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NOTES ON GLOBAL ATTRACTORS FOR A CLASS OF SEMILINEAR DEGENERATE PARABOLIC EQUATIONS

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Abstract. We study the regularity and fractal dimension estimates of global attractors for a class of semilinear degenerate parabolic equations in bounded domains.

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1 Introduction

The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to attack the problem for a dissipative dynamical system is to consider its global attractor. A first question is to study the existence of a global attractor. Once a global attractor is obtained, a next natural question is to study the most important properties of the global attractor from its fractal/Hausdorff dimension and dependence on parameters to its regularity and modes determining. In the last decades, many authors have paid attention

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to these problems and received many results for a large class of partial differential equations (see e.g. [4, 11] and references therein). However, to the best of our knowledge, little seems to be known for the asymptotic behavior of solutions to degenerate equations.

This work is a continuation of the paper [1] in which the authors proved the existence and upper semicontinuity of a global attractor in $L^2(\Omega)$ for the semigroup generated by the following semilinear degenerate parabolic equation with a variable, nonnegative coefficient, defined on a bounded domain $\Omega \subset \mathbb{R}^N, N \geq 2$, with smooth boundary $\partial\Omega$,

$$\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div}(\sigma(x)\nabla u) + f(u) &= g(x),\ x \in \Omega, t > 0, \\ u(x,t) &= 0,\ x \in \partial\Omega, t > 0, \\ u(x,0) &= u_0(x),\ x \in \Omega, \end{aligned} \tag{1.1}$$

where the coefficient diffusion σ, the nonlinearity f, and the external force g satisfy the following conditions:

$(\mathcal{H}_\alpha)$ σ is a nonnegative measurable function such that $\sigma \in L^1_{\text{loc}}(\Omega)$ and for some $\alpha \in (0,2)$, $\liminf_{x\to z} |x-z|^{-\alpha}\sigma(x) > 0$ for every $z \in \overline{\Omega}$;

(F) $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function satisfying

$$\begin{aligned} f(u)u &\geq C_1|u|^p - C_0, \\ |f'(u)| &\leq C_2(1+|u|^{p-2}), \\ f'(u) &\geq -C_3, \end{aligned} \tag{1.2}$$

for some $p \geq 2$, where C_0, C_1, C_2, C_3 are positive constants;

(G) $g \in L^2(\Omega)$.

Problem (1.1) can be derived as a simple model for neutron diffusion (feedback control of nuclear reactor) (see [6]); in this case u and σ stand for the neutron flux and neutron diffusion respectively. The assumption $(\mathcal{H}_\alpha)$ has a strong physical significance which is related to the existence of regions occupied by perfect insulators or perfect conductors [3, 7, 8]. The degeneracy of problem (1.1) is considered in the sense that the measurable, nonnegative diffusion coefficient $\sigma(\cdot)$, is allowed to have at most a finite number of (essential) zeroes at some points.

The long-time behavior of solutions to problem type (1.1) has been studied extensively in recent years (see e.g. [1, 2, 7, 8]). In particular, it is proved in [1] the existence of a global attractor in $L^2(\Omega)$ for the semigroup $S(t)$ generated by problem (1.1) by constructing a bounded absorbing set in $\mathcal{D}^1_0(\Omega,\sigma) \cap L^p(\Omega)$ and using the compactness of the embedding $\mathcal{D}^1_0(\Omega,\sigma) \hookrightarrow L^2(\Omega)$. The aim of this paper is to show that the global attractor obtained in [1] is in fact in $L^{2p-2}(\Omega) \cap \mathcal{D}^2_0(\Omega,\sigma)$ and to estimate its fractal dimension. As we know, if the external force g is only in $L^2(\Omega)$, then solutions of problem (1.1) are at most in $L^{2p-2}(\Omega) \cap \mathcal{D}^2_0(\Omega,\sigma)$ and have no higher regularity. Therefore, we cannot construct a bounded absorbing set in a more regular space, which is compactly embedded into $L^{2p-2}(\Omega) \cap \mathcal{D}^2_0(\Omega,\sigma)$. To overcome the difficulty caused by the lack of compactness of the embeddings, we exploit the asymptotic *a priori* estimate method introduced in [9, 12] to show the asymptotic compactness of $S(t)$ in $L^{2p-2}(\Omega)$ and $\mathcal{D}^2_0(\Omega,\sigma)$. As a result, we obtain the existence of global attractors in the spaces $L^{2p-2}(\Omega)$ and $\mathcal{D}^2_0(\Omega,\sigma)$. These global attractors and the global attractor obtained in [1] are of course the same object because the uniqueness of the global attractor of a semigroup. It is noticed that the obtained results seem to be optimal because any stationary to (1.1) belong to the global attractor and cannot belong to a smaller space than $L^{2p-2}(\Omega)\cap\mathcal{D}_0^2(\Omega,\sigma)$ if the forcing term $g\in L^2(\Omega)$. Finally, under a stronger assumption of the external force g, we prove the boundedness of the global attractor in $L^\infty(\Omega)$, and we use this boundedness to show that the global attractor has a finite fractal dimension.

The rest of the paper is organized as follows. In Section 2, we recall some results on function spaces and global attractors which we will use. Section 3 is devoted to the proof of the existence of the global attractor in $L^{2p-2}(\Omega)\cap\mathcal{D}_0^2(\Omega,\sigma)$ for the semigroup $S(t)$ generated by problem (1.1). In the last section, we give the estimates of the fractal dimension of the global attractor.

2 Preliminaries

2.1 Function spaces and operator

In order to study problem (1.1) we introduce some weighted spaces, namely $\mathcal{D}_0^1(\Omega,\sigma)$ and $\mathcal{D}_0^2(\Omega,\sigma)$, defined as the closures of $C_0^\infty(\Omega)$ with respect to the following norms

$$\|u\|_{\mathcal{D}_0^1(\Omega,\sigma)}:=\left(\int_\Omega \sigma(x)\,|\nabla u|^2\,\mathrm{d}x\right)^{\frac12},$$

$$\|u\|_{\mathcal{D}_0^2(\Omega,\sigma)}:=\left(\int_\Omega |\operatorname{div}(\sigma(x)\,\nabla u)|^2\,\mathrm{d}x\right)^{\frac12},$$

respectively. They are Hilbert spaces with respect to the following scalar products

$$(u,v)_{\mathcal{D}_0^1}:=\int_\Omega \sigma(x)\,\nabla u\nabla v\,\mathrm{d}x,$$

$$(u,v)_{\mathcal{D}_0^2}:=\int_\Omega \operatorname{div}(\sigma(x)\,\nabla u)\ \operatorname{div}(\sigma(x)\,\nabla v)\,\mathrm{d}x.$$

It is known (see e.g. [2]) that the operator $Au:=-\operatorname{div}(\sigma(x)\,\nabla u)$ with the homogeneous Dirichlet boundary condition in Ω has a family $\{e_n\}_{n=1}^\infty$ of eigenvectors, which forms an orthonormal basis of $L^2(\Omega)$, and a sequence of eigenvalues $\{\lambda_n\}_{n\ge1}$ such that $0<\lambda_1\le\ldots\le\lambda_n\le\ldots$ and $\lambda_n\to+\infty$ as $n\to+\infty$.

We recall some basic results of Caldiroli and Musina [3] related to the function space $\mathcal{D}_0^1(\Omega,\sigma)$.

Proposition 2.1 *Assume that Ω is a bounded domain in $\mathbb{R}^N (N\ge 2)$, and σ satisfies $(\mathcal{H}_\alpha)$. Then the following embeddings hold:*

(i) $\mathcal{D}_0^1(\Omega,\rho)\hookrightarrow L^{2^*_\alpha}(\Omega)$ continuously;

(ii) $\mathcal{D}_0^1(\Omega,\rho)\hookrightarrow L^p(\Omega)$ compactly if $p\in[1,2^*_\alpha)$, where $2^*_\alpha=\frac{2N}{N-2+\alpha}$.

The following result follows directly from the definitions of the spaces $\mathcal{D}_0^1(\Omega,\sigma)$, $\mathcal{D}_0^2(\Omega,\sigma)$ and the embedding $\mathcal{D}_0^1(\Omega,\sigma) \hookrightarrow L^2(\Omega)$ when σ satisfies $(\mathcal{H}_\alpha)$.

Proposition 2.2 *Assume that Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$), and σ satisfies $(\mathcal{H}_\alpha)$. Then $\mathcal{D}_0^2(\Omega,\sigma) \hookrightarrow \mathcal{D}_0^1(\Omega,\sigma)$ continuously.*

Proof. For any function $u \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}\|u\|^2_{\mathcal{D}_0^1(\Omega)} &= \int_\Omega \sigma|\nabla u|^2\,\mathrm{d}x = -\int_\Omega \mathrm{div}(\sigma\nabla u)u\,\mathrm{d}x \\ &\leq \Big(\int_\Omega |\,\mathrm{div}(\sigma\nabla u)|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega |u|^2\,\mathrm{d}x\Big)^{1/2} = \|u\|_{\mathcal{D}_0^2(\Omega)}\,\|u\|_{L^2(\Omega)}\,.\end{aligned}$$

Noting that $\|u\|_{L^2(\Omega)} \leq C\,\|u\|_{\mathcal{D}_0^1(\Omega)}$, where C is independent of u, we get the desired result. □

2.2 Global attractors

We recall some results in [12] which will be used later.

Proposition 2.3 *Let $\{S(t)\}_{t\geq 0}$ be a semigroup on $L^r(\Omega)$ and suppose that $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $L^r(\Omega)$. Then for any $\epsilon > 0$ and any bounded subset $B \subset L^r(\Omega)$, there exist two positive constants $T = T(B)$ and $M = M(\epsilon)$ such that*

$$meas(\Omega(|S(t)u_0| \geq M)) \leq \epsilon,$$

for all $u_0 \in B$ and $t \geq T$, where $meas(e)$ denotes the Lebesgue measure of $e \subset \Omega$ and $\Omega(|S(t)u_0| \geq M)) := \{x \in \Omega \mid |(S(t)u_0)(x)| \geq M\}$.

Definition 2.4 *Let X be a Banach space. The semigroup $\{S(t)\}_{t\geq 0}$ on X is called norm-to-weak continuous on X if for any $\{x_n\}_{n=1}^\infty \subset X$, $x_n \to x$, and $t_n \geq 0, t_n \to t$, we have $S(t_n)x_n \rightharpoonup S(t)x$ in X.*

The following result is useful for verifying that a semigroup is norm-to-weak continuous.

Proposition 2.5 *Let X, Y be two Banach spaces and X^*, Y^* be their respective dual spaces. We also assume that X is a dense subspace of Y, the injection $i : X \to Y$ is continuous and its adjoint $i^* : Y^* \to X^*$ is densely injective. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on X and Y, respectively, and assume furthermore that $S(t)$ is continuous or weak continuous on Y. Then $\{S(t)\}_{t\geq 0}$ is norm-to-weak continuous on X iff $\{S(t)\}_{t\geq 0}$ maps compact subsets of $X \times \mathbb{R}^+$ into bounded subsets of X.*

Theorem 2.6 *Let $\{S(t)\}_{t\geq 0}$ be a norm-to-weak continuous semigroup on $L^q(\Omega)$, and be continuous or weak continuous on $L^r(\Omega)$ for some $r \leq q$, and have a global attractor in $L^r(\Omega)$. Then $\{S(t)\}_{t\geq 0}$ has a global attractor in $L^q(\Omega)$ if and only if*

(i) $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $L^q(\Omega)$;

(ii) for any $\epsilon > 0$ and any bounded subset B of $L^q(\Omega)$, there exist positive constants $M = M(\epsilon, B)$ and $T = T(\epsilon, B)$ such that

$$\int_{\Omega(|S(t)u_0| \geq M)} |S(t)u_0|^q \, \mathrm{d}x < \epsilon, \tag{2.1}$$

for any $u_0 \in B$ and $t \geq T$.

Definition 2.7 *The semigroup $\{S(t)\}_{t\geq 0}$ is called satisfying Condition (C) in X if and only if for any bounded set B of X and for any $\epsilon > 0$, there exist a positive constant t_B and a finite-dimensional subspace X_1 of X, such that $\{PS(t)x | x \in B, t \geq t_B\}$ is bounded and*

$$|(I - P)\, S(t)\, x| \leq \epsilon \text{ for any } t \geq t_B \text{ and } x \in B,$$

where $P : X \to X_1$ is the canonical projector.

Theorem 2.8 *Let X be a Banach space and $\{S(t)\}_{t\geq 0}$ be a norm-to-weak continuous semigroup on X. Then $\{S(t)\}_{t\geq 0}$ has a global attractor in X provided that the following conditions hold:*

(i) $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in X,

(ii) $\{S(t)\}_{t\geq 0}$ satisfies Condition (C) in X.

2.3 Fractal dimensions of global attractors

Definition 2.9 *Let M be a compact set in a metric space X. Then its fractal dimension is defined by*

$$\dim_f M = \overline{\lim_{\epsilon \to 0}} \frac{\ln n(M, \epsilon)}{\ln(1/\epsilon)},$$

where $n(M, \epsilon)$ is the minimal number of closed balls of the radius ϵ which cover the set M.

The following result was given in [5].

Theorem 2.10 *Assume that M is a compact set in a Hilbert space H. Let V be a continuous mapping in H such that $M \subset V(M)$. Assume that there exists a finite dimensional projector P in the space H such that*

$$\|P(Vu_1 - Vu_2)\|_H \leq l \|u_1 - u_2\|_H, \quad u_1, u_2 \in M, \tag{2.2}$$
$$\|(I - P)(Vu_1 - Vu_2)\|_H \leq \delta \|u_1 - u_2\|_H, \quad u_1, u_2 \in M, \tag{2.3}$$

where $\delta < 1$. We also assume that $l \geq 1 - \delta$. Then the compact set M possesses a finite fractal dimension, specifically,

$$\dim_f(M) \leq \dim P. \ln \frac{9l}{1 - \delta} \big(\ln \frac{2}{1 + \delta} \big)^{-1}. \tag{2.4}$$

3 Regularity of the global attractor

In the paper [1] the authors constructed a continuous (nonlinear) semigroup $S(t) : L^2(\Omega) \to L^2(\Omega)$ associated to problem (1.1) as follows

$$S(t)u_0 := u(t),$$

where $u(t)$ is the unique weak solution of problem (1.1) with the initial datum u_0, and proved that the semigroup $S(t)$ possesses a compact connected global attractor $\mathcal{A}_{L^2}$ in $L^2(\Omega)$. In this section, we will show that the global attractor $\mathcal{A}_{L^2}$ is in fact in $L^{2p-2}(\Omega) \cap \mathcal{D}_0^2(\Omega, \sigma)$.

3.1 Existence of a global attractor in $L^{2p-2}(\Omega)$

Lemma 3.1 *Assume that assumptions* $(\mathcal{H}_\alpha)$*,* **(F)** *and* **(G)** *hold. Then for any bounded subset* B *in* $L^2(\Omega)$*, there exists a positive constant* $T = T(B)$ *such that*

$$\|u_t(s)\|^2_{L^2(\Omega)} \leq \rho_1 \text{ for any } u_0 \in B \text{ and } s \geq T,$$

where $u_t(s) = \frac{\mathrm{d}}{\mathrm{d}t}(S(t)u_0)|_{t=s}$ *and* ρ_1 *is a positive constant independent of* B.

Proof. We give here some formal caculations, the rigorous proof is done by use of Galerkin approximations and Lemma 11.2 in [10]. More precisely, we first derive some *a priori* estimates for the approximate Galerkin solutions u_m of the form

$$u_m(t) = \sum_{i=1}^{m} c_{im}(t)w_i,$$

where $\{w_i\}_{i=1}^{\infty}$ is a basis of $L^{2p-2}(\Omega) \cap \mathcal{D}_0^2(\Omega, \sigma)$. These solutions are smooth enough to justify the computations. Then we get the corresponding estimates for the solution u by taking limits and using Lemma 11.2 in [10].

By differentiating (1.1) in time and denoting $v = u_t$, we get

$$v_t - \operatorname{div}(\sigma(x)\nabla v) + f'(u)v = 0. \tag{3.1}$$

Multiplying the above equality by v, integrating over Ω and using **(F)**, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{L^2(\Omega)} + \int_\Omega \sigma(x)|\nabla v|^2\,\mathrm{d}x \leq C_3\|v\|^2_{L^2(\Omega)}, \tag{3.2}$$

hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{L^2(\Omega)} \leq 2C_3\|v\|^2_{L^2(\Omega)}. \tag{3.3}$$

On the other hand, it is proved in [1] that there exist a constant R and a time $t_0(\|u_0\|_{L^2(\Omega)})$ such that

$$\|u(t)\|^2_{\mathcal{D}_0^1(\Omega,\sigma)} + \|u(t)\|^p_{L^p(\Omega)} \leq R \quad \text{for all} \quad t \geq t_0(\|u_0\|_{L^2(\Omega)}). \tag{3.4}$$

Taking the inner product of (1.1) with u_t, we obtain

$$\|u_t\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2_{\mathcal{D}_0^1(\Omega,\sigma)} + 2\int_\Omega F(u)\,\mathrm{d}x\right) = \int_\Omega g u_t\,\mathrm{d}x \leq \frac{1}{2}\|g\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_t\|^2_{L^2(\Omega)}, \tag{3.5}$$

where $F(u) = \int_0^u f(\xi)\,\mathrm{d}\xi$, thus

$$\|u_t\|^2_{L^2(\Omega)} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2_{\mathcal{D}^1_0(\Omega,\sigma)} + 2\int_\Omega F(u)\,\mathrm{d}x\right) \le \|g\|^2_{L^2(\Omega)}. \tag{3.6}$$

Noting that from (**F**) we get that

$$C_4(|u|^p - 1) \le F(u) \le C_5(|u|^p + 1). \tag{3.7}$$

Integrating (3.6) from t to $t+1$ and then using (3.7), we get

$$\int_t^{t+1} \|u_t\|^2_{L^2(\Omega)} \le \|g\|^2_{L^2(\Omega)} + 2C_5|\Omega| + \|u(t)\|^2_{\mathcal{D}^1_0(\Omega,\sigma)} + 2\,C_5\,\|u(t)\|^p_{L^p(\Omega)}. \tag{3.8}$$

Since (3.4), there exists a constant C_6 which depends on $\|g\|_{L^2(\Omega)}$, C_4, C_5 and R such that

$$\int_t^{t+1} \|u_t\|^2_{L^2(\Omega)} \le C_6, \text{ for } t \ge t_0(\|u_0\|_{L^2(\Omega)})\,. \tag{3.9}$$

Combining (3.3) with (3.9), and using the uniform Gronwall inequality, we deduce that

$$\|u_t\|^2_{L^2(\Omega)} \le C(\|g\|_{L^2(\Omega)}, |\Omega|),$$

as t large enough. The proof is complete. □

Lemma 3.2 *The semigroup $\{S(t)\}_{t\ge0}$ has a bounded absorbing set in $L^{2p-2}(\Omega)$, i.e., there exists a positive constant ρ_{2p-2}, such that for any bounded subset $B \subset L^2(\Omega)$, there is a number $T = T(B) \ge 0$ such that*

$$\|u(t)\|_{L^{2p-2}(\Omega)} \le \rho_{2p-2}, \textit{ for any } t \ge T, u_0 \in B.$$

Proof. Taking $|u|^{p-2}u$ as a test function, we obtain

$$\int_\Omega |u|^{p-2}u.u_t\,\mathrm{d}x + \int_\Omega \sigma(x)|\nabla u|^2|u|^{p-2}\,\mathrm{d}x + \int_\Omega f(u)|u|^{p-2}u\,\mathrm{d}x = \int_\Omega g|u|^{p-2}u\,\mathrm{d}x.$$

Hence, using (1.2) and Cauchy's inequality, we obtain

$$\begin{aligned}&\int_\Omega \sigma(x)|\nabla u|^2|u|^{p-2}\,\mathrm{d}x + C_1\int_\Omega |u|^{2p-2}\,\mathrm{d}x\\ &\le C_0\int_\Omega |u|^{p-1}\,\mathrm{d}x + \frac{1}{C_1}\int_\Omega |g|^2\,\mathrm{d}x + \frac{C_1}{2}\int_\Omega |u|^{2p-2}\,\mathrm{d}x + \frac{1}{C_1}\int_\Omega |u_t|^2\,\mathrm{d}x.\end{aligned}$$

Using Cauchy's inequality once again, we arrive at

$$\frac{C_1}{4}\int_\Omega |u|^{2p-2}\,\mathrm{d}x \le \frac{1}{C_1}\|g\|^2_{L^2(\Omega)} + \frac{1}{C_1}\int_\Omega |u_t|^2\,\mathrm{d}x + C.$$

By Lemma 3.1, we can conclude that

$$\int_\Omega |u(t)|^{2p-2}\,\mathrm{d}x \le \rho_{2p-2}, \text{ for any } t \ge T, u_0 \in B,$$

where ρ_{2p-2} depends only on $C_0, C_1, C_2, \|g\|_{L^2(\Omega)}$. □

We now derive some estimates for the time derivatives of u by the well-known bootstrap technique. These estimates are useful for establishing asymptotic *a priori* estimates in $L^{2p-2}(\Omega)$.

Lemma 3.3 *For any $2 \leq r < \infty$ and any bounded subset $B \subset L^2(\Omega)$, there exists a positive constant T, which depends on r and the L^2-norm of B, such that*

$$\int_\Omega |u_t(s)|^r \,\mathrm{d}x \leq M \quad \textit{for any } u_0 \in B, s \geq T,$$

where the positive constant M depends on r but not on B, and $u_t(s) = \frac{\mathrm{d}}{\mathrm{d}t}(S(t)u_0)|_{t=s}$.

Proof. We prove by induction on k ($k = 0, 1, 2, \dots$) the existence of T_k, depending on k and B, such that

$$\int_\Omega |u_t(s)|^{2(\frac{N}{N-2+\alpha})^k} \,\mathrm{d}x \leq M_k \quad \text{for any } u_0 \in B, s \geq T_k, \qquad (A_k)$$

and

$$\int_t^{t+1} \Big(\int_\Omega |u_t(s)|^{2(\frac{N}{N-2+\alpha})^{k+1}} \,\mathrm{d}x \Big)^{\frac{N}{N-2+\alpha}} \mathrm{d}s \leq M_k \quad \text{for any } u_0 \in B, s \geq T_k, \qquad (B_k)$$

where M_k depends on k but not on B.

(i) Initialization of the induction ($k = 0$): The estimate (A_0) has been proved in Lemma 3.1, while (B_0) can be derived by integrating (3.2) from t to $t+1$ and using the embedding $\mathcal{D}_0^1(\Omega, \sigma) \hookrightarrow L^{\frac{2N}{N-2+\alpha}}(\Omega)$.

(ii) The induction argument: Assume that (A_k) and (B_k) hold for k, and we prove that they are true for $k+1$.
By differentiating (1.1) in time and denoting $v = u_t$, we have

$$v_t - \operatorname{div}(\sigma(x)\nabla v) + f'(u)v = 0. \tag{3.10}$$

Multiplying (3.10) by $|v|^{2(\frac{N}{N-2+\alpha})^{k+1}-2}.v$ and integrating over Ω, we obtain

$$C\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |v|^{2(\frac{N}{N-2+\alpha})^{k+1}} \,\mathrm{d}x + C\int_\Omega \sigma(x)|\nabla(v^{(\frac{N}{N-2+\alpha})^{k+1}})|^2 \,\mathrm{d}x \leq C_3 \int_\Omega |v|^{2(\frac{N}{N-2+\alpha})^{k+1}} \,\mathrm{d}x\,, \tag{3.11}$$

where the constant C depends on the spatial dimension N and k. Using (B_k) and the uniform Gronwall inequality, we infer from (3.11) that

$$\int_\Omega |v|^{2(\frac{N}{N-2+\alpha})^{k+1}} \,\mathrm{d}x \leq M_{k+1} \text{ for any } t \geq T_k\,, \tag{3.12}$$

which shows that (A_{k+1}) is true. For (B_{k+1}), we integrate (3.11) from t to $t+1$ and use (3.12) to get

$$\int_t^{t+1}\int_\Omega |\nabla(v^{(\frac{N}{N-2+\alpha})^{k+1}})|^2 \,\mathrm{d}x\,\mathrm{d}s \leq M_{k+1}\,. \tag{3.13}$$

Using the embedding $\mathcal{D}_0^1(\Omega, \sigma) \hookrightarrow L^{\frac{2N}{N-2+\alpha}}(\Omega)$, we obtain

$$\begin{aligned}\Big(\int_\Omega |v|^{(\frac{N}{N-2+\alpha})^{k+1}\frac{2N}{N-2+\alpha}} \,\mathrm{d}x \Big)^{\frac{N-2+\alpha}{N}} &= \|v^{(\frac{N}{N-2+\alpha})^{k+1}}\|^2_{L^{\frac{2N}{N-2+\alpha}}(\Omega)} \\ &\leq C\,\|\nabla v^{(\frac{N}{N-2+\alpha})^{k+1}}\|^2_{L^2(\Omega)}\,. \end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14), we deduce (B_{k+1}) immediately. Since $\frac{N}{N-2+\alpha} > 1$ ($N \geq 2$), we have $r \leq 2\,(\frac{N}{N-2+\alpha})^k$ provided that $k \leq \log_{\frac{N}{N-2+\alpha}} \frac{r}{2}$.

□

Lemma 3.4 *For any $\epsilon > 0$ and any bounded subset $B \subset L^2(\Omega)$, there exist $T \geq 0$ and $n_\epsilon \in \mathbb{N}$, such that*

$$\int_\Omega |v_2|^2 \,\mathrm{d}x \leq C\epsilon \quad \textit{for any } u_0 \in B,$$

provided that $t \geq T$ and $m \geq n_\epsilon$, where $v_2 = (I - P_m)\,v = (I - P_m)\,u_t$ and the constant C is independent of B and ϵ.

Proof. Multiplying (3.10) by v_2 and integrating over Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|v_2\|^2_{L^2(\Omega)} + \|v_2\|^2_{\mathcal{D}^1_0(\Omega,\sigma)} \leq \int_\Omega |f'(u)v|\,|v_2|\,\mathrm{d}x\,.$$

Hence,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|v_2\|^2_{L^2(\Omega)} + \lambda_m\|v_2\|^2_{L^2(\Omega)} \leq \int_\Omega |f'(u)v|\,|v_2|\,\mathrm{d}x\,, \tag{3.15}$$

where λ_m is the m^{th} eigenvalue of the operator $Au := -\operatorname{div}(\sigma(x)\nabla u)$ in Ω. From (**F**), Lemmas 3.2 and 3.3, we have

$$\int_\Omega |f'(u)v|^2\,\mathrm{d}x \leq \Big(\int_\Omega |f'(u)|^{2(\frac{p-1}{p-2})}\Big)^{\frac{p-2}{p-1}}\Big(\int_\Omega |v|^{2(p-1)}\Big)^{\frac{1}{p-1}} \leq M_0 \tag{3.16}$$

for any $u_0 \in B$ provided that $t \geq T$, where the constant M_0 is independent of B and the constant T depends only on B and p. Therefore, we infer from (3.15) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v_2\|^2_{L^2(\Omega)} + \lambda_m\,\|v_2\|^2_{L^2(\Omega)} \leq C\,.$$

If $t \geq T$, the last inequality shows that

$$\|v_2(t)\|^2_{L^2(\Omega)} \leq \|v_2(T)\|^2_{L^2(\Omega)}\,e^{-\lambda_m(t-T)} + \frac{C}{\lambda_m}\,(1 - e^{-\lambda_m(t-T)})\,.$$

This implies that the conclusion of the lemma is true provided that t and m are large enough. □

Choosing $Y = L^2(\Omega), X = L^{2p-2}(\Omega)$, by Proposition 2.5, we see that the semigroup $\{S(t)\}_{t\geq 0}$ is norm-to-weak continuous on $L^{2p-2}(\Omega)$. Thus, by Theorem 2.6, to prove the existence of a global attractor in $L^{2p-2}(\Omega)$, we only need to prove the following

Lemma 3.5 *For any $\epsilon > 0$ and any bounded subset $B \subset L^2(\Omega)$, there exist positive constants $M = M(B,\epsilon)$ and $T = T(B,\epsilon)$ such that*

$$\int_{\Omega(|u(t)|\geq M)} |u(t)|^{2p-2}\,\mathrm{d}x \leq C\epsilon \quad \textit{for any } u_0 \in B \textit{ as } t \geq T,$$

where the constant C is independent of B and ϵ.

Proof. For any fixed $\epsilon > 0$, by Lemma 2.3 and (**F**), there exist $M_1 = M_1(B, \epsilon) > 0$ and $T_1 = T_1(B, \epsilon) > 0$, such that the following estimates are valid for any $u_0 \in B$ and $t \geq T_1$:

$$\begin{aligned} &\int_{\Omega(|u(t)|\geq M_1)} |g|^2 \,\mathrm{d}x < \epsilon \text{ and } meas((\Omega|u(t)| \geq M_1)) < \epsilon, \\ &\int_{\Omega(|u(s)|\geq M_1)} |u_t(s)|^2 \,\mathrm{d}x < C\epsilon \quad \text{for } s \geq T_1, \end{aligned} \tag{3.17}$$

and $f(s) \geq 0$ for any $s \geq M_1$, $f(s) \leq 0$ for any $s \leq -M_1$. Denote $\Omega_{M_1} = \Omega(u(t) \geq M_1)$ and $\Omega_{2M_1} = \Omega(u(t) \geq 2M_1)$. Multiplying (1.1) by $(u - M_1)_+^{p-2}(u - M_1)_+$, where

$$(u - M_1)_+ = \begin{cases} u - M_1\,, & u \geq M_1\,, \\ 0\,, & u \leq M_1\,, \end{cases}$$

we have

$$\begin{aligned} \int_{\Omega_{M_1}} (u - M_1)_+^{p-1} u_t \,\mathrm{d}x + (p-1)\int_{\Omega_{M_1}} \sigma(x)(u - M_1)_+^{p-2}|\nabla u|^2 \,\mathrm{d}x + \int_{\Omega_{M_1}} f(u)(u - M_1)_+^{p-1} \,\mathrm{d}x \\ \leq \int_{\Omega_{M_1}} |g|^2 \,\mathrm{d}x \int_{\Omega_{M_1}} (u - M_1)_+^{2p-2} \,\mathrm{d}x\,. \end{aligned}$$

Hence and using (3.17), we have

$$\int_{\Omega_{M_1}} f(u)(u - M_1)^{p-1} \,\mathrm{d}x \leq C\epsilon.$$

Therefore, we have

$$\begin{aligned} \int_{\Omega_{2M_1}} f(u)u^{p-1}\frac{1}{2^{p-1}} \,\mathrm{d}x &\leq \int_{\Omega_{2M_1}} f(u)u^{p-2}\Big(1 - \frac{M_1}{u}\Big)^{p-1} \,\mathrm{d}x \\ &\leq \int_{\Omega_{M_1}} f(u)(u - M_1)^{p-1} \,\mathrm{d}x \leq C\epsilon. \end{aligned}$$

Noticing that $meas(\Omega_{2M_1}) \leq \epsilon$ and (**F**), the above inequality implies that

$$\int_{\Omega_{2M_1}} u^{2p-2} \,\mathrm{d}x \leq C\epsilon \text{ as } t \geq T_1. \tag{3.18}$$

Now taking $|(u + M_1)_-|^{p-2}(u + M_1)_-$ as a test function, where

$$(u + M_1)_- = \begin{cases} u + M_1, & u \geq -M_1 \\ 0, & u \leq -M_1, \end{cases}$$

we have in the same in fashion as above that

$$\int_{\Omega(u(t)\leq -2M_1)} |u(t)|^{2p-2} \,\mathrm{d}x \leq C\epsilon, \text{ as } t \geq T_1. \tag{3.19}$$

Combining (3.18) and (3.19), we have

$$\int_{\Omega(|u(t)|\geq 2M_1)} |u(t)|^{2p-2} \,\mathrm{d}x \leq C\epsilon, \text{ for any } u_0 \in B, t \geq T_1.$$

This completes the proof. □

Therefore, by Theorem 2.6, we have

Theorem 3.6 *Under the conditions* $(\mathcal{H}_\alpha)$*,* **(F)** *and* **(G)***, the semigroup* $\{S(t)\}_{t\geq 0}$ *generated by problem* (1.1) *has a global attractor* $\mathcal{A}_{L^{2p-2}}$ *in* $L^{2p-2}(\Omega)$*, that is,* $\mathcal{A}_{L^{2p-2}}$ *is compact, invariant in* $L^{2p-2}(\Omega)$ *and attracts every bounded set of* $L^2(\Omega)$ *in the topology of* $L^{2p-2}(\Omega)$*.*

3.2 Existence of a global attractor in $\mathcal{D}_0^2(\Omega,\sigma)$

First, we show the existence of a bounded absorbing set in $\mathcal{D}_0^2(\Omega,\sigma)$.

Lemma 3.7 *The semigroup* $\{S(t)\}_{t\geq 0}$ *has a bounded absorbing set in* $\mathcal{D}_0^2(\Omega,\sigma)$*, i.e., there exists a constant* $\rho_A > 0$ *such that for any bounded subset* $B \subset L^2(\Omega)$*, there is a* $T_B > 0$ *such that*

$$\|\operatorname{div}(\sigma(x)\nabla u(t))\|_{L^2(\Omega)} \leq \rho_A, \text{ for any } t \geq T_B, u_0 \in B.$$

Proof. Taking the L^2-inner product of (1.1) with $-\operatorname{div}(\sigma(x)\nabla u)$, we have

$$\begin{aligned}&\|\operatorname{div}(\sigma(x)\nabla u)\|^2_{L^2(\Omega)}\\ &\leq \int_\Omega u_t \operatorname{div}(\sigma(x)\nabla u)\,\mathrm{d}x + \int_\Omega f'(u)\sigma(x)|\nabla u|^2\,\mathrm{d}x - \int_\Omega g(x)\operatorname{div}(\sigma(x)\nabla u)\,\mathrm{d}x.\end{aligned}$$

By the Hölder inequality and assumption (**F**) we have

$$\|\operatorname{div}(\sigma(x)\nabla u)\|^2_{L^2(\Omega)} \leq C(\|u_t\|^2_{L^2(\Omega)} + \|u\|^2_{\mathcal{D}_0^1(\Omega,\sigma)} + \|g\|^2_{L^2(\Omega)}). \tag{3.20}$$

Hence, from Lemma 3.1 and the fact that $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $\mathcal{D}_0^1(\Omega,\sigma)$, we have

$$\|\operatorname{div}(\sigma(x)\nabla u(t))\|_{L^2(\Omega)} \leq \rho_A$$

for t large enough. This completes the proof. □

Let $\mathcal{K}(A)$ be the Kuratowski measure of noncompactness in $L^2(\Omega)$ of A defined by

$$\mathcal{K}(A) = \inf\{\delta > 0 \mid A \text{ has a finite open cover of sets of diameter } < \delta\}.$$

We have the following lemma in [12].

Lemma 3.8 *Assume* $f(.)$ *satisfies conditions* **(F)***. Then for any subset* $A \subset L^{2p-2}(\Omega)$*, if* $\mathcal{K}(A) < \epsilon$ *in* $L^{2p-2}(\Omega)$*, then we have*

$$\mathcal{K}(f(A)) < C\epsilon \;\text{ in }\; L^2(\Omega),$$

where $f(A) = \{f(u) \mid u \in A\}$ *and the constant* C *depends on the* L^{2p-2}*-norm of* A*, the Lebesgue measure of* Ω *and the coefficients* C_0, C_1, C_2 *in* **(F)***.*

Let $H_m = \operatorname{span}\{e_1, e_2, ..., e_m\}$ in $L^2(\Omega)$, where $\{e_j\}_{j=1}^\infty$ are eigenvectors of the operator $Au = -\operatorname{div}(\sigma(x)\nabla u)$ with the homogeneous Dirichlet boundary condition in Ω and $P_m : L^2(\Omega) \to H_m$ be the orthogonal projection. We now verify that $\{S(t)\}_{t\geq 0}$ satisfies Condition (C) in $\mathcal{D}_0^2(\Omega,\sigma)$.

Lemma 3.9 *For any $\epsilon > 0$ and any bounded subset $B \subset L^2(\Omega)$, there exist $T = T(\epsilon, B) \geq 0$ and $n_\epsilon \in \mathbb{N}$, such that*

$$\int_\Omega |(I - P_m)\,\mathrm{div}(\sigma(x)\nabla u)|^2\,\mathrm{d}x \leq \epsilon \text{ for any } u_0 \in B,$$

provided that $t \geq T$ and $m \geq n_\epsilon$.

Proof. Denoting $u_2 = (I - P_m)\,u$, and multiplying (1.1) by $-\,\mathrm{div}(\sigma(x)\nabla u_2)$, we have

$$\begin{aligned}&\int_\Omega |(I - P_m)\,\mathrm{div}(\sigma(x)\nabla u)|^2\,\mathrm{d}x\\ &\leq \int_\Omega u_t\,\mathrm{div}(\sigma(x)\nabla u_2)\,\mathrm{d}x + \int_\Omega f(u)\,\mathrm{div}(\sigma(x)\nabla u_2)\,\mathrm{d}x - \int_\Omega g(x)\,\mathrm{div}(\sigma(x)\nabla u_2)\,\mathrm{d}x\,.\end{aligned}$$

By Cauchy's inequality, we have

$$\int_\Omega |(I-P_m)\,\mathrm{div}(\sigma(x)\,\nabla u)|^2\,\mathrm{d}x \leq \frac{1}{2}\int_\Omega |(I-P_m)\,u_t|^2\,\mathrm{d}x + \int_\Omega |f(u)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |(I-P_m)\,g|^2\,\mathrm{d}x\,.$$

From Lemmas 3.4 and 3.8, we have

$$\int_\Omega |(I - P_m)\,\mathrm{div}(\sigma(x)\,\nabla u)|^2\,\mathrm{d}x \leq \epsilon \text{ for any } u_0 \in B,\ t \geq T,\ m \geq n_\epsilon\,.$$

□

From Lemmas 3.7, 3.9 and Theorem 2.8, we obtain the following result.

Theorem 3.10 *Assume conditions $(\mathcal{H}_\alpha)$, **(F)** and **(G)** hold. Then the semigroup $\{S(t)\}_{t\geq 0}$ generated by problem (1.1) has a global attractor $\mathcal{A}_{\mathcal{D}_0^2}$ in $\mathcal{D}_0^2(\Omega, \sigma)$, that is, $\mathcal{A}_{\mathcal{D}_0^2}$ is compact, invariant in $\mathcal{D}_0^2(\Omega, \sigma)$ and attracts every bounded set of $L^2(\Omega)$ in the topology of $\mathcal{D}_0^2(\Omega, \sigma)$.*

Remark 3.11 The global attractors $\mathcal{A}_{L^{2p-2}}$ and $\mathcal{A}_{\mathcal{D}_0^2}$ obtained in Theorems 3.6 and 3.10 are of course the same object and are equal to the global attractor $\mathcal{A}_{L^2}$ obtained in [1]. From now on, we will denote by $\mathcal{A}$ the global attractor of the semigroup associated to problem (1.1). In particular, we have that $\mathcal{A}$ is a compact set in $L^{2p-2}(\Omega) \cap \mathcal{D}_0^2(\Omega, \sigma)$ and is connected in $L^{2p-2}(\Omega)$ and $\mathcal{D}_0^2(\Omega, \sigma)$.

4 Fractal dimension estimates of the global attractor

In this section, instead of $(\mathbf{G})$, we assume the external force g satisfies a stronger condition:

$(\mathbf{G}')$ $g \in L^\infty(\Omega)$.

Lemma 4.1 *Under conditions $(\mathbf{F})$ and $(\mathbf{G}')$, the global attractor $\mathcal{A}$ is uniformly bounded in $L^\infty(\Omega)$.*

Proof. We multiply the first equation in (1.1) by $(u-M)_+$ and integrate over Ω, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-M)_+^2\,\mathrm{d}x + \int_\Omega \sigma(x)|\nabla(u-M)_+|^2\,\mathrm{d}x + \int_\Omega f(u)(u-M)_+\,\mathrm{d}x = \int_\Omega g(u-M)_+\,\mathrm{d}x.$$

Using the embedding $\mathcal{D}_0^1(\Omega,\sigma)\subset L^2(\Omega)$, we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-M)_+^2\,\mathrm{d}x + \lambda\int_\Omega (u-M)_+^2\,\mathrm{d}x \le \int_\Omega (g-f(u))(u-M)_+\,\mathrm{d}x.$$

By hypothesis (**F**), $f(u)\to+\infty$ as $u\to+\infty$, so we can choose M large enough such that $f(u)\ge\|g\|_{L^\infty(\Omega)}$ when $u\ge M$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-M)_+^2\,\mathrm{d}x + 2\lambda\int_\Omega (u-M)_+^2\,\mathrm{d}x \le 0.$$

By Gronwall's inequality, we have

$$\int_\Omega (u-M)_+^2\,\mathrm{d}x \le e^{-2\lambda t}\int_\Omega (u_0-M)^2\,\mathrm{d}x \to 0 \text{ as } t\to+\infty.$$

Since the attractor is bounded in $L^2(\Omega)$ and for any $v\in\mathcal{A}$ there exists a u_0 such that $v=S(t)u_0$, we have

$$\int_\Omega (u-M)_+^2\,\mathrm{d}x = 0 \tag{4.1}$$

for all $u\in\mathcal{A}$. Repeating the same step above, just taking $(u+M)_-$ instead of $(u-M)_+$, we deduce that

$$\int_\Omega (u+M)_-^2\,\mathrm{d}x = 0. \tag{4.2}$$

Taking into account the definitions of $(u-M)_+$ and $(u+M)_-$ (see the proof of Lemma 3.5), it follows from (4.1) and (4.2) that $|u(x)|\le M$ for a.e. $x\in\Omega$, that is, $\|u\|_{L^\infty(\Omega)}\le M$. □

Theorem 4.2 *Assume that assumptions* $(\mathcal{H}_\alpha)$, **(F)** *and* **(G)** *hold. Then the global attractor* $\mathcal{A}$ *of the semigroup associated to problem* (1.1) *possesses a finite fractal dimension in* $L^2(\Omega)$*, specifically,*

$$\dim_f\mathcal{A}\le m\,\ln\frac{9\,e^{C_3}}{1-\delta}\left(\ln\frac{2}{1+\delta}\right)^{-1},$$

where $\delta=e^{-2\,\lambda_m}+\frac{C}{C_3+\lambda_m}$ *for some* $C>0$ *and* m *is large enough such that* $\delta<1$.

Proof. Let $u_{01},u_{02}\in\mathcal{A}$ arbitrary, and let $u_1(t)=S(t)\,u_{01}$ and $u_2(t)=S(t)\,u_{02}$ be solutions to problem (1.1) with initial data u_{01},u_{02}. Since $S(t)\mathcal{A}=\mathcal{A}$ for all $t\ge0$ and, by Lemma 4.1, $\mathcal{A}$ is a bounded set in $L^\infty(\Omega)$, there exists $M>0$ such that

$$\|u_i(t)\|_{L^\infty(\Omega)}\le M, i=1,2,\ \text{for all } t\ge0. \tag{4.3}$$

Putting $w(t) = u_1(t) - u_2(t)$, from (1.1) we have

$$w_t - \mathrm{div}(\sigma(x)\nabla w) + f(u_1) - f(u_2) = 0. \tag{4.4}$$

Taking the inner product of (4.4) with $w(t)$ in $L^2(\Omega)$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{L^2(\Omega)} + \|w\|^2_{\mathcal{D}^1_0(\Omega,\sigma)} + (f(u_1) - f(u_2), w) = 0.$$

Using hypothesis (**F**), in particular, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{L^2(\Omega)} \le 2C_3\|w\|^2_{L^2(\Omega)},$$

hence,

$$\|w(t)\|^2_{L^2(\Omega)} \le e^{2C_3 t}\|w(0)\|^2_{L^2(\Omega)}.$$

Let $w(t) = w_1(t) + w_2(t)$, where $w_1(t)$ is the projection of $w(t)$ in $P_m L^2(\Omega)$, then

$$\|w_1(t)\|^2_{L^2(\Omega)} \le e^{2C_3 t}\|w(0)\|^2_{L^2(\Omega)}. \tag{4.5}$$

On the other hand, taking the inner product of (4.4) with $w_2(t)$ in $L^2(\Omega)$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_2\|^2_{L^2(\Omega)} + \|w_2\|^2_{\mathcal{D}^1_0(\Omega,\sigma)} + (f(u_1) - f(u_2), w_2) = 0\,.$$

Since

$$\begin{aligned}\Big|\int_\Omega (f(u_1) - f(u_2))\, w_2\, \mathrm{d}x\Big| &\le \int_\Omega |f'(u_1 + \theta(u_2 - u_1))|\, |w|\, |w_2|\, \mathrm{d}x \\ &\le C\int_\Omega (1 + |u_1|^{p-2} + |u_2|^{p-2})\, |w|\, |w_2|\, \mathrm{d}x \\ &\le C\, \|w_2\|_{L^2(\Omega)}\, \|w\|_{L^2(\Omega)} \Big(1 + \|u_1\|^{p-2}_{L^\infty(\Omega)} + \|u_2\|^{p-2}_{L^\infty(\Omega)}\Big), \\ &\le C\, \|w\|^2_{L^2(\Omega)} \ \text{ because } \ \|u_i\|_{L^\infty(\Omega)} \le M,\ i = 1, 2\,,\end{aligned}$$

and $\|w_2\|^2_{\mathcal{D}^1_0(\Omega,\sigma)} \ge \lambda_m\, \|w_2\|^2_{L^2(\Omega)}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_2\|^2_{L^2(\Omega)} + 2\,\lambda_m\, \|w_2\|^2_{L^2(\Omega)} \le C\, \|w\|^2_{L^2(\Omega)}.$$

Hence, using Gronwall's inequality we have

$$\begin{aligned}\|w_2(t)\|^2_{L^2(\Omega)} &\le e^{-2\lambda_m t}\|w_2(0)\|^2_{L^2(\Omega)} + C\, e^{-2\lambda_m t}\int_0^t e^{2\lambda_m s}\, \|w(s)\|^2_{L^2(\Omega)}\, \mathrm{d}s \\ &\le e^{-2\lambda_m t}\|w_2(0)\|^2_{L^2(\Omega)} + C\, e^{-2\lambda_m t}\int_0^t e^{2\lambda_m s} e^{2C_3 s}\|w(0)\|^2_{L^2(\Omega)}\, \mathrm{d}s \\ &\le \left(e^{-2\lambda_m t} + \frac{Ce^{2C_3 t}}{\lambda_m + C_3}\right) \|w(0)\|^2_{L^2(\Omega)}\,. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), in particular, we have

$$\|w_1(1)\|^2_{L^2(\Omega)} \le e^{2C_3}\, \|w(0)\,\|^2_{L^2(\Omega)}, \quad \|w_2(1)\|^2_{L^2(\Omega)} \le \delta\, \|w(0)\|^2_{L^2(\Omega)}\,,$$

where $\delta = e^{-2\lambda_m} + \frac{C}{\lambda_m + C_3} < 1$ if m is sufficiently large. Now, applying Theorem 2.8 with $M = \mathcal{A}$, $V = S(1)$, $l = e^{2C_3}$, and δ as above, we get the desired result. □

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