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INTEGRAL MANIFOLDS OF NONAUTONOMOUS BOUNDARY CAUCHY PROBLEMS

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Abstract. The existence of integral manifolds for nonlinear boundary Cauchy problems is established using an extension of the variation of constants formula recently established in [4]. Examples include nonautonomous structured population equations and nonautonomous retarded differential equations.

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1 Introduction

Consider the nonlinear boundary Cauchy problem for arbitrary $\tau \in \mathbb{R}_+ = [0, \infty)$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t) = A_{\max}(t)u(t), \quad t \in [\tau, \infty), \\ L(t)u(t) = f(t, u(t)), \quad t \in [\tau, \infty), \\ u(\tau) = x, \end{cases} \tag{1.1}$$

where $A_{\max}(t)$ is a closed operator on a Banach space X endowed with a maximal domain $D(A_{\max}(t))$, and $L(t) : D(A_{\max}(t)) \to \partial X$, with a 'boundary space' ∂X and a function $f : \mathbb{R}_+ \times X \to \partial X$, the solution $u : [\tau, \infty) \to X$ takes the initial value $x \in X$ at time τ. This type of equation has recently been suggested and investigated as a model class with various applications like population equations, retarded differential (difference) equations, heat equations and boundary control problems (see e.g. [1, 3] and the references therein). The corresponding linear boundary Cauchy problem of (1.1) is given by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t) = A_{\max}(t)u(t), \quad t \in [\tau, \infty), \\ L(t)u(t) = 0, \quad t \in [\tau, \infty), \\ u(\tau) = x. \end{cases} \tag{1.2}$$

In the autonomous case these abstract Cauchy problems were first studied by Greiner [6, 7, 8] and Thieme [13], e.g. by using perturbation results for the domains of semigroups.

The homogeneous boundary Cauchy problem (1.2) has been investigated by Kellermann [9] and Nguyen Lan [10]. In these papers, the authors proved the existence of solutions to these problems and generation of an evolution family.

In [4] the authors have studied the boundary Cauchy problem in the case that the first equation in (1.1) is replaced by an inhomogeneous equation $\frac{\mathrm{d}}{\mathrm{d}t} u(t) = A_{\max}(t)u(t) + g(t)$ and f in the second equation is replaced by $f(t, u(t)) \equiv f(t)$. For this type of equation they established a variation of constants formula which can be easily extended to a variation of constants formula for (1.1) using the contraction fixed point theorem. Utilizing the variation of constants formula we will follow the Lyapunov-Perron approach to develop an invariant manifold theory for the class of equations (1.1).

The structure of the paper is as follows: In Section 2 we list natural assumptions for well-posedness of equation (1.1), the concepts of mild solution and exponential splitting. Moreover, we cite two examples illustrating our abstract problem and general assumptions. Section 3 is devoted to an invariant manifold theorem for (1.1) which yields sufficient conditions for the existence of e.g. a stable or unstable manifold.

To conclude the introductory section, we collect notation used in this paper. For Banach spaces X, Y, let $\mathcal{L}(X, Y)$ denote the space of all linear bounded operators from X to Y, define $\mathcal{L}(X) := \mathcal{L}(X, X)$. We denote by id_X the identity map defined on X.

By $C_b(\mathbb{R}_+, X)$ we denote the space of all continuous and bounded functions from $\mathbb{R}_+$ into X.

Let $A : D(A) \subset X \to X$ be a closed linear operator, we denote by

$$\rho(A) := \{\lambda \in \mathbb{C} \mid \lambda\,\mathrm{id}_X - A : D(A) \to X \text{ is bijective}\}$$

the *resolvent set* of A. For $\lambda \in \rho(A)$, the operator $R(\lambda, A) := (\lambda\,\mathrm{id}_X - A)^{-1}$ is called the *resolvent* of A.

Finally, for a measurable set $\Omega \subset \mathbb{R}^n$ and $1 \leq p < \infty$, let $L^p(\Omega)$ denote the space of all measurable functions from Ω to $\mathbb{R}^n$ satisfying that

$$\|u\|_p := \left(\int_\Omega |u(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

Let $L^\infty(\Omega)$ denote the space of all essentially bounded measurable functions. The Sobolev space $W^{1,1}(\Omega)$ is given by

$$W^{1,1}(\Omega) = \left\{ u \in L^1(\Omega) \mid u' \in L^1(\Omega) \right\},$$

where the derivative u' is defined in the weak sense. Let $W^{1,1}(\Omega)$ be endowed with the norm

$$\|u\|_{W^{1,1}} := \|u\|_1 + \|u'\|_1.$$

2 Preliminaries

In this section we recall some definitions and results, formulate assumptions and discuss some examples.

2.1 Linear nonautonomous boundary Cauchy problems

A family of linear (unbounded) operators $(A(t))_{0 \leq t \leq T}$ defined on a Banach space X is called a *stable family* if there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A(t))$ for all $0 \leq t \leq T$ and

$$\Big\| \prod_{i=1}^{k} R(\lambda, A(t_i)) \Big\| \leq M(\lambda - \omega)^{-k}$$

for $\lambda > \omega$ and any finite sequence $0 \leq t_1 \leq \cdots \leq t_k \leq T$.

Remark 1 *In the autonomous case ($A(t) = A$), suppose that A generates a strongly continuous semigroup. Then, by the Hille-Yosida Theorem (see [5, Theorem II.3.8]) A is stable.*

A family of linear bounded operators $(U(t,s))_{t \geq s \in J}$, $J := \mathbb{R}_+$ or $\mathbb{R}$, on a Banach space X is called *evolution family* if

(1) $U(t,s) = U(t,r)U(r,s)$ and $U(s,s) = \mathrm{id}_X$ for all $t \geq r \geq s \in J$,

(2) the mapping $\{(t,s) \in J \times J \, : t \geq s\} \ni (t,s) \mapsto U(t,s) \in \mathcal{L}(X)$ is strongly continuous.

The *growth bound* of $(U(t,s))_{t \geq s \geq 0}$ is defined by

$$\omega(U) := \inf \left\{ \omega \in \mathbb{R} : \exists \, M_\omega \geq 1 \text{ with } \|U(t,s)\| \leq M_\omega e^{\omega(t-s)} \; \forall \, t \geq s \in J \right\}.$$

The evolution family $(U(t,s))_{t \geq s \geq 0}$ is called *exponentially bounded* provided that $\omega(U) < \infty$. We now turn to the notion of exponential splitting for an evolution family.

Definition 2 (Exponential Splitting) *An evolution family $(U(t,s))_{t\geq s\geq 0}$ on a Banach space X has an* exponential splitting *with exponents $\alpha<\beta$, if there exist projections $P(t), t\in\mathbb{R}_+$, being uniformly bounded and strongly continuous and a constant $N\geq 1$ such that*

(1) $P(t)U(t,s)=U(t,s)P(s)$ *for all $t\geq s\geq 0$,*

(2) the restriction $U_Q(t,s): Q(s)X\to Q(t)X$ is invertible for $t\geq s\geq 0$ and we set $U(s,t):=[U_Q(t,s)]^{-1}$, where $Q(t):=\mathrm{id}_X-P(t)$.

(3) $\|U(t,s)P(s)\|\leq Ne^{\alpha(t-s)}$ *and* $\|[U_Q(t,s)Q(s)]^{-1}\|\leq Ne^{-\beta(t-s)}$ *for all $t\geq s\geq 0$.*

Let $X, D, \partial X$ be Banach spaces such that D is dense and continuously embedded in X. On these spaces, the operators $A_{\max}(t)\in\mathcal{L}(D,X), L(t)\in\mathcal{L}(D,\partial X)$, for $t\geq 0$, are supposed to satisfy the following hypotheses:

(H1) There are positive constants C_1, C_2 such that

$$C_1\|x\|_D\leq\|x\|+\|A_{\max}(t)x\|\leq C_2\|x\|_D$$

for all $x\in D$ and $t\geq 0$;

(H2) for each $x\in D$ the mapping $\mathbb{R}_+\ni t\mapsto A_{\max}(t)x\in X$ is continuously differentiable;

(H3) the operators $L(t): D\to\partial X, t\geq 0$, are surjective;

(H4) for each $x\in D$ the mapping $\mathbb{R}_+\ni t\mapsto L(t)x\in\partial X$ is continuously differentiable;

(H5) there exist constants $\gamma>0$ and $\omega\in\mathbb{R}$ such that

$$||L(t)x||_{\partial X}\geq\gamma^{-1}(\lambda-\omega)||x||_X,$$

for $x\in\ker(\lambda\,\mathrm{id}_X-A_{\max}(t)), \lambda>\omega$ and $t\geq 0$;

(H6) the family of operators $(A(t))_{t\geq 0}$, $A(t):=A_{\max}(t)|_{\ker L(t)}$, generates an evolution family $(U(t,s))_{t\geq s\geq 0}$.

In the following lemma, we cite consequences of the above assumptions from [6, Lemma 1.2] which will be needed below.

Lemma 3 *The restriction $L(t)|_{\ker(\lambda\,\mathrm{id}_X-A_{\max}(t))}$ is an isomorphism from $\ker(\lambda\,\mathrm{id}_X-A_{\max}(t))$ into ∂X and its inverse $L_{\lambda,t}:=[L(t)|_{\ker(\lambda\,\mathrm{id}_X-A_{\max}(t))}]^{-1}:\partial X\to\ker(\lambda\,\mathrm{id}_X-A_{\max}(t))$ satisfies*

$$\|L_{\lambda,t}\|\leq\gamma(\lambda-\omega)^{-1}\quad\textit{for } \lambda>\omega \textit{ and } t\geq 0.$$

To illustrate sufficient and natural conditions which imply the assumptions (H1)–(H6) for application-relevant classes of boundary Cauchy problems we discuss examples of a nonautonomous structured population equation and a nonautonomous functional differential equation.

Example 4 (Nonautonomous Structured Population Equation) *Consider a nonautonomous population equation*

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,a,x) = -\dfrac{\partial}{\partial a}u(t,a,x) - \mu(a,x)u(t,a,x) + A(a,x)u(t,a,x), \\ \hfill t \geq s, a \geq 0, x \in \Omega, \\ u(t,0,x) = \displaystyle\int_0^\infty \beta(t,a,x)u(t,a,x)\,\mathrm{d}a, \quad t \geq s, x \in \Omega, \\ u(s,a,x) = f(a,x), \quad a \geq 0, x \in \Omega. \end{cases} \tag{2.1}$$

Here Ω is a bounded domain in $\mathbb{R}^n$, the function $u(t,a,x)$ represents the density of individuals of the population of age a and size x at time t. The functions μ and β correspond to the aging and the birth rates, respectively. Finally, we note that this equation is a special case of the very general nonautonomous population equation with diffusion treated by Rhandi and Schnaubelt in [12].

We impose the following conditions:

(i) $A(a,\cdot) \in L^\infty(\Omega)$ *for all* $a \geq 0$ *and* $A(\cdot,\cdot) \in C_b(\mathbb{R}_+, L^\infty(\Omega))$. *Moreover,* $(A(a,\cdot))_{a\geq 0}$ *is a family of operators generating an exponentially bounded evolution family* $U(a,r)_{a\geq r\geq 0}$ *on the Banach space* $L^1(\Omega)$.

(ii) $0 \leq \mu \in C_b(\mathbb{R}_+, L^\infty(\Omega))$.

(iii) $0 \leq \beta \in C^1(\mathbb{R}_+, L^\infty(\mathbb{R}_+ \times \Omega) \cap L^1(\mathbb{R}_+ \times \Omega))$ *the space of continuously differentiable functions from* $\mathbb{R}_+$ *into* $L^\infty(\mathbb{R}_+ \times \Omega) \cap L^1(\mathbb{R}_+ \times \Omega)$.

Our aim is to write equation (2.1) *as a boundary Cauchy problem of the form* (1.2) *satisfying the hypotheses (H1)–(H6). For this purpose, we define the Banach spaces*

$$\partial X := L^1(\Omega),\ X := L^1(\mathbb{R}_+, \partial X) \simeq L^1(\mathbb{R}_+ \times \Omega) \text{ and } D := W^{1,1}(\mathbb{R}_+, \partial X),$$

and for each $t \geq 0$ *the operator* $A_{\max}(t) : X \to X$ *by* $D(A_{\max}(t)) = D$ *and*

$$(A_{\max}(t)\varphi)(a) = -\frac{\partial}{\partial a}\varphi(a,\cdot) + B(a,\cdot)\varphi(a,\cdot) \tag{2.2}$$

for all $\varphi \in D$, *where*

$$B(a,\cdot)\varphi(a,\cdot) := A(a,\cdot)\varphi(a,\cdot) - \mu(a,\cdot)\varphi(a,\cdot). \tag{2.3}$$

For each $t \geq 0$, *we define* $L(t) : D \longrightarrow \partial X$ *by*

$$L(t)\varphi = \varphi(0,\cdot) - \Phi(t)\varphi \quad \text{for all } \varphi \in D, \tag{2.4}$$

where $\Phi(t) : X \to \partial X$ *given by*

$$\Phi(t)\varphi := \int_0^\infty \beta(t,a,\cdot)\varphi(a,\cdot)\,\mathrm{d}a.$$

It is obvious to see that $\Phi(t) \in \mathcal{L}(X, \partial X)$. *We show now that the hypotheses (H1)–(H6) are satisfied:*

Verification of (H1): Since $A(\cdot) \in C_b(\mathbb{R}_+, L^\infty(\Omega))$ and $\mu(\cdot) \in C_b(\mathbb{R}_+, L^\infty(\Omega))$ it follows that

$$A_\infty := \sup_{a\geq 0} \|A(a)\|_\infty < \infty \quad \textit{and} \quad \mu_\infty := \sup_{a\geq 0} \|\mu(a)\|_\infty < \infty.$$

Let $\varphi \in D$ be arbitrary. From the definition of $\|\cdot\|_D$, we have

$$\begin{aligned} \|\varphi\|_D &= \int_0^\infty \|\varphi(a)\|_1 \,\mathrm{d}a + \int_0^\infty \|\varphi'(a)\|_1 \,\mathrm{d}a \\ &\leq \|\varphi\|_X + \|A_{\max}(t)\varphi\|_X + \int_0^{+\infty} \|A(a)\varphi(a) - \mu(a)\varphi(a)\|_1 \,\mathrm{d}a \\ &\leq (1 + A_\infty + \mu_\infty)(\|\varphi\|_X + \|A_{\max}(t)\varphi\|_X). \end{aligned}$$

On the other hand,

$$\begin{aligned} \|\varphi\|_X + \|A_{\max}(t)\varphi\|_X &= \|\varphi\|_X + \|\varphi' + A(\cdot)\varphi - \mu(\cdot)\varphi\|_X \\ &\leq (1 + A_\infty + \mu_\infty)\|\varphi\|_D. \end{aligned}$$

This shows the assumption (H1) with $C_1 = (1 + A_\infty + \mu_\infty)^{-1}$ and $C_2 = (1 + A_\infty + \mu_\infty)$.

Verification of (H2): From (2.2)*, we derive that $A_{\max}(t)$ is independent of t. Therefore, the map $t \mapsto A_{\max}(t)\varphi$ is continuously differentiable for each fixed $\varphi \in X$.*

Verification of (H3): See Appendix, Lemma 11.

Verification of (H4): From continuous differentiability of β, we derive that for each $\varphi \in D$ the mapping from $\mathbb{R}_+ \to \partial X, t \mapsto L(t)\varphi$ defined as in (2.4) *is also continuously differentiable.*

Verification of (H5): Define a family of linear operators $(C(a))_{a\geq 0}$ on ∂X by

$$C(a)\varphi := -\mu(a, \cdot)\varphi.$$

From (ii), we have $\mu \in C_b(\mathbb{R}_+, L^\infty(\Omega))$ and therefore

$$\sup_{a\in\mathbb{R}_+} \|C(a)\|_1 \leq \sup_{a\in\mathbb{R}_+} \|\mu(a, \cdot)\|_\infty < \infty,$$

which together with (i) implies that the family of operators $(B(a, \cdot))_{a\geq 0}$, given by (2.4)*, generates on ∂X an exponentially bounded evolution family $(V(t, s))_{t\geq s\geq 0}$ given by*

$$V(t, s)\varphi = U(t, s)\varphi + \int_s^t U(t, \sigma)C(\sigma)V(\sigma, s)\,\mathrm{d}\sigma,$$

for all $t \geq s \geq 0$ and $\varphi \in \partial X$. Then (H5) is an application of Appendix, Lemma 12.

Verification of (H6): The corresponding evolution semigroup $(T(t))_{t\geq 0}$ of the evolution family $(V(a, r))_{a\geq r\geq 0}$ is given by

$$(T(t)\varphi)(a) = \begin{cases} V(a, a-t)\varphi(a-t) & a \geq t, \\ 0, & a < t, \end{cases} \tag{2.5}$$

for $\varphi \in X$. One can show that $(T(t))_{t\geq 0}$ is a strongly continuous semigroup. Its generator denoted by A_0 is a restriction of $A_{\max}(t)$ with

$$D(A_0) = \{\varphi \in D |\ \varphi(0) = 0\}.$$

Thus, according to Remark 1 we obtain that A_0 is stable. Since $A(t) := A_{\max}(t)|_{\ker L(t)}$ is a bounded perturbation of A_0 with

$$A(t)\varphi = A_0\varphi, \quad D(A(t)) = \{\varphi \in D |\ \varphi(0) = \Phi(t)\varphi\}$$

it follows together with [10, Theorem 2.3] that $A(t)$ generates an evolution family. Hence, (H6) is satisfied.

Example 5 (Nonautonomous functional differential equation)

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x(t) = B(t)x(t), \quad t \geq s \geq 0, \\ x_s = \varphi \in C([-r,0],E). \end{cases} \tag{2.6}$$

Here $B(t)$ is defined on a Banach space E. Furthermore, $r \geq 0$, $\varphi \in C([-r,0],E)$ and the retarded function x_s is defined as $x_s(\tau) := x(s+\tau)$ for $\tau \in [-r,0]$. We assume the following conditions:

(i) The family of linear operators $B(t), t \geq 0$, is stable and generates an evolution family $(U(t,s))_{t\geq s\geq 0}$ satisfying

$$\|U(t,s)\| \leq Me^{\omega(t-s)}, \quad t \geq s \geq 0;$$

(ii) the domain $D(B(t)) := D_B$ is independent of t, $B(0)$ is a closed operator in E and the function $t \mapsto B(t)x$ is continuously differentiable for all $x \in E$.

Define the Banach spaces

$$X := C([-r,0],E), \quad \partial X := E$$

and

$$D := \left\{\varphi \in C^1([-r,0],E) \text{ such that } \varphi(0) \in D_B\right\}$$

endowed with the norm $|\varphi| := \|\varphi\|_{C^1([-r,0],E])} + \|B(0)\varphi(0)\|$.

For each $t \geq 0$, we define the operator $A_{\max}(t) : X \to X$ with $D(A_{\max}) = D$ by

$$A_{\max}(t)\varphi = \frac{\partial}{\partial x}\varphi \quad \text{for all } \varphi \in D,$$

and the operator $L(t) : D(A_{\max}(t)) \longrightarrow \partial X$ by

$$L(t)\varphi = \varphi'(0) - B(t)\varphi(0) \quad \text{for all } \varphi \in D.$$

Then, the above retarded differential equation (2.6) can be written as a linear boundary Cauchy problem (1.2). For more details, we refer the reader to [10].

2.2 Nonlinear boundary Cauchy problems

In case $f \equiv 0$ the boundary Cauchy problem (1.1) reduces to the linear boundary Cauchy problem (1.2) which was studied in the last subsection under the assumptions (H1)–(H6). In particular, let $(U(t,s))_{t\geq s\geq 0}$ denote the evolution family from (H6). We want to study nonlinear perturbations (1.1) of (1.2) and therefore assume that the nonlinearity f is not too far away from 0:

(H7) The nonlinear part $f : \mathbb{R}_+ \times X \to \partial X$ is assumed to be continuous, satisfies that $f(t,0) = 0$ for all $t \in \mathbb{R}_+$ and there exists a positive constant ℓ such that one has the global Lipschitz estimate

$$\|f(t,x) - f(t,\bar{x})\| \leq \ell \|x - \bar{x}\| \quad \text{for all } x, \bar{x} \in X, t \in \mathbb{R}_+.$$

Under the assumptions (H1)–(H7) the semilinear boundary Cauchy problem (1.1) admits a unique mild solution. For $\tau \in \mathbb{R}_+$, $x \in X$, a function $u = u(\cdot,\tau,x) : [\tau,\infty) \to X$ is called *mild solution of* (1.1) if it satisfies the integral equation

$$u(t,\tau,x) = U(t,\tau)x + \lim_{\lambda\to\infty} \int_\tau^\infty U(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma, u(\sigma,\tau,x))\,\mathrm{d}\sigma, \quad t \geq \tau. \tag{2.7}$$

The unique existence follows with the usual contraction arguments (see e.g. [3, 7, 11]) and uses the *variation of constants formula* from [4] for solutions $v : [\tau,\infty) \to X$ of inhomogeneous boundary Cauchy problems, i.e. systems (1.1) with $f(t,u(t)) \equiv g(t)$ independent of $u(t)$

$$v(t) = U(t,\tau)x + \lim_{\lambda\to\infty} \int_\tau^\infty U(t,\sigma)\lambda L_{\lambda,\sigma} g(\sigma)\,\mathrm{d}\sigma, \quad t \geq \tau,$$

where $L_{\lambda,\sigma}$ is defined as in Lemma 3.

3 Integral Manifolds of Nonlinear Boundary Cauchy Problems

In this section, we consider the following system

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t) = A_{\max}(t)u(t), & t \in [0,\infty), \\ L(t)u(t) = f(t,u(t)), & t \in [0,\infty), \end{cases} \tag{3.1}$$

where $A_{\max}(t), L(t), f(t,x)$ are assumed to satisfy assumptions (H1)–(H6). For $\tau \in \mathbb{R}_+$ and $x \in X$ let $u(\cdot,\tau,x)$ denote the mild solution of (3.1) satisfying that $u(\tau) = x$. In case the evolution family $(U(t,s))_{t\geq s\geq 0}$ of the corresponding linear boundary Cauchy problem has an exponential splitting with exponents $\alpha < \beta$, projections $P(\cdot)$, $Q(\cdot) = I - P(\cdot)$ and constant N then for all $\zeta \in (\alpha,\beta)$ the two sets

$$\begin{aligned} \mathcal{M}_\zeta^s &= \Big\{(\tau,\xi) \in \mathbb{R}_+ \times X : \tau \in \mathbb{R}_+, \xi \in P(\tau)X\Big\}, \\ \mathcal{M}_\zeta^u &= \Big\{(\tau,\xi) \in \mathbb{R}_+ \times X : \tau \in \mathbb{R}_+, \xi \in Q(\tau)X\Big\}, \end{aligned}$$

are called *(pseudo)-stable* and *(pseudo)-unstable vector bundles* or *manifolds*, they consist of solutions which are exponentially bounded from above and below, respectively, in the sense of Definition 2. In case $\alpha < 0 < \beta$ they are called *stable* and *unstable*, respectively.

Our aim in this section is to construct nonlinear analogues of $\mathcal{M}_\zeta^s$ and $\mathcal{M}_\zeta^u$ by using the Lyapunov-Perron approach as e.g. in [2]. Since solution curves are sometimes also called *integral curves* and because $\mathcal{M}_\zeta^s$ and $\mathcal{M}_\zeta^u$ are invariant manifolds, i.e. consist of solution curves, they are also called *integral manifolds*.

3.1 Pseudo-Stable Manifolds

Consider the nonlinear boundary Cauchy problem (3.1) which satisfies additionally assumption (H7). By (H7) equation (3.1) has the zero solution. We show that for each fixed $\tau \in \mathbb{R}_+$ the set of mild solutions $\varphi \in C([\tau, \infty), X)$ of (3.1) which converge to zero exponentially fast as $t \to \infty$ forms a so-called stable manifold which is the graph of a Lipschitz-continuous chart. In fact, we prove more generally, that sets of solutions which are exponentially bounded by $Ne^{\zeta(t-\tau)}$ for $t \geq \tau$ form manifolds, so-called ζ-pseudo-stable manifolds. To this end choose $\zeta \in \mathbb{R}, \tau \in \mathbb{R}_+$. Then the set

$$\mathcal{X}^+_{\tau,\zeta}(X) := \left\{ \varphi \in C([\tau, \infty), X) : \sup_{t \geq \tau} e^{\zeta(\tau - t)} \|\varphi(t)\| < \infty \right\}$$

is a Banach space with respect to the norm

$$\|\varphi\|_{\tau,\zeta} := \sup_{t \geq \tau} e^{\zeta(\tau-t)} \|\varphi(t)\|.$$

Our overall approach is to characterize stable manifolds as a fixed point problem in $\mathcal{X}^+_{\tau,\zeta}(X)$. Thereto we define the *Lyapunov-Perron Operators* $\mathcal{T}^+ : \mathcal{X}^+_{\tau,\zeta}(X) \times X \to \mathcal{X}^+_{\tau,\zeta}(X)$ by

$$\mathcal{T}^+(\varphi, x)(t) := U(t,\tau)P(\tau)x + \lim_{\lambda \to \infty} \int_\tau^\infty G(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma, \varphi(\sigma))\, \mathrm{d}\sigma, \tag{3.2}$$

where the *Green's function* G is defined by

$$G(t,s) := \begin{cases} U(t,s)P(s) & \text{for } t \geq s, \\ -U(t,s)Q(s) & \text{for } t \leq s. \end{cases}$$

We also write $\mathcal{T}^+(t, \varphi, x) = \mathcal{T}^+(\varphi, x)(t)$ for all $(t, x) \in [\tau, \infty) \times X$. Some fundamental properties of the operator $\mathcal{T}^+$ are established in the following proposition.

Proposition 6 *Suppose that the evolution family $(U(t,s))_{t \geq s \geq 0}$ has an exponential splitting with exponents $\alpha < \beta$, projections $P(\cdot)$ and constant N. Then for all $\tau \in [0, \infty)$ the following assertions hold:*

(i) For any $\zeta \in (\alpha, \beta)$, the Lyapunov-Perron operator $\mathcal{T}^+ : \mathcal{X}^+_{\zeta,\tau}(X) \times X \to \mathcal{X}^+_{\zeta,\tau}(X)$ defined as in (3.2) *is well-defined.*

(ii) For any $\zeta \in (\alpha, \beta)$, let $\varphi \in \mathcal{X}^+_{\tau,\zeta}(X)$ and $\xi \in \operatorname{im} P(\tau)$. Then, the following statements are equivalent:

(a) φ is the mild solution of (3.1) *with $P(\tau)\varphi(\tau) = \xi$,*

(b) φ is the fixed point of the Lyapunov-Perron operator $\mathcal{T}^+(\cdot, \xi) : \mathcal{X}^+_{\tau,\zeta}(X) \to \mathcal{X}^+_{\tau,\zeta}(X)$ defined as in (3.2).

(iii) Suppose that $N\ell\gamma < \frac{\beta-\alpha}{2}$ and choose and fix $\eta \in (\frac{N\ell\gamma}{2}, \frac{\beta-\alpha}{2})$. Then, for any $\zeta \in [\alpha+\eta, \beta-\eta]$ the Lyapunov-Perron operator is uniformly contractive in the first component. More precisely, for all $\varphi_1, \varphi_2 \in \mathcal{X}^+_{\tau,\zeta}(X)$ and $\xi_1, \xi_2 \in X$ we have

$$\|\mathcal{T}^+(\cdot, \varphi_1, \xi_1) - \mathcal{T}^+(\cdot, \varphi_2, \xi_2)\|_{\tau,\zeta} \leq N\|P(\tau)(\xi_1 - \xi_2)\| + \frac{2N\ell\gamma}{\eta}\|\varphi_1 - \varphi_2\|_{\tau,\zeta}. \tag{3.3}$$

Proof. (i) Let $\varphi \in \mathcal{X}^+_{\tau,\zeta}(X)$ and $\xi \in X$. An elementary computation yields that for all $t \geq \tau$

$$e^{-\zeta(t-\tau)}\|\mathcal{T}^+(t,\varphi,\xi)\| \leq N\|P(\tau)\xi\| + N\ell\gamma\left(\frac{1}{\zeta-\alpha}+\frac{1}{\beta-\zeta}\right)\|\varphi\|_{\tau,\zeta},$$

where we use the fact that $\lim_{\lambda\to\infty}\|\lambda L_{\lambda,\sigma}\| \leq \gamma$, see Lemma 3. Therefore, the operator $\mathcal{T}^+$ is well-defined.

(ii) (a) $\Rightarrow$ (b): Since φ is a mild solution of (3.1) it follows that

$$\varphi(t) = U(t,\tau)\varphi(\tau) + \lim_{\lambda\to\infty}\int_\tau^\infty U(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma,\varphi(\sigma))\,\mathrm{d}\sigma.$$

Together with $P(\tau)\varphi(\tau) = \xi$ we get

$$\begin{aligned}\varphi(t) \;&=\; U(t,\tau)\xi + \lim_{\lambda\to\infty}\int_\tau^t U(t,\sigma)P(\sigma)\lambda L_{\lambda,\sigma} f(\sigma,\varphi(\sigma))\,\mathrm{d}\sigma + \\ &\quad U(t,\tau)Q(\tau)\left(\varphi(\tau) + \lim_{\lambda\to\infty}\int_\tau^t U(\tau,\sigma)Q(\sigma)\lambda L_{\lambda,\sigma} f(\sigma,\varphi(\sigma))\,\mathrm{d}\sigma\right).\end{aligned}$$

Hence, $\varphi \in \mathcal{X}^+_{\tau,\zeta}$ with $\zeta \in (\alpha,\beta)$ implies that

$$\varphi(\tau) + \lim_{\lambda\to\infty}\int_\tau^\infty U(\tau,\sigma)Q(\sigma)\lambda L_{\lambda,\sigma} f(\sigma,\varphi(\sigma))\,\mathrm{d}\sigma = 0,$$

which concludes that $\varphi = \mathcal{T}^+(\varphi,\xi)$ and the first implication is proved.

(b) $\Rightarrow$ (a): Since φ is the fixed point of $\mathcal{T}^+(\cdot,\xi)$ it follows that

$$\varphi(t) = U(t,\tau)P(\tau)\xi + \lim_{\lambda\to\infty}\int_\tau^\infty G(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma,\varphi(\sigma))\,\mathrm{d}\sigma.$$

Replacing t by τ in the above equality yields that

$$\varphi(\tau) = \xi - \lim_{\lambda\to\infty}\int_\tau^\infty U(\tau,\sigma)Q(\sigma)\lambda L_{\lambda,\sigma} f(\sigma,\varphi(\sigma))\,\mathrm{d}\sigma.$$

Therefore, we get

$$\varphi(t) = U(t,\tau)\varphi(\tau) + \lim_{\lambda\to\infty}\int_\tau^t U(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma,\varphi(\sigma))\,\mathrm{d}\sigma,$$

which completes the proof of this part.

(iii) From (3.2), we derive that

$$\begin{aligned}\mathcal{T}^+(t,\varphi_1,\xi_1) - \mathcal{T}^+(t,\varphi_2,\xi_2) \;&=\; U(t,\tau)P(\tau)(\xi_1-\xi_2) + \\ &\quad + \lim_{\lambda\to\infty}\int_\tau^\infty G(t,\sigma)\lambda L_{\lambda,\sigma}(f(\sigma,\varphi_1(\sigma)) - f(\sigma,\varphi_2(\sigma)))\,\mathrm{d}\sigma.\end{aligned}$$

This together with the fact that the evolution family $(U(t,s))_{t\geq s\geq 0}$ has an exponential splitting with exponents $\alpha < \beta$, projections $P(\cdot)$ and constant N and the Lipschitz continuity of f implies that

$$\begin{aligned}e^{\zeta(\tau-t)}\|\mathcal{T}^+(t,\varphi_1,\xi) - \mathcal{T}^+(t,\varphi_2,\xi)\| \;&\leq\; N\|P(\tau)(\xi_1-\xi_2)\| + \\ &\quad + N\ell\gamma\left[\int_\tau^t e^{\eta(\sigma-t)}\,\mathrm{d}\sigma + \int_t^\infty e^{\eta(t-\sigma)}\,\mathrm{d}\sigma\right]\|\varphi_1-\varphi_2\|_{\tau,\xi}.\end{aligned}$$

Therefore,

$$\sup_{t\geq\tau} e^{\zeta(\tau-t)}\|\mathcal{T}^+(t,\varphi_1,\xi)-\mathcal{T}^+(t,\varphi_2,\xi)\| \leq N\|P(\tau)(\xi_1-\xi_2)\| + \frac{2N\ell\gamma}{\eta}\|\varphi_1-\varphi_2\|_{\tau,\xi},$$

which completes the proof. □

Definition 7 (Pseudo-Stable Manifolds) *For any $\zeta\in\mathbb{R}$, the ζ-*pseudo stable manifold *is defined as*

$$\mathcal{W}^s_\zeta := \Big\{(\tau,x)\in\mathbb{R}_+\times X : \varphi\in\mathcal{X}^+_{\tau,\zeta}(X) \text{ is mild sol. of (3.1) and } P(\tau)\varphi(\tau)=x\Big\}.$$

We are now ready to state the main result on the existence of pseudo-stable manifolds for nonlinear boundary Cauchy problems.

Theorem 8 (Pseudo-stable Manifold Theorem) *Assume that* (3.1) *satisfies the assumptions (H1)–(H7) and suppose that the corresponding evolution family $(U(t,s))_{t\geq s\geq 0}$ has an exponential splitting with exponents $\alpha<\beta$, projections $P(\cdot)$ and constant N. Furthermore, we assume that $N\ell\gamma<\beta-\alpha$. Choose and fix $\eta\in(\frac{N\ell\gamma}{2},\frac{\beta-\alpha}{2})$. Then for any $\zeta\in[\alpha+\eta,\beta-\eta]$, the ζ-pseudo stable manifold $\mathcal{W}^s_\zeta$ has the following representation*

$$\mathcal{W}^s_\zeta = \Big\{(\tau,\xi+s^+(\tau,\xi))\in\mathbb{R}_+\times X : \tau\in\mathbb{R}_+,\xi\in P(\tau)X\Big\}, \tag{3.4}$$

with for each $\tau\in\mathbb{R}_+$ the uniquely determined continuous mapping $s^+(\tau,\cdot):P(\tau)X\to X$ given by

$$s^+(\tau,\xi) = \lim_{\lambda\to\infty}\int_\tau^\infty G(\tau,\sigma)\lambda L_{\lambda,\sigma} f(\sigma,\varphi(\sigma))\,\mathrm{d}\sigma, \tag{3.5}$$

where φ is the unique fixed point of $\mathcal{T}^+(\cdot,\xi)$. Furthermore, for each $\tau\in\mathbb{R}_+$ the function $s^+(\tau,\cdot)$ satisfies

$$s^+(\tau,0)\equiv 0 \quad \textit{and} \quad \textit{Lip}(s^+(\tau,\cdot)) \leq \frac{N^2\ell\gamma}{\eta-2N\ell\gamma}.$$

Proof. Let $(\tau,x)\in\mathcal{W}^s_\zeta$, where $\tau\in\mathbb{R}_+$ and $x\in X$. Define $\xi=P(\tau)x$. According to Definition 7 and Proposition 6(ii), we obtain that the mild solution φ of (3.1) with $P(\tau)\varphi(\tau)=x$ is the unique fixed point of the Lyapunov-Perron operator $\mathcal{T}^+(\cdot,\xi)$. Therefore,

$$x = \xi + \lim_{\lambda\to\infty}\int_\tau^\infty G(\tau,\sigma)\lambda L_{\lambda,\sigma} f(\sigma,\varphi(\sigma))\,\mathrm{d}\sigma.$$

Then, $x=\xi+s^+(\tau,\xi)$. Conversely, let $\xi\in P(\tau)X$, where $\tau\in\mathbb{R}_+$. We will show that $\xi+s^+(\tau,\xi)\in\mathcal{W}^s_\zeta$. In light of Proposition 6 (iii), the Lyapunov-Perron operator $\mathcal{T}^+(\cdot,\xi)$ is contractive and thus has a unique fixed point in $\mathcal{X}^+_{\tau,\xi}(X)$ denoted by φ. This together with Proposition 6 (ii) implies that φ is a solution of (3.1) and therefore

$$\varphi(\tau) = \xi + \lim_{\lambda\to\infty}\int_\tau^\infty G(\tau,\sigma)\lambda L_{\lambda,\sigma} f(\sigma,\varphi(\sigma))\,\mathrm{d}\sigma = \xi + s^+(\tau,\xi),$$

which proves (3.4). Since 0 is the fixed point of $\mathcal{T}^+(\cdot,0)$ it follows that $s^+(\tau,0)=0$ for all $\tau\in\mathbb{R}_+$. To conclude the proof, we prove the Lipschitz continuity of s with respect to the second argument. Let $\xi_1, \xi_2 \in P(\tau)(X)$ for a $\tau \in \mathbb{R}_+$. Let $\varphi_1, \varphi_2 \in \mathcal{X}^+_{\tau,\zeta}(X)$ denote the fixed point of $\mathcal{T}^+(\cdot, \xi_1), \mathcal{T}^+(\cdot, \xi_2)$, respectively. Using (3.3), we obtain that

$$\|\varphi_1 - \varphi_2\|_{\tau,\zeta} \le N\|\xi_1 - \xi_2\| + \frac{2N\ell\gamma}{\eta}\|\varphi_1 - \varphi_2\|_{\tau,\zeta}.$$

Therefore,

$$\|\varphi_1 - \varphi_2\|_{\tau,\zeta} \le \frac{N\eta}{\eta - 2N\ell\gamma}\|\xi_1 - \xi_2\|.$$

This together with (3.5) implies that

$$\|s^+(\tau, \xi_1) - s^+(\tau, \xi_2)\| \le \frac{N^2\ell\gamma}{\eta - 2N\ell\gamma}\|\xi_1 - \xi_2\|,$$

which completes the proof. □

3.2 Pseudo-Unstable Manifolds

In order to provide the definition of pseudo-unstable manifolds, we introduce the following space: for a given $\zeta \in \mathbb{R}, \tau \in \mathbb{R}_+$, the set

$$\mathcal{X}^-_{\tau,\zeta}(X) := \left\{\varphi \in C((-\infty, \tau], X) : \sup_{t \le \tau} e^{\zeta(\tau - t)}\|\varphi(t)\| < \infty\right\}$$

is a Banach space with respect to the norm

$$\|\varphi\|_{\tau,\zeta} := \sup_{t \le \tau} e^{\zeta(\tau - t)}\|\varphi(t)\|.$$

Definition 9 (Pseudo-unstable Manifolds) *For any $\zeta \in \mathbb{R}$, the ζ-*pseudo unstable manifold *$\mathcal{W}^u_\zeta$ is the set of all $(\tau, x) \in \mathbb{R}_+ \times X$ satisfying the following conditions:*

(i) For any $t \le \tau$, there exists a $y \in X$ (and hence unique due to uniqueness of solution) which is denoted by $u(t, \tau, x)$ such that $u(\tau, t, y) = x$.

(ii) $u(\cdot, \tau, x) \in \mathcal{X}^-_{\tau,\zeta}(X)$.

The existence of pseudo-unstable manifold for nonlinear boundary Cauchy problem is stated and proved in the following theorem.

Theorem 10 (Pseudo-unstable Manifold Theorem) *Suppose that the evolution family $(U(t, s))_{t \ge s \ge 0}$ associated with the corresponding linear system of* (3.1) *has an exponential splitting with exponents $\alpha < \beta$, projections $P(\cdot)$ and constant N and the nonlinear part satisfies (H7). Furthermore, we assume that $N\ell\gamma < \beta - \alpha$. Choose and fix $\eta \in (\frac{N\ell\gamma}{2}, \frac{\beta - \alpha}{2})$. Then for any $\zeta \in [\alpha + \eta, \beta - \eta]$, the ζ-pseudo unstable manifold $\mathcal{W}^u_\zeta$ has the following presentation*

$$\mathcal{W}^u_\zeta = \left\{(\tau, \xi + s^-(\tau, \xi)) \in \mathbb{R}_+ \times X : \tau \in \mathbb{R}_+, \xi \in Q(\tau)X\right\},$$

with for each $\tau \in \mathbb{R}_+$ *the uniquely determined continuous mapping* $s^-(\tau,\cdot) : Q(\tau)X \to X$ *given by*

$$s^-(\tau,\xi) = \lim_{\lambda\to\infty} \int_{-\infty}^{\tau} G(\tau,\sigma)\lambda L_{\lambda,\sigma} f(\sigma,\varphi(\sigma))\,\mathrm{d}\sigma,$$

where φ *is the unique fixed point of* $\mathcal{T}^-(\cdot,\xi)$. *Furthermore, for each* $\tau \in \mathbb{R}_+$ *the function* $s^-(\tau,\cdot)$ *satisfies*

$$s^-(\tau,0) \equiv 0 \quad \textit{and} \quad \textit{Lip}(s^-(\tau,\cdot)) \le \frac{N^2\ell\gamma}{\eta - 2N\ell\gamma}.$$

Proof. Analog to the proof of Theorem 8. □

4 Appendix

Let Ω be a bounded set of $\mathbb{R}^n$ and $\beta : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ satisfying that

$$\operatorname*{esssup}_{(a,x)\in\mathbb{R}_+\times\Omega} \beta(a,x) < \infty \quad \text{and} \quad \int_{\mathbb{R}_+\times\Omega} \beta(a,x)\,\mathrm{d}a\,\mathrm{d}x < \infty. \tag{4.1}$$

Set

$$\partial X := L^1(\Omega),\ X := L^1(\mathbb{R}_+,\partial X) \quad \text{ and } D := W^{1,1}(\mathbb{R}_+,\partial X).$$

Define the linear operator $L : D \to \partial X$ by

$$Lu(x) := u(0,x) - \int_0^\infty \beta(a,x)u(a,x)\,\mathrm{d}a \qquad \text{for all } x \in \Omega. \tag{4.2}$$

In the following lemma, we state and prove some fundamental properties of the operator L.

Lemma 11 *The operator* L *defined as in* (4.2) *is bounded and surjective.*

Proof. We first show the boundedness of L. Since $\varphi(0,\cdot) = -\int_0^\infty \frac{\partial}{\partial a}\varphi(a,\cdot)\,\mathrm{d}a$ for all $\varphi \in D$ it follows that

$$\begin{aligned} \|L(t)\varphi\|_1 &= \left\|\varphi(0,\cdot) - \int_0^\infty \beta(a,\cdot)\varphi(a,\cdot)\,\mathrm{d}a\right\|_1 \\ &\le \|\varphi(0,\cdot)\|_1 + \left\|\int_0^\infty \beta(a,\cdot)\varphi(a,\cdot)\,\mathrm{d}a\right\|_1 \\ &\le (1+\|\beta\|_\infty)\|\varphi\|_{W^{1,1}}. \end{aligned}$$

To prove the surjectivity of L, let $f \in \partial X$ be arbitrary. Define

$$u(a,x) := \frac{2f(x)}{1+e^{-2\int_0^\infty \beta(t,x)\,\mathrm{d}t}} e^{-2\int_0^a \beta(t,x)\,\mathrm{d}t} \qquad \text{for all } (a,x) \in \mathbb{R}_+\times\Omega.$$

We have $u \in D$. Furthermore, from (4.2) it is easy to see that $Lu = f$ and therefore L is surjective. □

Lemma 12 *Let $(B(a))_{a\geq 0}$ be a family of operators on ∂X which generates an evolution family $(V(a,r))_{a\geq r\geq 0}$ with a growth bound $\omega(V) < +\infty$. Define an operator $A_{\max} : X \to X$ by*

$$(A_{\max}\varphi)(a) = -\frac{\partial}{\partial a}\varphi(a) + B(a)\varphi(a) \tag{4.3}$$

with the domain

$$D(A_{\max}) = \{\varphi \in D : \varphi(a) \in D(B(a)) \text{ for a.e. } a \in \mathbb{R}_+,\ B(\cdot)\varphi(\cdot) \in X\}.$$

Then, the following statements hold:

(i) For all $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega(V)$, we have

$$\ker(\lambda\,\mathrm{id}_X - A_{\max}) = \left\{e^{-\lambda\cdot}V(\cdot,0)f |\ f \in \partial X\right\}.$$

(ii) There exist constants $\gamma > 0$ and $\widetilde{\omega} \in \mathbb{R}$ such that for $\lambda \in \mathbb{C}$ with $\Re\lambda > \widetilde{\omega}$ we have

$$||L\varphi||_{\partial X} \geq \gamma^{-1}(\Re\lambda - \widetilde{\omega})||\varphi||_X \quad \text{for all } \varphi \in \ker(\lambda\,\mathrm{id}_X - A_{\max}).$$

Proof. (i) See [12].

(ii) Since the evolution family $(V(a,r))_{a\geq r\geq 0}$ is exponentially bounded with the growth bound $\omega(V)$, it follows that for each $\omega > \omega(V)$ there exists $M_\omega \geq 1$ such that

$$\|V(a,r)\| \leq M_\omega e^{\omega(a-r)} \quad \text{for all } a \geq r \geq 0. \tag{4.4}$$

Take $\lambda \in \mathbb{C}$ such that $\Re\lambda > \omega + M_\omega\|\beta\|_\infty$ and $\varphi \in \ker(\lambda\,\mathrm{id}_X - A_{\max})$. From part (i), we get that $\varphi(\cdot) = e^{-\lambda\cdot}V(\cdot,0)\varphi(0)$. This together with (4.4) implies that

$$\begin{aligned}\|\varphi\|_X &= \int_0^\infty \|e^{-\lambda a}V(a,0)\varphi(0)\|_{\partial X}\,\mathrm{d}a\\ &\leq \int_0^\infty M_\omega e^{(\omega-\Re\lambda)a}\|\varphi(0)\|_{\partial X}\,\mathrm{d}a\\ &= \frac{M_\omega}{\Re\lambda-\omega}\|\varphi(0)\|_{\partial X}.\end{aligned} \tag{4.5}$$

On the other hand, we have

$$\begin{aligned}\|\varphi(0)\|_{\partial X} &\leq \left\|\varphi(0) - \int_0^\infty \beta(a,\cdot)\varphi(a)\,\mathrm{d}a\right\|_{\partial X} + \left\|\int_0^\infty \beta(a,\cdot)\varphi(a)\,\mathrm{d}a\right\|_{\partial X}\\ &\leq \|L\varphi\|_{\partial X} + \|\beta\|_\infty\|\varphi\|_X,\end{aligned}$$

which together with (4.5) implies that

$$\|\varphi\|_X \leq \frac{M_\omega}{\Re\lambda - (\omega + M_\omega\|\beta\|_\infty)}\|L\varphi\|_{\partial X},$$

which proves the lemma with $\gamma = M_\omega$ and $\widetilde{\omega} = \omega + M_\omega\|\beta\|_\infty$. □

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