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2} \in X$, are absolutely continuous and $z(0) = z(\pi) = 0\}$. $Au = \frac{\partial^2 u}{\partial x^2}$. Then

$$Az = -\sum_{n=1}^{\infty} n^2 (z, z_n) z_n,\ z \in D(A),$$

where $z_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx),\ n \in \mathbb{N}$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ in X and that is given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2 t}(z, z_n) z_n,\ \text{for all } z \in X,\ \text{and every } t > 0.$$

From these expression it follows that $(T(t))_{t\geq 0}$ is uniformly bounded compact semigroup, so that, $R(\lambda, A) = (\lambda I - A)^{-1}$ is a compact operator for $\lambda \in \rho(A)$ i.e. $A \in A^\alpha(\theta_0, \omega_0)$.

(4) $B : U \to X$ is any bounded linear operator.

(5) $g : X \to X$ is any Lipschitz function satisfying assumption (H1). For example

$$g(y) = \sum_{i=1}^{n} c_i y(b_i),\quad \text{for } y \in X,$$

where c_i $(i = 1, \ldots n)$ are given constants and b_i $(i = 1, \ldots n)$ be given real numbers such that $0 < b_1 < \cdots < b_n \leq T$.

(6) Functions $a_i,\ i = 1, \ldots, m$, may be taken as $a_i(t) = k_i t^p\quad (i = 1, \ldots, m)$ for $t \in [0,T]$, where $k_i \in (0,1]$ $(i = 1, \ldots, m),\ n \in \mathbb{N}$.

(7) $I_1 : X \to X$ is any function satisfying assumption (H3), *e.g.* $I_1(y) = \frac{|y|}{3+|y|}$.

(8) $f : [0, T] \times X^{m+1} \to X, T > 0$ is any function satisfying assumption (H2).

Hence we may apply the results of Theorem 3.1 to this example.

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