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EXISTENCE OF ALMOST AUTOMORPHIC SOLUTIONS OF NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we provide a set of sufficient conditions for the existence and uniqueness of an almost automorphic mild solution to a first order nonautonomous partial neutral functional differential equations. As an application, a first-order boundary value problem arising in control systems is considered.

Keywords: Almost automorphic function; evolution semigroup; neutral functional differential equation; mild solution.

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1 Introduction

Let $(X, \|.\|)$ be a complex Banach space. This paper deals with the existence and uniqueness of an almost automorphic solution of the following nonautonomous neutral functional-differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}[u(t) + f(t, u_t)] = A(t)u(t) + g(t, u_t), \quad t \in [\sigma, \sigma + b), \tag{1.1}$$

$$u_\sigma = \phi \in \mathcal{B} \tag{1.2}$$

where $A(t) : D(A(t)) \subset X \to X$ is a family of densely defined closed linear operators on a common domain $\mathbb{D} = D(A(t))$, which is independent of t and generates a family of exponentially

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stable evolution operators $\{U(t,s) : t \geq s > -\infty\}$, in the sense that there exist constants $M > 0$, $\zeta > 0$ such that

$$\|U(t,s)\| \leq Me^{-\zeta(t-s)}.$$

Note that for each $x \in X$, the function $(t,s) \to U(t,s)x$ is continuous and $U(t,s) \in \mathcal{L}(X,\mathbb{D})$ for every $t > s$, (cf. [17]). The history $u_t : (-\infty, 0] \to X$, defined by $u_t(\theta) := u(t+\theta)$ for each $\theta \in (-\infty, 0]$, belongs to an abstract phase space $\mathcal{B}$, that is defined axiomatically. The functions f and g are continuous functions from $\mathbb{R} \times \mathcal{B}$ to X.

To the best of our knowledge, N'Guérékata [7] was first to study the exponentially stable almost automorphic equations and subsequently Goldstein and N'Guérékata [6] generalized the concept.

In this paper, we prove the existence and uniqueness of an almost automorphic solution of (1.1) and (1.2). However, Diagana et al. [4] also have considered the same equation as (1.1) and provided the sufficient conditions for the existence and uniqueness of a pseudo almost periodic solution.

The almost automorphy is the generalization of the almost periodicity. The notion of almost automorphic functions was introduced by Bochner [1], in his landmark paper and asymptotically almost automorphic functions were introduced by N'Guérékata [9]. Almost automorphic functions together with compact almost automorphic functions were extensively studied by Veech and Zaki [18, 19, 20].

The existence of almost automorphic solutions has been considered by many authors, see e.g. [16, 14, 15, 2, 5]. Recently authors [16] have proved the existence and uniqueness of almost automorphic solution of a neutral functional differential equation. For more on neutral functional differential equations, we refer [10].

Recently Diagana, Henriquez and Hernandez [3] have considered the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}[u(t) + f(t,u_t)] = Au(t) + g(t,u_t), \quad t \in [\sigma, \sigma + b), \tag{1.3}$$

and have given a set of sufficient conditions for the existence and uniqueness of an almost automorphic mild solutions of (1.3). This paper generalizes the results of [3], to the case of a nonautonomous evolution equation.

2 Preliminaries

In this section we recall certain definitions and lemmas to be used subsequently in this paper.

Definition 2.1 *A strongly continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ of $\{s'_n\}_{n\in\mathbb{N}}$ such that*

$$g(t) = \lim_{n\to\infty} f(t + s_n),$$

is well defined for each $t \in \mathbb{R}$ and

$$f(t) = \lim_{n\to\infty} g(t - s_n),$$

for each $t \in \mathbb{R}$.

We denote by $AA(X)$, the set of all such functions.

Remark 2.2 *The range of an almost automorphic function is relatively compact on X, therefore it is bounded.*

Lemma 2.3 *(N'Guérékata [8]), $(AA(X), \|.\|_{AA(X)})$ is a Banach space with the supremum norm given by,*

$$\|f\|_{AA(X)} = \sup_{t\in\mathbb{R}} \|f(t)\|.$$

Let Z and W be two Banach spaces.

Definition 2.4 *A continuous function $f : \mathbb{R} \times Z \to W$ is said to be almost automorphic, in $t \in \mathbb{R}$ for each $z \in Z$ if for every real sequence (s'_n), there exists a subsequence (s_n) such that*

$$g(t, z) := \lim_{n\to\infty} f(t + s_n, z) \text{ in } W,$$

is well defined in W, for each $t \in \mathbb{R}$ and each $z \in Z$ and

$$f(t, z) := \lim_{n\to\infty} g(t - s_n, z) \text{ in } W,$$

is well defined for each $t \in \mathbb{R}$ each $z \in Z$.

We denote by $AA(\mathbb{R} \times Z, W)$, the set of all such functions.

Theorem 2.5 *Let $f : \mathbb{R} \times Z \to W$ be an almost automorphic function in $t \in \mathbb{R}$ for each $z \in Z$ and assume that F satisfies Lipschitz condition in z uniformly in $t \in \mathbb{R}$. Let $\phi : \mathbb{R} \to Z$ be almost automorphic. Then the function $\Phi : \mathbb{R} \to W$ defined by $\Phi(t) = f(t, \phi(t))$ is almost automorphic.*

Proof. For a proof of the theorem, we refer to N'Guérékata [9]. □

Definition 2.6 *A continuous function $f : \mathbb{R} \to Z$ is said to be compact almost automorphic, if for every sequence of real numbers (s'_n), there exists a subsequence (s_n) such that $g(t) := \lim_{n\to\infty} f(t + s_n)$ and $f(t) = \lim_{n\to\infty} g(t - s_n)$ uniformly on compact subsets of $\mathbb{R}$. The collection of those functions will be denoted by $AA_c(Z)$.*

Let us denote by $AP(Z)$, the set of all functions $\{f; f : \mathbb{R} \to Z\}$ such that f is almost periodic. $AP(Z), AA_c(Z)$ and $AA(Z)$ are closed subsets of $(BC(\mathbb{R}, Z), \|.\|_\infty)$ with

$$AP(Z) \subset AA_c(Z) \subset AA(Z) \subset BC(\mathbb{R}, Z)$$

A proof of next lemma follows in view of the above statement.

Lemma 2.7 *N'Guérékata [8], $AA_c(Z)$ endowed with sup norm is a Banach space.*

Definition 2.8 *A continuous function $f : \mathbb{R}\times Z \to W, (t, u) \to f(t, u)$ is said to be compact almost automorphic, if for every sequence of real numbers (s'_n) there exists a subsequence (s_n) such that $g(t, z) := \lim_{n\to\infty} f(t + s_n, z)$ and $f(t, z) = \lim_{n\to\infty} g(t - s_n, z)$ in W, uniformly on compact subsets of $\mathbb{R}$, for each $z \in Z$. The collection of those functions will be denoted by $AA_c(Z, W)$.*

Now we will give a definition of fading memory space (phase space) $\mathcal{B}$, axiomatically using ideas and notations developed in [13]. More precisely, $\mathcal{B}$ will denote the vector space of functions $x_t : (-\infty, 0] \to X$, defined as $x_t(s) = x(t+s)$ for $s \in \mathbb{R}^-$, endowed with a seminorm denoted by $\|.\|_{\mathcal{B}}$. A Banach space $(\mathcal{B}, \|.\|_{\mathcal{B}})$ which consists of such functions $\psi : (-\infty, 0] \to X$, is called a "fading memory space", if it satisfies the following axioms.

(A) If $x : (-\infty, \sigma + a) \to X$ with $a > 0$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for each $t \in [\sigma, \sigma + a)$ the following conditions hold:
  (i) x_t is in $\mathcal{B}$,
  (ii) $\|x(t)\| \le H\|x_t\|_{\mathcal{B}}$,
  (iii) $\|x_t\|_{\mathcal{B}} \le K(t-\sigma)\sup\{\|x(s)\| : \sigma \le s \le t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$,
  where $H > 0$ is a constant, and $K, M : [0, \infty) \to [1, \infty)$ are functions such that $K(.)$ and $M(.)$ are respectively continuous and locally bounded, and H, K, M are independent of $x(.)$.

$(A1)$ If $x(.)$ is a function as in (A), then x_t is a $\mathcal{B}$ valued continuous function on $[\sigma, \sigma + a)$.

(B) The space $\mathcal{B}$ is complete.

$(C2)$ If $(\phi^n)_{n\in\mathbb{N}}$ is a sequence of continuous functions with compact support defined from $(-\infty, 0]$ into X, which converges to ϕ uniformly on compact subsets of $(-\infty, 0]$ and if $\{\phi^n\}$ is a cauchy sequence in $\mathcal{B}$, then $\phi \in \mathcal{B}$ and $\phi^n \to \phi$ in $\mathcal{B}$.

Remark 2.9 *Throughout the rest of the paper, by axiom $(C2)$, there exists a constant $\mathfrak{L} > 0$ such that $\|\phi\|_{\mathcal{B}} \le \mathfrak{L}\sup_{\theta \le 0}\|\phi\|_\theta$ for every $\phi \in BC((-\infty, 0]; X)$, see proposition 7.1.1 of [13].*

Definition 2.10 *Let $S(t) : \mathcal{B} \to \mathcal{B}$ be a C_0 semigroup defined by $S(t)\phi(\theta) = \phi(0)$ on $[-t, 0]$ and $S(t)\phi(\theta) = \phi(t+\theta)$ on $(-\infty, -t]$. The phase space $\mathcal{B}$ is called a fading memory space if $\|S(t)\phi\|_{\mathcal{B}} \to 0$ as $t \to \infty$ for each $\phi \in \mathcal{B}$ with $\phi(0) = 0$.*

Remark 2.11 *Throughout this paper we suppose the existence of a constant $\mathcal{K} > 0$, such that $\max\{K(t), M(t)\} \le \mathcal{K}$ for each $t \ge 0$. We observe that this condition is verified, for example if $\mathcal{B}$ is a fading memory space, see, e.g., proposition 7.1.5 of [13] for details.*

3 Main results

In this section, we prove the existence and uniqueness of an almost automorphic solution of system (1.1), (1.2). For that we first mention some assumptions, then prove certain lemmas, required for the proof of our main result.

We shall assume the following conditions:

(H1) We assume that the operators $A(t)$ and $U(r, s)$ commute that is,

$$A(t)U(r, s) = U(r, s)A(t).$$

(H2) The function $s \to A(s)U(t,s)$, defined from $(-\infty, t)$ into $\mathcal{L}(X)$ is strongly measurable and there exists a nonincreasing function $H : [0,\infty) \to [0,\infty)$ and $\gamma > 0$ with $e^{-\delta s}H(s) \in L^1([0,\infty))$ such that

$$\|A(s)U(t,s)\| \le e^{-\gamma(t-s)}H(t-s), \quad t > s.$$

(H3) $U(t,s) \in bAA(X)$, that is for every sequence (s'_n) of real numbers, there is a subsequence (s_n) such that

$$\|U(t+s_n, s+s_n) - U(t,s)\| < \epsilon.$$

(H4) The functions $f : \mathbb{R} \times \mathcal{B} \to X,\ g : \mathbb{R} \times \mathcal{B} \to X$ are almost automorphic with respect to the first variable and are Lipschitz continuous with respect to the second variable, that is there exist continuous and bounded functions $L_f, L_g : \mathbb{R} \to (0,\infty)$ such that

$$\|f(t,x) - f(t,y)\| \le L_f(t).\|x-y\|,$$

$$\|g(t,x) - g(t,y)\| \le L_g(t).\|x-y\|.$$

Now we give a definition of the mild solution of system (1.1) and (1.2).

Definition 3.1 *A continuous function $u : [\sigma, \sigma+b) \to X,\ b > 0$, is called mild solution for the neutral system (1.1) and (1.2) on $[\sigma,\sigma+b)$, if $u_s \in \mathcal{B}$ for every $s \in \mathbb{R}$ and the function $s \to A(s)U(t,s)f(s,u_s)$ is integrable on $[\sigma,t)$ for every $\sigma < t < \sigma+b$, and*

$$u(t) = U(t,\sigma)(\phi(0) + f(\sigma,\phi)) - f(t,u_t) - \int_\sigma^t A(s)\,U(t,s)\,f(s,u_s)\,\mathrm{d}s + \int_\sigma^t U(t,s)\,g(s,u_s)\,\mathrm{d}s, \quad t \in [\sigma,\sigma+b).$$

Remark 3.2 *In general the operator function $s \to A(s)U(t,s)$ is not integrable over $(-\infty,t)$. If $\mathfrak{L}$ and $\mathcal{K}$ are the constants appearing in the Remarks 2.9 and 2.11 respectively and f satisfies (H3), then from Bochner's criterion of integrability of functions and the following estimate*

$$\begin{aligned}\|A(s)U(t,s)f(s,u_s)\| &\le \|A(s)U(t,s)\|_{\mathcal{L}(X)}.\|f(s,u_s)\|_X \\ &\le e^{-\gamma(t-s)}H(t-s)\Bigg(L_f(s)(M(s)\|u_s\|_{\mathcal{B}} + K(s)\|u\|_\infty) + \|f(s,0)\|\Bigg) \\ &\le e^{-\gamma(t-s)}H(t-s)\Bigg(L_f(s)(\mathfrak{L}+1)\mathcal{K}\|u\|_\infty + \|f(s,0)\|\Bigg),\end{aligned}$$

it follows that the function $s \to A(s)\,U(t,s)\,f(s,u_s)$ is integrable on $(-\infty,t)$ for each $t > 0$.

Now the following definition of a mild solution of equations (1.1) and (1.2) is well defined.

Definition 3.3 *A function $u \in AA_c(X)$ is an almost automorphic solution to neutral system (1.1) and (1.2) provided that the function $s \to A(s)U(t,s)f(s,u_s)$, is integrable on $(-\infty,t)$ for each $t \in \mathbb{R}$ and the following holds:*

$$u(t) = -f(t,u_t) - \int_{-\infty}^t A(s)\,U(t,s)\,f(s,u_s)\,\mathrm{d}s + \int_{-\infty}^t U(t,s)\,g(s,u_s)\,\mathrm{d}s,\ t \in \mathbb{R}.$$

Lemma 3.4 *For $u \in AA_c(X)$, the function Fu defined by*

$$(Fu)(t) := \int_{-\infty}^{t} U(t,s)\, u(s)\, \mathrm{d}s,$$

is also compact almost automorphic.

Proof. First we observe that Fu is bounded. Since u is compact almost automorphic, so is bounded (say $\|u\| \le M_1$). Hence

$$\begin{aligned} \|(Fu)(t)\| &\le \int_{-\infty}^{t} \|U(t,s)\|\, \|u(s)\|\, \mathrm{d}s \\ &\le \int_{-\infty}^{t} Me^{-\zeta(t-s)}\, M_1\, \mathrm{d}s \\ &\le \frac{MM_1}{\zeta}. \end{aligned}$$

Thus Fu is bounded. Now we show that $(Fu)(t)$ is compact almost automorphic with respect to $t \in \mathbb{R}$. Since $u \in AA_c(X)$, for any sequence $(s'_n)_{n\in\mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n\in\mathbb{N}}$ and $v \in BC(\mathbb{R}, X)$ such that

$$u(t+s_n) \to v(t) \ \text{ as } n \to \infty \text{ and } v(t-s_n) \to u(t) \ \text{ as } n \to \infty,$$

holds on compact subsets of $\mathbb{R}$. Now from Bochner's criterion of integrability of functions and the following estimate

$$\|U(t,s)u(s)\| \le Me^{-\zeta(t-s)}\|u\|_\infty, \tag{3.1}$$

we have that the function $s \to U(t,s)u(s)$ is integrable over $(-\infty, t)$, $t \in \mathbb{R}$. Further by the Lebesgue dominated convergence theorem and (3.1), it follows that $(Fu)(t+s_n)$ converges to

$$(Gv)(t) := \int_{-\infty}^{t} U(t,s)\, v(s)\, \mathrm{d}s,$$

for all $t \in \mathbb{R}$. To show that the convergence is uniform on compact subsets of $\mathbb{R}$, let $K \subset \mathbb{R}$ be an arbitrary compact set and let $\epsilon > 0$. Fix $L > 0$, such that $K \subset [\frac{-L}{2}, \frac{L}{2}]$, with $\|u(s+s_n) - v(s)\| < \epsilon$ and $\|U(t+s_n, s+s_n) - U(t,s)\| \le \epsilon$.

For each $t \in K$, one has

$$\begin{aligned} \|(Fu)(t+s_n) - (Gv)(t)\| &= \|\int_{-\infty}^{t} U(t+s_n, s+s_n)\, u(s+s_n)\, \mathrm{d}s - \int_{-\infty}^{t} U(t,s)\, v(s)\, \mathrm{d}s\| \\ &\le \int_{-\infty}^{t} \|U(t+s_n, s+s_n)\|\, \|u(s+s_n) - v(s)\|\, \mathrm{d}s \\ &\quad + \int_{-\infty}^{t} \Big(\|U(t+s_n, s+s_n) - U(t,s)\|\Big) \|v(s)\|\, \mathrm{d}s \\ &\le \int_{-\infty}^{t} M\, e^{-\zeta(t-s)}\epsilon\, \mathrm{d}s + \int_{-\infty}^{t} \epsilon\|u\|_\infty\, \mathrm{d}s\,. \end{aligned} \tag{3.2}$$

This implies that $(Fu)(t+s_n) \to (Gv)(t)$ as $n \to \infty$, uniformly on K. One can similarly prove that $(Gu)(t-s_n) \to (Fv)(t)$ as $n \to \infty$. This completes the proof of the lemma. □

Lemma 3.5 *For $u \in AA_c(X)$, the function Fu, defined by*

$$(Fu)(t) := \int_{-\infty}^{t} A(s)\, U(t,s)\, u(s)\, \mathrm{d}s,$$

is also compact almost automorphic.

Proof. It can be easily verified that, Fu is bounded in a similar way as in Lemma 3.4. Now we show that Fu is compact almost automorphic. Since u is a compact almost automorphic function on X, for a given sequence $(s'_n)_{n\in\mathbb{N}}$ of real numbers, we fix a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $v \in BC(\mathbb{R}, X)$ such that

$$u(t+s_n) \to v(t) \text{ as } n\to\infty,\ v(t-s_n) \to u(t) \text{ as } n\to\infty,$$

uniformly on compact subsets of $\mathbb{R}$. Using Bochner's criterion for integrable functions and the following estimate

$$\|A(s)U(t,s)u(s)\| \le e^{-\gamma(t-s)} H(t-s)\|u(s)\|, \tag{3.3}$$

we have the function $s \to A(s)U(t,s)u(s)$ is integrable over $(-\infty, t)$, for each $t \in \mathbb{R}$. Now, the Lebesgue dominated convergence theorem and the estimate (3.3), implies that

$$(Fu)(t+s_n) = \int_{-\infty}^{t} A(s+s_n)\, U(t+s_n, s+s_n)\, u(s+s_n)\, \mathrm{d}s \quad t\in\mathbb{R},\ n\in\mathbb{N},$$

converges to

$$Gv(t) := \int_{-\infty}^{t} A(s)\, U(t,s)\, v(s)\, \mathrm{d}s, \quad t \in \mathbb{R}.$$

Now, we show that convergence is uniform on compact subsets of $\mathbb{R}$. Let $\epsilon > 0$ and $K \subset \mathbb{R}$ be an arbitrary compact subset. Fix $L > 0$, $N_\epsilon \in \mathbb{N}$ such that $K \subset [\frac{-L}{2}, \frac{L}{2}]$ and the following holds,

$$\int_{\frac{L}{2}}^{\infty} e^{-\gamma s}\, H(s)\, \mathrm{d}s < \epsilon,$$
$$\|u(s+s_n) - v(s)\| \le \epsilon, \quad n \ge N_\epsilon,\ s\in[-L, L].$$

Here $\|u\|_\infty := \sup_{t\in\mathbb{R}} \|u(t)\|$: For each $t \in K$, we have,

$$\begin{aligned}
&\|(Fu)(t+s_n) - (Gv)(t)\| \\
&\le \int_{-\infty}^{t} \|A(s+s_n)\, U(t+s_n, s+s_n)\|\, \|u(s+s_n) - v(s)\|\, \mathrm{d}s \\
&\quad + \int_{-\infty}^{t} \Big(\|A(s+s_n)\, U(t+s_n, s+s_n)\| + \|A(s)U(t,s)\| \Big) \|v(s)\|\, \mathrm{d}s \\
&\le \int_{-\infty}^{-L} 2\, e^{-\gamma(t-s)}\, H(t-s) \left(\|u(s+s_n) - v(s)\| + \|u\|_\infty\right) \mathrm{d}s \\
&\quad + \int_{-L}^{t} 2e^{-\gamma(t-s)}\, H(t-s) \left(\|u(s+s_n) - v(s)\| + \|u\|_\infty\right) \mathrm{d}s \\
&\le \int_{t+L}^{\infty} 2\, e^{-\gamma s}\, H(s) \left(2\,\|u\|_\infty + \|u\|_\infty\right) \mathrm{d}s + \int_{0}^{t+L} 2\, e^{-\gamma s} H(s) \left(\epsilon + \|u\|_\infty\right) \mathrm{d}s \\
&\le 6\|u\|_\infty \int_{\frac{L}{2}}^{\infty} e^{-\gamma s}\, H(s)\, \mathrm{d}s + 2\,\epsilon \int_0^\infty e^{-\gamma s}\, H(s)\, \mathrm{d}s + 2\|u\|_\infty \int_{\frac{L}{2}}^{\infty} e^{-\gamma s}\, H(s)\, \mathrm{d}s \\
&\le 2\epsilon\left(4\|u\|_\infty + \int_0^\infty e^{-\gamma s}\, H(s)\, \mathrm{d}s\right).
\end{aligned}$$

This proves the convergence is uniform on K. In a similar manner one can also show that $(Gv)(t - s_n) \to (Fu)(t)$ as $n \to \infty$. This completes the proof of lemma. $\square$

Lemma 3.6 *See [3], Let $u \in AA_c(X)$, then the function $s \to u_s$ belongs to $AA_c(\mathcal{B})$.*

Proof. A proof of the lemma goes along the same line as in [3]. For the sake of convenience we give a proof here again. For a given sequence $(s'_n)_{n\in\mathbb{N}}$ of real numbers, we fix a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $v \in BC(\mathbb{R}, X)$ such that $u(s + s_n) \to v(s)$ uniformly on compact subsets of $\mathbb{R}$. Since $\mathcal{B}$ satisfies axiom $(C2)$ in proposition 7.1.1 of [12], we infer that $u_{s+s_n} \to v_s$ in $\mathcal{B}$ for each $s \in \mathbb{R}$. Let $K \subset \mathbb{R}$ be an arbitrary compact set and let $L > 0$ such that $K \subset [-L, L]$. For $\epsilon > 0$, we fix $N_{\epsilon,L} \in \mathbb{N}$ such that

$$\begin{aligned} &\|u(s + s_n) - v(s)\| \le \epsilon, \quad s \in [-L, L], \\ &\|u_{-L+s_n} - v_{-L}\| \le \epsilon, \end{aligned}$$

whenever $n \ge N_{\epsilon,L}$. $\square$

Proposition 3.7 *Under the assumptions (H1)–(H4), there exists unique almost automorphic mild solution of equations (1.1) and (1.2), whenever $\Theta < 1$, where*

$$\Theta = \left(L_f + \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\gamma(t-s)}\, H(t-s)\, L_f(s)\,\mathrm{d}s + M \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\zeta(t-s)}\, L_g(s)\ \mathrm{d}s \right) \mathfrak{L},$$

where $L_f = \sup_{t\in\mathbb{R}} L_f(t)$ and $\mathfrak{L}$ is the constant appearing in Remark 2.9.

Proof. First, we define the operator $\Gamma : AA_c(X) \to C(\mathbb{R}, X)$ by

$$\Gamma u(t) = -f(t, u_t) - \int_{-\infty}^{t} A(s)\, U(t, s)\, f(s, u_s)\,\mathrm{d}s + \int_{-\infty}^{t} U(t, s)\, g(s, u_s)\,\mathrm{d}s,\ t \in \mathbb{R}.$$

From the previous assumptions and the lemmas one can easily see that Γu is well defined and continuous. We show that the map Γ is strict contraction. For $u, v \in AA_c(X)$, we have

$$\begin{aligned} \|\Gamma u(t) - \Gamma v(t)\| &\le L_f(t)\|u_t - v_t\|_{\mathcal{B}} \\ &\quad + \int_{-\infty}^{t} L_f(s) e^{-\gamma(t-s)}\, H(t-s)\, \|u_s - v_s\|_{\mathcal{B}}\,\mathrm{d}s \\ &\quad + M \int_{-\infty}^{t} e^{-\zeta(t-s)}\, L_g(s)\|u_s - v_s\|_{\mathcal{B}}\,\mathrm{d}s \\ &\le \Big(L_f + \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\gamma(t-s)}\, H(t-s)\, L_f(s)\,\mathrm{d}s \\ &\quad + M \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\zeta(t-s)}\, L_g(s)\,\mathrm{d}s \Big) \mathfrak{L}\|u - v\|_\infty \\ &\le \Theta\, \|u - v\|_\infty. \end{aligned}$$

From our assertion, Γ is a strict contraction. By the Banach contraction theorem, Γ has a unique fixed point in $AA_c(X)$.

4 Example

In this section we present the example considered by [3], we show the existence and uniqueness of almost automorphic solution of the following partial differential equation.

$$\frac{\partial}{\partial t}\left[u(t,x)+\int_{-\infty}^{0}\int_{0}^{\pi} b(s,\eta,x)\,u(t+s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s\right]$$
$$=\frac{\partial^2}{\partial x^2}u(t,x)+a_0(t)\,u(t,x)+\int_{-\infty}^{0} a(s)\,u(t+s,x)\,\mathrm{d}s, \tag{4.1}$$
$$u(t,0)=u(t,\pi)=0, \tag{4.2}$$

for $(t,x)\in\mathbb{R}\times[0,\pi]$. □

Equations of the type (4.1) and (4.2) (see [11]), arise in control systems described by abstract retarded functional differential equations with the feedback control governed by a proportional integro-differential law. Now we introduce some related background.

Let $X=(L^2[0,\pi];\|.\|_2)$, with $\|f\|_2:=(\int_0^\pi |f(x)|^2\,\mathrm{d}x)^{\frac{1}{2}}$. Define the linear operator A by

$$D(A):=\{u\in X: u''\in X, u(0)=u(\pi)=0\},\ \text{and } Au:=u'',$$

for all $u\in D(A)$. Where A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X, such that A has a discrete spectrum with the eigenvalues of the form $-n^2,\ n\in\mathbb{N}$; whose corresponding (normalized) eigenfunctions are given by $z_n(\xi):=\sqrt{\frac{2}{\pi}}\sin(n\xi)$. Also the following properties hold true.

(a) $\{z_n : n\in\mathbb{N}\}$ is an orthonormal basis for X.

(b) For $u\in X,\ T(t)u=\sum_{n=1}^{\infty} e^{-n^2t}\langle u,z_n\rangle z_n$ and $Au=-\sum_{n=1}^{\infty} n^2\langle u,z_n\rangle z_n$ for all $u\in D(A)$.

(c) It is possible to define the fractional power $(-A)^\alpha, 0<\alpha\leq 1$ of A, as a closed linear operator over its domain $D(-A)^\alpha$. Precisely, the operator $(-A)^\alpha: D(-A)^\alpha\subseteq X\to X$ is given by $(-A)^\alpha u=\sum_{n=1}^{\infty} n^{2\alpha}\langle u,z_n\rangle z_n$, for all $u\in D((-A)^\alpha)$, where

$$D((-A)^\alpha)=\left\{u(.)\in X:\sum_{n=1}^{\infty} n^{2\alpha}\langle u,z_n\rangle z_n\in X\right\}.$$

(d) If X_α denotes the space $D((-A)^\alpha)$ endowed with the graph norm $\|.\|_\alpha$, then X_α is a Banach space. Moreover the injection $X_\alpha\to X_\beta$ is continuous for $0<\beta\leq\alpha\leq 1$ and there exist some constants $C_\alpha,\delta_\alpha>0$ such that

$$\|T(t)\|_{\mathcal{L}(X_\alpha,X)}\leq\frac{C_\alpha e^{-\delta_\alpha t}}{t^\alpha},$$

for $t>0$.

For the phase space, we choose the space $\mathcal{B}=C_r\times L^p(\rho,X),\ r\geq 0,\ 1\leq p<\infty$. Hereby we follow the terminology of [13], where g is replaced by ρ to avoid confusion with the function g

that we previously considered. This illustrates that the function $\rho : (-\infty, -r) \to \mathbb{R}$ is a positive (Lebesgue) integrable function and that there exists a nonnegative locally bounded function γ on $(-\infty, 0]$ such that $\rho(\xi + \theta) \leq \gamma(\xi)\rho(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r)\backslash N_\xi$, where $N_\xi \subset (-\infty, -r)$ is a set whose Lebesgue measure is zero. The space $C_r \times L^p(\rho, X)$ denotes the collection of functions $\phi : (-\infty, 0] \to X$, such that $\phi(.)$ is continuous on $[-r, 0]$, (Lebesgue) measurable and $\rho\|\phi\|_p{}^p$ is (Lebesgue) integrable on $(-\infty, -r)$. The seminorm on $\mathcal{B}$ is defined by

$$\|\phi\|_{\mathcal{B}} := \sup_{\theta\in[-r,0]} \|\phi(\theta)\|_2 + \left(\int_{-\infty}^{-r} \rho(\theta)\|\phi(\theta)\|_2{}^p \,\mathrm{d}\theta \right)^{\frac{1}{p}}.$$

In what follows we assume that $\rho(.)$ is a continuous function satisfying assumptions (g-5)–(g-7), Theorem 1.3.8 in [13]. Under the previous assumptions, $\mathcal{B}$ is a fading memory space, which satisfies the axioms (A), $(A1)$, (B) and $(C2)$, see Theorem 1.3.8 in [13] for details.

Let us mention that equations of type (4.1) and (4.2), arise for instance in control systems described by an abstract retarded functional differential equation with feedback control governed by the proportional integro-differential law, (see example 4.2 of [11]) for details.

Throughout the rest of this section we assume that $r = 0, p = 2$ and use the notations of the Remarks 2.9 and 2.11. In addition to the above mentioned assumptions, we suppose the following conditions hold.

(i) The functions $b(.)$, $\frac{\partial^i}{\partial\zeta^i} b(\tau, \eta, \zeta)$, $i = 1, 2$ are Lebesgue measurable with $b(\tau, \eta, \pi) = 0$, $b(\tau, \eta, 0) = 0$ for every (τ, η) and

$$N_1 := \max\left\{ \int_0^\pi \int_{-\infty}^0 \int_0^\pi \rho^{-1}(\tau)\left(\frac{\partial^i}{\partial\zeta^i} b(\tau, \eta, \zeta)\right)^2 \mathrm{d}\eta\,\mathrm{d}\tau\,\mathrm{d}\zeta : i = 0, 1 \right\} < \infty.$$

(ii) The function $a(.), i = 1, 2$ is continuous with $L = \left(\int_{-\infty}^0 \frac{a^2(s)}{\rho(s)}\,\mathrm{d}s \right)^{\frac{1}{2}} < \infty$.

Under the previous conditions, we define the operators $f, g : \mathbb{R} \times \mathcal{B} \to X$ by

$$f(t, \psi)(x) := \int_{-\infty}^0 \int_0^\pi b(s, \eta, x)\,\psi(s, \eta)\ \mathrm{d}\eta ds,$$
$$g(t, \psi)(x) := a_0(t)\psi(x) + \int_{-\infty}^0 a(s)\,\psi(s, x)\,\mathrm{d}s,$$

which enables us to transform the system (4.1) and (4.2) into the abstract system (1.1) and (1.2). Obviously f, g are continuous. Moreover, with the help of (i), it is easy to see that f has values in $Y = [D(-A)^{\frac{1}{2}}]$ and that $f(t, .) : \mathcal{B} \to X_{\frac{1}{2}}$ is a bounded linear operator with $\|f(t, .)\|_{\mathcal{L}(Y,X)} \leq \sqrt{N_1}$ for each $t \in \mathbb{R}$. It is to be noted that $Y \hookrightarrow X$ is continuously embedded. Furthermore for each $t \in \mathbb{R}$, $g(t, .)$ is a bounded linear operator with $\|g(t, .)\| \leq L$. The following result is an easy consequence of Proposition 3.7.

Theorem 4.1 *Assume the assumptions of the Proposition 3.7, that is (H1)-(H4) hold and all the conditions/assumptions, mentioned in the Example 4, hold true. Then the system (4.1) and (4.2) has a unique almost automorphic solution whenever;*

$$\Theta = \sqrt{N_1}\left(1 + 2C_{\frac{1}{2}} + \delta_{\frac{1}{2}}^{-1}\right) + L < 1.$$

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